# Topic 3: Introduction to Regular Surfaces

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Math 142:
Differential Geometry

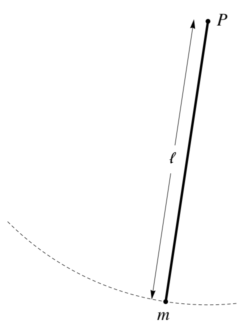
# A Motivational Example

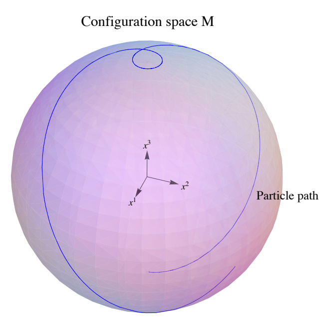
## Mathematical Models and Physical Systems

When we wish to describe a physical system in a "mathematical" way we try to construct some sort of mathematical structure which, in some sense, "represents" those aspects of the system which are of interest to us. This structure is then a "mathematical model" of the physical system.

## Example

A mass $m$ is fixed on the end of a rigid rod of negligible mass having length $\ell$. One end of the rod is fixed at a point $P$ in space so that the mass can move about about $P$ subject to the condition that it always be a distance $\ell$ from $P$. The sphere $M$ (a *regular surface* or *manifold*) of all possible positions for $m$ is called the *configuration space* of the system.



# A Motivational Example

## Example (cont'd)

Suppose we are only interested in the motion of the particle. Then we take, as the state of the particle, the pair of three-dimensional vectors $(x, v)$, $x = (x^1, x^2, x^3)$, $v = (v^1, v^2, v^3)$, where $x$ is the position vector of $m$ and $v$ is the velocity vector of $m$ (with respect to some Cartesian coordinate system).

Since the mass must stay on the sphere $M$, we see $v$ must be tangent to $M$. Thus our *state space* $S$ does not consist of all pairs of 3-vectors but, instead, we have the *tangent bundle* of $M$ (which can also be viewed as a manifold);

$$S = \{(x, v) \mid x \in M \text{ and } v \text{ is tangent to } M \text{ at } x\}.$$

Although $S$ is not a Euclidean space, nor an open set in one, we shall see that $S$ is a space on which notions such as tangent vector, vector field, and time-dependent vector field have meaning. If we have a force field then the force field will determine a vector field on the state space $S$.

## Definitions and Examples

### Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood $V$ in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).

2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)

3. $\mathbf{x}$ is regular (so we can use linear algebra)

### Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* $S$ of $\mathbb{R}^3$, and not as a map. This is achieved by covering $S$ with the traces of parametrizations which satisfy conditions 1, 2, and 3.

# Definitions and Examples

### $\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$

the functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ have continuous partial derivatives of all orders.

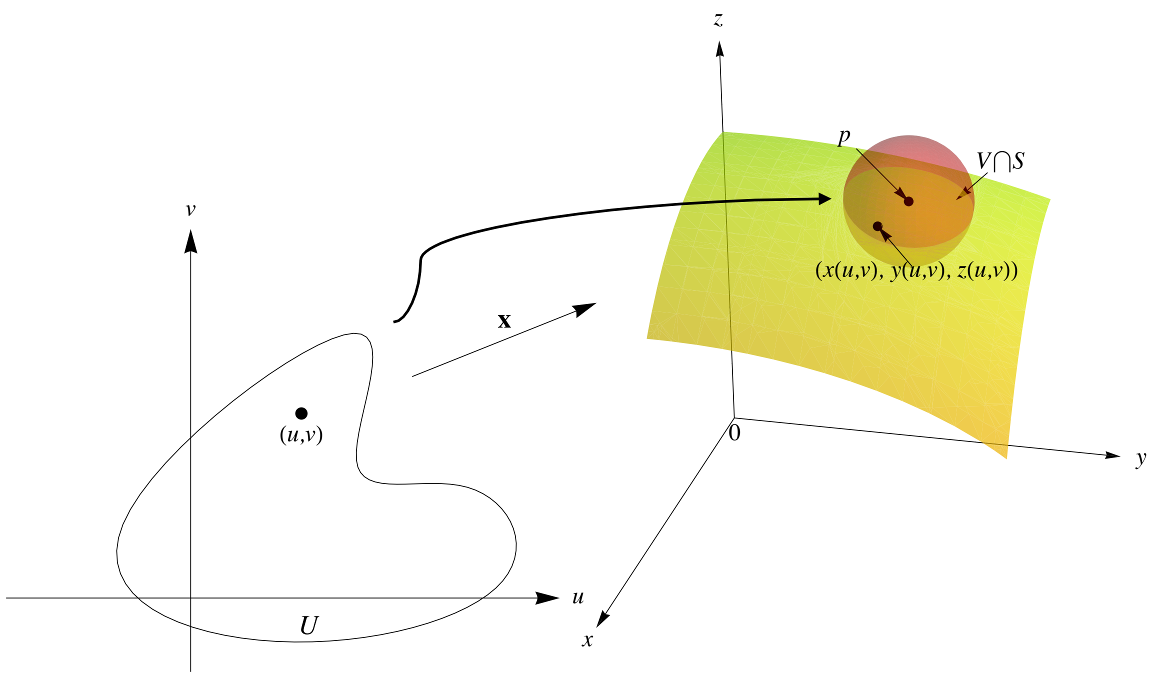
### $\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set $W$ containing $V \cap S$.

### $\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

# Definitions and Examples



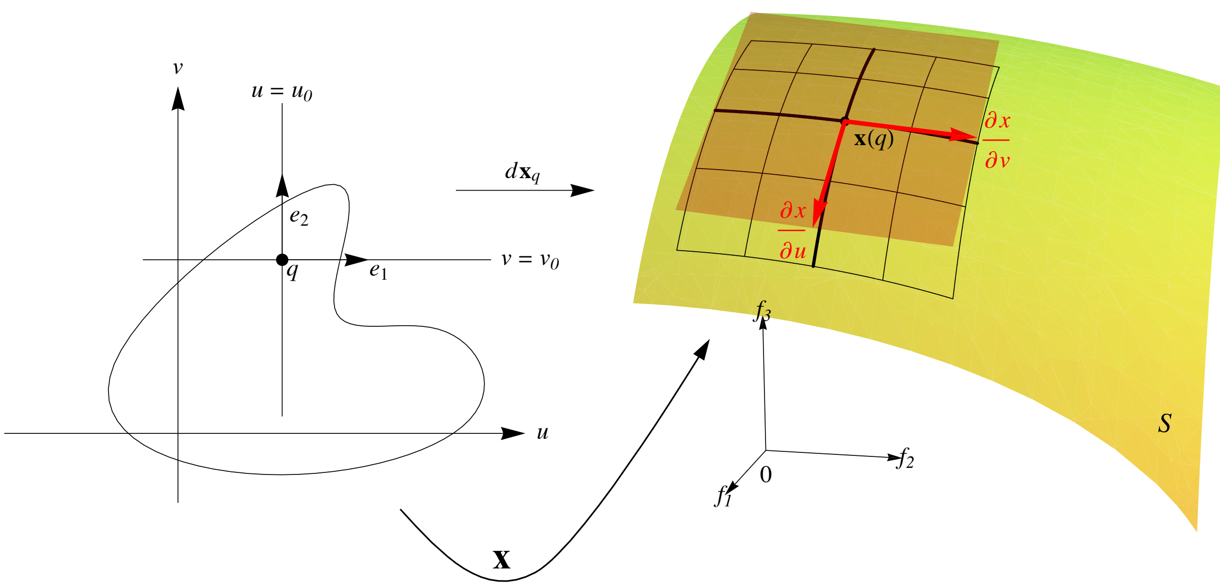

## Definition

The mapping **x** is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) $p$. The neighborhood $V \cap S$ of $p$ in $S$ is called a *coordinate neighborhood*.

# The Regularity Condition

## An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1,0)$, $e_2 = (0,1)$ of $\mathbb{R}^2$ with coordinates $u, v$ and $f_1 = (1,0,0)$, $f_2 = (0,1,0)$, $f_3 = (0,0,1)$ of $\mathbb{R}^3$, with coordinates $(x, y, z)$.



# The Regularity Condition

## An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial\mathbf{x}/\partial u \wedge \partial\mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at $q$.

## The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on $S$.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set $S$ itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of $S$.

# Proving that a Set is a Regular Surface

## Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

## Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of $S^2$.

## Proving that a Set is a Regular Surface

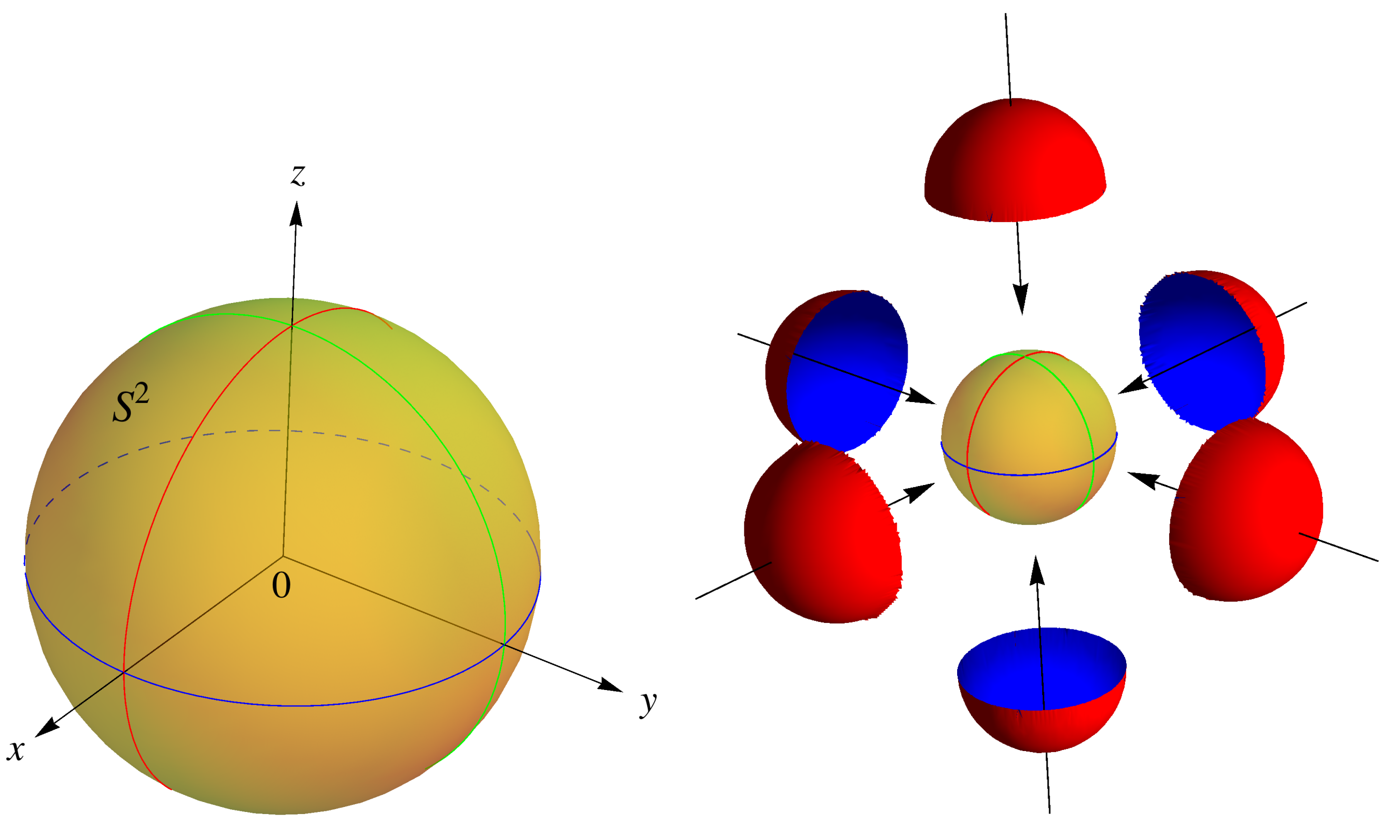
We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers $S^2$ minus the equator $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$. Then, using the $xz$ and $zy$ planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z), \\
\mathbf{x}_5(y, z) &= (+\sqrt{1 - (y^2 + z^2)}), y, z), \\
\mathbf{x}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover $S^2$ completely and shows that $S^2$ is a regular surface.



z
$S^2$
0
x
y

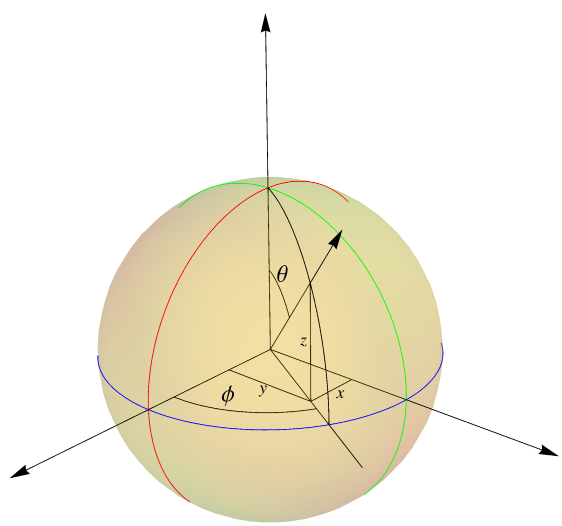
# Proving that a Set is a Regular Surface

## Method 2: Using Spherical Coordinates

For most applications, it is convenient to relate parametrizations to the geographical coordinates on $S^2$. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



# Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of $S^2$.

Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where $C$ is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, $\theta$ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing $\theta$, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines $\varphi$ uniquely $(0 < \varphi < 2\pi)$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set $S$ is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of $S^2$ (including the two poles) and that $S^2$ can be covered with the coordinate neighborhoods of two parametrizations of this type.