Math 143 Problem Set 2

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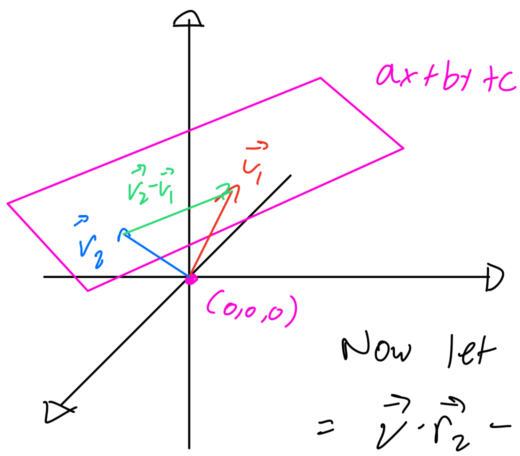
A) Cross Product Review

***2.** A plane P contained in R^3 is given by the equation $ax + by + cz + d = 0$. Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.



Choose 2 vectors in the plane

$\vec{r}_1 = \langle x_1, y_1, z_1 \rangle$

$\vec{r}_2 = \langle x_2, y_2, z_2 \rangle$

Let $\vec{r} = \vec{r}_2 - \vec{r}_1 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, which lies parallel to the plane.

Now let $\vec{v} = \langle a, b, c \rangle$. Note tha $\vec{v} \cdot \vec{r} = \vec{v} \cdot (\vec{r}_2 - \vec{r}_1)$

$$= \vec{v} \cdot \vec{r}_2 - \vec{v} \cdot \vec{r}_1 = ax_1 + by_1 + cz_1 - ax_2 - by_2 - cz_2$$

$$= -d - (-d) = \cancel{d} + d = 0$$

Therefore, $\vec{v} \perp$ the plane.

$$\min_{(x,y,z)} \|(x,y,z)\| \quad \text{s.t. } ax + by + cz - d = 0$$

$$\min_{(x,y,z,\lambda)} L = \sqrt{x^2+y^2+z^2} - \lambda(ax + by + cz - d)$$

$$\frac{\partial L}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \lambda a \qquad \frac{\partial L}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \lambda b \qquad \frac{\partial L}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \lambda c$$

$$x = \lambda a\sqrt{x^2+y^2+z^2} \qquad y = \lambda a\sqrt{x^2+y^2+z^2} \qquad z = \lambda c\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial L}{\partial \lambda} = -ax - by - cz + d \longrightarrow ax + by + cz = d$$

$$\lambda\sqrt{x^2+y^2+z^2}\,(a^2+b^2+c^2) = d$$

$$\lambda = \frac{d}{(a^2+b^2+c^2)\sqrt{x^2+y^2+z^2}} \longrightarrow x = \frac{ad}{a^2+b^2+c^2} \qquad y = \frac{bd}{a^2+b^2+c^2} \qquad z = \frac{cd}{a^2+b^2+c^2}$$

$$\|(x,y,z)\| = \sqrt{\frac{a^2d^2}{(a^2+b^2+c^2)^2} + \frac{b^2d^2}{(a^2+b^2+c^2)^2} + \frac{c^2d^2}{(a^2+b^2+c^2)^2}} = \frac{|d|}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}$$

$$= \boxed{\frac{d}{\sqrt{a^2+b^2+c^2}}}$$

5. Show that the equation of a plane passing through three noncolinear points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ is given by

$$(p - p_1) \wedge (p - p_2) \cdot (p - p_3) = 0,$$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1$, for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

$\Leftarrow$) Assume the equation is true. We now show p is in the plane. Let the vectors $(p-p_1)$, $(p-p_2)$ and $(p-p_3)$ in the plane be the columns of a matrix D. Then note that

$$|D| = |(p-p_1) \ (p-p_2) \ (p-p_3)| = (p-p_1) \wedge (p-p_2) \cdot (p-p_3) = 0 \quad \text{(by given formula)}$$

Since $\det D = 0$, these vectors are not linearly independent. Therefore, they are coplanar. ✓

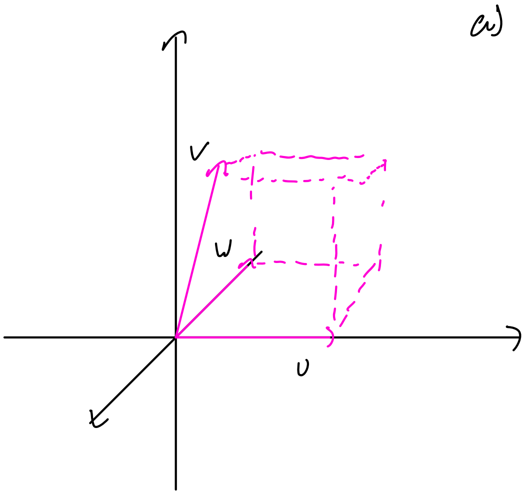
$\Rightarrow$) Assume p_1, p_2, p_3, p are in the plane. Then $(p-p_1)$, $(p-p_2)$, and $(p-p_3)$ are coplanar. Therefore, $\det(D) = 0$. But,

$$\det(D) = |(p-p_1) \ (p-p_2) \ (p-p_3)| = (p-p_1) \wedge (p-p_2) \cdot (p-p_3) = 0 \quad ✓ \blacksquare$$

11. a. Show that the volume V of a parallelepiped generated by three linearly independent vectors $u, v, w \in R^3$ is given by $V = |(u \wedge v) \cdot w|$, and introduce an *oriented volume* in R^3.

b. Prove that

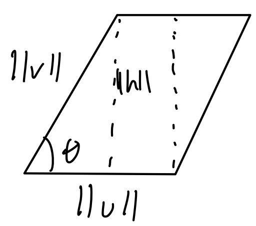
$$V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix}.$$



a) Consider the parallelogram formed by $\vec{u}, \vec{v}$.
The area is $\|u\| \|v\| \sin\theta = \|\vec{u} \times \vec{v}\|$.
The volume is then $\|\vec{u} \times \vec{v}\|$ multiplied by the depth, which is the component of w perpendicular to the parallelogram. The length of this component is

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = A \|\vec{w}\| \cos\theta = V ✓.$$

b) Since $V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$, let $D = [\vec{u} \ \vec{v} \ \vec{w}]$.
Then $|\operatorname{Det}(D)| = |[\vec{u} \ \vec{v} \ \vec{w}]| = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = V$.



$\sin\theta = \|H\| / \|v\|$

$\|h\| = \|v\| \sin\theta$

$\Rightarrow A = \|v\| \|u\| \sin\theta$

$A = \|\vec{u} \wedge \vec{v}\|$

Since $V = \det D$,

$$VV = V^2 = \det(D) \det(D) = \det(D^2)$$

$$D^2 = \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ \vec{w}^T \end{bmatrix} [\vec{u} \ \vec{v} \ \vec{w}] = \begin{bmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{bmatrix}$$

So, $\boxed{V^2 = \det(D^2) = \begin{vmatrix} u\cdot u & u\cdot v & u\cdot w \\ v\cdot u & v\cdot v & v\cdot w \\ w\cdot u & w\cdot v & w\cdot w \end{vmatrix}}$ ■

13. Let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval (a, b) into R^3. If the derivatives $u'(t)$ and $v'(t)$ satisfy the conditions

$$u'(t) = au(t) + bv(t), \qquad v'(t) = cu(t) - av(t),$$

where a, b, and c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

To show $u(t) \wedge v(t)$ is constant, we can show $(u(t) \wedge v(t))' = 0$.

$u'(t) \wedge v(t) + u(t) \wedge v'(t)$

$= (au(t) + bv(t)) \wedge v(t) + u(t) \wedge (cu(t) - av(t))$

$= a\,u(t) \wedge v(t) + \underbrace{b\,v(t) \wedge v(t)}_{=0} + \underbrace{c\,u(t) \wedge u(t)}_{=0} - a\,u(t) \wedge v(t)$

$= a(u(t) \wedge v(t) - u(t) \wedge v(t)) = a \cdot 0 = \boxed{0}$

Therefore, $u(t) \wedge v(t)$ is constant in time. ■

B) Problems from Lectures

a) Find the length of the curve obtained by intersecting the sphere $x^2+y^2+z^2=4$ and the cylinder $(x-1)^2+y^2=1$ in $\mathbb{R}^3$.

Let $\begin{cases} (x-1) = \cos(t) \\ y = \sin(t) \\ z = s \end{cases}$. $\quad t \in [0, 2\pi]$

Then we can plug these parameterizations into the sphere eqn:

$(\cos(t)+1)^2 + \sin^2(t) + s^2 = 4 \qquad (\cos(t)+1)(\cos(t)+1) = \cos^2 t + 2\cos(t) + 1$

$s^2 = 4 - \sin^2(t) - \cos^2(t) - 2\cos(t) - 1$

$s^2 = 2(1-\cos(t)) = 4\sin^2(t/2)$

$s = 2\sin(t/2)$.

This gives the curve $\alpha(t) = \begin{cases} x(t) = 1 + \cos(t) \\ y(t) = \sin(t) \\ z(t) = 2\sin(t/2) \end{cases}$

$\alpha'(t) = (-\sin(t), \cos(t), \cos(t/2))$

$||\alpha'(t)|| = \sqrt{\sin^2(t)+\cos^2(t) + \cos^2(t/2)} = \sqrt{1+\cos^2(t/2)}.$

Note that we only look at the first octant. Looking at $y = \sin(t)$, we know this occurs from $t=0$ to $t=\pi$.

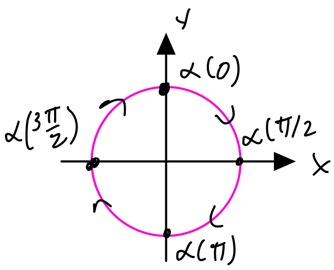
To find arc length:

$$l = \int_0^{\pi} \sqrt{1+\cos^2(t/2)}\, dt = \boxed{3.8202} \text{ (Wolfram Alpha)}$$

C) Other Problems

1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Let $\alpha(t) = (\sin(t), \cos(t))$, $t \in [0, 2\pi]$



$\alpha(\frac{\pi}{2}) = (\sin(\frac{\pi}{2}), \cos(\frac{\pi}{2})) = (1, 0)$

$\alpha(\pi) = (\sin(\pi), \cos(\pi)) = (0, -1)$

$\alpha(3\pi/2) = (\sin(3\pi/2), \cos(3\pi/2)) = (-1, 0)$

3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

$\alpha(t)$ has zero curvature ($||\alpha''(t)|| = 0$). Thus, α must be a line.

4. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve and let $v \in R^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Let $\alpha(t) \cdot \vec{v} = k(t)$. Then we know

$\alpha'(t) \cdot \vec{v} = k'(t) = 0.$

$k'(t) = 0 \implies k(t) = c.$ Since $k(0) = 0$, $k(t) = 0.$

Therefore, $\alpha(t) \perp \vec{v} \ \forall t \in I.$ ∎

5. Let $\alpha: I \longrightarrow R^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

$\implies$) Assume $\alpha(t) \perp \alpha'(t)$ $\forall t \in I$. Then $\alpha(t) \cdot \alpha'(t) = 0$

$\alpha(t) \cdot \alpha(t) = ||\alpha(t)||^2$

$\frac{d}{dt} ||\alpha(t)||^2 = 2\alpha'(t) \cdot \alpha(t) = 0 \implies \frac{d}{dt} ||\alpha(t)||^2 = 0$

$\|\alpha(t)\|^2 = k$, where k is a constant. k cannot be zero since that would contradict the claim that $\alpha'(t) \neq 0$ $\forall t \in I$.

$\Leftarrow$) Assume $\|\alpha(t)\| = k$, a nonzero constant. Then

$$\alpha(t) \cdot \alpha(t) = k^2$$

$$\alpha'(t) \cdot \alpha(t) + \alpha(t) \cdot \alpha'(t) = 0$$

$$2\alpha(t) \cdot \alpha'(t) = 0 \implies \alpha(t) \perp \alpha'(t) \quad \checkmark$$

Therefore, $\|\alpha(t)\|$ is a nonzero constant iff $\alpha(t) \perp \alpha'(t)$ $\forall t \in I$.