Welcome to Math 142

Professor Weiqing Gu

https://math142hmc.github.io

Lecture 1

- Introduction to the course.
- What is a Manifold?
- Examples of needs of using geometric methods to extract information from big data including data from physics, economics, politics, culture, and business.
- Least square methods in geometric view.
- Why differential geometry?
- Why manifolds?
- Review of Linear Algebra.

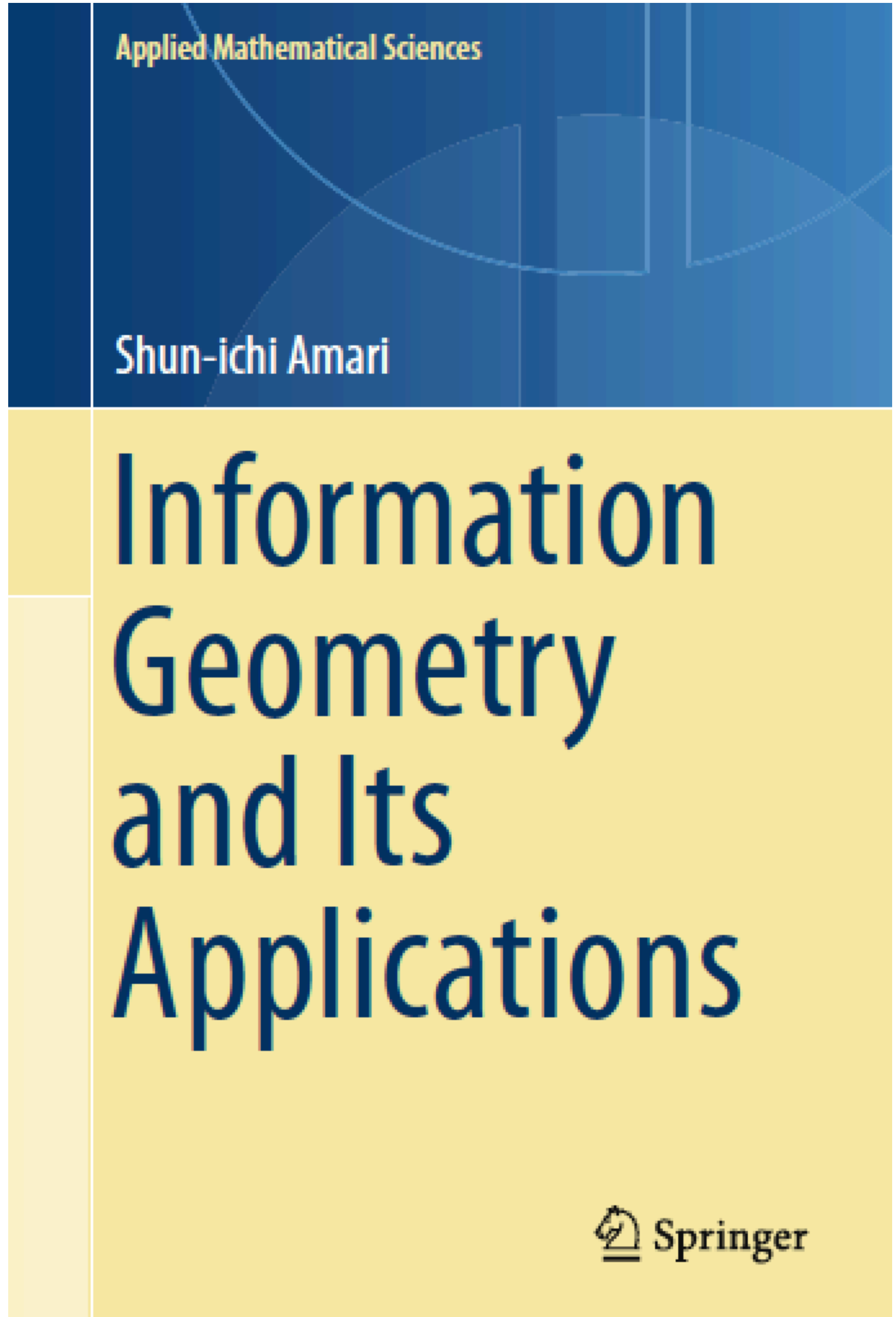
Math 142, Differential Geometry, Course Schedule, Fall 2017

- Instructor: Prof. Weiqing Gu
- Text: Differential Geometry of Curves and Surfaces by Do Carmo
- Lectures: MW 1:15 PM - 2:30 PM, SHAN 3465
- Office and Phone: SHAN 3420, Ext 18929.
- E-mail: gu@g.hmc.edu
- Grader/Tutor: Paul David: paul.david@cgu.edu
- Teaching Assistant: Natchanon Suaysom: nsuaysom@g.hmc.edu
- (Responsible for the course webpage updates etc.)
- Tutoring Hours and location: TBD

Office Hours and Problem Sessions

- Office Hours:
  - Monday: 4:00 PM – 5:00 PM
  - Wednesday: 5:00 PM - 6:00 PM
  - Or by appointment.

Additional Text Book:



Homework

- Homework will be assigned once a week on Wednesday and will be due the next Wednesday before the lecture. Discussions and collaborations are encouraged. But you must write up your own solutions. Before tackling any problems, read the relevant section of the text and review your lecture notes. Consult with the faculty, the tutor or your classmates about matters that are unclear to you. NEVER LET YOURSELF FALL BEHIND.

Midterm Exam

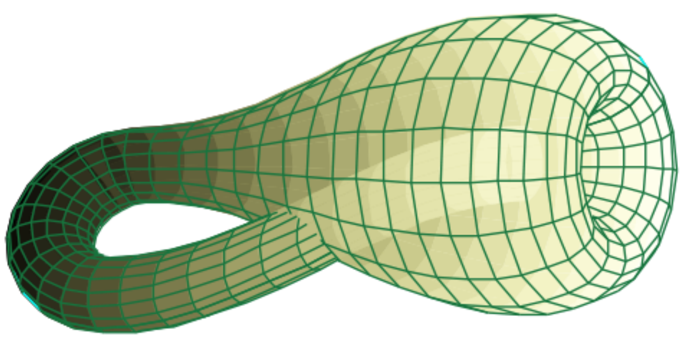
- Take Home
- October 18, Wednesday

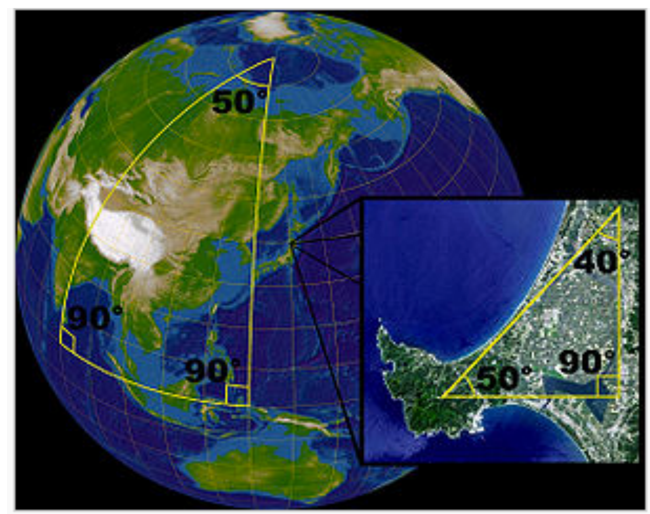
Grading Policy

- 40% - Homework
- 20% - Midterm Exam
- 40% - Final Term Project

What is a manifold?

- An n-dimensional manifold locally "looks like" a piece of $\mathbf{R}^n$.
- For examples, sphere, torus, and Klein Bottle.
- **Key features of a manifold: curved**



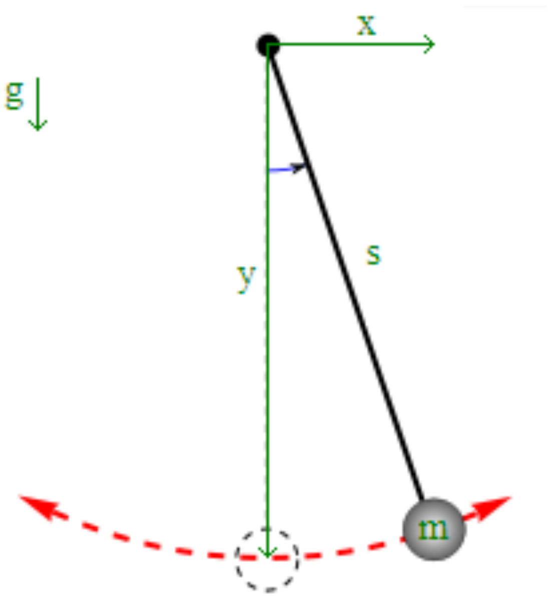


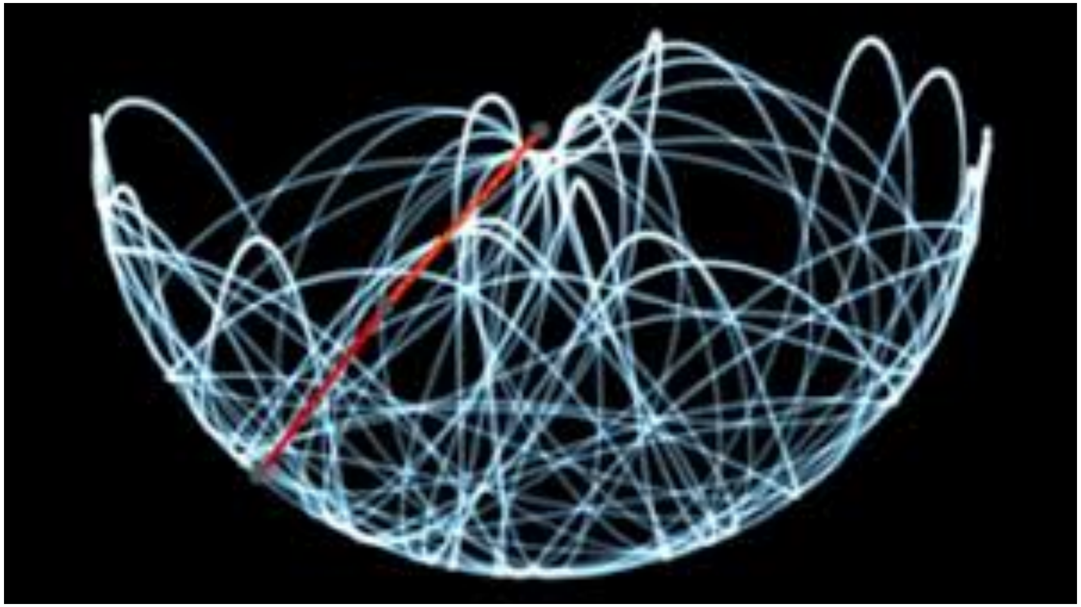
The sphere (surface of a ball) is a two-dimensional manifold since it can be represented by a collection of two-dimensional maps.

The simplest manifold is a circle which is 1-dimensional manifold, next simple example is a sphere.

- There are many different ways to write down a circle or a sphere mathematically. What are those?
- The importance of a sphere $\mathbf{S}^n$ (viewed as a manifold) is to use it to do modeling for complex real world problems, just like a vector space $\mathbf{R}^n$.
- Can you think of some of the examples?

Example:
Data Created by a Pendulum Motion





Even for 300 years of earthquake detection big data sets using a Pendulum, they all lie on a sphere, which is intrinsically only 2-dimentional!

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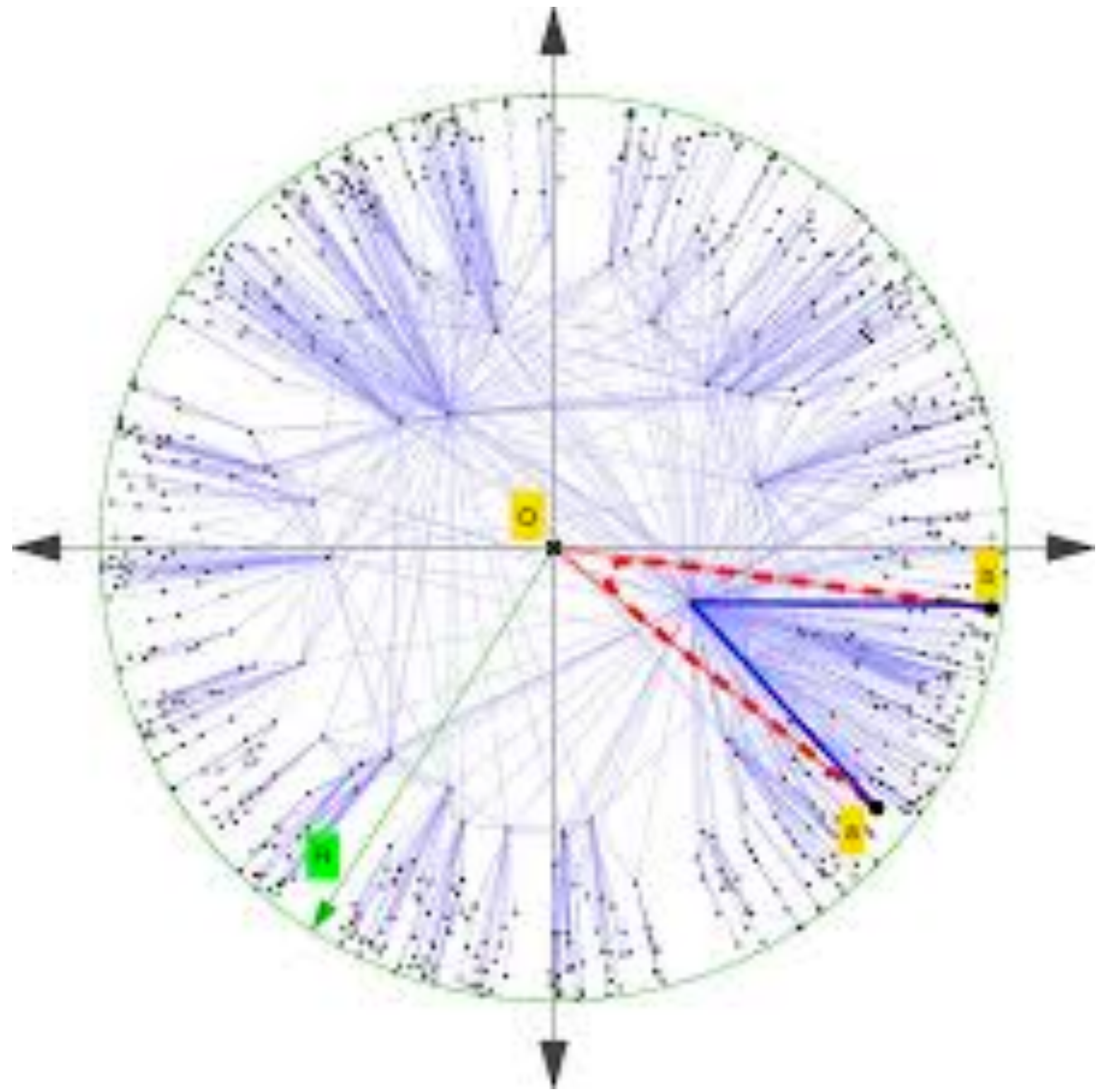
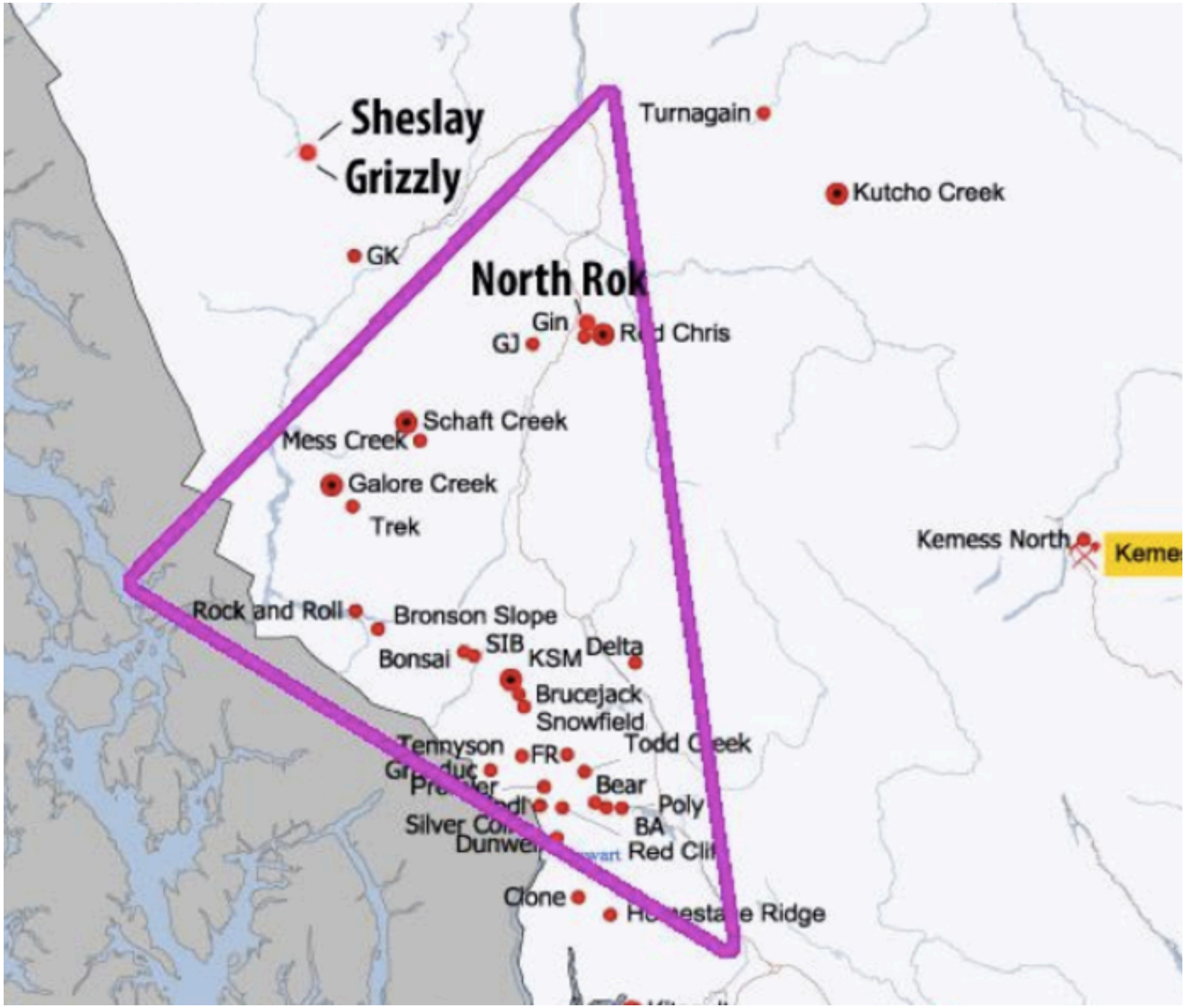
Example 1: The modelled network illustrates the connection between geometry and scale-free topology of complex networks.



https://www.nature.com/articles/ncomms1063

Example 2: How to teach the machine auto finding the boundary of the triangular below including some important data? According to the B.C. Geological Survey Minfile database, more than 900 documented mineral occurrences have been identified within the Golden Triangle.



How to teach the machine auto finding the boundary of the red (culture) data below?

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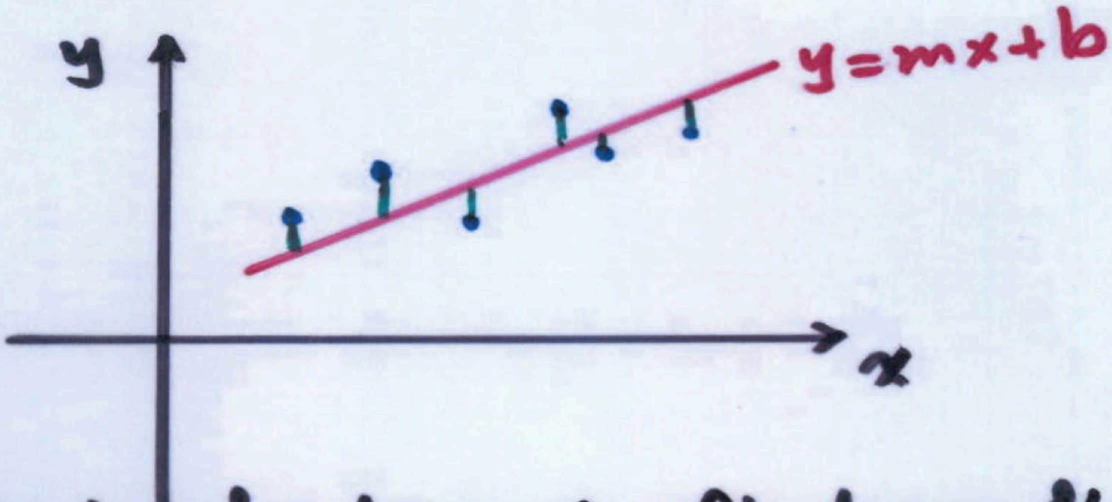
Recall: Least Square Approximation

Spse we have data pts (x_i, y_i) and want to find the line $y = mx + b$ which best describes the data.



The problem boils down to find m & b.

The error between one point and the line is

$$e_i = y_i - (mx_i + b)$$

Our objective is minimizing the total error.

- However, the errors e_i, some could be positive and some could be negative. A simple sum of the errors would not work well.
- Can you think about an example why not working well?
- How to fix this problem?
- Instead we consider the following **objective or cost function**:

L_2 norm

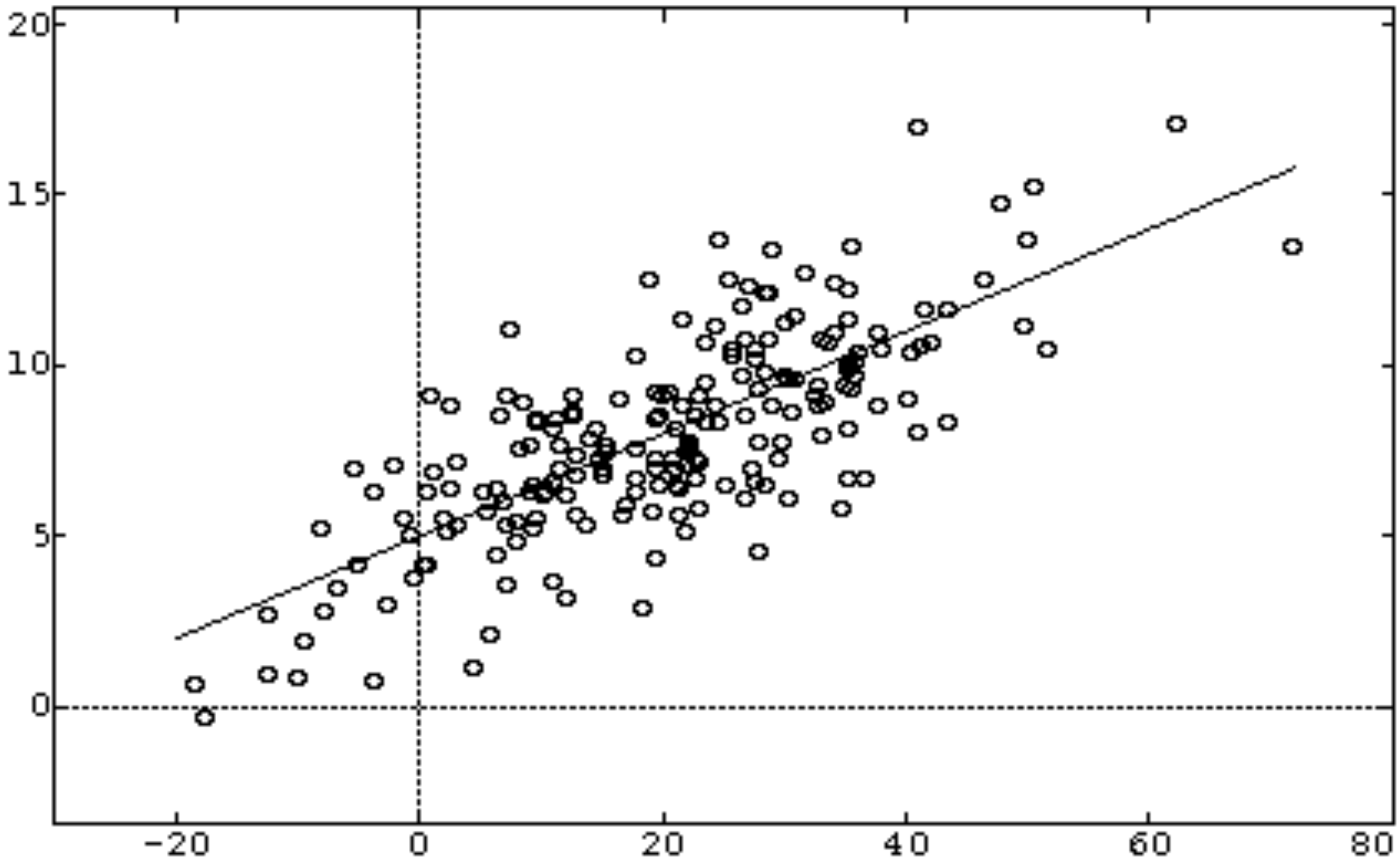
- $\boxed{J(m,b) = \sum (e_i)^2 = \sum (y_i - mx_i - b)^2}$

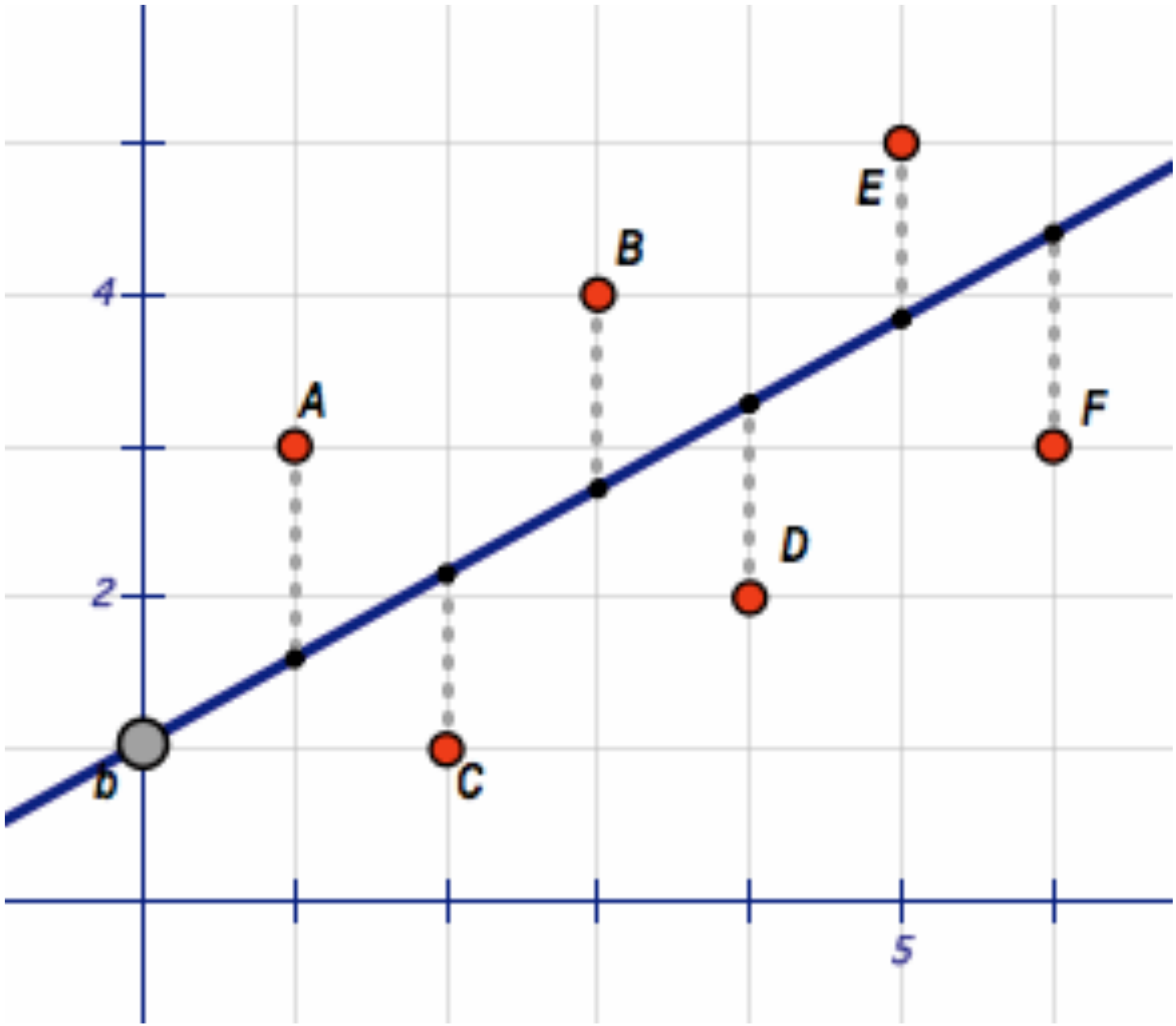
- Can we use $\sum |e_i|$ instead?

L_1 norm

Linear Regression

Given some data: $D = \{x_i, y_i\}$





4
2
5
A
B
C
D
E
F
b

Key: See the Problem From Dual Point View

- The coefficients of the line are changing.
- We think the coefficients as parameters in the model.
- We need to find “best coefficients” for fit the data.
- Can we guarantee the existence of such “best coefficients”?
- What are the configuration of these parameters?
- Is it compact?

It is crucial to develop analytic thinking for creating new algorithms for physics modeling and big data analytics!

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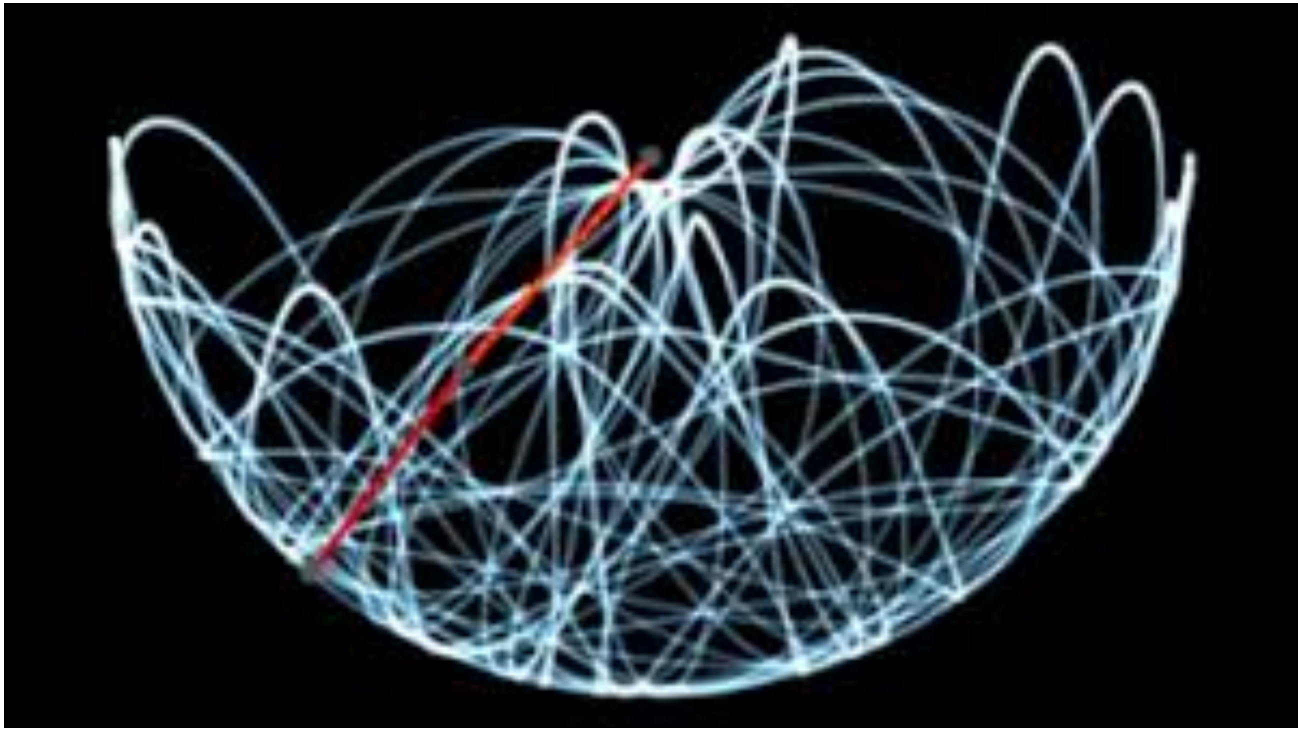
Example: Velocity of a Pendulum

- Tangent bundle of a sphere



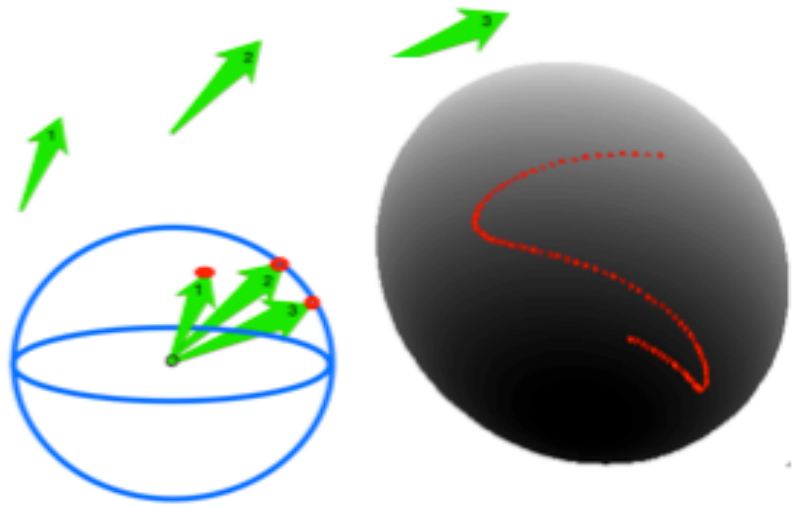
Why Manifolds?

- **Example 1:**
- Consider the set of all planes through the origin in $\mathbf{R}^3$

Dynamics and Kinematics of a UAV-- Applications in Physics and Dynamic Data

Only consider UAV heading directions here, but works for any other UAV characteristics

- **Example: Only look at UAV "headings"**
- **All possible headings for all UAVs form a sphere.**

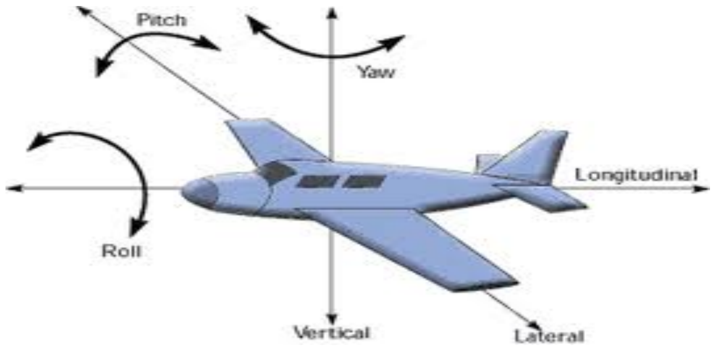


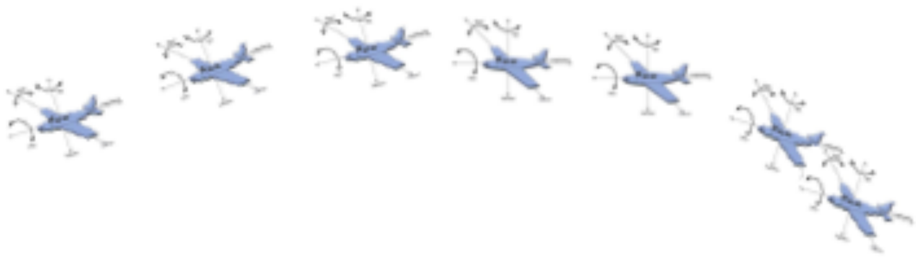
- ***Example: Developed a dimension-reduction technique for nonlinear data***

Example: Only Manifolds can be used to capture and model UAV nonlinear dynamical behaviors

Question: How to take the derivative of a moving frames?

Differential Geometry!





Rotations

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} | RR^T = I, \mathtt{det}(R) = +1\}$$

Representing a rotational trajectory:

$$R(t) : t \to SO(3)$$

Rotational velocity:

$$\dot{R}(t)R^T(t)$$

is a skew-symmetric matrix

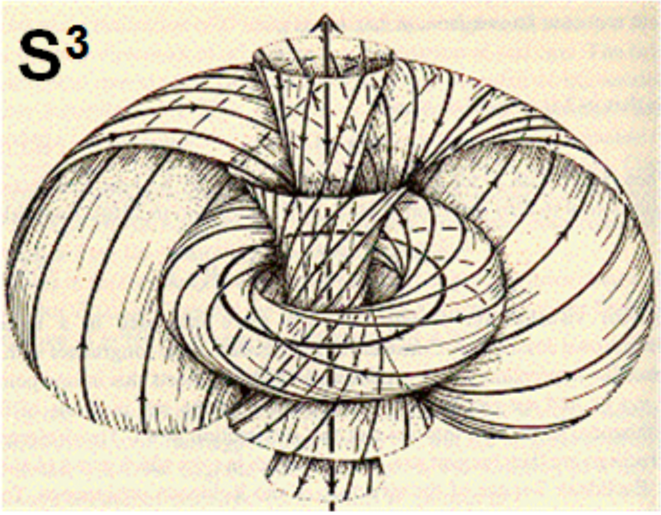
Tangent Plan of SO(3)

The tangent space to $SO(3)$ at the identity element is the space of skew-symmetric matrices, known as its Lie Algebra:

$$so(3) = \{\hat{w} \in \mathbb{R}^{3\times 3} | w \in \mathbb{R}^3\}$$

Often Need to Applied Manifold and Differential Geometric Techniques. *They are powerful! Why?*

- Work out details with the students on board if time permits.



By identify antipodal points

$M = SO(3) = RP^3$

Tangent Bundle of $SO(3) \times R^3$

Rigid Body Motion

Key:
Reduced a differential geometric problem to a linear algebra problem.
Fast computation.
Used in auto controlling UAVs.
But globally need to worry about "gimbal effect." (Later)

$SE(3)$:

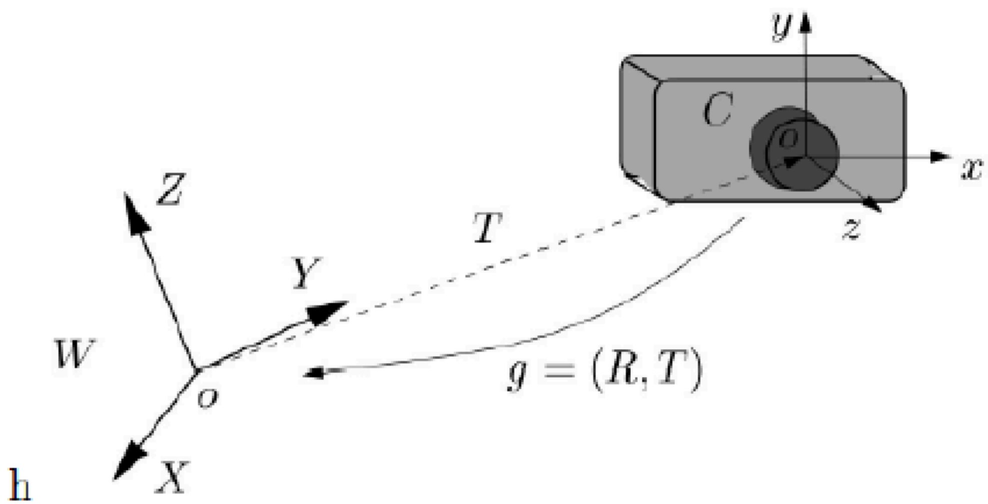
$$SE(3) = \{g = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} | R \in SO(3), T \in \mathbb{R}^3\}$$

The tangent space is the Lie Algebra of this group:

$$se(3) = \{\xi = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} | \hat{w} \in so(3), v \in \mathbb{R}^3\}$$

Another example of Why Differential Geometry and manifolds?

Example: In computer vision



We want to find a map that takes translation and rotation into account.

Applications in Computer Vision

Details: Later

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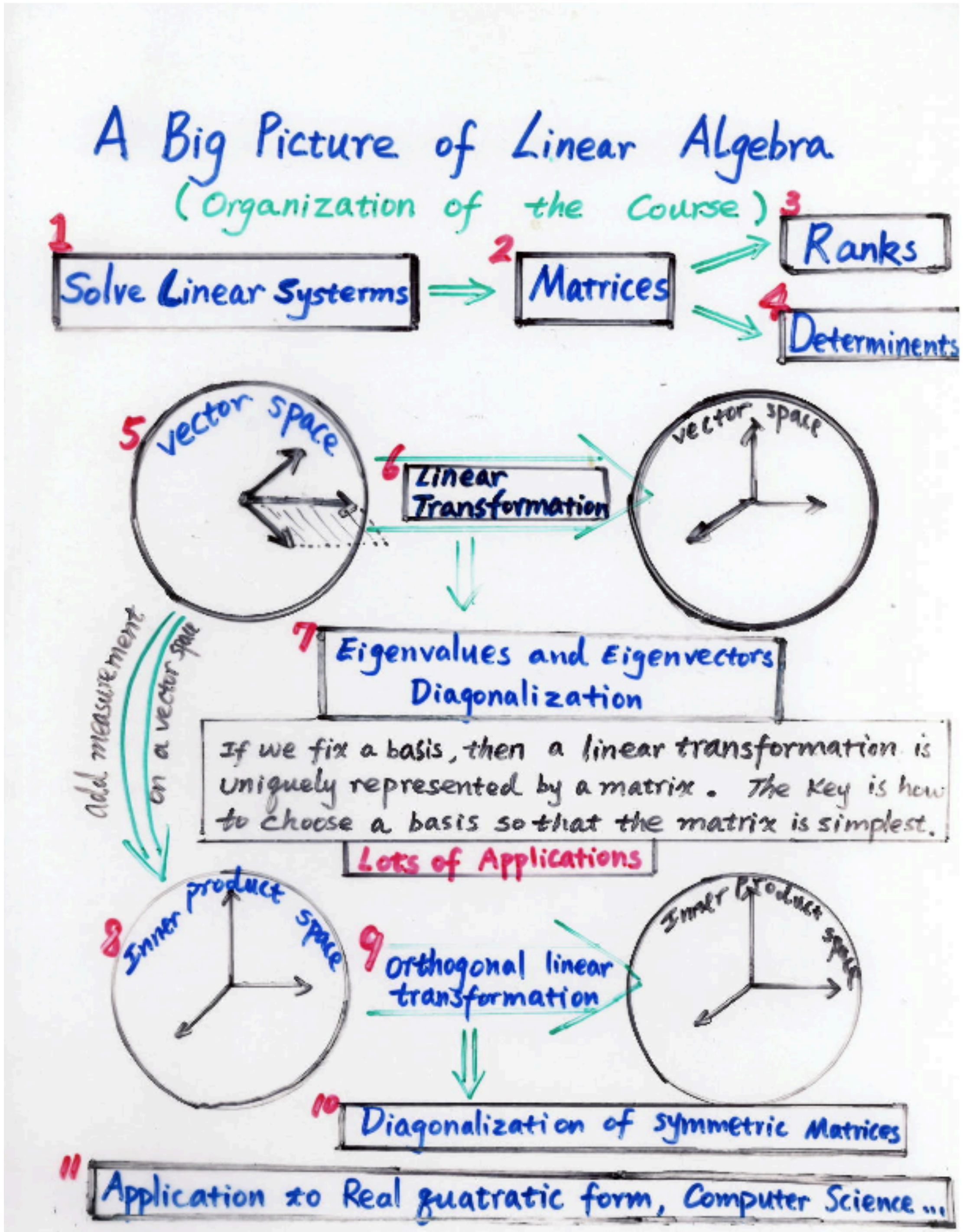
From Regular Surface to Abstract Manifold:

Mimic the linear algebra philosophy.

Form rank and trace to geometric and topologic invariants.



From viewing a matrix multiplication as an action to understand an instanton. An instanton is a classical solution to equations of motion with a finite, non-zero action, either in quantum mechanics or in quantum field theory.