Math 142 Problem Set 8

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C) Other Problems

5. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \qquad F = 0.$$

b. The coefficients of the second fundamental form are

$$e = 2, \qquad g = -2, \qquad f = 0.$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \qquad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

d. The lines of curvature are the coordinate curves.

Parts (a-c) are done computationally. See (and run) PS8.py.

d) The lines of curvature are coordinate curves because $f = F = 0$. (Source: Section 3-3 in Do Carmo).

1. Prove that the differentiability of a vector field does not depend on the choice of a coordinate system.

1) A change of coordinates for a vector v can be represented by $v' = Mv$, where M is a transformation matrix. Since a matrix multiplication is simply a combination of addition and multiplication operations, and differentiation is linear, the resulting vector is also differentiable. That is, assume $v = (v_1, v_2, \ldots, v_n)$ is differentiable. Then an arbitrary

$$v_i' = \sum_{j=1}^{n} M_{ij} v_j$$

And so

$$\frac{\partial}{\partial x} v_i' = \sum_{j=1}^{n} \frac{\partial}{\partial x} M_{ij} v_j = \sum_{j=1}^{n} M_{kj} \frac{\partial}{\partial x} v_j$$

is differentiable.

2. Prove that the vector field obtained on the torus by parametrizing all its meridians by arc length and taking their tangent vectors (Example 1) is differentiable.

The tangent vector for a curve parameterized by arc length is just the derivative of the curve. Therefore, we just need to show that the meridians parameterized by arc length are twice differentiable. Reparameterizing doesn't change differentiability. So, take any twice-differentiable parameterization of a meridian $\alpha(a,b) = \sqrt{a^2+b^2}$.

5. Let S be a surface and $\mathbf{x}\colon U \longrightarrow S$ be a parametrization of S. If $ac - b^2 < 0$, show that

$$a(u,v)(u')^2 + 2b(u,v)u'v' + c(u,v)(v')^2 = 0$$

can be factored into two distinct equations, each of which determines a field of directions on $\mathbf{x}(U) \subset S$. Prove that these two fields of directions are orthogonal if and only if

$$Ec - 2Fb + Ga = 0.$$

Let $x = u'$ and $y = v'$. Then, following the book, we can factor this as

$$(Ax+By)(Ax+Dy) \quad \text{such that} \quad A^2 = a \Rightarrow A = \sqrt{a}$$

$$BD = c$$

$$A(B+D) = 2b \Rightarrow B+D = \frac{2b}{\sqrt{a}}$$

$$B = \frac{2b}{\sqrt{a}} - D \Rightarrow \left(\frac{2b}{\sqrt{a}} - D\right)D = c \Rightarrow \frac{2b}{\sqrt{a}}D - D^2 = c \Rightarrow D^2 - \frac{2b}{\sqrt{a}}D + c = 0$$

$$D = \frac{\frac{2b}{\sqrt{a}} \pm \sqrt{\frac{4b^2}{a} - 4c}}{2} = \frac{b}{\sqrt{a}} \pm \sqrt{\frac{b^2}{a} - c}$$

$$ac - b^2 < 0 \Rightarrow -\frac{b^2}{a} + c < 0$$

$$\frac{b^2}{a} > c$$

Choose $D = \frac{b}{\sqrt{a}} + \sqrt{\frac{b^2}{a} - c}$

$$B + D = \frac{2b}{\sqrt{a}} \Rightarrow B = \frac{2b}{\sqrt{a}} - D = \frac{b}{\sqrt{a}} - \sqrt{\frac{b^2}{a} - c}$$

So, we have

$$\boxed{\left[\sqrt{a}\,u' + \left(\frac{b}{\sqrt{a}} - \sqrt{\frac{b^2}{a^2} - c}\right)v'\right]\left[\sqrt{a}\,u' + \left(\frac{b}{\sqrt{a}} + \sqrt{\frac{b^2}{a^2} - c}\right)v'\right] = 0}$$

8. Show that if w is a differentiable vector field on a surface S and $w(p) \neq 0$ for some $p \in S$, then it is possible to parametrize a neighborhood of p by $\mathbf{x}(u, v)$ in such a way that $\mathbf{x}_u = w$.

Let $M: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Let $y = x(M(u,v))$. Then we have $y_u = M x_u$. Since $w \neq 0$ and $x_u \neq 0$, $\exists$ some M such that $Mx_u = w$. So, by choosing the right M we can always guarantee a parameterization of S where $x_u = w$.