Math 142 Problem Set 4

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a) Show that the set of rigid motions $E(3)$ forms a group. (Later, we will see that $E(3)$ is in fact a Lie group.)

Rigid motion: Orthogonal map composed w/ a translation.

Let $\rho \in E(3)$. Then, for $x \in \mathbb{R}^3$, $\rho(x) = Ax + b$, A is a 3×3 orthogonal matrix, b is a 3×1 vector.

Identity: Let $e(x) = Ax + b$ $\quad A = I_3, \quad b = (0,0,0)$

$\Rightarrow e(x) = Ax + b = x + 0 = x$ ✓

Closure: let $\rho_1 = A_1x + b_1$, $\rho_2 = A_2x + b_2$. $A_1, A_2 \in O(3)$ and $b_1, b_2 \in \mathbb{R}^3$.

$$\rho_1 \circ \rho_2 (x) = \rho_1(A_2x + b_2) = A_1A_2x + A_1b_2 + b_1 = (A_1A_2)x + (A_1b_2 + b_1)$$

Since $A_1, A_2 \in O(3)$, then $A_1A_2 \in O(3)$. Furthermore, $A_1b_2 + b_1$ is a vector in $\mathbb{R}^3$ which is a translation. So, $\rho_1 \circ \rho_2 \in E(3)$.

Associativity: Let $\rho_n = A_nx + b_n$, $A_n \in O(3)$ $b_n \in \mathbb{R}^3$. Then

$$\begin{aligned}
\rho_1 \circ (\rho_2 \circ \rho_3)(x) &= A_1(A_2(A_3x + b_3) + b_2) + b_1 \\
&= A_1A_2A_3x + A_1(A_2b_3 + b_2) + b_1 \\
&= (A_1A_2)A_3x + (A_1A_2)b_3 + (A_1b_2 + b_1) \\
&= (\rho_1 \circ \rho_2) \circ \rho_3 \quad ✓
\end{aligned}$$

Inverse: Let $\rho = Ax + b$. Then $\rho^{-1} = A^Tx - A^Tb$:

$$\begin{aligned}
\rho \circ \rho^{-1}(x) &= A(A^Tx - A^Tb) + b \\
&= AA^Tx - AA^Tb + b \\
&= x - b + b \qquad (A \in O(3) \Rightarrow A^{-1} = A^T) \\
&= x = e(x). \quad ✓
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho^{-1} \circ \rho &= A^T(Ax + b) - A^Tb \\
&= A^TAx + A^Tb - A^Tb = x = e(x). \quad ✓
\end{aligned}$$

Since $E(3)$ satisfies all 4 group axioms, it is a group.

B) Problems from Lecture

a) Show that of all simple closed curves in the plane with given length l, a circle bounds the largest area.

From the proof of the isoperimetric inequality, we know that, for a curve of length l,

$$l^2 - 4\pi A \geq 0.$$

So, if the curve is a circle, we can show a circle bounds the maximum area by showing equality holds for a circle. That is, if $l = 2\pi r$ and $A = \pi r^2$, where r is the circle's radius. We then have:

$$l^2 - 4\pi A = (2\pi r)^2 - 4\pi(\pi r^2) = 4\pi^2 r^2 - 4\pi^2 r^2 = 0$$

Thus, $l^2 - 4\pi A = 0$ and we see that a circle bounds the largest area.

C) Other problems

1-6 Problem 2:

2. Let $\alpha: I \longrightarrow R^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that

***a.** The osculating plane at s is the limit position of the plane passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \longrightarrow 0$.

b. The limit position of the circle passing through $\alpha(s)$, $\alpha(s + h_1)$, $\alpha(s + h_2)$ when $h_1, h_2 \longrightarrow 0$ is a circle in the osculating plane at s, the center of which is on the line that contains $n(s)$ and the radius of which is the radius of curvature $1/k(s)$; this circle is called the *osculating circle* at s.

2) a) The osculating plane has to contain $\alpha(s)$, $\alpha(s+h_1)$ and $\alpha(s+h_2)$, and it is unique. Call this plane $P(h_1, h_2)$. We know that $\alpha(s) \in P(h_1, h_2)$

We first show that $\alpha(s) + \alpha'(s) \in P(h_1, h_2)$. Since we operate in the limit as $h_1, h_2 \to 0$:

$$\alpha(s) + \alpha'(s) = \alpha(s) + \frac{1}{h_1}\left(\alpha(s+h_1) - \alpha(s)\right) \in P(h_1, h_2)$$

(since this is an affine combination of two points in the plane)

Now we show $\alpha(s) + \alpha''(s) \in P(h_1, h_2)$:

$$\alpha(s) + \alpha''(s) = \alpha(s) + \frac{1}{h_2}\left(\alpha'(s+h_2) - \alpha'(s)\right)$$

$$= \alpha(s) + \frac{1}{h_2}\left(\frac{\alpha(s+h_2) - \alpha(s)}{h_2} - \frac{\alpha(s+h_1) - \alpha(s)}{h_1}\right) \in P(h_1, h_2)$$

(once again, an affine combination of points in the plane).

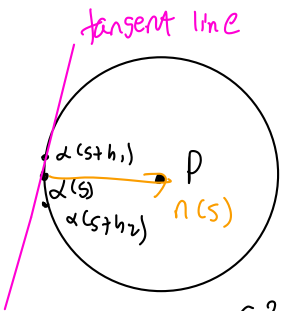
Since $P(h_1, h_2)$ contains $\alpha(s)$, $\alpha(s)+\alpha'(s)$, and $\alpha(s)+\alpha''(s)$, it fits the definition of the osculating plane at $\alpha(s)$.

b) Call the point p the center of the circle. The radius r is then:

$$r = \|\alpha(s) - p\| = \|\alpha(s+h_1) - p\| = \|\alpha(s+h_2) - p\|$$

Since $h_1, h_2 \to 0$, we know the line connecting $\alpha(s)$ and $\alpha(s+h_1)$, as well as the line connecting $\alpha(s)$ with $\alpha(s+h_2)$, are tangent to the circle.

Since $n(s)$ is in the osculating plane and orthogonal to said tangent line, it must point towards p.



Parameterize the circle by

$$\beta(t') = (r\cos(t'), r\sin(t'))$$

we want to parameterize β by arc length. Note that

$$\beta'(t') = \sqrt{r^2\cos^2(t') + r^2\sin^2(t')} = r$$

So, $l = \int_{t=0}^{2\pi} r = 2\pi r \Rightarrow$ reparameterize $t = rt' \Rightarrow t' = t/r$

So, $\beta(t) = (r\cos(t/r), r\sin(t/r))$. Now,

$$\beta'(t) = (-\sin(t/r), \cos(t/r))$$

$$\beta''(t) = \left(-\frac{1}{r}\cos(t/r), -\frac{1}{r}\sin(t/r)\right)$$

And its curvature

$\|\beta''(t)\| = k(s)$ (shares 3 points with $\alpha(s)$)

$$= \sqrt{\frac{1}{r^2}\cos^2(t/r) + \frac{1}{r^2}\sin^2(t/r)} = \frac{1}{r} = k(s) \Rightarrow \boxed{r = \frac{1}{k(s)}}$$ ✓

Problem 1, 1-7

*1. Is there a simple closed curve in the plane with length equal to 6 feet and bounding an area of 3 square feet?

1) Check isoperimetric inequality: $A = 3$, $\ell = 6$

$\ell^2 - 4\pi A = 36 - 4\pi(3) = 36 - 12\pi < 0$ $(\pi > 3)$

$\boxed{\text{No}}$. This would violate the isoperimetric inequality

Problem 2, 1-7

*2. Let $\overline{AB}$ be a segment of straight line and let $l >$ length of AB. Show that the curve C joining A and B, with length l, and such that together with $\overline{AB}$ bounds the largest possible area is an arc of a circle passing through A and B (Fig. 1-35).

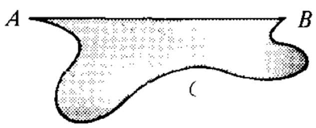


Figure 1-35

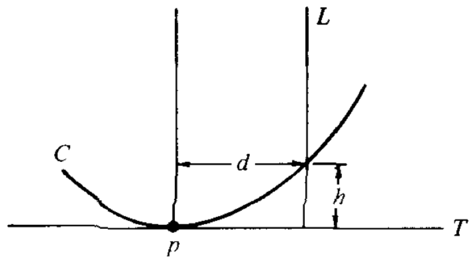


Figure 1-36

2) From the isoperimetric inequality, we know the maximum area is bounded by a circle — anything else would either violate the inequality or not bound the maximum area (as proved in part B).

Problem 3, 2-2

3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = (x, y, \sqrt{x^2+y^2})$

Consider the case $f(x,y,z) = 0$. Note that

$f_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$ $\qquad f_y = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$

f is therefore not differentiable at $(0,0)$. Therefore, the two-sheeted cone is not a regular surface since it is not differentiable at $f(0,0) = (0,0,0)$.

Problem 5, 2-2

*5. Let $P = \{(x, y, z) \in R^3; x = y\}$ (a plane) and let $\mathbf{x}: U \subset R^2 \to R^3$ be given by

$$\mathbf{x}(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in R^2; u > v\}$. Clearly, $\mathbf{x}(U) \subset P$. Is $\mathbf{x}$ a parametrization of P?

$\vec{x}$ is surjective to the neighborhood $V = B_1(1,1,1)$, so it is a parameterization.

Problem 10, 2-2

10. Let C be a figure "8" in the xy plane and let S be the cylindrical surface over C (Fig. 2-11); that is,

$$S = \{(x, y, z) \in R^3; (x, y) \in C\}.$$

Is the set S a regular surface?

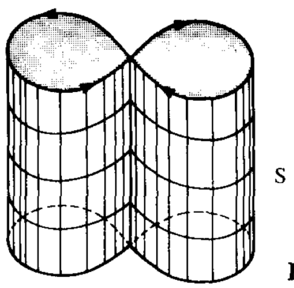


Figure 2-11

$\boxed{\text{No}}$. The intersection is a critical value of the map $f: C \to \mathbb{R}^3$ defined by $f(x,y) = (x, y, h)$, where h is the cylinder's height. Since f has a critical point, S is not a regular surface.

2-2 Problem 16

16. One way to define a system of coordinates for the sphere S^2, given by $x^2 + y^2 + (z-1)^2 = 1$, is to consider the so-called *stereographic projection* $\pi: S^2 \sim \{N\} \to R^2$ which carries a point $p = (x, y, z)$ of the sphere S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy plane with the straight line which connects N to p (Fig. 2-12). Let $(u, v) = \pi(x, y, z)$, where $(x, y, z) \in S^2 \sim \{N\}$ and $(u, v) \in xy$ plane.

a. Show that $\pi^{-1}: R^2 \to S^2$ is given by

$$\pi^{-1}\begin{cases} x = \dfrac{4u}{u^2+v^2+4}, \\ y = \dfrac{4v}{u^2+v^2+4}, \\ z = \dfrac{2(u^2+v^2)}{u^2+v^2+4}. \end{cases}$$

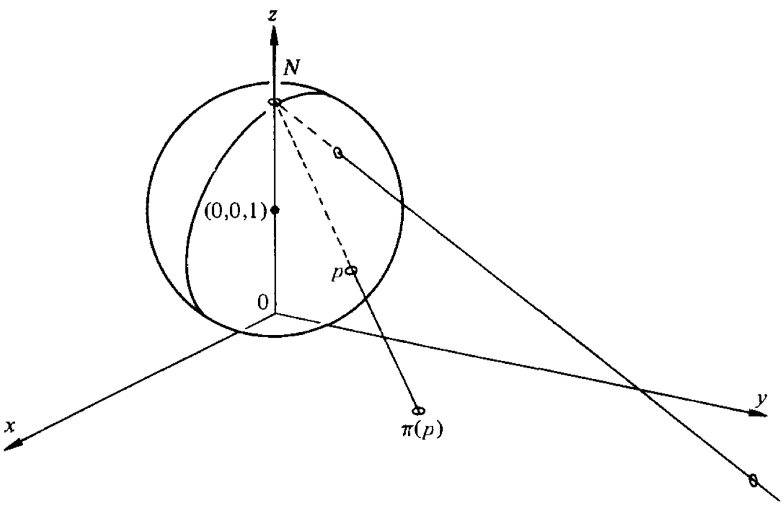


Figure 2-12. The stereographic projection.

b. Show that it is possible, using stereographic projection, to cover the sphere with two coordinate neighborhoods.

a) Since the sphere is of radius 1, we want to find a mapping $\pi^{-1}(u,v)$ such that

$$\|\pi^{-1}(u,v) - (0,0,1)\| = 1. \text{ Therefore,}$$

$\exists a$ s.t. $(0,0,2) + a(\pi^{-1}(u,v) - (0,0,2)) = (u,v,0)$

So,

$$\pi^{-1}(u,v) = \frac{1}{a}(u,v,-2) + (0,0,2)$$

$$\|\pi^{-1}(u,v) - (0,0,1)\| = \|\tfrac{1}{a}(u,v,-2) + (0,0,2) - (0,0,1)\|$$

$$= \|\tfrac{1}{a}(u,v,-2) + (0,0,1)\| = \|(\tfrac{u}{a}, \tfrac{v}{a}, -\tfrac{2}{a}+1)\| = 1$$

$$\sqrt{\frac{u^2+v^2}{a^2} + \left(\frac{-2}{a}+1\right)^2} = 1 \Rightarrow \frac{u^2+v^2}{a^2} + \left(\frac{4}{a^2} - \frac{4}{a} + \cancel{1}\right) = \cancel{1}$$

$$\frac{u^2+v^2+4}{a^2} = \frac{4}{a} \implies \frac{u^2+v^2+4}{a} = 4 \implies a = \frac{u^2+v^2+4}{4}$$

So, we have

$$\pi^{-1}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{-2\cdot\cancel{4}}{u^2+v^2+4} + \frac{2(u^2+v^2+\cancel{4})}{u^2+v^2+4} \right)$$

$$= \boxed{\left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right)}$$

b) The first neighborhood $V_1 = S^2 \setminus \{N\}$ is covered by π^{-1}. To cover N, we need to define another sterographic projection ϕ, but with the South pole $S=(0,0,-2)$ instead of N. So, by the process above, we can deduce ϕ^{-1} will cover $S^2 \setminus \{S\}$ — the two together cover the whole sphere!

Extra Credit

Assume for the sake of contradiction that $\exists A'$ such that $A' > A$ for a given length l. We know, then, that

$$l^2 - 4\pi A' < l^2 - 4\pi A.$$

But, by the isoperimetric inequality, we must have

$$l^2 - 4\pi A = (2\pi r)^2 - 4\pi(\pi r^2) = 4\pi^2 r^2 - 4\pi^2 r^2 = 0$$

Since

$l^2 - 4\pi A' < l^2 - 4\pi A = 0$, we have $l^2 - 4\pi A' < 0$.

This is a contradiction, so A' cannot exist.