Lecture 13: The Gauss Map in Local Coordinates

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Math 142:
Differential Geometry

The Gauss Map in Local Coordinates

Defining Coordinates

Let $N = \dfrac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}$, and write

$$II_p(\alpha') = e(u')^2 + 2fu'v' + g(v')^2.$$

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Then

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

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That is,

$$a_{11} = \frac{fF - eG}{EG - F^2}, \quad a_{12} = \frac{gF - fG}{EG - F^2},$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \quad a_{22} = \frac{fF - gE}{EG - F^2}.$$

Curvature in Local Coordinates

Gaussian, Mean, and Principal Curvature

By definition, we have

$$K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2}$$
$$H = -\frac{1}{2}(a_{11} + a_{22} = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}$$
$$k = H \pm \sqrt{H^2 - K}$$

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\end{aligned}
$$

Remark

If a parametrization of a regular surface is such that $F = f = 0$, then the principal curvatures are given by e/E and g/G. In fact, in this case, the Gaussian and the mean curvatures are given by

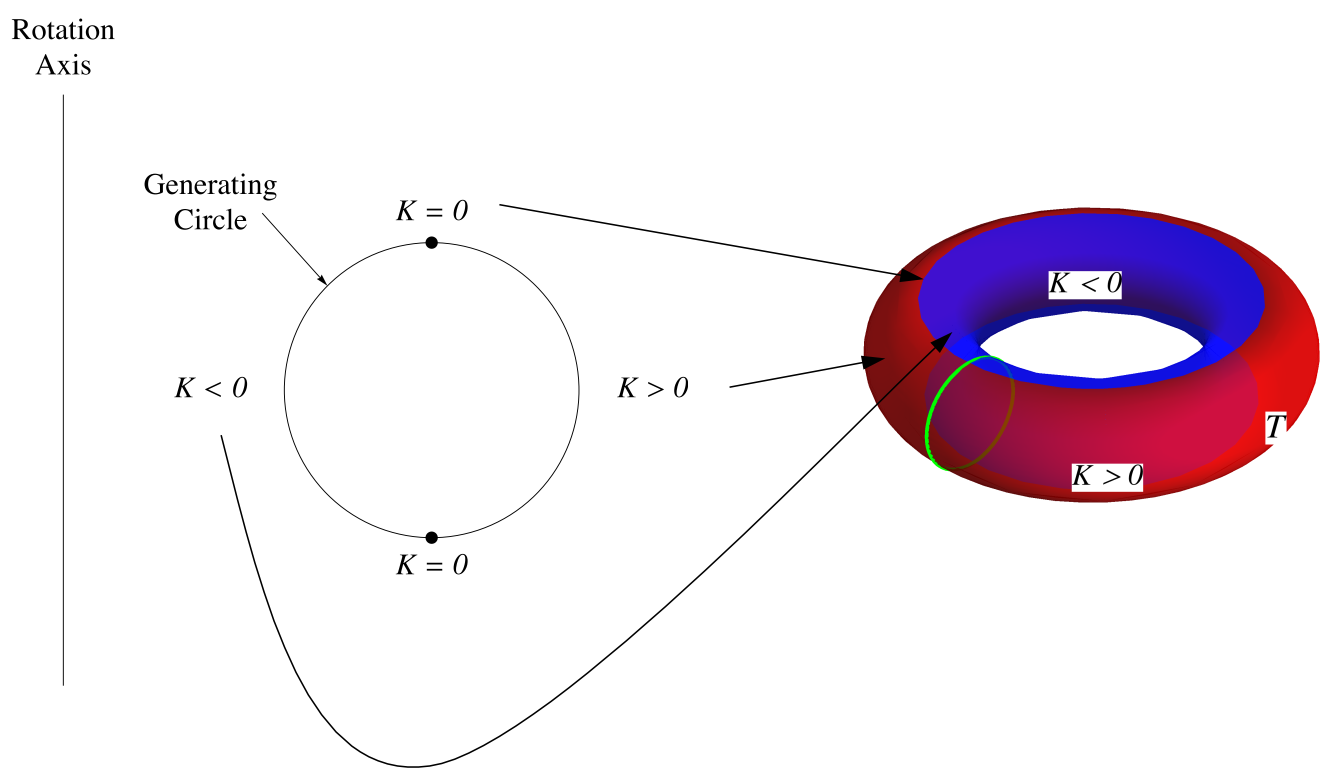
$$
K = \frac{eg}{EG}, \qquad H = \frac{1}{2}\frac{eG + gE}{EG}.
$$

Examples

Example

We shall compute the Gaussian curvature of the points of the torus covered by the parametrization (with $0 < u < 2\pi, 0 < v < 2\pi$)

$$\mathbf{x}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$


Rotation
Axis
Generating
Circle
K = 0
K < 0
K > 0
K = 0
K < 0
T
K > 0

Examples

Example (Surfaces of Revolution)

Consider a surface of revolution parametrized by

$$\begin{aligned} \mathbf{x}(u, v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)), \\ 0 < u < 2\pi, \quad a < v < b, \varphi(v) \neq 0. \end{aligned}$$

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The coefficients of the first fundamental form are given by

$$E = \varphi^2, \qquad F = 0, \qquad G = (\varphi')^2 + (\psi')^2.$$

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It is convenient to assume that the rotating curve is parametrized by arc length, that is, that

$$(\varphi')^2 + (\psi')^2 = G = 1.$$

Examples

Example (cont'd)

Thus, we compute

$$
\begin{aligned}
K &= -\frac{\psi'(\psi'\varphi'' - \psi''\varphi')}{\varphi} = -\frac{(\psi')^2\varphi'' + (\varphi')^2\varphi''}{\varphi} = -\frac{\varphi''}{\varphi},\\
H &= \frac{1}{2}\frac{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')}{\varphi},\\
\frac{e}{E} &= -\frac{\psi'\varphi}{\varphi^2} = -\frac{\psi'}{\varphi},\\
\frac{g}{G} &= \psi'\varphi'' - \psi''\varphi'.
\end{aligned}
$$

Examples

Example

Very often a surface is given as the graph of a differentiable function $z = h(x, y)$, where (x, y) belong to an open set $U \subset \mathbb{R}^2$. In this case, the Gauss map is

$$N(x, y) = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}},$$

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$$N(x, y) = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}},$$

the second fundamental form is given by

$$e = \frac{h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}}, \quad f = \frac{h_{xy}}{\sqrt{1 + h_x^2 + h_y^2}}, \quad g = \frac{h_{yy}}{\sqrt{1 + h_x^2 + h_y^2}},$$

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and the curvature is

$$K = \frac{h_{xx} h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}, \quad 2H = \frac{(1 + h_x^2) h_{yy} - 2 h_x h_y h_{xy} + (1 + h_y^2) h_{xx}}{(1 + h_x^2 + h_y^2)^{3/2}}.$$

Graphs of Functions

Given a point p of a surface S, we can choose the coordinate axis of $\mathbb{R}^3$ so that the origin O of the coordinates is at p and the z axis is directed along the positive normal of S at p (thus, the xy plane agrees with $T_p(S)$). It follows that a neighborhood of p in S can be represented in the form $z = h(x, y)$, $(x, y) \in U \subset \mathbb{R}^2$, where U is an open set and h is a differentiable function with $h(0, 0) = h_x(0, 0) = h_y(0, 0) = 0$.

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The second fundamental form of S at p applied to the vector $(x, y) \in \mathbb{R}^2$ becomes, in this case,

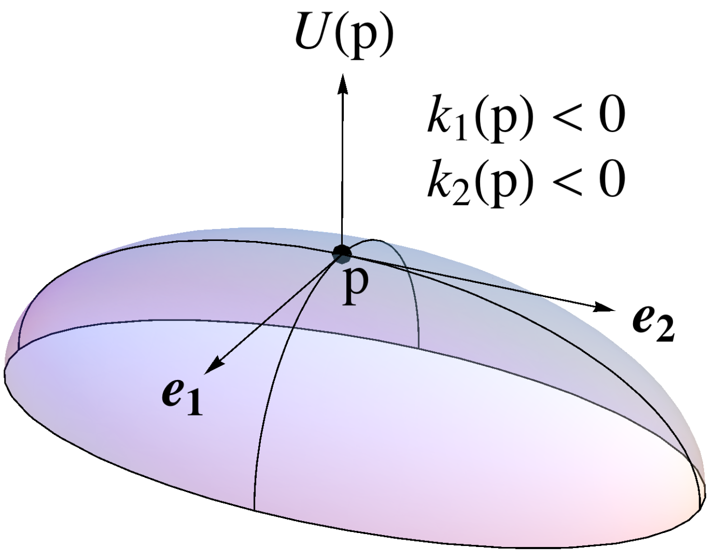
$$h_x x(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2.$$

In elementary calculus of two variables, the above quadratic form is known as the *Hessian* of h at $0, 0$. Thus, the Hessian of h at $(0, 0)$ is the second fundamental form of S at p.

The Sign of the Gaussian Curvature

Remark

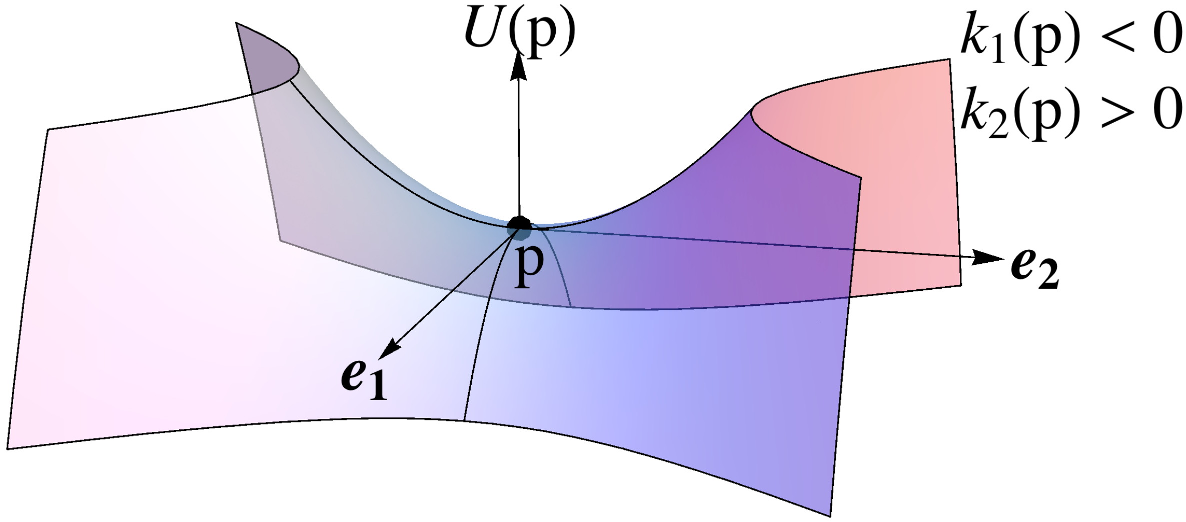
1. *Positive.* If $K(p) > 0$, then



The Sign of the Gaussian Curvature

Remark

2. *Negative.* If $K(p) < 0$, then



The Sign of the Gaussian Curvature

Remark

3. *Zero.* If $K(p) = 0$, then
   (a) Only one principal curvature is zero, say
   $$k_1(p) \neq 0, \quad k_2(p) = 0.$$
   (b) Both principal curvatures are zero:
   $$k_1(p) = k_2(p) = 0.$$