Math 142 Problem Set 7

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A) Problems on Reviewing Self-Adjoint Linear Maps

THEOREM. *Let $A: V \to V$ be a self-adjoint linear map. Then there exists an orthonormal basis $\{e_1, e_2\}$ of V such that $A(e_1) = \lambda_1 e_1$, $A(e_2) = \lambda_2 e_2$ (that is, e_1 and e_2 are eigenvectors, and λ_1, λ_2 are eigenvalues of A). In the basis $\{e_1, e_2\}$, the matrix of A is clearly diagonal and the elements λ_1, λ_2, $\lambda_1 \geq \lambda_2$, on the diagonal are the maximum and the minimum, respectively, of the quadratic form $Q(v) = \langle Av, v\rangle$ on the unit circle of V.*

Consider a quadratic form $Q(v) = \langle Av, v\rangle$. An earlier proposition proved there exists an orthonormal basis $\{e_1, e_2\}$ of V w/ $Q(e_1) = \lambda_1$, $Q(e_2) = \lambda_2 \leq \lambda_1$. λ_1, λ_2 are the max and min, respectively, of Q in the unit circle. Now, we show e_1, e_2 are the eigenvalues of A:

Since $B(e_1, e_2) = \langle Ae_1, e_2\rangle = 0$ and $e_2 \neq 0$, we have Ae_1 is either parallel to e_1 or $Ae_1 = 0$. If Ae_1 is parallel to e_1, then $Ae_1 = \alpha e_1$, and since $\langle Ae_1, e_1\rangle = \lambda_1 = \langle \alpha e_1, e_1\rangle = \alpha$, we conclude $Ae_1 = \lambda_1 e_1$. Similarly, we can prove $A_2 e_2 = \lambda_2 e_2$. ∎

B) Problems for Course Materials

Surfaces are often expressed as graphs of differentiable functions $z = h(x,y)$, $(x,y) \in U \subset \mathbb{R}^2$. We parameterize the surface as

$$X(u,v) = (u, v, h(u,v)), \quad (u,v) \in U \qquad u = x,\ v = y.$$

We see that

$$X_u = (1, 0, h_u) \quad X_v = (0, 1, h_v) \quad X_{uu} = (0, 0, h_{uu}) \quad X_{uv} = (0, 0, h_{uv}) \quad X_{vv} = (0, 0, h_{vv})$$

Thus $$N(x,y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}}$$ is a unit normal on the surface.

The coefs. of the second fundamental form are given by

$$e = \frac{h_{xx}}{(1+h_x^2+h_y^2)^{1/2}} \qquad f = \frac{h_{xy}}{(1+h_x^2+h_y^2)^{1/2}} \qquad g = \frac{h_{yy}}{(1+h_x^2+h_y^2)^{1/2}}$$

From these, we can compute

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1+h_x^2+h_y^2)^2} \qquad H = \frac{(1+h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1+h_y^2)h_{xx}}{(1+h_x^2+h_y^2)^{3/2}}$$

The second fundamental form of S at p applied to $(x,y) \in \mathbb{R}^2$ becomes

$$\mathrm{II}_p = \left.\begin{matrix} h_{xx}(0,0)x^2 + 2h_{xy}(0,0)xy + h_{yy}(0,0)y^2 \end{matrix}\right\} \text{Hessian of } h \text{ at } (0,0)$$

Now let's apply this to the Dupin Indicatrix. Let $\varepsilon > 0$ such that

$$C = \{ (x,y) \in T_p(S) \,;\, h(x,y) = \varepsilon \} \text{ is a regular curve.}$$

We want to show that if p is not a planar point, the curve C is "approximately" similar to the Dupin indicatrix.

Assume x and y axes are along the principal directions, w/ the x axis along the direction of maximum principal curvature. Thus, $f = h_{xy}(0,0) = 0$ and

$$k_1(p) = \frac{e}{E} = h_{xx}(0,0) \qquad k_2(p) = \frac{g}{G} = h_{yy}(0,0)$$

Taylor expand $h(x,y)$ around $(0,0)$ and note that $h_x(0,0) = 0 = h_y(0,0)$:

$$h(x,y) = \frac{1}{2}\left(h_{xx}(0,0)x^2 + h_{xy}(0,0)xy + h_{yy}(0,0)y^2 \right) + R \qquad \left[\lim_{(x,y)\to(0,0)} \frac{R}{x^2+y^2} = 0 \right]$$

$$= \frac{1}{2}\left(k_1 x^2 + k_2 y^2 \right) + R$$

Thus, C is given by $k_1 x^2 + k_2 y^2 + 2R = 2\varepsilon$.

If p is not planar, we can consider $k_1 x^2 + k_2 y^2 = 2\varepsilon$ as a first-order approximation of C. Now, let

$$x = \bar{x}\sqrt{2\varepsilon} \qquad y = \bar{y}\sqrt{2\varepsilon},$$

we have $k_1 \bar{x}^2 + k_2 \bar{y}^2 = 1 \implies$ Dupin indicatrix at p.

So, if p is a nonplanar point, the intersection with S of a plane parallel to $T_p(S)$ and close to p is, in a first-order approximation, a curve similar to the Dupin indicatrix at p.

C) Other Problems

a. Problem 2, p. 151

2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

Let $N_p(S)$ be the normal of a point $p \in T_p(S)$ of a regular surface S. Denote the curve as $\alpha(t)$ in the intersection of S and the plane $T_p(S)$. For any $\alpha(t_0)$, the tangent plane stays the same $\Rightarrow dN(\alpha(t_0))_{\alpha'(t_0)} = 0$. Now let

$a = \dfrac{\alpha''(t_0)}{|\alpha''(t_0)|}$, a unit vector in $T_{\alpha(t_0)}(S)$ perpendicular to $\alpha'(t_0)$. Then let $b = dN(\alpha(t_0))_a$.

The gaussian curvature is $|dN(\alpha(t_0))| = 0 \Rightarrow$ curve is parabolic or planar.

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is constant.

Let x_u, x_v be an orthonormal basis for $T_p(S)$. Then let $a = a_1 x_u + a_2 x_v$, $b = b_1 x_u + b_2 x_v$ be orthogonal. The sum of their normal curvatures is:

$-\langle dN_p(a), a\rangle - \langle dN_p(b), b\rangle$

$= -\langle dN_p(a_1 x_u + a_2 x_v), a_1 x_u + a_2 x_v\rangle - \langle dN_p(b_1 x_u + b_2 x_v), b_1 x_u + b_2 x_v\rangle$

$= -a_1^2 \langle dN_p(x_u), x_u\rangle - a_1 a_2 \langle dN_p(x_u), x_v\rangle$

$- a_1 a_2 \langle dN_p(x_v), x_u\rangle - a_2^2 \langle dN_p(x_v), x_v\rangle$

$- b_1^2 \langle dN_p(x_u), x_u\rangle - b_1 b_2 \langle dN_p(x_u), x_v\rangle$

$b_1 b_2 \langle dN_p(x_v), x_u\rangle - b_2^2 \langle dN_p(x_v), x_v\rangle$

$\langle dN_p(x_u), x_v\rangle = \langle x_u, dN_p(x_v)\rangle$

$= -(a_1^2 + b_1^2)\langle dN_p(x_u), x_u\rangle - 2(a_1 a_2 + b_1 b_2)\langle dN_p(x_u), x_v\rangle - (a_2^2 + b_2^2)\langle dN_p(x_v), x_v\rangle$

$= -(a_1^2 + a_2^2)\langle dN_p(x_u), x_u\rangle - 2\cancel{(a_1 a_2 - a_2 a_1)}\langle dN_p(x_u), x_v\rangle - (a_2^2 + a_1^2)\langle dN_p(x_v), x_v\rangle$

$= -(1 \cdot \langle dN_p(x_u), x_u\rangle + 1 \cdot \langle dN_p(x_v), x_v\rangle)$

$= \boxed{-(\langle dN_p(x_u), x_u\rangle + \langle dN_p(x_v), x_v\rangle)}$

8. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:

a. Paraboloid of revolution $z = x^2 + y^2$.

b. Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.

c. Catenoid $x^2 + y^2 = \cosh^2 z$.

a) Parameterize the surface as $x(u,v) = (u, v, u^2+v^2)$

$x_u = (1, 0, 2u) \qquad x_v = (0, 1, 2v)$

$$x_u \wedge x_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1) \qquad |x_u \wedge x_v| = \sqrt{4u^2+4v^2+1}$$

$$\boxed{N(q) = \frac{\langle -2u, -2v, 1\rangle}{\sqrt{4u^2+4v^2+1}}} \Rightarrow \text{The top hemisphere}$$

b) Parameterize the surface as $x(u,v) = (u, v, \sqrt{u^2+v^2-1})$

$x_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2-1}}\right) \qquad x_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2-1}}\right)$

$$x_u \wedge x_v = \begin{vmatrix} i & j & j \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2-1}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2-1}} \end{vmatrix} = \left(-\frac{u}{\sqrt{u^2+v^2-1}}, -\frac{v}{\sqrt{u^2+v^2-1}}, 1\right)$$

$\Downarrow$

$\boxed{\text{The top hemisphere}}$

c) Parameterize the surface as ¯_(ツ)_/¯ ♡

17. Show that if $H \equiv 0$ on S and S has no planar points, then the Gauss map $N: S \longrightarrow S^2$ has the following property:

$$\langle dN_p(w_1), dN_p(w_2)\rangle = -K(p)\langle w_1, w_2\rangle$$

for all $p \in S$ and all $w_1, w_2 \in T_p(S)$. Show that the above condition implies that the angle of two intersecting curves on S^2 and the angle of their spherical images (cf. Exercise 9) are equal up to a sign.

If $H=0$, $k_1=-k_2$ $\quad K = k_1 k_2$ $\quad dN_p \neq 0$ $\quad \forall p \in S$ $\quad K = -k_1^2 \Rightarrow -K = k_1^2$

If we express w_1, w_2 in the basis $\{e_1, e_2\}$ where $dN_p(e_1) = k_1 e_1$ and $dN_p(e_2) = k_2 e_2$ (eigen basis of dN_p), we get

$$w_1 = a_1 e_1 + a_2 b_2 \qquad W_2 = b_1 e_1 + b_2 e_2$$

$$\langle dN_p(w_1), dN_p(w_2)\rangle = \langle dN_p(a_1 e_1 + a_2 b_2), dN_p(b_1 e_1 + b_2 e_2)\rangle$$

$$= \langle a_1 k_1 e_1 + a_2 k_2 e_2, b_1 k_1 e_1 + b_2 k_2 e_2\rangle$$

$$= \langle a_1 k_1 e_1 + a_2 k_2 e_2, b_1 k_1 e_1\rangle + \langle a_1 k_1 e_1 + a_2 k_2 e_2, b_2 k_2 e_2\rangle$$

$$= \langle a_1 k_1 e_1, b_1 k_1 e_1\rangle + \underbrace{\langle a_2 k_2 \cancel{e_2}, b_1 k_1 e_1\rangle}_{=0} + \underbrace{\langle a_1 k_1 \cancel{e_1}, b_2 k_2 e_2\rangle}_{=0} + \langle a_2 k_2 e_2, b_2 k_2 e_2\rangle$$

$$= k_1^2\langle a_1 e_1, b_1 e_1\rangle + k_2^2\langle a_2 e_2, b_2 e_2\rangle$$

$$= k_1^2 a_1 b_1 + k_2^2 a_2 b_2 = k^2(a_1 b_1 + a_2 b_2) = k_1^2\langle w_1, w_2\rangle = \boxed{-K\langle w_1, w_2\rangle}$$