Q: How to study differential map? study its differential.

Q: How to study a manifold? by study its tangent vectors and tangent spaces.

15

Topic: Tangent vector (by comparing with smaller mfld), Tangent space, on Manifolds

Differential of a differentiable mapping.

for regular surface: V (velocity of α in $\mathbb{R}^3$)

But we don't have embient space

Tangent vector:

How to define a tangent vector to a (Diff ← start to omit) manifold?

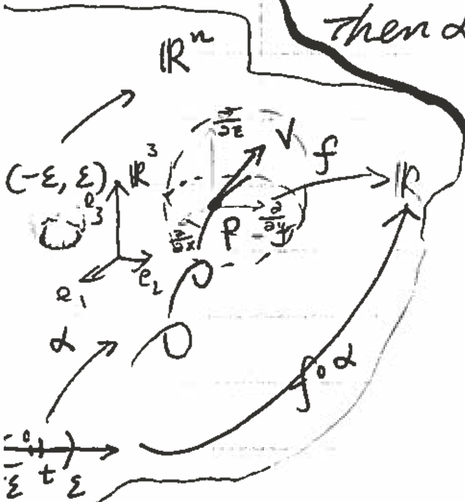
(Note: Our mfld is not embedded in any Euclidean space!

Let's find characteristic property of a tangent vector $\in \mathbb{R}^n$.

Recall: Let $\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$, with $\alpha(0) = p$. Write

$$\alpha(t) = (x_1(t), \dots, x_n(t)), \quad t \in (-\varepsilon, \varepsilon), \quad (x_1, \dots, x_n) \in \mathbb{R}^n$$

Then $\alpha'(0) = (x_1'(0), \dots, x_n'(0)) = v \in \mathbb{R}^n$



Now let f be a differentiable function defined in a nbhd of p. We can restrict f to the curve α and express the directional derivative with respect to the vector $V \in \mathbb{R}^n$ as

$$\left.\frac{d\, f\circ\alpha}{dt}\right|_{t=0} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{t=0} \left.\frac{dx_i}{dt}\right|_{t=0} = \Big(\underbrace{\sum x_i'(0) \frac{\partial}{\partial x_i}}_{\text{does not depend on which curve}}\Big) f$$

☆ Characteristic property: the directional derivative with respect to V is an operator on differentiable functions that depends uniquely on V.

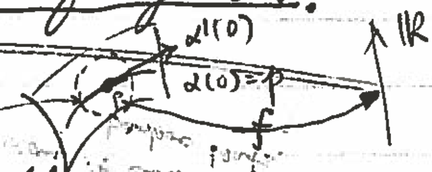
$\mathbb{R}^n \longrightarrow \mathbb{R}$

$f : (x_1, x_2 \cdots x_n) \longmapsto f(x_1 \cdots x_n)$

$f \circ \alpha(t) = f(x_1(t), \cdots x_n(t))$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

draw a picture instead!



Definition: Let M be a differentiable manifold. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a (differentiable) curve in M with $\alpha(0) = p \in M$.

The tangent vector to the curve α at $t=0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$, given by

$$\boxed{\alpha'(0) f = \left.\frac{d(f\circ\alpha)}{dt}\right|_{t=0}, \quad f \in \mathcal{D}.}$$

$\mathcal{D}$ = set of functions which are differentiable at p.

$\alpha'(0)$ is function defined on set of functions which are differentiable at p.

see life analogy on notebook.

Q. Can we view $\alpha'(0) \in (T_pM)^{**}$? — we view f is a 0-form.

A tangent vector at p of M is the tangent vector at $t=0$ of some curve $: (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$.

Tangent space

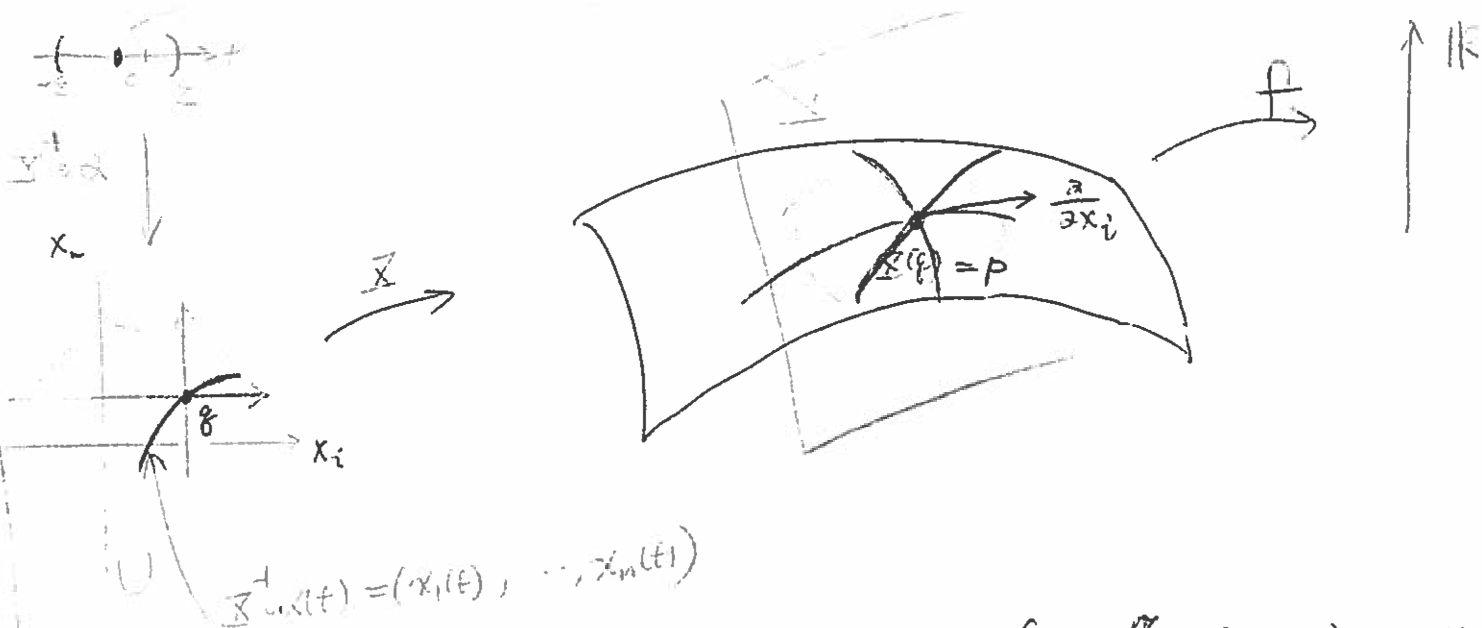
Let $T_pM \triangleq$ all tangent vectors to M at p.

Claim: T_pM is a vector space. Moreover if we choose a param[etrization] $\mathbf{X}: U \to M$, then T_pM has a basis $\left\{ \left(\frac{\partial}{\partial x_1}\right)_0, \cdots, \left(\frac{\partial}{\partial x_n}\right)_0 \right\}$ associat[ed] to $\mathbf{X}$

If we choose a parametrization $\mathbf{X}: U \to M^n$ at $p = \mathbf{X}$ We can express the function f and the curve α in this parametrization by

$$f \circ \mathbf{X}(q) = f(x_1, \cdots, x_n), \quad q = (x_1, \cdots, x_n) \in U$$

$$\mathbf{X}^{-1} \circ \alpha(t) = (x_1(t), x_2(t) \cdots, x_n(t))$$



Then restricting f to α, $\Rightarrow f \circ \alpha = f(x_1(t), x_2(t), \cdots, x_n(t))$

Then by definition

$$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt} f(x_1(t), \cdots, x_n(t))\Big|_{t=0}$$

$$= \sum_{i=1}^{n} x_i'(0) \frac{\partial f}{\partial x_i} = \left(\sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_0 \right) f$$

$\Rightarrow \alpha'(0)$ can be expressed in the parametrization $\mathbf{X}$ by

$$\alpha'(0) = \sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)_0 \quad \text{--- } (*)$$

<u>Claim</u>: $\left(\frac{\partial}{\partial x_i}\right)_0$ = the tangent vector at p of the "coordinate curve".

<u>Pf</u>: $\alpha_i : x_i \longrightarrow \mathbf{X}(0, 0, \cdots, x_i, 0, \cdots 0)$

$f \circ \alpha_i(t) = f(0, 0, \cdots, x_i(t), \cdots 0) \qquad x_i(t) = t$

$$\alpha_i'(0)f = \frac{d}{dt} f\circ\alpha_i(t) = \frac{\partial f}{\partial x_1}\cdot 0 + \cdots + \frac{\partial f}{\partial x_i}\cdot 1 + \cdots + \frac{\partial f}{\partial x_n}\cdot 0\Big|_{t=0} = \left(\frac{\partial}{\partial x_i}\right)_0 (f)$$

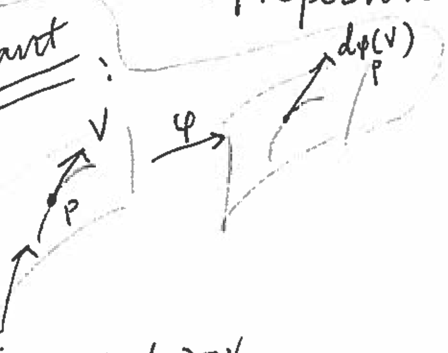
$$\Rightarrow \alpha_i'(0) = \left(\frac{\partial}{\partial x_i}\right)_0$$

Remark: $(*)$ says that every vector (on T_pM) is a linear combination of $\left\{\left(\frac{\partial}{\partial x_1}\right)_0, \ldots, \left(\frac{\partial}{\partial x_n}\right)_0\right\}$, thus T_pM is a vector space.
(It is clearly that vector structure does not depend on the choice of $\mathbf{X}$.)

easy to see

$\boxed{\text{Differential of } \varphi:}$ ← just draw the picture!

want:



choose s.t $\alpha'(0) = v$

$-\varepsilon \quad \varepsilon$

Let $\beta = \varphi\circ\alpha$

$\frac{d\beta}{dt}\Big| = \beta'(0) \triangleq d\varphi_p(v)$

Proposition: Let M_1^n and M_2^m be differentiable manifolds and $\varphi : M_1 \rightarrow M_2$ be a differentiable mapping. For every $p \in M_1$ and every $v \in T_pM_1$ choose a differentiable curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$. Take $\beta = \varphi\circ\alpha$. The mapping $d\varphi_p : T_pM_1 \rightarrow T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α.

... this definition is independent of choice of α

<u>Definition</u>: The linear mapping $d\varphi_p$ defined by this proposition is called ~~the~~ <u>differential</u> of φ at p.

Pf of proposition: Let

Motivation: We defined a tangent plane $T_p(M)$ at each point $p \in M$ and we want to develop differential geometry around p as the study of the variation of $T_p(M)$. (Local geometry).

Pf: Let $X : U \to M_1$ be a parametrization at p
$\bar{Y} : V \to M_2$ [illegible] at $\varphi(p)$

Let $v \in T_p M_1$

choose a curve $\alpha : (-\varepsilon, \varepsilon) \to M_1$ with ① $\alpha(0)$ [illegible]

Take $\beta = \varphi \circ \alpha$. Then β is a curve in M_2 with $\beta(0) = \varphi\circ\alpha(0) = \varphi(p)$, $\beta'(0) \in T_{\varphi(p)} M_2$

Define $d\varphi_p : T_p M_1 \to T_p M_2$
$v \longmapsto \beta'(0)$

WTS ① $d\varphi_p$ is linear

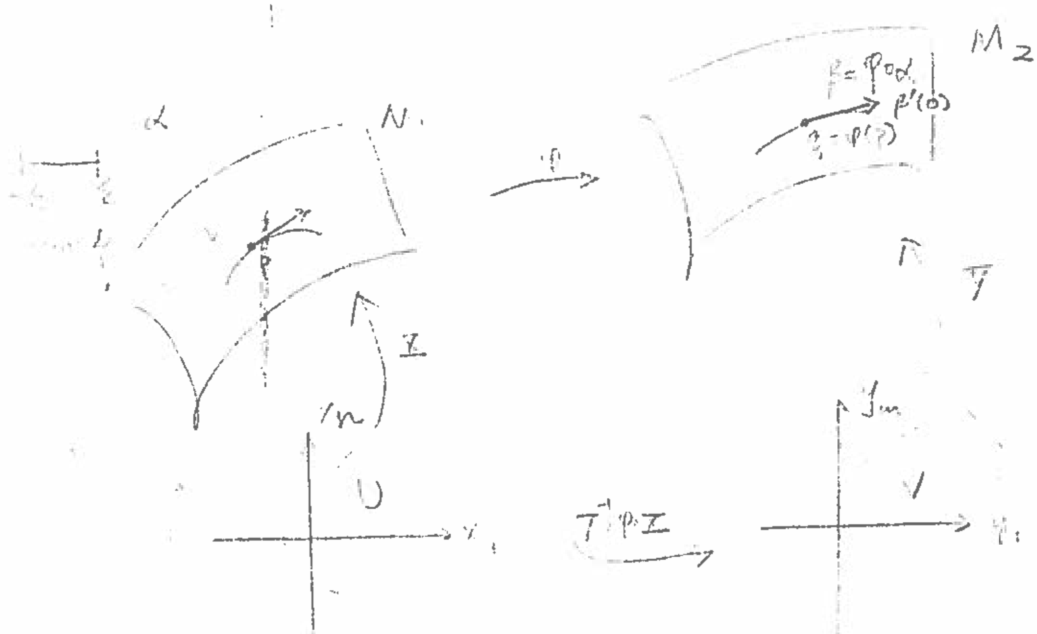
Express φ in the [illegible] parametrizations, we have

$$\bar{Y}^{-1} \circ \varphi \circ X(q) = (y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n))$$

$$q = (x_1, \ldots, x_n) \in U, \ (y_1, \ldots, y_m) \in V$$

Express α in the parametrization X, we obtain

$$X^{-1} \circ \alpha(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$



Define
$d\varphi_p(v) = \beta'(0)$

<u>Claim</u> ① $d\varphi_p$ is linear [illegible] from $T_p M_1$ to T_q [illegible]

② $d\varphi_p$ does not depend on the choice of α

Topic on Vector Fields

(require the background of differential equations (existence, uniqueness, dependence on the initial conditions.))

§1.1 vector fields (~~Definition~~) and local flow

① A Vector field in an open set $U \subseteq \mathbb{R}^n$.
(Say $n=2$)

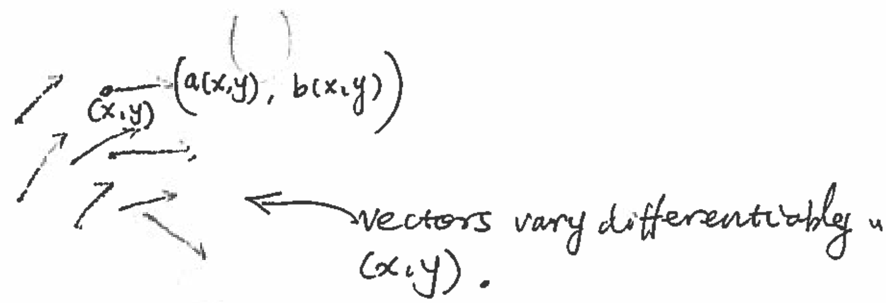
In general a vector field is a map: $\mathbb{R}^n \to \mathbb{R}^n$

A <u>vector field</u> in an open set $U \subseteq \mathbb{R}^2$ is a map which assigns to each $q \in U$ a vector $W(q) \in \mathbb{R}^2$.

$$W: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$q=(x,y) \longmapsto (a(x,y), b(x,y)) = W(q)$$

The vector field W is said to be <u>differentiable</u> if the function $a(x,y)$, $b(x,y)$ is differentiable on U.

<u>Geometrically,</u>



<u>Example</u>: ① $W(x,y) = (x,y)$
② $W(x,y) = (y,-x)$

Given a vector field W, it is natural to ask whether there exists a <u>trajectory</u> of this field, that is, whether there exist a differentiable parametrized curve $\alpha(t) = (x(t), y(t))$, $t \in I$, s.t. $\alpha'(t) = W(\alpha(t))$ $\Rightarrow \begin{cases} x'(t) = a(x(t), y(t)) \\ y'(t) = b(x(t), y(t)) \end{cases}$

For ①, a trajectory, passing through the point (x_0, y_0) of the vector field $W(x,y)=(x,y)$ is the straight line $\alpha(t) = (x_0 e^t, y_0 e^t)$, $t \in \mathbb{R}$.

$\Rightarrow \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$

For ②, a trajectory [illegible] $(y, -x)$ [illegible]

is the circle [illegible]

Note That is: the vector field W determines a system of differential equations

$$\begin{cases} \frac{dx}{dt} = a(x,y) \\ \frac{dy}{dt} = b(x,y) \end{cases} \quad \cdots (1) \quad \left(\text{here } \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right)$$

and that a trajectory of W is a solution to equation (1).

The fundamental thm of (local) existence and uniqueness of solutions of Eq(1) is equivalent to the following statement on trajectories.

I, J open interval of $\mathbb{R}$ containing origin $0 \in \mathbb{R}$.

Thm: Let W be a differentiable vector field in an open set $U \subseteq \mathbb{R}^2$. Given $p \in U$, there exists a traject[ory] $\alpha : I \to U$ of W (i.e. $\alpha'(t) = W(\alpha(t))$, $t \in I$) with $\alpha(0) =$ [p]. This trajectory is unique in the following sense: Any oth[er] trajectory $\beta : J \to U$ with $\beta(0) = p$ agrees with α in I ∩ [J]

☆ Important fact: trajectory passing through p " varies differ[entiably] with p". precisely speaking:

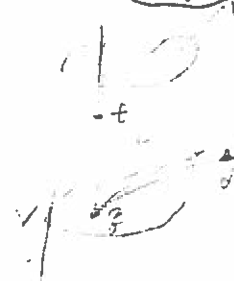
Thm: Let W be a vector field in an open set U ⊂ [$\mathbb{R}^2$]. For each $p \in U$, there exist a nbhd $V \subset U$ of p, an interv[al] I, and a mapping $\varphi : V \times I \to U$ such that

1. For a fixed $q \in V$, the curve $\varphi(q, t)$, $t \in I$ is the trajectory of W passing through q, that is

$$\begin{cases} \varphi(q, 0) = q \\ \frac{\partial \varphi}{\partial t}(q,t) = W(\varphi(q,t)) \end{cases}$$

2. φ is differentiable.

flow like water wind, any flow
map $\varphi(q,t)$ $t \in I$ to
$\alpha(q,t)$
$V \times I$



Remark. Geometrically this means that all trajectories which pass, for $t = 0$, in a certain nbhd V of p may be "collected" into a single differentiable map.

③ A Vector field on a regular surface:

Def.: A vector field W is in an open set $U \subset S$ of a regular surface S is a correspondence which assigns to each $p \in U$ a vector $W(p) \in T_p(S)$. The vector field W is differentiable at $p \in U$ if, for some parametrization $X(u,v)$ at p, the function $a(u,v)$ and $b(u,v)$ given by

$$W(p) = a(u,v)X_u + b(u,v)X_v$$

are differentiable functions at p. (It is clear that the definition does not depend on the choice of X.)

Example 1): A vector field on torus T^2: $(|W(p)|=1)$

2): A vector field on $S^2 \setminus \{N,S\}$ $(|W(p)|=1)$

" " " " S^2:

reparametrize all the semi-meridians by the same parameter t, $-1<t<1$, and define $V(p)=(1-t^2)W(p)$ for $p \in S^2 \setminus \{N,S\}$ and $V(N)=V(S)=0$.

③ A vector field on a Manifold:

<u>Def</u>: <u>A vector field X on a differentiable manifold M</u> is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_p(M)$. In terms of mappings, X is a mapping of M into the tangent bundle TM. The field is <u>differentiable</u> if the mapping $X: M \to TM$ is differentiable.

$\mathbb{R}^n \hookrightarrow TM$, $\pi \downarrow \uparrow s$, M — a section of the tangent bundle; $\pi \circ s = p$

<u>Local expression:</u> Consider a parametrization $\mathbf{x}: U \subset \mathbb{R}^n \to M$. We can write $X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$ where each $a_i: U \to \mathbb{R}$ is a function on U and $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ is the basis associated to $\mathbf{x}$.

talk about it after flow

<u>Note</u>: X is differentiable $\Leftrightarrow$ functions a_i are differentiable for some (and, therefore, for any) parametrization.

<u>Remark</u>: A vector field is a differential operator

$$X: \mathcal{D} \to \mathcal{F}$$
$$f \mapsto (Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p)$$

f denotes, by abuse of notation, the expression of f in parametrization $\mathbf{x}$.

It is [illegible], the function Xf does not depend on the choice [illegible]

$\mathcal{D}$ = set of differentiable functions on M.

$\mathcal{E}$ = [illegible] functions on M.

Claim: X is differentiable iff $X: \mathcal{D} \to \mathcal{E}$, that is $Xf \in \mathcal{E}$ for all $f \in \mathcal{D}$.

<u>Remark</u>: Since a differentiable manifold is locally d[illegible] to $\mathbb{R}^n$, the fundamental theorem on existence, uniqueness and dependence on initial conditions of [illegible] (which is [illegible] local [illegible]) extends naturally to differentiable manifolds.

X — vector field

$\underline{X}$ — local parametrization

Thm: Let X be a diff. vector field on a differentiable manifold M, and let $p \in M$. Then there exist a nbhd $U \subset M$ of p, an interval $(-\delta, \delta)$, $\delta > 0$, and a <u>differentiable mapping</u> $\varphi : (-\delta, \delta) \times U \to M$ s.t. the curve $t \to \varphi(t, q)$, $t \in (-\delta, \delta)$, $q \in U$, is the <u>unique</u> curve which satisfy $\frac{\partial \varphi}{\partial t} = X(\varphi(t, q))$ and $\varphi(0, q) = q$.

φ is called a local flow of X.

for each q, there is a curve

It is common to use the notation $\varphi_t(q) = \varphi(t, q)$ and call $\varphi_t : U \to M$ the local flow of X.

§1.2 (Lie) Brackets and Lie derivatives

The interpretation of X as an operator on $\mathcal{D}$ permits us to consider the iterates of X.

$\Downarrow$

we will have a special kind of algebra structure on set of vector fields on M.

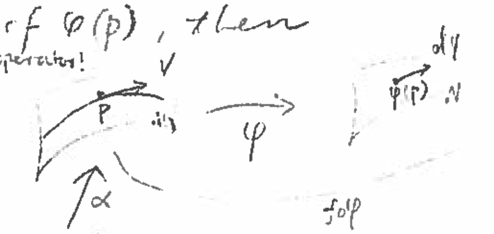
$\Downarrow$

more structures $\Rightarrow$ more manipulations $\Rightarrow$ more informations

<u>Recall</u>: $\varphi : M \to M$ is a diffeo, $V \in T_pM$ and f is a differentiable function in a nbhd of $\varphi(p)$, then

Diff. operator acting on f! — also a differential operator!

$$\Big(\underbrace{d\varphi(V)}_{\text{is a tangent vector at } \varphi(p) \text{ which is a differential operator!}} f\Big)(\varphi(p)) = V(f \circ \varphi)(p)$$



Indeed, let $\alpha : (-\varepsilon, \varepsilon) \to M$ a differentiable curve with $\alpha'(0) = V$, $\alpha(0) = p$. Then

$$\big(d\varphi(V) f\big)(\varphi(p)) = \frac{d}{dt}\big(f \circ \underbrace{\varphi \circ \overset{\text{a curve pass } p}{\alpha}}_{\beta \text{ a curve pass } \varphi(p)}\big)\Big|_{t=0} = V(f \circ \varphi)(p)$$

$\Rightarrow$ Let X and Y are differentiable fields on M and $f : M \to \mathbb{R}$ is a differentiable function, we can consider the function $\underline{X(Yf)}$ and $\underline{Y(Xf)}$

$X(Yf)$: not a vector field

$Y(Xf)$: not a vector field

Since they involves derivatives of higher order (not a first order diff. operator)

But $XY - YX$ is.

Lemma: Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z on M such that, for all $f \in \mathcal{D}$, $Zf = (XY - YX)f$.

Pf. Proof of uniqueness:

Spse that Z exists, want to show Z is unique.

Let $p \in M$ and let $\mathbf{x} : U \to M$ be a parametrization at p, and let

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$$

be the expression for X and Y in these parametrization

then for $f \in \mathcal{D}$

$$XYf = X\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \left(\frac{\partial}{\partial x_i}\left(b_j \frac{\partial f}{\partial x_j}\right)\right)$$

$$= \sum_{i,j=1}^{n} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$YXf = Y\left(\sum_i a_i \frac{\partial f}{\partial x_i}\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} b_j \frac{\partial}{\partial x_j}\left(a_i \frac{\partial f}{\partial x_i}\right)$$

$$= \sum_{i,j=1}^{n} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + b_j a_i \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Therefore, Z is given in the parametrization $\mathbf{x}$, by

$$Zf = XYf - YXf = \sum_{i,j}\left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j} \quad \cdots \; (*)$$

$$\Rightarrow Z \text{ is unique: } \mathbf{Z} = \sum_{i,j}\left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}$$

Proof of existence:

Define Z_α in each coordinate nbhd $\mathbf{x}_\alpha(U_\alpha)$ of differentiable structure $\{(U_\alpha, \mathbf{x}_\alpha)\}$ on M by $(*)$. By uniqueness $Z_\alpha = Z_\beta$ on $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) \neq \emptyset$, which allow us to define Z over the entire manifold M.

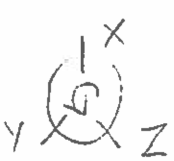
Q.E.D.

Definition: The vector field Z given by above Lemma is called the <u>(Lie) bracket</u>: $[X,Y] = XY - YX$ of X and Y.

<u>Note</u> $Z = [X,Y]$ is differentiable

The bracket operation has the following properties:

<u>Proposition</u>: If X, Y and Z are differentiable vector fields on M, $a, b \in \mathbb{R}$ and f, g are differentiable functions then:



(a) $[X,Y] = -[Y,X]$

(b) $[aX+bY, Z] = a[X,Z] + b[Y,Z]$

(c) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)

(d) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$

Pf. (a) and (b) are immediate.

(c): $[[X,Y],Z] = [XY - YX, Z] = (XY-YX)Z - Z(XY-YX)$

$= XYZ - YXZ - ZXY + ZYX$

$[X,[Y,Z]] = [X,(YZ-ZY)] = X(YZ-ZY) - (YZ-ZY)X$

$= XYZ - \cancel{XZY} - \cancel{YZX} + ZYX$

$[Y,[Z,X]] = Y(ZX - XZ) - (ZX - XZ)Y$

$= \cancel{YZX} - YXZ - ZXY + \cancel{XZY}$

(d) $[fX, gY] = fX(gY) - gY(fX) = fX(g)Y + fgXY$

$- gY(f)X - gfYX$

$= fg[X,Y] + fX(g)Y - gY(f)X$