Math 143 Problem Set 1

Problem 2a, sample final exam)

$$A=\begin{bmatrix}1&0&-4\\0&5&4\\-4&4&3\end{bmatrix} \qquad |A-\lambda I|=0 \implies \begin{vmatrix}1-\lambda&0&-4\\0&5-\lambda&4\\-4&4&3-\lambda\end{vmatrix} = -\lambda^3+9\lambda^2+9\lambda-81=0$$

$$\implies \boxed{\lambda_1=3,\ \lambda_2=9,\ \lambda_3=-3}$$

Find $\vec{v}_1$: $(A-\lambda I)\vec{v}=\vec{0}$

$$A-3I=\begin{bmatrix}-2&0&-4\\0&2&4\\-4&4&0\end{bmatrix} \xrightarrow[-\frac{1}{2}R_1\to R_1]{-2R_1+R_3\to R_3} \begin{bmatrix}1&0&2\\0&2&4\\0&4&8\end{bmatrix} \xrightarrow[\frac{1}{2}R_2\to R_2]{-2R_2+R_3\to R_3} \begin{bmatrix}1&0&2\\0&1&2\\0&0&0\end{bmatrix}$$

$v_{11}+2v_{13}=0 \implies v_{11}=-2v_{13}$

$v_{12}+2v_{13}=0 \implies v_{12}=-2v_{13}$

$$\vec{v}_1=v_{13}\begin{bmatrix}-2\\-2\\1\end{bmatrix}=\boxed{\begin{bmatrix}-2\\-2\\1\end{bmatrix}}$$

Find $\vec{v}_2$:

$$A-9I=\begin{bmatrix}-8&0&-4\\0&-4&4\\-4&4&-6\end{bmatrix} \xrightarrow[-\frac{1}{8}R_1\to R_1]{-\frac{1}{2}R_1+R_3\to R_3} \begin{bmatrix}1&0&\frac{1}{2}\\0&-4&4\\0&4&-4\end{bmatrix} \xrightarrow[-\frac{1}{4}R_2\to R_2]{R_2+R_3\to R_3} \begin{bmatrix}1&0&\frac{1}{2}\\0&1&-1\\0&0&0\end{bmatrix}$$

$v_{22}=v_{23} \qquad v_{21}=-\frac{1}{2}v_{23} \implies \vec{v}_2=v_{23}\begin{bmatrix}-1/2\\1\\1\end{bmatrix} \implies \boxed{\vec{v}_2=\begin{bmatrix}1\\-2\\-2\end{bmatrix}}$

Find $\vec{v}_3$:

$$A+3I=\begin{bmatrix}4&0&-4\\0&8&4\\-4&4&6\end{bmatrix} \xrightarrow[\frac{1}{4}R_1\to R_1]{R_1+R_3\to R_3} \begin{bmatrix}1&0&-1\\0&8&4\\0&4&2\end{bmatrix} \xrightarrow[\frac{1}{8}R_2\to R_2]{-\frac{1}{2}R_2+R_3\to R_3} \begin{bmatrix}1&0&-1\\0&1&1/2\\0&0&0\end{bmatrix}$$

$v_{31}=v_{33} \qquad v_{32}=-\frac{1}{2}v_{33} \qquad \boxed{\vec{v}_3=v_{33}\begin{bmatrix}1\\-\frac{1}{2}\\1\end{bmatrix}=\begin{bmatrix}-2\\1\\-2\end{bmatrix}}$

Problem 3a, sample final exam:

$$L\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2a-c\\a+b-c\\c\end{bmatrix}$$

$$\boxed{L=\begin{array}{c}\begin{array}{ccc}2&t&t^2\end{array}\\ \begin{bmatrix}2&0&-1\\1&1&-1\\0&0&1\end{bmatrix}\end{array}}$$

Problem 7a, Sample final exam:

Prove that similar matrices have the same eigenvalues.

Let A, B be similar matrices. That is, $\exists$ an invertible matrix P such that $B = P^{-1}AP$.

So, $B - \lambda I = P^{-1}AP - \lambda I \Rightarrow |B - \lambda I| = |P^{-1}AP - \lambda I|$

$$= |P^{-1}AP - \lambda P^{-1} I P|$$

$$= |P^{-1}(A - \lambda I)P| = |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |A - \lambda I| \quad \left(|P^{-1}| = \frac{1}{|P|}\right)$$

So, $|B - \lambda I| = |A - \lambda I| \Rightarrow A, B$ have the same characteristic polynomials. Therefore, they must have the same eigenvalues.

Problem 10, sample final exam:

Define $c_{ij} = \langle v_i, v_j \rangle$ where $S = \{v_1, \dots, v_n\}$ is an ordered set making a basis for a vector space V. Then define $C = [c_{ij}]$ and

$$\langle v, w \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j = \vec{v}^T C \vec{w}$$

$\Rightarrow$) Assume this defines an inner product on V. Then 4 assumptions are satisfied:

1) $\langle v, v \rangle \geq 0$, $= 0$ only if $v = 0$
2) $\langle v, w \rangle = \langle w, v \rangle$
3) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4) $\langle cv, w \rangle = c \langle v, w \rangle$

So, applying property 1, note that

$$\langle v, v \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i c_{ij} v_j > 0 \quad \text{for } v \text{ nonzero}$$

Note we can equivalently represent this as

$$\vec{v}^T C \vec{v} > 0 \Rightarrow C \text{ is positive definite} \checkmark$$

$\Leftarrow$) Assume C is positive definite. Then

1) $\langle v, v \rangle = \vec{v}^T C \vec{v}$. Since C is positive definite, $\vec{v}^T C \vec{v} \geq 0$, and equal only if $\vec{v} = \vec{0}$.

2) Since C is symmetric,

$$\langle v,w\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} b_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ji} b_j = \sum_{i=1}^{n} b_j c_{ji} a_i = \vec{w}^T C\vec{v} = \langle w,v\rangle \checkmark$$

3) Let $x = d_1 v_1 + d_2 v_2 + \dots + d_n v_n$. Then

$$\langle v+x,w\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} (a_i+d_i) c_{ij} b_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} b_j + \sum_{i=1}^{n}\sum_{j=1}^{n} d_i c_{ij} b_j$$

$$= \langle v,w\rangle + \langle x,w\rangle \quad \checkmark$$

4) Let $k\in\mathbb{R}$. Then

$$\langle kv,w\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} k a_i c_{ij} b_j = k\sum_{i=1}^{n}\sum_{j=1}^{n} a_i c_{ij} b_j = k\langle v,w\rangle.$$

Therefore, $\langle v,w\rangle$ is an inner product.

Thus, we see that $\langle v,w\rangle$ is an inner product if and only if C is positive definite. ■

Problem 16b, sample final exam:

Let $A = PBP^{-1}$ and $k\in\mathbb{Z}^+$. We proceed with a proof by induction.

1) Base case: $n=1$

$A^1 = PB^1P^{-1}$ ✓

2) Assume true for $n=k$

$A^k = PB^kP^{-1}$

3) Show for $n=k+1$

$$A^{k+1} = A^k A = (PB^kP^{-1})(PBP^{-1}) = PB^k(P^{-1}P)BP^{-1} = PB^k I BP^{-1} = PB^k B P^{-1}$$

$$= \boxed{PB^{k+1}P^{-1}}$$

Problem 19a, sample final exam:

$q(\vec{x}) = 3x_1^2 - 3x_2^2 - 3x_3^2 + 4x_2x_3$

$q(\vec{x}) = \vec{x}^T A\vec{x}$, $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & -3 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -3-\lambda & 2 \\ 0 & 2 & -3-\lambda \end{vmatrix} = (-\lambda+3)(\lambda^2+6\lambda+5) = -\lambda^3-6\lambda^2-5\lambda + 3\lambda^2+18\lambda+15$$

$$= -\lambda^3 - 3\lambda^2 + 13\lambda + 15 = 0$$

$\lambda_1 = -5 \qquad \lambda_2 = -1 \qquad \lambda_3 = 3$

<u>Find $\vec{v}_1$:</u>

$$A-\lambda_1 I = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} v_{11} = 0 \\ v_{12} = -v_{13} \end{matrix} \qquad \vec{v}_1 = \begin{bmatrix} 0 \\ -v_{13} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} v_{13}$$

<u>Find $\vec{v}_2$:</u>

$$A-\lambda_2 I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} v_{21} = 0 \\ v_{22} = v_{23} \end{matrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v_{22}$$

<u>Find $\vec{v}_3$:</u>

$$A-\lambda_3 I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_2 \to R_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -6 \end{bmatrix} \xrightarrow{-2R_2+R_3 \to R_3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

$-2v_{33} = 0 \qquad v_{32} = 2v_{33} \Rightarrow v_{32} = 0 \qquad \vec{v}_3 = v_{31}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Apply Gram-Schmidt to orthonormalize $\{v_1, v_2, v_3\} \to \{u_1, u_2, u_3\}$

$P = [u_1\ u_2\ u_3]$

So, $P^TAP = D = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} A [u_1\ u_2\ u_3] = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} [\lambda_1 u_1\ \ \lambda_2 u_2\ \ \lambda_3 u_3]$

$$= \begin{bmatrix} \lambda u_1^T u_1 & 0 & 0 \\ 0 & \lambda_2 u_2^T u_2 & 0 \\ 0 & 0 & \lambda_3 u_3^T u_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

So, quadratic form of D:

$\vec{x}^T P^T D P \vec{x} \to$ let $\vec{y} = P\vec{x}$

$\vec{y}^T D \vec{y} = g(\vec{y})$

The quadratic form is $g(\vec{y}) = \vec{y}^T D \vec{y}$ with $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$$\boxed{\operatorname{rank}(g) = 3, \quad \operatorname{signature}(g) = 1 - 2 = -1}$$