Lecture 9: Global Theory

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Math 142:
Differential Geometry

Global Theory

1. Definitions
   Closed Curve
   - Simple
   - Nonsimple

   Interior of a curve (region)
   - Positively oriented
   - Negatively oriented
2. A formula to find the area of a region bounded by a curve
3. Solving the traditional Isoperimetric Problem

Definitions and an Area Formula

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We usually consider the curve $\alpha : [0, \ell] \to \mathbb{R}^2$ parametrized by arc length s; hence, ℓ is the length of α. Sometimes we refer to a simple closed curve C, meaning the trace of such an object. The curvature of α will be taken with a sign (see next slide).

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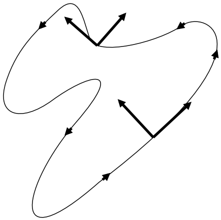
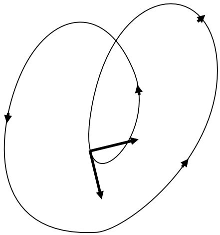
We assume that a *simple closed curve C in the plane bounds a region of the plane* that is called the *interior* of C.

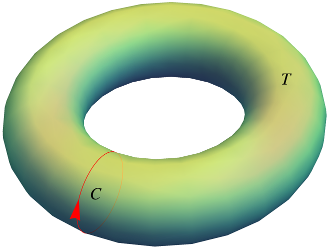
Illustrations



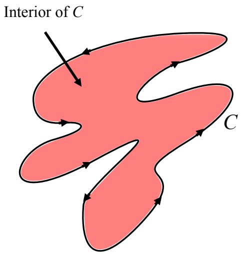
(a) A simple closed curve



(b) A (nonsimple) closed curve



(c) A simple closed curve C on a torus T; C bounds no region on T

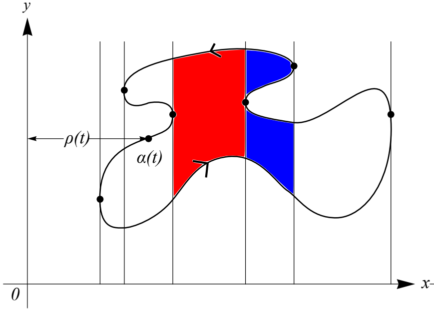


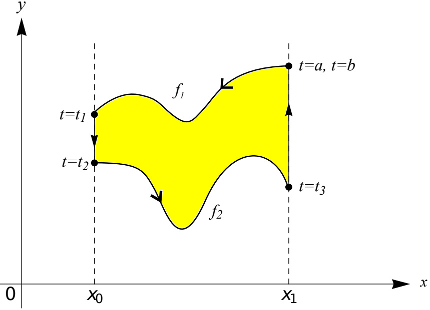
(d) C is positively oriented

A Formula to Find the Area of a Region bounded by C

We shall make use of the following formula for the area A bounded by a positively oriented simple closed curve $\alpha(t) = (x(t), y(t))$, where $t \in [a, b]$ is an arbitrary parameter:

$$A = -\int_a^b y(t)x'(t)\,dt = \int_a^b x(t)y'(t)\,dt = \frac{1}{2}\int_a^b (xy' - yx')\,dt.$$





The Isoperimetric Problem

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Theorem (The Isoperimetric Inequality)

Let C *be a simple closed plane curve with length* ℓ*, and let* A *be the area of the region bounded by* C*. Then*

$$\ell^2 - 4\pi A \geq 0,$$

and equality holds if and only if C *is a circle.*

The Isoperimetric Inequality

Proof

Let E and E' be two parallel lines which do not meet the closed curve C, and move them together until they first meet C. We thus obtain two parallel tangent lines to C, L and L', so that the curve is entirely contained in the strip bounded by L and L'. Consider a circle S^1 which is tangent to both L and L' and does not meet C. Let O be the center of S^1 and take a coordinate system with origin at O and the x axis perpendicular to L and L'.

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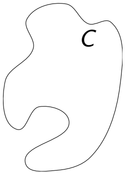

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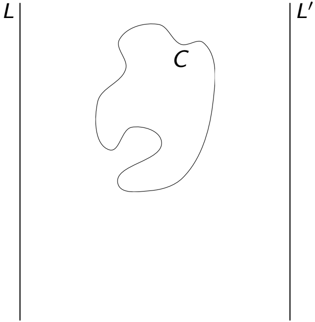

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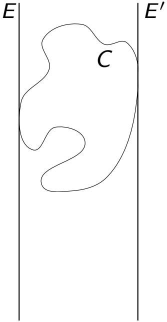

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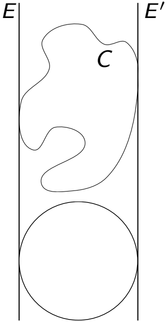

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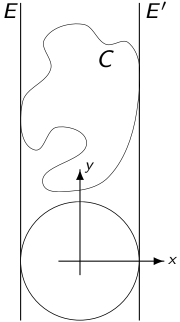

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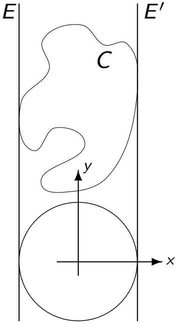
Proof (cont'd)

Parametrize C by arc length, $\alpha(s) = (x(s), y(s))$, so that it is positively oriented and the tangency points of L and L' are $s = 0$ and $s = s_1$, respectively.



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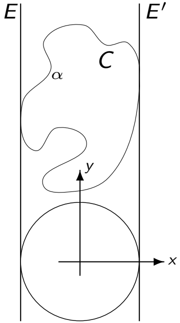

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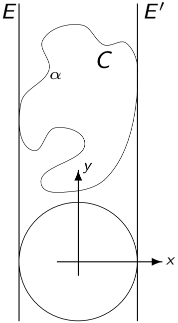

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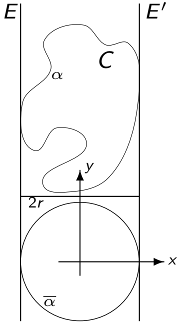

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Remark

The isoperimetric inequality holds true for a wide class of curves. Direct proofs have been found that work as long as we can define arc length and area for the curves under consideration. For the applications, it is convenient to remark that the theorem holds for *piecewise* C^1 *curves*, That is, continuous curves that are made up by a finite number of C^1 arcs. These curves can have a finite number of corners, where the tangent is discontinuous.