Math 142 Reading Week 2

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1 Section 3.1: Linear Filtering

Linear Filters are the most common type of neighborhood operator, where output pixel values are defined as a weighted sum of input pixel values:

$$g(i,j) = \sum_{k,l} f(i+k, j+l)h(k,l) = f \otimes h \tag{1}$$

A common variant of the above **correlation operator** is the **convolution operator**, which switches the sign of the pixel offsets:

$$g(i,j) = \sum_{k,l} f(i-k, j-l,)h(k,l) = \sum_{k,l} f(k,l)h(i-k, j-l) = f * h, \tag{2}$$

where h is called the **impulse response function**.If we convert the two-dimensional images $f(i,j), g(i,j)$ into raster-ordered vectors, we can write correlation and convolution as matrix operators:

$$\vec{g} = H\vec{f}, \tag{3}$$

where the sparse matrix H contains the convolution kernels. One may notice that, on the boundaries of the image, we will always see darkening due to the fact that the filter size reaches out of the image. To remedy this, we can **pad** the image. There are multiple methods, such as zero-padding, constant-padding, clamp / replicate-padding, cyclic wrapping, mirroring, or extending.

1.1 3.2.1 Separable Filtering

Convolutions require K^2 (K is the size of the filter) operations per pixel to perform, which can be sped up to $2K$ operations per pixel by convolving one-dimensionally along both the horizontal and vertical axes. If a convolutional kernel allows this, it is called a **separable kernel**. This two-dimensional kernel K is the outer product of a horizontal kernel h and a vertical kernel v:

$$K = \vec{v}\vec{h}^T \tag{4}$$

We can tell if a kernel is separable by taking the 2-dimensional kernel matrix and taking its SVD:

$$K = \sum_i \sigma_i \vec{u}_i \vec{v}_i^T \tag{5}$$

If only the *first* singular value σ_0 is non-zero, then the kernel is separable and the vertical and horizontal kernels are $\sqrt{\sigma_0}\vec{u}_0$ and $\sqrt{\sigma_0}\vec{v}_0$, respectively. Examples of using convolutions in image processing include using smoothing kernels to filter out high-frequency noise (low-pass filters)