Math 142 Reading Week 4

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1 Section 3.4: Fourier Transforms

Fourier transforms can be used to analyze the frequency characteristics of various filters. We can achieve this by passing a sinusoid of a known frequency through a filter and observing how much it attenuates. Define

$$s(x) = \sin(2 * \pi f x + \phi_i) = \sin(\omega x + \phi_i) \tag{1}$$

be the sinusoid with *frequency* f and *angular frequency* ω which a *phase shift* ϕ_i. We can convolve this sinusoidal signal with a filter that has an impulse response $h(x)$ to get another sinusoid $o(x)$ with the same frequency but different magnitude A and phase ϕ_o:

$$o() = h(x) * s(x) = A \sin(\omega x + \phi_o) \tag{2}$$

We call the new magnitude the **gain** of the filter, and the phase difference $\Delta\phi = \phi_o - \phi_i$ is called the **shift**. If we consider a complex-valued sinusoid, then we write

$$s(x) = \exp(j\omega x) \implies o(x) = h(x) * s(x) = A \exp(j\omega x + \phi) \tag{3}$$

Then, the **Fourier transform** is a tabulation of the magnitude and phase response at each frequency:

$$H(\omega) = \mathcal{F}\{h(x)\} = A \exp(j\phi) \tag{4}$$

That is, it's the response to a complex sinusoid of frequency ω passed through a filter $h(x)$. The closed-form evaluations of the Fourier transform are written differently for the continuous and discrete domains. For the continuous domain, we have:

$$H(\omega) = \int_{-\infty}^{\infty} h(x) \exp(-j\omega x) dx \tag{5}$$

Whereas in the discrete domain we have the **Discrete Fourier Transform (DFT)**:

$$H(k) = \frac{1}{N} \sum_{x=0}^{N-1} h(x) \exp(-j \frac{2\pi k x}{N} \tag{6}$$

The DFT, at face value, takes $O(N^2)$ operations to compute but can be sped up via the **Fast Fourier Transform (FFT)** which computes in $O(N \log_2 N$ operations.

Some useful properties of the fourier transform are:

- Superposition – FT of a sum of signals is the sum of their FTs $\implies$ the FT is a linear operator
- Shift – The FT of a shifted signal is the transform of the original signal multiplied by a linear phase shift

- Convolution – the FT of a pair of convolved signals is the product of their transformations