Math 142 Reading Week 6

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1 3.4.3 Wiener Filtering

The Fourier transform is a useful tool for analyzing frequency characteristics of a filter kernel or image. It can also be used to analyze the frequency spectrum of a whole *class* of images.

A simple model for an image is assuming they are random noise fields with expected magnitude and frequency given by the *power spectrum* $P_s(\omega_x, \omega_y)$:

$$\langle [S(\omega_x, \omega_y)]^2 \rangle = P_s(\omega_x, \omega_y), \tag{1}$$

where the angle brackets represent the expected value of a random variable. We can generate images with Gaussian noise $S(\omega_x, \omega_y)$ where each pixel is a zero-mean Gaussian with variance $P_s(\omega_x, \omega_y)$.

The observation that signal spectra capture a first-order description of spatial statistics is widely used in signal and image processing. Assuming that an image is sampled from a correlated Gaussian random noise field combined with a statistical model of the measurement process yields an optimum restoration filter known as the **Wiener Filter**.

To derive it, we can analyze each frequencyt component of a signal's Fourier transform independently. The noisy image formation process can then be represented as

$$o(x, y) = s(x, y) + n(x, y), \tag{2}$$

where $s(x, y)$ is the unknown image we are trying to recover, $n(x, y)$ is the additive noise signal, and $o(x, y)$ is the observed image. The linearity of the Fourier transform allows us to write

$$O(\omega_x, \omega_y) = S(\omega_x, \omega_y) + N(\omega_x, \omega_y), \tag{3}$$

where each of these quantities corresponds to the Fourier transform of the corresponding image.

At each frequency we know the unknown transformation component $S(\omega_x, \omega_y)$ has a prior distribution which is a zero-mean Gaussian with variance $P_s(\omega_x, \omega_y)$. We also have noise measurement with variance σ_n^2. We have a prior distribution

$$p(S) = \exp -\frac{(S-\mu)^2}{2P_S}, \tag{4}$$

where μ is zero everywhere except the origin. So we have

$$P(S|O) = \exp -\frac{(S-O)^2}{2P_n} \tag{5}$$

If we take the negative logarithm of both sides and solve for the optimal (minimizing) S, we get

$$S_{opt} = \frac{1}{1+\frac{P_n}{P_S}} O. \tag{6}$$

The quantity

$$W(\omega_x, \omega_y) = \frac{1}{1 + \frac{\sigma_n^2}{P_S(\omega_x, \omega_y)}} \tag{7}$$

is the Fourier transform of the optimum *Wiener Filter* needed to remove noise from an image whose power spectrum is $P_S(\omega_x, \omega_y)$.