Math 142 Problem Set 5

A) Problems on Reviewing Continuity and Differentiability

DEFINITION 1. *Let* $F: U \subset R^n \to R^m$ *be a differentiable map. To each* $p \in U$ *we associate a linear map* $dF_p: R^n \to R^m$ *which is called the* differential *of* F *at* p *and is defined as follows. Let* $w \in R^n$ *and let* $\alpha: (-\epsilon, \epsilon) \to U$ *be a differentiable curve such that* $\alpha(0) = p$, $\alpha'(0) = w$. *By the chain rule, the curve* $\beta = F \circ \alpha: (-\epsilon, \epsilon) \to R^m$ *is also differentiable.* Then (Fig. A2-5)

$$dF_p(w) = \beta'(0).$$

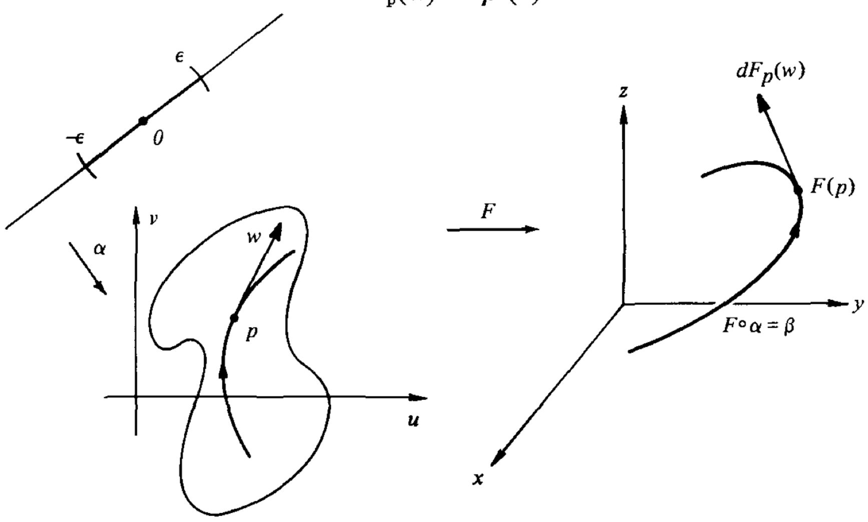


Figure A2-5

PROPOSITION 7. *The above definition of* dF_p *does not depend on the choice of the curve which passes through* p *with tangent vector* w, *and* dF_p *is, in fact, a linear map.*

a) Proof: Let e_i denote the i-th column in the $n \times n$ identity matrix. Likewise, denote f_i as the i-th column in the $m \times m$ identity matrix. Then we have bases for $\mathbb{R}^n, \mathbb{R}^m$ as $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^m$ respectively.

Now, let $w \in \mathbb{R}^n$, $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. Also, let $\alpha: (-\varepsilon, \varepsilon) \to U$ be a differentiable curve such that $\alpha(0) = p$, $\alpha'(0) = w$. By chain rule, we have $\beta = F \circ \alpha: (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ is also differentiable.

Assume α takes the form

$$\alpha(t) = (u_1(t), u_2(t), \dots, u_n(t))$$

and F takes the form

$$F(u_1, \dots, u_n) = (F_1((u_1(t), u_2(t), \dots, u_n(t)), \dots, F_m((u_1(t), u_2(t), \dots, u_n(t)))$$

Then $\beta(t) = (F_1(\alpha(t)), F_2(\alpha(t)), \dots, F_m(\alpha(t)))$

So $\beta'(t) = \left(\sum_{i=1}^n \frac{\partial F_1}{\partial u_i} u_i'(t), \dots, \sum_{i=1}^n \frac{\partial F_m}{\partial u_i} u_i'(t)\right)$

In other words,

$$\beta'(t) = \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial u_1} & \frac{\partial F_m}{\partial u_2} & \cdots & \frac{\partial F_m}{\partial u_n} \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} u_1'(t) \\ u_2'(t) \\ \vdots \\ u_n'(t) \end{bmatrix}}_{\alpha'(t)} \Rightarrow \beta'(0) = J\alpha(0) = Jw,$$

regardless of what α is b/c $\alpha'(0) = w$ no matter what α is chosen to be.

The above equation also shows that $\beta'(0) = \beta'(\alpha(0)) = dF_p = J$ is an $m \times n$ matrix, i.e., a linear map from $\mathbb{R}^n \to \mathbb{R}^m$. ✓

PROPOSITION 8 (The Chain Rule for Maps). *Let* $F: U \subset R^n \longrightarrow R^m$ *and* $G: V \subset R^m \longrightarrow R^k$ *be differentiable maps, where* U *and* V *are open sets such that* $F(U) \subset V$. *Then* $G \circ F: U \longrightarrow R^k$ *is a differentiable map, and*

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p, \qquad p \in U.$$

b) Proof. The fact that $G \circ F$ is differentiable is a consequence of the chain rule for functions. Now, let $w_1 \in \mathbb{R}^n$ and let's consider a curve $\alpha: (-\varepsilon_2, \varepsilon_2) \to U$ with $\alpha(0) = p$, $\alpha'(0) = w_1$. Set $dF_p(w_1) = w_2$ and observe

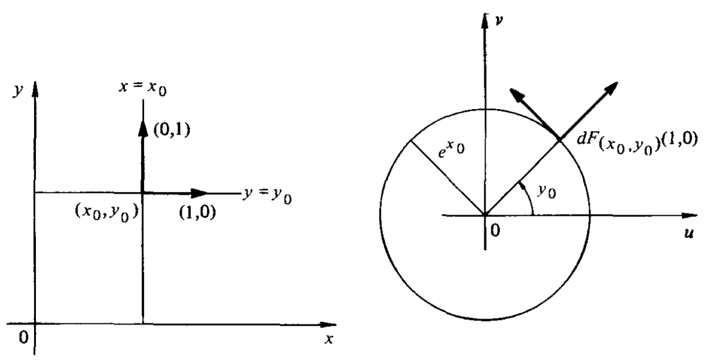
$$dG_{F(p)}(w_2) = \frac{d}{dt}(G \circ F \circ \alpha)_{t=0} = dG_{F(p)}(w_2) = dG_{F(p)} \circ dF_p(w_1). \quad ✓$$

c) **Rewrite example 11.**

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (e^x \cos y, e^x \sin y)$, $(x,y) \in \mathbb{R}^2$.

The component functions $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$ have continuous partial derivatives of all orders, so F is differentiable.

See how the line $x = x_0$ maps to the circle $(e^{x_0}\cos y, e^{x_0}\sin y)$ and the horizontal line $y = y_0$ maps to the half line $(e^x \cos y_0, e^x \sin y_0)$ w/ slope $\tan y_0$.



It follows that

$$dF_{(x_0,y_0)}(1,0) = \frac{d}{dx}(e^x \cos y_0, e^x \sin y_0)\Big|_{x=x_0} = (e^{x_0}\cos y_0, e^{x_0}\sin y_0)$$

$$dF_{(x_0,y_0)}(1,0) = \frac{d}{dy}(e^{x_0}\cos y, e^{x_0}\sin y)\Big|_{y=y_0} = (-e^{x_0}\sin y_0, e^{x_0}\cos y_0)$$

We can check by evaluating the Jacobian of F:

$$dF_{(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

Applying this to $(1,0)$ and $(0,1)$ at (x_0, y_0) yields the same results.

Note that $\det(dF_{(x,y)}) \neq 0 \;\; \forall (x,y) \in \mathbb{R}^2$. Therefore, dF_p is nonsingular and we can apply the inverse function theorem to conclude F is a local diffeomorphism. ∎

The inverse function theorem is true only in a neighborhood of p because differentiability of a function at a point is defined only in a neighborhood of that point. A function that is not differentiable at all points may still be invertible in neighborhoods where the derivative is continuous and non-zero.

d) Show that an infinite cylinder after deleting a vertical line is diffeomorphic to a plane.

d) Consider the cylinder of radius r $\alpha: [0, 2\pi] \times [0, \infty) \to \mathbb{R}^3$ be defined by $\alpha(\theta, h) = (r\cos\theta, r\sin\theta, h)$, a parameterization of the cylinder (the first two coordinates are a circle, and the last is the height). Now, let $x: (a,b) \to \left(2\pi \frac{|a|}{|a|+1}, b\right)$. Note that x is a bijection, so we simply need to show it is differentiable and has a differentiable inverse. To do this, we show dx_p is invertible for all points $p \in \mathbb{R}^3$.

$$dx_p = \begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} \end{bmatrix} = \begin{bmatrix} \frac{2\pi}{(1+|a|)^2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(dx_p) = \frac{2\pi}{(1+|a|)^2} > 0 \quad \forall a, b \in \mathbb{R}.$$ So, x is differentiable and has a differentiable inverse. Therefore it is diffeomorphic to the cylinder.

B) Problems From Lecture

PROPOSITION 2. *If* $f: U \subset R^3 \to R$ *is a differentiable function and* $a \in f(U)$ *is a regular value of* f, *then* $f^{-1}(a)$ *is a regular surface in* R^3.

a) Proof. Let $p = (x_0, y_0, z_0)$ be a point of $f^{-1}(a)$. Since a is a regular value, we can assume $f_z \neq 0$ at p. Define $F: U \subset \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (x, y, f(x,y,z)) = (u, v, t)$$

Indicate (u,v,t) as points where F takes its values. Then

$$dF_p = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix}$$

So long as $f_z \neq 0$, $\det(dF_p) \neq 0$ and so we can use the inverse function theorem to note the existence V of p and W of $F(p)$ such that $F: V \to W$ is invertible and $F^{-1}: W \to V$ is differentiable. So, the coordinate functions

$$x = u \qquad y = z \qquad z = g(u,v,t) \qquad (u,v,t) \in W$$

are differentiable. In particular, $g(u,v,a) = h(x,y)$ is a differentiable function defined in the projection of V onto the xy plane. Since

$$F(f^{-1}(a) \cap V) = W \cap \{(u,v,t) : t = a\}$$

we conclude that the graph of h is $f^{-1}(a) \cap V$. By proposition 1 of section 2-3, $f^{-1}(a) \cap V$ is a coordinate neighborhood of p. Therefore, every $p \in f^{-1}(a)$ can be covered by a coordinate neighborhood, so $f^{-1}(a)$ is a regular surface.

PROPOSITION 4. *Let* p ∈ S *be a point of a regular surface* S *and let* **x**: U ⊂ R² → R³ *be a map with* p ∈ **x**(U) *such that conditions* 1 *and* 3 *of Def.* 1 *hold. Assume that* **x** *is one-to-one. Then* **x**⁻¹ *is continuous.*

b) Proof. Write $\vec{x}(u,v) = (x(u,v), y(u,v), z(u,v))$, $(u,v) \in U$ and let $q \in U$. We can assume by conds. 1 & 3 that $\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$. Now, let $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(x,y,z) \mapsto (x,y)$. From the inverse function theorem, we have neighborhoods V_1 of q in U and V_2 of $\pi \circ \vec{x}(q)$ in $\mathbb{R}^2$ such that $\pi \circ \vec{x}$ maps V_1 diffeomorphically into V_2.

Assume now $\vec{x}$ is one-to-one. Then, restricted to $\vec{x}(V_1)$,

$$\vec{x}^{-1} = (\pi \circ \vec{x})^{-1} \circ \pi$$

Thus, $\vec{x}^{-1}$ is continuous since it's a composition of continuous functions. Since q is arbitrary, $\vec{x}^{-1}$ is continuous in $\vec{x}(U)$.

C) Other Problems

a) Problem 7 on page 66:

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

a) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 2x + 2y + 2z - 2 = 0 \quad \Rightarrow \quad x + y + z = 1$

Critical points all lie on the plane $\boxed{x+y+z-1=0.}$

Critical values are $V_c = \{ (x+y+z-1)^2 : x+y+z-1 = 0 \}$

$$\boxed{V_c = \{0\}}$$

b) $f(x,y,z) = c \Rightarrow (x+y+z-1)^2 = c$

$f^{-1}(c) = \{ (x,y,z) : (x+y+z-1)^2 = c \}$

$f(x,y,z) = c$ is a regular surface so long as $f(x,y,z)$ is not a critical value. That is, so long as $x+y+z-1 \neq 0 \Rightarrow x+y+z \neq 1$.

So, $f(x,y,z) = c$ is a regular surface as long as $\boxed{c \neq 0.}$

$f(x,y,z) = (x+y+z-1)^2 = c$

$(1-1)^2 = c \quad \Rightarrow c = 0$

c) $f(x,y,z) = xyz^2$

$\frac{\partial f}{\partial x} = yz^2 \qquad \frac{\partial f}{\partial y} = xz^2 \qquad \frac{\partial f}{\partial z} = 2xyz$

Critical points: the x and y axes as well as the origin

Critical values: 0

$f(x,y,z) = xyz^2 = c = x(yz^2) = 0 \Rightarrow f(x,y,z) = c$ is not a regular surface for any choice of c.

b) Problem 11, section 2-2

11. Show that the set $S = \{(x, y, z) \in R^3; z = x^2 - y^2\}$ is a regular surface and check that parts a and b are parametrizations for S:

a. $\mathbf{x}(u, v) = (u + v, u - v, 4uv)$, $(u, v) \in R^2$.

***b.** $\mathbf{x}(u, v) = (u \cosh v, u \sinh v, u^2)$, $(u, v) \in R^2$, $u \neq 0$.

Which parts of S do these parametrizations cover?

a) First, we show $\vec{x}$ has an invertible differential:

$$d\vec{x}(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 4v & 4u \end{bmatrix}$$

, which is invertible everywhere because it contains $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which is invertible.

For all $(u,v) \in \mathbb{R}^2$: $(u+v)^2 - (u-v)^2 = u^2 + v^2 + 2uv - u^2 - v^2 + 2uv = 4uv$.
So, $z = x^2 - y^2$. Therefore $\vec{x}(\mathbb{R}^2) \subset S$.

Finally, we show $\vec{x}$ is bijective, meaning it is homeomorphic. Suppose $\vec{x}(u_1, v_1) = \vec{x}(u_2, v_2)$.
Then

$$A\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = A\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \qquad (A \text{ invertible})$$

Therefore, $\vec{x}$ is a parameterization of S.

b) First, we show $\vec{x}$ has an invertible differential:

$$d\vec{x}(u,v) = \begin{bmatrix} \cosh v & u\sinh v \\ \sinh v & u\cosh v \\ 4v & 4u \end{bmatrix}$$

, which has linearly ind. cols b/c $u \neq 0$. Therefore, the map is always invertible.

Now note that

$$x^2 - y^2 = u^2(\cosh^2 v - \sinh^2 v) = u^2 = z \quad \checkmark$$

So, $\vec{x}(\mathbb{R}^2) \subset S$. Finally, to show $\vec{x}$ is one-to-one, observe that z is determined from $\pm u$. Since $\cosh v > 0$, $\text{sign}(u) = \text{sign}(x)$. Thus, $\sinh v$ (and therefore v) is determined.

Therefore, $\vec{x}$ is a homeomorphism and a parameterization of S.

Problem 1, Section 2-3

*1. Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A: S^2 \longrightarrow S^2$ be the (*antipodal*) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

c) Note that

$$dA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

, which is invertible everywhere. Also, note that A is a bijection. Thus, A is a diffeomorphism.

Problem 8, section 2-3

*8. Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ and $H = \{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 1\}$. Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively, and let $F: S^2 - \{N\} \cup \{S\} \longrightarrow H$ be defined as follows: For each $p \in S^2 - \{N\} \cup \{S\}$ let the perpendicular from p to the z axis meet $0z$ at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (Fig. 2-20). Prove that F is differentiable.

d) Given a point $p = (x, y, z) \in S^2$ we find q by projection onto the z-axis, so $q = (0, 0, z)$. The half-line joining q to p is parameterized by (tx, ty, z) where $0 \le t$. This line intersects H when

$$t^2x^2 + t^2y^2 - z^2 = 1$$

Solving for t we get

$$t = \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}$$

So

$$F(p) = \left(\frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}} x, \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}} y, z \right)$$

Let $V = \mathbb{R}^3 \setminus \{(x, y, z) : x = y = 0\}$, then V is an open subset of $\mathbb{R}^3$ and F has continuous partial derivatives on V. $\therefore$ F is invertible on V. Since $S^2 \setminus (\{N\} \cup \{S\}) \subset V$ and S^2 and H are regular surfaces, $F|S^2 : S^2 \to H$ is differentiable.

Problem 10, Section 2-3

10. Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q (Fig. 2-21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?

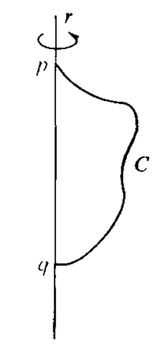


Figure 2-21

e) The surface generated needs to have parameterizations at the points p and q. Also, C should have no self-intersections. These conditions will be met if the curve formed by joining C w/ its reflection over r is a simple closed regular curve.

More formally, let C' be the curve given by the reflection of C over r. We require that C satisfies the condition that $C \cup C'$ is a regular closed curve.

Problem 15, Section 2-3

15. Let C be a regular curve and let $\alpha: I \subset R \longrightarrow C$, $\beta: J \subset R \longrightarrow C$ be two parametrizations of C in a neighborhood of $p \in \alpha(I) \cap \beta(I) = W$. Let

$$h = \alpha^{-1} \circ \beta: \beta^{-1}(W) \longrightarrow \alpha^{-1}(W)$$

be the change of parameters. Prove that

a. h is a diffeomorphism.

b. The absolute value of the arc length of C in W does not depend on which parametrization is chosen to define it, that is,

$$\left|\int_{t_0}^{t} |\alpha'(t)|\,dt\right| = \left|\int_{\tau_0}^{\tau} |\beta'(\tau)|\,d\tau\right|, \qquad t = h(\tau), t \in I, \tau \in J.$$

a) This proof is offered in the textbook:

Proof: $h = \alpha^{-1} \circ \beta$ is a homeomorphism b/c it is the composition of homeomorphisms. It is not possible to conclude it is differentiable since α^{-1} is defined in an open subset of C.

We proceed as follows: let $r \in \beta^{-1}(W)$ and set $q = h(r)$. Since $\alpha(u,v) = (x(u,v), y(u,v), z(u,v))$ is a parameterization, we can assume that

$$\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$$

We extend α to a map $F: I \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$F(u,v,t) = (x(u,v), y(u,v), z(u,v)+t) \qquad (u,v) \in I,\ t \in \mathbb{R}$$

It is clear that F is differentiable and the restriction $F|U \times \{0\} = \vec{x}$.
Calculating the determinant of the differential dF_q:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & 1 \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0$$

We can then apply the Inverse function theorem: $\exists$ a neighborhood M of $\alpha(q) \subset \mathbb{R}^3$ such that F^{-1} exists and is differentiable in M.

By continuity of β, there exists a neighborhood N of r in J such that $y(N) \subset M$. Notice that, restricted to N, $h|N = F^{-1} \circ \vec{y}\,|N$ is a composition of differentiable maps. We can thus apply the chain rule for maps and conclude that h is differentiable at r. $\therefore h$ differentiable on $\beta^{-1}(W)$.

Same argument can be used to show h^{-1} differentiable $\Rightarrow h$ is diffeomorphism.

b) $\left|\int_{t_0}^{t} |\beta'(\tau)|\, d\tau\right| = \left|\int_{t_0}^{t} |(\alpha \circ h)'(\tau)|\, d\tau\right| = \left|\int_{t_0}^{t} |\alpha'(\tau)|\, d\tau\right|$ ✓