Lecture 16 Part A: Moving Frames

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Math 143

Moving Frame Continued

We have seen several kind of moving frames:

- UAV moving frames
- Moving frames on surfaces:
- **Key**: the derivatives of moving from can be written as a linear combination of this moving frame again. The coefficients are Christoffel symbols.
- **Today: Moving frame on a 3D curve:**
- **Frenet Frame**

Recall: Curvature of a 3D curve

Curvature

Geometric Meaning

Let $\alpha : I = (a, b) \to \mathbb{R}^3$ be a curve parametrized by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $\|\alpha''(s)\|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $\|\alpha''(s)\|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s.

Definition

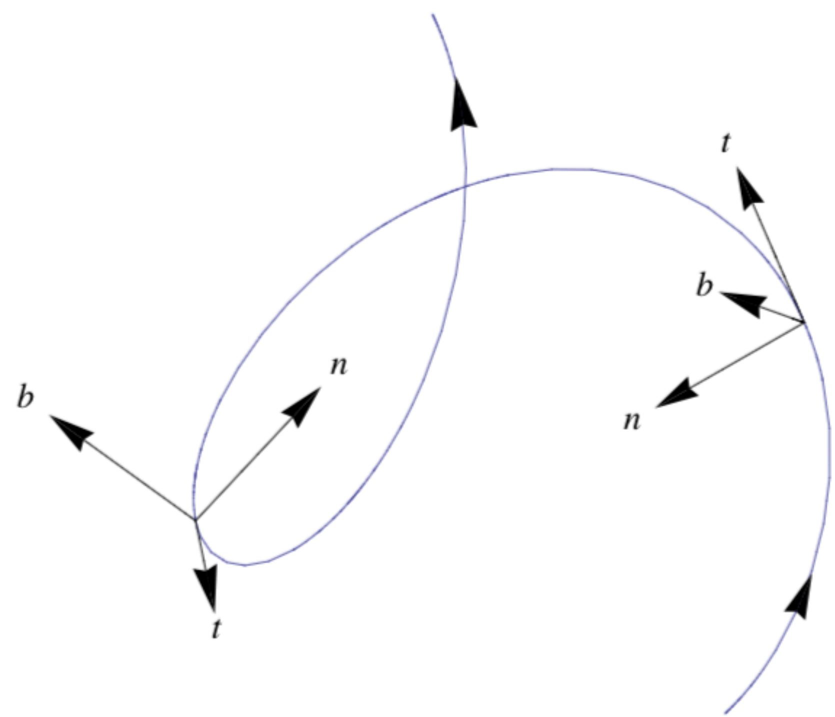
Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $\|\alpha''(s)\| = k(s)$ is called the *curvature* of α at s.

Today: We learn torsion of a 3D curve

Torsion

Geometric Meaning

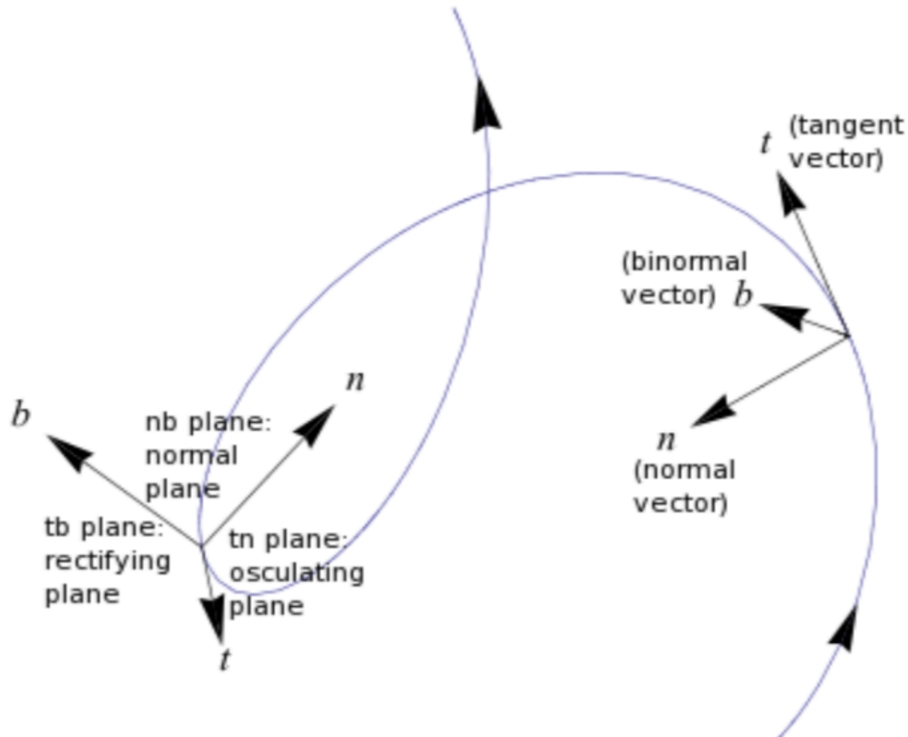
Since $b(s)$ is a unit vector, the length $\|b'(s)\|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



Frenet Frame

- Working out details with the students on the board.

$$\alpha'(s) \stackrel{\triangle}{=} t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$



Frenet Formula

- Derive Frenet Formulas with students on the board.

$$\begin{cases} t' = kn, \\ n' = -kt - \tau b, \\ b' = \tau n \end{cases}$$

Fundamental Theorem of the Local Theory of Curves

Theorem

Given differentiable functions $k(s) > 0$ *and* $\tau(s), s \in I$*, there exists a regular parametrized curve* $\alpha : I \to \mathbb{R}^3$ *such that* s *is the arc length,* $k(s)$ *is the curvature, and* $\tau(s)$ *is the torsion of* α *Moreover, any other curve* $\overline{\alpha}$ *satisfying the same conditions differs from* α *by a rigid motion; that is, there exists an orthogonal map* ρ *of* $\mathbb{R}^3$*, with positive determinant, and a vector* c *such that* $\overline{\alpha} = \rho \circ \alpha + c$.