Lecture 8: Local Canonical Form

Prof. Weiqing Gu

Math 142:
Differential Geometry

Local Canonical Form

One of the most effective methods of solving problems in geometry consists of finding a coordinate system which is adapted to the problem. In the study of local properties of a curve, in the neighborhood of the point s, we have a natural coordinate system, namely the Frenet trihedron at s. It is therefore convenient to refer the curve to this trihedron.

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length without singular points of order 1 (that is, $\alpha(s) \neq 0$ for all $s \in I$). We shall write the equations of the curve, in a neighborhoods of s_0, using the trihedron $t(s_0)$, $n(s_0)$, $b(s_0)$ as a basis for $\mathbb{R}^3$.

Local Canonical Form

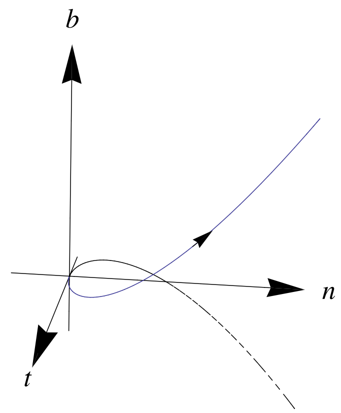
Equations

Let us now take the system $0xyz$ in such a way that the origin 0 agrees with $\alpha(0)$ and that $t = (1, 0, 0)$, $n = (0, 1, 0)$, and $b = (0, 0, 1)$. Under these conditions, $\alpha(s) = (x(s), y(s), z(s))$ is given by

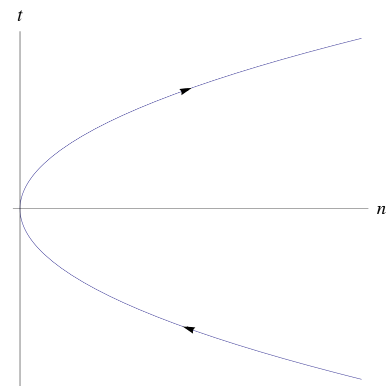
$$\begin{cases} x(s) = s - \dfrac{k^2 s^3}{6} + R_x, \\ y(s) = \dfrac{ks^2}{2} + \dfrac{k' s^3}{6} + R_y, \\ z(s) = -\dfrac{k\tau s^3}{6} + R_z, \end{cases} \tag{1}$$

where $R = (R_x, R_y, R_z)$. The representation (1) is called the *local canonical form* of α, in a neighborhood of $s = 0$.

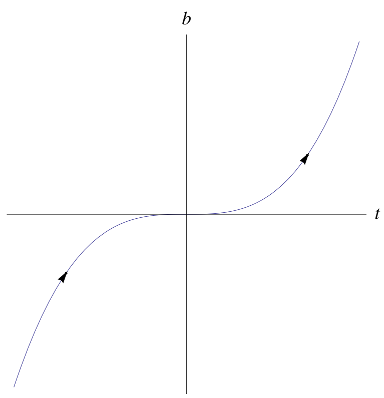
A Sketch of projections of the trace of α, for small s, in the tn, tb, and nb planes:



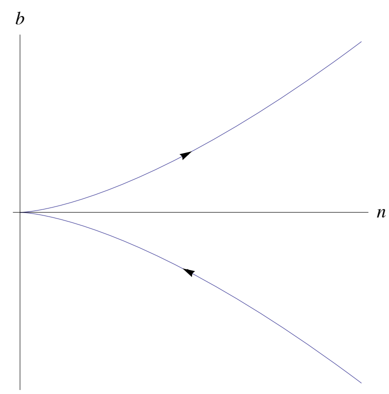
A Curve in $\mathbb{R}^3$



Projection over the plane tn



Projection over the plane tb



Projection over the plane nb

Cautions and Subtleties

1. We must distinguish a curve from its trace!

Example

The two distinct parametrized curves

$$\alpha(t) = (\cos t, \sin t),$$
$$\beta(t) = (\cos 2t, \sin 2t),$$

where $t \in (0 - \epsilon, 2\pi + \epsilon), \epsilon > 0$, have the same trace, namely, the circle $x^2 + y^2 = 1$. Notice that the velocity vector of the second curve is the double of the first one.

Cautions and Subtleties

2. Changing the orientation of a curve.

- Both curvature and torsion are invariant under the change of orientation, while the tangent vector changes its orientation. The normal vector is invariant under a change of orientation and the binormal changes orientation.

Cautions and Subtleties

3. $k(s) = 0 \Rightarrow \alpha(s)$ is a straight line (homework)
$\tau(s) \not\Rightarrow \alpha(s)$ is a plane curve.

Example

Encouraged To Do

Do Carmo p. 25 #10

Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t > 0 \\ (0, 0, 0), & t = 0 \end{cases}$$

a. Prove that α is a differentiable curve

b. Prove that α is regular for all t and that the curvature $k(t) \neq 0$ for $t \neq 0, t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c. Show that the limit of the osculating planes as $t \to 0$, $t > 0$, is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0$, $t < 0$, is the plane $z = 0$ (this implies that the normal vector is discontinuous at $t = 0$ and shows why we excluded points where $k = 0$).

d. Show that τ can be defined so that $\tau = 0$, even though α is not a plane curve.

Cautions and Subtleties

4. While $k(s) \geq 0$, $\tau(s)$ may be either positive or negative.

From the third equation of (1) it follows that if $\tau < 0$ and s is sufficiently small, then $z(s)$ increases with s. Let us make the convention of calling the "positive side" of the osculating plane that side toward which b is pointing. Then, since $z(0) = 0$, when we describe the curve in the direction of increasing arc length, the curve will cross the osculating plane at $s = 0$, pointing toward the positive side. If, on the contrary, $\tau > 0$, the curve (described in the direction of increasing arc length) will cross the osculating plane pointing to the side opposite the positive side.

Positive and Negative Torsion

Example

The helix of Exercise 1 of Sec. 1-5 (Do Carmo) has negative torsion. An example of a curve with positive torsion is the helix

$$\alpha(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, -b \frac{s}{c}\right)$$

obtained from the first one by a reflection in the xz plane.

Remark

It is also usual to define torsion by $b' = -\tau n$. With such a definition, the torsion of the helix of Exercise 1 becomes positive.

Exercise Problems

Do Carmo pg. 47 #2a

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that the osculating plane at s is the limit position of the plane passing through $\alpha(s)$, $\alpha(s+h_1)$, $\alpha(s+h_2)$ when $h_1, h_2 \to 0$.

Solution

Consider the local canonical form at s. Without loss of generality, we may assume that $s = 0$, and we construct our coordinate system so that $e_1 = \vec{t}(0)$, $e_2 = \vec{n}(0)$, and $e_3 = \vec{b}(0)$.

Consider a plane passing through $\alpha(0)$, $\alpha(h_1)$, and $\alpha(h_2)$. Say the plane equation is $ax + by + cz = 0$. Here we may assume that the normal vector

$$N = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

to the plane is a unit vector.

Exercise Problems

Solution

Now for any point $\alpha(s)$ on the curve which is close to $\alpha(0)$, let us consider the length $F(s)$ of the normal projection of $\alpha(s)$ to N. Then

$$F(s) = \alpha(s) \cdot N = ax(s) + by(s) + cz(s). \tag{2}$$

Since $\alpha(h_1)$, $\alpha(h_2)$, and $\alpha(0)$ are on the plane, their projection to N is 0. Hence, $F(0) = F(h_1) = F(h_2) = 0$.

By differentiating, we see that $F'(s) = \alpha'(s) \cdot N$. Therefore

$$F'(0) = \alpha'(0) \cdot N = t(0) \cdot N = (1, 0, 0) \cdot (a, b, c) = a. \tag{3}$$

Similarly, $F''(s) = \alpha''(s) \cdot N$, so $F''(0) = k(0)b$.

Note: You can also find $F'(s)$ by using local canonical form to calculate $x'(s), y'(s)$, and $z'(s)$ and get $x'(0) = 1, y'(0) = z'(0) = 0$. Similarly, to find $F''(s)$ you can find $x''(0) = 0$, $y''(0) = k$, and $z''(0) = 0$.

Exercise Problems

Solution

However,

$$F'(0) = \lim_{h_1 \to 0} \frac{\cancelto{0}{F(h_1)} - \cancelto{0}{F(0)}}{h_1} = \lim_{h_1 \to 0} \frac{0}{h_1} = 0,$$

So by Equation (2), $a = 0$.

Similarly,

$$F''(0) = \lim_{h_2 \to 0} \frac{F'(h_2) - F'(0)}{h_2} = \lim_{h_2 \to 0} \frac{\displaystyle\lim_{h_1 \to h_2} \frac{\cancelto{0}{F(h_1)} - \cancelto{0}{F(h_2)}}{h_1 - h_2} - 0}{h_2} = 0,$$

so by Equation (3), $k(0)b = 0$. *Since* $k(0) \neq 0$, *it follows that* $b = 0$.

Exercise Problems

Solution

Thus, as $h_1, h_2 \to 0$*, the equation of the plane becomes* $cz = 0$*. Since* $a^2 + b^2 + c^2 = 1$ *by assumption, it follows that* $c = \pm 1$*. Therefore, the equation of the limit position of the plane passing through* $\alpha(0)$*,* $\alpha(h_1)$*,* $\alpha(h_2)$ *is* $z = 0$*. Since this is precisely the osculating plane at* 0*, the assertion holds.*

Exercise Problems

Do Carmo pg. 47 #2b

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that the limit position of the circle passing through $\alpha(s)$, $\alpha(s+h_1)$, $\alpha(s+h_2)$ when $h_1, h_2 \to 0$ is a circle in the osculating plane at s, the center of which is on the line that contains $n(s)$ and the radius of which is the radius of curvature $1/k(s)$; this circle is called the *osculating circle* at s.

Solution

We have shown that the limit position of the plane $P_{h_1 h_2}$ passing through $\alpha(0)$, $\alpha(h_1)$, and $\alpha(h_2)$ as $h_1, h_2 \to 0$ is the osculating plane P at $s = 0$.

If a circle passes through $\alpha(0)$, $\alpha(h_1)$, and $\alpha(h_2)$, then it must lie on the plane $P_{h_1 h_2}$. As $h_1, h_2 \to 0$, $P_{h_1 h_2} \to P$, so the circle $C_{h_1 h_2}$ tends to a limit circle C in the plane P with radius r. Note that r could be 0.

Exercise Problems

Solution

Since the circle passes through the origin at $\alpha(0)$, we can write the circle's equation as

$$(x - x_0)^2 + (y - y_0)^2 = x_0^2 + y_0^2,$$

or more simply,

$$x^2 - 2x_0x + y^2 - 2y_0y = 0.$$

Notice that, at $\alpha(0)$, the limiting circle and the curve have the same tangent. Let the circle be parametrized by h_1:

$$x(h_1)^2 - 2x_0x(h_1) + y(h_1)^2 - 2y_0y(h_1) = 0. \tag{4}$$

Exercise Problems

Solution

As h_1 approaches zero, the point on the circle can be viewed as the point on the curve up to the derivative of order 1. Thus, by the local canonical form,

$$\begin{cases} x(h_1) = h_1 - \frac{k^2}{6} h_1^3 + R_x \\ y(h_1) = \frac{k}{2} h_1^2 + \frac{k'}{6} h_1^3 + R_y \\ z(h_1) = -\frac{k\tau}{6} h_1^3 + R_z \end{cases} \tag{5}$$

Plugging (5) into (4), dividing both sides by h_1, and taking the limit as $h_1 \to 0$, we find that $x_0 = 0$. Hence, (4) becomes

$$x(h_1)^2 + y(h_1)^2 - 2y_0 y(h_1) = 0. \tag{6}$$

Dividing both sides of (6) by h_1^2 and taking the limit as $h_1 \to 0$, we find that $y_0 = 1/k$. Thus, the circle is centered on the y axis (the line containing the $\vec{n}(s)$ by construction) and has radius $1/k$, as desired.