Math 142 Problem Set 3

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A) Review of Orthogonal Transformations

a) Let ρ and τ be two orthogonal transformations on an Euclidean space $(V^n, <,>)$. Prove that the composition of ρ and τ is again an orthogonal transformation of $(V^n, <,>)$. So is the inverse of ρ.

a) ρ, τ orthogonal $\iff \rho^{-1} = \rho^T$, $\tau^{-1} = \tau^T \iff \rho\rho^T = \rho^T\rho = \tau\tau^T = \tau^T\tau = I$

Note that the composition of ρ and τ is $\rho\tau$. To show $\rho\tau$ is orthogonal, we show that $(\rho\tau)^{-1} = (\rho\tau)^T$. That is,

$$(\rho\tau)^T\rho\tau = \tau^T\rho^T\rho\tau = \tau^T(\rho^T\rho)\tau = \tau^T I \tau = \tau^T\tau = I.$$

Since $(\rho\tau)^T = (\rho\tau)^{-1}$, $\rho\tau$ is an orthogonal map. A similar argument can be made for $\tau\rho$.

So, $\rho\tau : V \to V$ is orthogonal. The same argument can be reversed for ρ^T to show $\rho^{-1} = \rho^T$ is orthogonal.

b) **6.** A *translation* by a vector v in R^3 is the map $A: R^3 \longrightarrow R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map $\rho: R^3 \longrightarrow R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors $u, v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

a) Let $v \in \mathbb{R}^3$ and let $\rho: \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal map w/ positive determinant. Then:

$\|\rho v\|^2 = \rho v \cdot \rho v = v \cdot v = \|v\|^2$ (defn of orthogonality)

$\Rightarrow \|\rho v\| = \|v\| \Rightarrow \|v\|$ is invariant under an orthogonal transformation.

Now let $w \in \mathbb{R}^3$. Then

$$\|v \cdot w\| = \|v\|\,\|w\| \cos\theta(v,w) \Rightarrow \cos\theta(v,w) = \frac{\|v \cdot w\|}{\|v\|\,\|w\|}$$

Now look at $\cos\theta(\rho v, \rho w)$:

$$\cos\theta(\rho v, \rho w) = \frac{\|\rho v \cdot \rho w\|}{\|\rho v\| \, \|\rho w\|} = \frac{\|\rho v \cdot \rho w\|}{\|v\| \, \|w\|} \quad \text{(from above)}$$

$$= \frac{\|v \cdot w\|}{\|v\| \, \|w\|} \quad \text{(def'n of orthogonal transformation)}$$

$$= \cos\theta(w, v)$$

Thus, the angle θ between 2 vectors is preserved by an orthogonal map so long as it is in between 0 and π (inclusive)

b) Let $\rho: \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal transformation and $v, w \in \mathbb{R}^3$. Then

$$\rho v \times \rho w = (\det \rho)(\rho^{-1})^T (v \times w) = (\det \rho)(\rho^T)^T (v \times w) \quad \text{(def'n of orthogonality)}$$

$$= \rho (\det \rho)(v \times w)$$

$$= (\det \rho)\, \rho(v \times w) \Rightarrow \boxed{\text{not equivalent!}}$$

But, look at $\|v \times w\| = \|v\| \, \|w\| \cdot \sin(\theta)$

$$\|\rho v \times \rho w\| = \|\rho v\| \|\rho w\| \cdot \sin(\theta)$$

We know from above, the θ above are equivalent.

$$\|\rho v \times \rho w\| = \|v\| \, \|w\| \cdot \sin(\theta) = \|v \times w\|$$ $\Rightarrow$ norm is preserved, direction is <u>not</u> regardless of $\det(\rho)$'s value.

c) Let $\rho: \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid transformation and $\alpha: I \to \mathbb{R}^3$ be a curve parameterized by arc length. Then

$$s(t) = \int_0^t \|\alpha'(t')\| \, dt' \qquad k(s) = \|\alpha''(s)\| \qquad \tau = \frac{1}{\|n(s)\|} \qquad n(s) = \frac{\alpha''(s)}{k(s)}$$

Let $\rho(v) = Av + b$, where A is an orthogonal map w/ $\det(A) > 0$ and b is a constant vector.

Note that A does not change with s.

$$(\rho\alpha(s))' = (A\alpha(s) + b)' = A\alpha'(s)$$

Now,

$$s(t) = \int_{t_0}^{t} \|(\rho\alpha(t'))'\|\, dt' = \int_{t_0}^{t} \|A\alpha'(t')\|\, dt' = \int_{t_0}^{t} \|\alpha'(t')\| dt'$$

(norm invariant under orthogonal transformations)

$\Rightarrow$ arc length is <u>invariant</u> under ρ.

Now, $k(s) = \|\alpha''(s)\|$. So,

$$(\rho\alpha(s))' = A\alpha'(s)$$

$$(\rho\alpha(s))'' = (A\alpha'(s))' = A\alpha''(s)$$

$$\|(\rho\alpha(s))''\| = \|A\alpha''(s)\| = \|\alpha''(s)\| = k(s)$$ (norm invariant under orthogonal transformations)

Finally, $\tau(s)$. First, we need $n(s)$ and $b(s)$:

$$n(s) = \frac{\alpha''(s)}{k(s)} \qquad b(s) = \frac{1}{k(s)}(\alpha'(s) \times \alpha''(s))$$

$$b'(s) = \tau(s)\, n(s) \qquad \tau(s) = \frac{\|b'(s)\|}{\|n(s)\|} = \frac{\|t(s) \times n(s)\|}{1} = \|t(s)\|\,\|n(s)\|\,\sin(\theta)$$

$$= \|\alpha'(s)\|\, \left\|\frac{1}{k(s)} \cdot \alpha''(s)\right\|$$

So, note that

$$\|(\rho\alpha(s))'\|\, \left\|\frac{1}{k(s)}(\rho\alpha(s))''\right\| \sin\theta = \|\alpha'(s)\|\, \left\|\frac{1}{k(s)}\alpha''(s)\right\| \sin\theta = \tau(s),$$

which uses many properties derived above.

Therefore, arc length, curvature, and torsion are invariant under rigid motion

B) Lecture Problems

a) Show SO(n) is a group with respect to the usual matrix multiplication. (Later, we will see that SO(n) is in fact a Lie group.)

a) $SO(n)$: set of all matrices $M \in \mathbb{R}^{n\times n}$ such that $M^T M = I$ and $\det M = 1$

<u>Identity</u>: $I^T I = I I = I$. Let $A \in SO(n)$. Then $AI = IA = A$. ✓

<u>Associativity</u>: The operation is matrix multiplication, so $SO(n)$ inherits this from generic matrix multiplication.

<u>Inverse</u>: Since $M^T M = M M^T = I$, each element has an inverse equal to its transpose.

<u>Closure</u>: Let $A, B \in SO(n)$. Then $A^T A = B^T B = I$. So,

$$(AB)^T (AB) = (B^T A^T)(AB) = B^T (A^T A) B = B^T I B = B^T B = I$$

Since $(AB)^T = (AB)^{-1}$, AB is orthogonal. Additionally,

$$\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$$

So, $\det(AB) = 1$, so $AB \in SO(n)$.

$SO(n)$ satisfies all group axioms, and is therefore a group. ■

b) Show that the mirror reflection τ (as defined in the lecture) is an orthogonal transformation and $\tau^2 = id$, where id is the identity transformation.

b) $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad \tau(x) = Mx = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$

$$\tau^T = \tau \Rightarrow \tau^T \tau = \tau^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = id$$

$$\tau \tau^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = id$$

Since $\tau \tau^T = \tau^T \tau = I$, τ is an orthogonal transformation. Also, since $\tau^T = \tau$, we know that $\tau^2 = id$.

C1) Other Problems, Group 1

5. Let $\alpha: (-1, +\infty) \longrightarrow R^2$ be given by

$$\alpha(t) = \left(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\right).$$

Prove that:

a. For $t = 0$, α is tangent to the x axis.

b. As $t \longrightarrow +\infty$, $\alpha(t) \longrightarrow (0, 0)$ and $\alpha'(t) \longrightarrow (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \longrightarrow -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of α in such a way that it becomes symmetric relative to the line $y = x$ is called the *folium of Descartes* (see Fig. 1-10).

5) a) $\alpha(t) = \left(\dfrac{3at}{1+t^3}, \dfrac{3at^2}{1+t^3}\right)$

$$\alpha'(t) = \left(\frac{3a(1+t^3) - 3t^2(3at)}{(1+t^3)^2}, \frac{6at(1+t^3) - 3t^2(3at^2)}{(1+t^3)^2}\right)$$

$$= \left(\frac{3a + 3at^3 - 9at^3}{(1+t^3)^2}, \frac{6at + 6at^4 - 9at^4}{(1+t^3)^2}\right) = \left(\frac{3a - 6at^3}{(1+t^3)^2}, \frac{6at - 3at^4}{(1+t^3)^2}\right)$$

$\alpha'(0) = \left(\dfrac{3a}{1}, 0\right) = (3a, 0)$. This is parallel to the x-axis, so α is tangent to the x-axis at $t=0$.

b) $$\lim_{t\to\infty} \alpha(t) = \left(\lim_{t\to\infty} \frac{3at}{1+t^3}, \lim_{t\to\infty} \frac{3at^2}{1+t^3}\right) = \left(\lim_{t\to\infty} \frac{3a}{3t^2}, \lim_{t\to\infty} \frac{6at}{3t^2}\right)$$

$$= \left(0, \lim_{t\to\infty} \frac{6a}{6t}\right) = (0,0) \checkmark$$

$$\lim_{t\to\infty} \alpha'(t) = \left(\lim_{t\to\infty} \frac{3a-6at^3}{(1+t^3)^2}, \lim_{t\to\infty} \frac{6at-3at^4}{(1+t^3)^2}\right) = \left(\lim_{t\to\infty} \frac{-18at^2}{2(1+t^3)\cdot 3t^2}, \lim_{t\to\infty} \frac{6a-12at^3}{2(1+t^3)\cdot 3t^2}\right)$$

$$= \left(\lim_{t\to\infty} \frac{-36at}{\frac{d}{dt}\left[6t^2+6t^5\right]}, \lim_{t\to\infty} \frac{-36at^2}{\frac{d}{dt}\left[6t^2+6t^5\right]}\right)$$

$$= \left(\lim_{t\to\infty} \frac{-36at}{12t+30t^4}, \lim_{t\to\infty} \frac{-36at^2}{12t+30t^4}\right)$$

$= (0,0)$ (order of denominator > order of numerator)

c) Consider the expression $x(t) + y(t) + a$. We show this approaches 0 as $t \to -1$:

$$\lim_{t \to -1} \alpha(t) = \left(\lim_{t \to -1} \frac{3at}{1+t^3}, \lim_{t \to -1} \frac{3at^2}{1+t^3} \right) = \left(\lim_{t \to -1} \frac{3a}{3t^2}, \lim_{t \to -1} \frac{6a\cancel{t}}{3t^{\cancel{2}}} \right)$$

$= (a, -2a)$

So, we have $\lim_{t \to -1} x(t) + y(t) + a = a - 2a + a = 0.$

Thus, $\alpha(t)$ approaches the line $x + y + a = 0$ as $t \to -1$.

Now consider $\alpha'(t) = \left(\frac{3a - 6at^3}{(1+t^3)^2}, \frac{6at - 3at^4}{(1+t^3)^2} \right)$

$$\lim_{t \to -1} x'(t) + y'(t) = \lim_{t \to -1} \frac{-3at^4 - 6at^3 + 6at + 3a}{(1+t^3)^2} = \lim_{t \to -1} \frac{-12at^3 - 18at^2 + 6a}{6t^2(1+t^3)}$$

$$= \lim_{t \to -1} \frac{-36at^2 - 36at}{12t(1+t^3) + 3t^2 \cdot 6t^2} = -36a + 36a$$

To show that the slope approaches -1 as $t \to -1$, we show that

$\lim_{t \to -1} \frac{y(t)}{x(t)} = -1$:

$$\frac{y(t)}{x(t)} = \frac{3at^2}{1+t^3} \cdot \frac{1+t^3}{3at} = \frac{3at^2}{3at} = \frac{t}{1} = t$$

$\lim_{t \to -1} t = -1 \to$ the slope α' approaches the line $x(t) + y(t) + a = 0$ as well.

6. Let $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t)$, $t \in R$, a and b constants, $a > 0$, $b < 0$, be a parametrized curve.

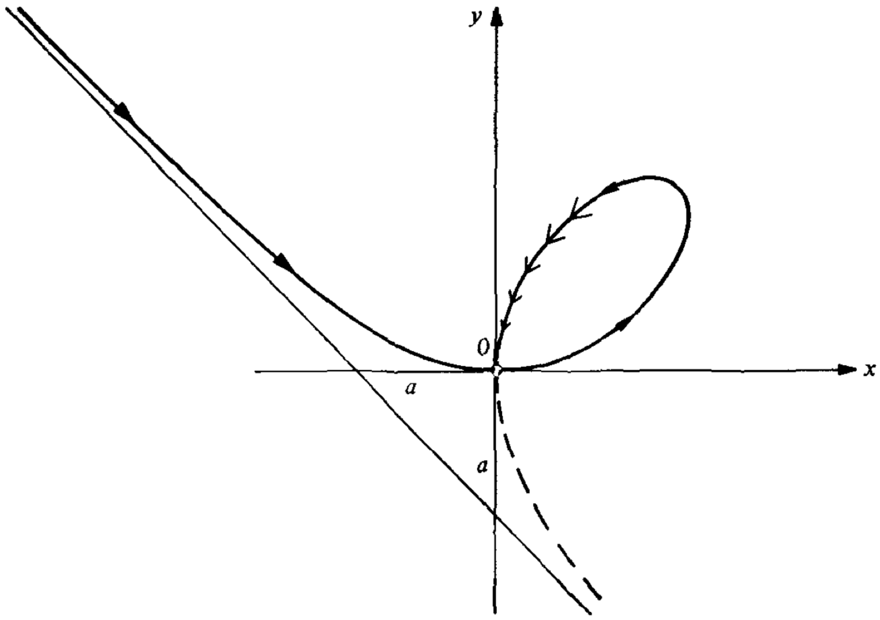

Figure 1-10. Folium of Descartes.

a. Show that as $t \to +\infty$, $\alpha(t)$ approaches the origin 0, spiraling around it (because of this, the trace of α is called the *logarithmic spiral*; see Fig. 1-11).

b. Show that $\alpha'(t) \to (0, 0)$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty} \int_{t_0}^{t} |\alpha'(t)|\, dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

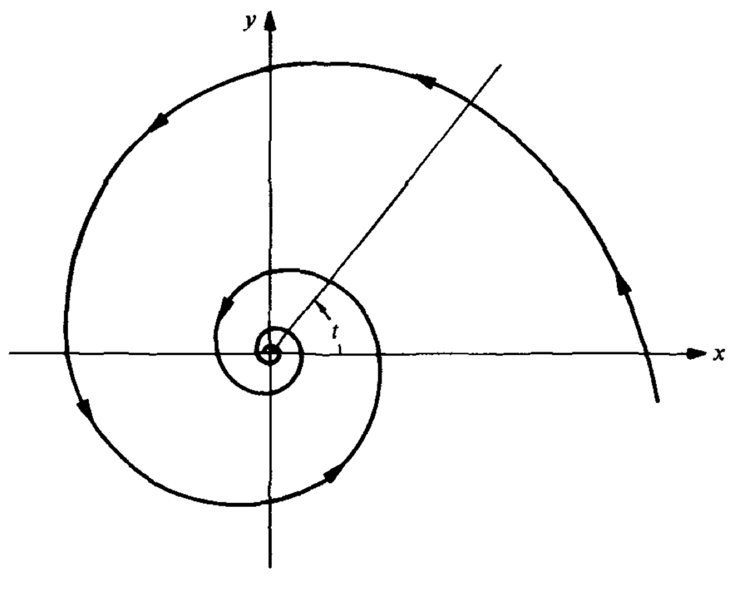

Figure 1-11. Logarithmic spiral.

a) $\alpha(t) = \left(ae^{bt}\cos(t), ae^{bt}\sin(t)\right)$ $\quad b<0,\ a>0$

$\lim_{t\to\infty} \alpha(t) = \left(\lim_{t\to\infty} ae^{bt}\cos(t), \lim_{t\to\infty} ae^{bt}\sin(t)\right) = (0,0)$ (b/c $b<0$, $e^{-x}\to 0$ as $x\to\infty$)

b) $\alpha'(t) = \left(a\left[be^{bt}\cos(t) - e^{bt}\sin(t)\right], a\left[be^{bt}\sin(t) + e^{bt}\cos(t)\right]\right)$

$$\lim_{t\to\infty} \alpha'(t) = \left(a\left[\lim_{t\to\infty} be^{bt}\cos(t) - \lim_{t\to\infty} e^{bt}\sin(t)\right], a\left[\lim_{t\to\infty} be^{bt}\sin(t) + \lim_{t\to\infty} e^{bt}\cos(t)\right]\right)$$

$$= \left(a[0-0], a[0+0]\right) = \boxed{(0,0)} \quad \checkmark$$

$$\|\alpha'(t)\| = a\sqrt{\left(be^{bt}\cos(t) - e^{bt}\sin(t)\right)^2 + \left(be^{bt}\sin(t) + e^{bt}\cos(t)\right)}$$

$$= a\sqrt{b^2e^{2bt}\cos^2(t) + e^{2bt}\sin^2(t) - \cancel{2be^{2bt}\cos(t)\sin(t)} + b^2e^{2bt}\sin^2(t) + e^{2bt}\cos^2(t) + \cancel{2be^{2bt}\sin(t)\cos(t)}}$$

$$= a\sqrt{b^2e^{2bt} + e^{2bt}} = a\sqrt{e^{2bt}(b^2+1)} = ae^{bt}\sqrt{b^2+1}$$

$$s(t) = \int_{t_0}^{t} \|\alpha'(x)\|\,dx = \int_{t_0}^{t} ae^{bx}\sqrt{b^2+1}\,dx = a\sqrt{b^2+1}\int_{t_0}^{t} e^{bx}\,dx$$

$$= \frac{a\sqrt{b^2+1}}{b} e^{bx}\Big|_{t_0}^{t} = \frac{a}{b}\sqrt{b^2+1}\left(e^{bt} - e^{t_0 x}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{a}{b}\sqrt{b^2+1}\left(\lim_{t\to\infty} e^{bt} - e^{t_0 x}\right) = \boxed{\frac{-a}{b}\sqrt{b^2+1}\; e^{t_0 x}}$$

Since $b<0$, this is a finite, positive arc length.

C2) Other Problems, Group 2

Unless explicity stated, $\alpha: I \longrightarrow R^3$ is a curve parametrized by arc length s, with curvature $k(s) \neq 0$, for all $s \in I$.

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

1) a) $\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\cdot\frac{s}{c}\right) \quad s \in \mathbb{R}$

$\alpha'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$

$\|\alpha'(s)\| = \sqrt{\frac{a^2}{c^2}\sin^2\left(\frac{s}{c}\right) + \frac{a^2}{c^2}\cos^2\left(\frac{s}{c}\right) + \frac{b^2}{c^2}} = \sqrt{\frac{a^2+b^2}{c^2}} = \sqrt{1} = 1$

$\ell = \int_0^s \|\alpha'(k)\|\, dk = \int_0^s 1 = \boxed{s}$ ✓

b) $k(s) = \|\alpha''(s)\| \qquad \alpha''(s) = \left(-\frac{a}{c^2}\sin\left(\frac{s}{c}\right), -\frac{a}{c^2}\cos\left(\frac{s}{c}\right), 0\right)$

$k(s) = \sqrt{\frac{a^2}{c^4}\sin^2\left(\frac{s}{c}\right) + \frac{a^2}{c^4}\cos^2\left(\frac{s}{c}\right)} = \sqrt{\frac{a^2}{c^4}} = \boxed{\frac{a}{c^2}}$

$n(s) = \frac{\alpha''(s)}{k(s)} = \frac{c^2}{a}\left(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0\right) = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$

$t(s) = \alpha'(s) = \left(\frac{-a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$

$b(s) = t(s) \wedge n(s) = \left(\frac{-a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right) \wedge \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right)$

$$= \begin{vmatrix} i & j & k \\ -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right) & -\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{a}{c}\cos\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\sin\left(\frac{s}{c}\right) & 0 \end{vmatrix} i - \begin{vmatrix} -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{b}{c} \\ -\cos\left(\frac{s}{c}\right) & 0 \end{vmatrix} j + \begin{vmatrix} -\frac{a}{c}\sin\left(\frac{s}{c}\right) & \frac{a}{c}\cos\left(\frac{s}{c}\right) \\ -\cos\left(\frac{s}{c}\right) & -\sin\left(\frac{s}{c}\right) \end{vmatrix}$$

$$= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{-b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\sin^2\left(\frac{s}{c}\right) + \frac{a}{c}\cos^2\left(\frac{s}{c}\right)\right)$$

$$= \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{-b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right)$$

$$b'(s) = \left(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{c}\right), 0\right) = \frac{-b}{c^2}\, n(s) \implies \boxed{\tau(s) = \tau = \frac{-b}{c^2}}$$

c) The oscdating plane is defined as the plane formed by the vectors $t(s)$ and $n(s)$ — that is, all vectors $x \in \mathbb{R}^3$ normal to $b(s)$:

$P = \{x \in \mathbb{R}^3 : (\alpha(s) - x)\cdot b(s) = 0\}$. Let $x = (x_1, x_2, x_3)$. This gives

$$\left(a\cos\left(\frac{s}{c}\right) - x_1, a\sin\left(\frac{s}{c}\right) - x_2, b\frac{s}{c} - x_3\right)\cdot\left(\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{-b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right) = 0$$

$$\left(\cancel{a\cos\left(\frac{s}{c}\right)} - x_1\right)\frac{b}{c}\sin\left(\frac{s}{c}\right) + \left(\cancel{a\sin\frac{s}{c}} - x_2\right)\cdot\frac{-b}{c}\cos\left(\frac{s}{c}\right) + \left(b\cdot\frac{s}{c} - x_3\right)\frac{a}{c} = 0$$

$$-\frac{b}{c}\sin\left(\frac{s}{c}\right)x_1 + \frac{b}{c}\cos\left(\frac{s}{c}\right)x_2 - \frac{ab}{c^2}x_3 + ab\cdot\frac{s}{c^2} = 0$$

$$\boxed{\frac{b}{c}\sin\left(\frac{s}{c}\right)x_1 - \frac{b}{c}\cos\left(\frac{s}{c}\right)x_2 + \frac{a}{c}x_3 = \frac{ab}{c^2}s}$$

d) We first show that a line that contains $n(s)$ and $\alpha(s)$ intersects the z axis. For scalars $d(s), f(s)$, we want to know if:

$$d(s)n(s) + \alpha(s) = \hat{z}f(s) \qquad \hat{z} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

$$d(s)n(s) - f(s)\hat{z} = -\alpha(s)$$

This translates to the system

$$\begin{bmatrix}-\cos\left(\frac{s}{c}\right) & 0\\ -\sin\left(\frac{s}{c}\right) & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}d(s)\\ f(s)\end{bmatrix} = \begin{bmatrix}-a\cos\left(\frac{s}{c}\right)\\ -a\sin\left(\frac{s}{c}\right)\\ -b\cdot\frac{s}{c}\end{bmatrix}$$

which will always hold true for $d(s) = a$, $f(s) = b\cdot\frac{s}{c}$. Thus the intersection always occurs. Now,

$$n\cdot\hat{z} = 0 + 0 + 0 = 0 \implies \boxed{\theta(n,\hat{z}) = \pi/2,\ 3\pi/2 \text{ (constant)}}$$

e) Note that $t(s) = \alpha'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$

$$t(s)\cdot\hat{z} = \frac{b}{c} \Rightarrow \|t(s)\cdot\hat{z}\| = \|t(s)\|\,\|\hat{z}\|\cos\theta$$

$$\frac{b}{c} = 1\cdot 1\cdot\cos\theta \Rightarrow \boxed{\theta = \arccos\left(\frac{b}{c}\right)} \quad \text{(constant!)}$$

Thus, $\alpha'(s)$ always makes a constant angle w.r.t. the z-axis, which is expected b/c $\alpha'(s)_z$ is constant,

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s)\cdot\alpha'''(s)}{|k(s)|^2}.$$

Note that
$$\alpha' = t$$
$$\alpha'' = kn$$
$$\alpha''' = k'n + n'k = k'n + (-kt - \tau b)k$$
$$= k'n - k^2 t - k\tau b$$

Let's show this formula directly:

$$\frac{-(t \wedge kn)\cdot(k'n - k^2 t - k\tau b)}{|k(s)|^2}$$

Now evaluate the products using the Frenet frame as a basis $\begin{bmatrix} t \\ n \\ b \end{bmatrix}$:

$$-(t\wedge kn) = -k\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -k \end{bmatrix}$$

$$k'n - k^2 t - k\tau b = \begin{bmatrix} -k^2 \\ k' \\ -k\tau \end{bmatrix}$$

$$-(t\wedge kn)\cdot(k'n - k^2 t - k\tau b) = k^2\tau$$

$$\frac{k(s)^2}{|k(s)|^2}\tau = \boxed{\tau} \quad ✓$$

5. A regular parametrized curve α has the property that all its tangent lines pass through a fixed point.

a. Prove that the trace of α is a (segment of a) straight line.

b. Does the conclusion in part a still hold if α is not regular?

a) Let p be the aforementioned fixed point and let $\alpha(s)$ be parameterized by arc length. The tangent line at s is given by

$$\ell(x) = \alpha(s) + \alpha'(s)\, x$$

passes through $\alpha(s)$

slope

Now, $\forall s \in I$, $\exists\, x(s)$ such that $\ell(s) = \alpha(s) + \alpha'(s)\, x(s) = p$

Differentiate this w.r.t. s:

$$\alpha'(s) + \alpha''(s)\, x(s) + \alpha'(s)\, x'(s) = 0$$

$$(1 + x'(s))\, \alpha'(s) + \alpha''(s)\, x(s) = 0$$

We know this is a linearly independent system because $\alpha'(s) \perp \alpha''(s)$ for a regular curve parameterized by arc length. Also, since $\alpha(s)$ is regular, $\alpha'(s) \neq 0$ $\forall s \in I$. Thus, the system below holds only in two cases:

① $1 + x'(s) = 0$ and $\alpha''(s) = 0$

or

② $1 + x'(s) = 0$ and $x(s) = 0$ } cannot be true for all s

① is the only plausible solution, so we know $\alpha''(s) = 0$ and therefore α is a segment of a straight line.

b) No, if $\alpha(s)$ is not regular then $\alpha'(s)$ can be 0 as well and we cannot rule out all but one case as we did before.

D) Extra Credit

8. The trace of the parametrized curve (arbitrary parameter)

$$\alpha(t) = (t, \cosh t), \qquad t \in R,$$

is called the *catenary*.

a. Show that the signed curvature (cf. Remark 1) of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}.$$

b. Show that the evolute (cf. Exercise 7) of the catenary is

$$\beta(t) = (t - \sinh t \cosh t,\ 2 \cosh t).$$

a) $\alpha : \mathbb{R} \to \mathbb{R}^2$

Since α is not parameterized by arc length, let s be the parameter for which it is.

$$\alpha'(t) = (1, \sinh(t))$$

$$\|\alpha'(t)\| = \sqrt{1+\sinh^2(t)} = \cosh(t)$$

$$T(t) = \frac{1}{\|\alpha'(t)\|}\alpha'(t) = \left(\frac{1}{\cosh(t)}, \frac{\sinh(t)}{\cosh(t)}\right)$$

$$\frac{dT}{dt} = \left(-\frac{\sinh(t)}{\cosh^2(t)}, \frac{\cosh^2(t)-\sinh^2(t)}{\cosh^2(t)}\right) = \left(\frac{-\sinh(t)}{\cosh^2(t)}, \frac{1}{\cosh^2(t)}\right)$$

$$= \frac{1}{\cosh(t)}\left(\frac{-\sinh(t)}{\cosh(t)}, \frac{1}{\cosh(t)}\right)$$

Now,

$$\frac{dT}{ds} = \frac{dT}{dt}\cdot\frac{dt}{ds} = kn. \text{ Since } \frac{ds}{dt} = \|\alpha'(t)\|,\ \frac{dt}{ds} = \frac{1}{\|\alpha'(t)\|} = \frac{1}{\cosh(t)}$$

$$k\left(\frac{-\sinh(t)}{\cosh(t)}, \frac{1}{\cosh(t)}\right) = \frac{1}{\cosh(t)}\cdot\left(\frac{-\sinh(t)}{\cosh^2(t)}, \frac{1}{\cosh^2(t)}\right)$$

$$k\left(\cancel{\frac{-\sinh(t)}{\cosh^2(t)}}, \frac{1}{\cosh^2(t)}\right) = \frac{1}{\cosh^2(t)}\cdot\left(\cancel{\frac{-\sinh(t)}{\cosh^2(t)}}, \frac{1}{\cosh^2(t)}\right) \Rightarrow \boxed{k = \frac{1}{\cosh^2(t)}}$$

b) $\beta(t) = \alpha(t) + \frac{1}{\kappa(t)} n(t)$

$$= (t, \cosh(t)) + \cosh^2(t)\left(-\frac{\sinh(t)}{\cosh(t)}, \frac{1}{\cosh(t)}\right)$$

$$= (t, \cosh(t)) + (-\sinh(t)\cosh(t), \cosh(t))$$

$$\boxed{\beta(t) = (t - \sinh(t)\cosh(t), 2\cosh(t))}$$