Lecture 19-Geodesic Equations and Exponential Map

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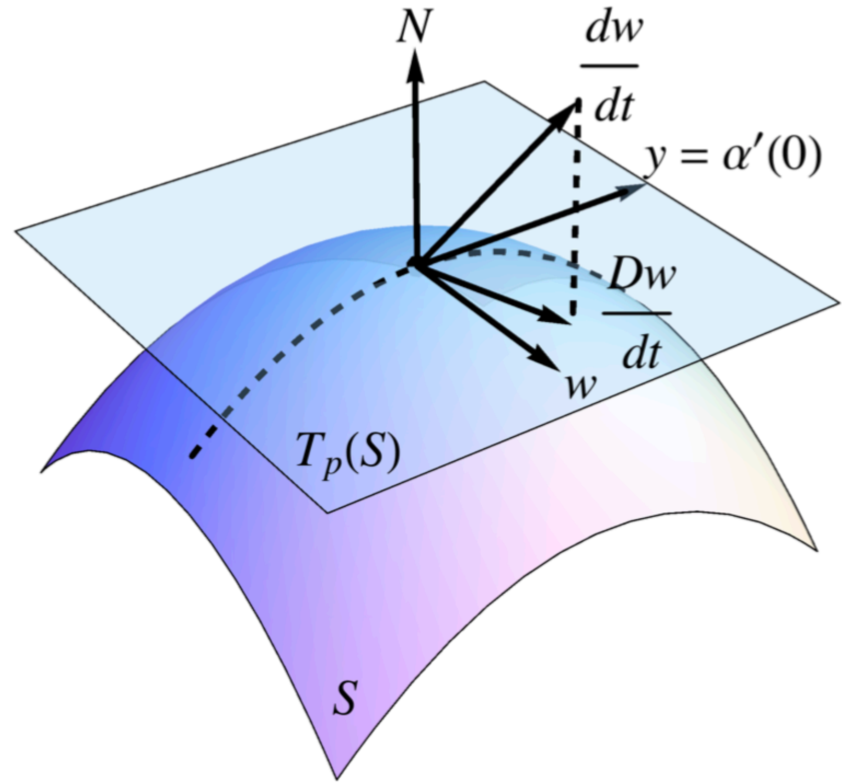
Recall: what is covariant derivative?



Let w be a differentiable vector field in
Let $y \in T_p(S)$. Consider a parametrizec

$$\alpha : (-\epsilon, \epsilon) \to U,$$

with $\alpha(0) = p$ and $\alpha'(0) = y$, and let $w(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field w to the curve α. The vector field obtained by the normal projection of $w'(0)$ onto the plane $T_p(S)$ is called the *covariant derivative* at p of the vector field w relative to the vector y. This covariant derivative is denoted by $(Dw/dt)(0)$ or $(D_y w)(p)$.

The Covariant Derivative in Local Coordinates

The above definition makes use of the normal vector of S and of a particular curve α, tangent to y at p. To show that covariant differentiation is a concept of the intrinsic geometry and that it does not depend on the choice of the curve α, we shall obtain its expression in terms of a parametrization $\mathbf{x}(u, v)$ of S in p.

- Working out details with the students on the board

Covariant derivative equation

$$
\begin{aligned}
\frac{Dw}{dt} &= (a' + \Gamma_{11}^1 au' + \Gamma_{12}^1 av' + \Gamma_{12}^1 bu' + \Gamma_{22}^1 bv')\mathbf{x}_u \\
&\quad + (b' + \Gamma_{11}^2 au' + \Gamma_{12}^2 av' + \Gamma_{12}^2 bu' + \Gamma_{22}^2 bv')\mathbf{x}_v. \qquad (1)
\end{aligned}
$$

Obtained by throwing away the normal components of dw/dt.

Recall: Geodesics

A nonconstant, parametrized curve $\gamma : I \to S$ is said to be *geodesic* at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is,

$$\frac{D\gamma'(t)}{dt} = 0;$$

γ is a *parametrized geodesic* if it is geodesic for all $t \in I$.

Recall:

A vector field w along a parametrized curve $\alpha : I \to S$ is said to be *parallel* if $Dw/dt = 0$ for every $t \in I$.

Example: Geodesic on a Sphere

The great circles of a sphere S^2 are geodesics. Indeed, the great circles C are obtained by intersecting the sphere with a plane that passes through the center O of the sphere. The principal normal at a point $p \in C$ lies in the direction of the line that connects p to O because C is a circle of center O. Since S^2 is a sphere, the normal lies in the same direction, which verifies our assertion.

- Recall we have proved:
  1. $\|\gamma'(t)\|$ is constant.
  2. A parametrized geodesic may admit self-intersections.

Geodesic in local coordinates

Let $\mathbf{x}(u(t), v(t))$, $t \in J$, be the expression of $\gamma : J \to S$ in the parametrization $\mathbf{x}$. Then, the tangent vector field $\gamma'(t)$, $t \in J$, is given by

$$w = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v.$$

Note: Plug a(t) = u'(t), b(t) = v'(t) into the covariant derivative equation (1) on slide 4.

Therefore, the fact that w is parallel is equivalent to the the system of differential equations

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0,$$
$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0,$$

obtained by equating to zero the coefficients of $\mathbf{x}_u$ and $\mathbf{x}_v$.

Geodesic Equation

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0,$$
$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0,$$

- Where the solutions (u(t), v(t)) will be geodesic in local coordinates. An the following curve will be the geodesic on the surface S:

$\mathbf{x}(u(t), v(t))$, $t \in J$, be the expression of $\gamma : J \to S$

Homework

Let us study locally the geodesics of a surface of revolution with the parametrization

$$x = f(v)\cos u, \qquad y = f(v)\sin u, \qquad z = g(v).$$

See page 255, baby Do Carmo, Example 5 for more details

Example 5. We shall use system (4) to study locally the geodesics of a surface of revolution (cf. Example 4, Sec. 2-3) with the parametrization

$$x = f(v)\cos u, \qquad y = f(v)\sin u, \qquad z = g(v).$$

By Example 1 of Sec. 4-1, the Christoffel symbols are given by

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = -\frac{ff'}{(f')^2 + (g')^2}, \qquad \Gamma^1_{12} = \frac{ff'}{f^2},$$

$$\Gamma^2_{12} = 0, \qquad \Gamma^1_{22} = 0, \qquad \Gamma^2_{22} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

With the values above, system (4) becomes

$$\begin{aligned} u'' + \frac{2ff'}{f^2}u'v' &= 0, \\ v'' - \frac{ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + f'g''}{(f')^2 + (g')^2}(v')^2 &= 0. \end{aligned} \tag{4a}$$

We are going to obtain some conclusions from these equations.

First, as expected, the meridians $u = \text{const.}$ and $v = v(s)$, parametrized by arc length s, are geodesics. Indeed, the first equation of (4a) is trivially satisfied by $u = \text{const.}$ The second equation becomes

Geodesic on manifold

Definition 7.1.2 Let (M, g) be a Riemannian manifold. A curve, $\gamma\colon I \to M$, (where $I \subseteq \mathbb{R}$ is any interval) is a *geodesic* iff $\gamma'(t)$ is parallel along γ, that is, iff

$$\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = 0.$$

If M was embedded in $\mathbb{R}^d$, a geodesic would be a curve, γ, such that the acceleration vector, $\gamma'' = \frac{D\gamma'}{dt}$, is normal to $T_{\gamma(t)}M$.

By Proposition 6.4.6, $\|\gamma'(t)\| = \sqrt{g(\gamma'(t), \gamma'(t))}$ is constant, say $\|\gamma'(t)\| = c$.

- Same definition for regular surface or a manifold!

Geodesic equation for manifold

In a local chart, (U, φ), since a geodesic is characterized by the fact that its velocity vector field, $\gamma'(t)$, along γ is parallel, by Proposition 6.3.4, it is the solution of the following system of second-order ODE's in the unknowns, u_k:

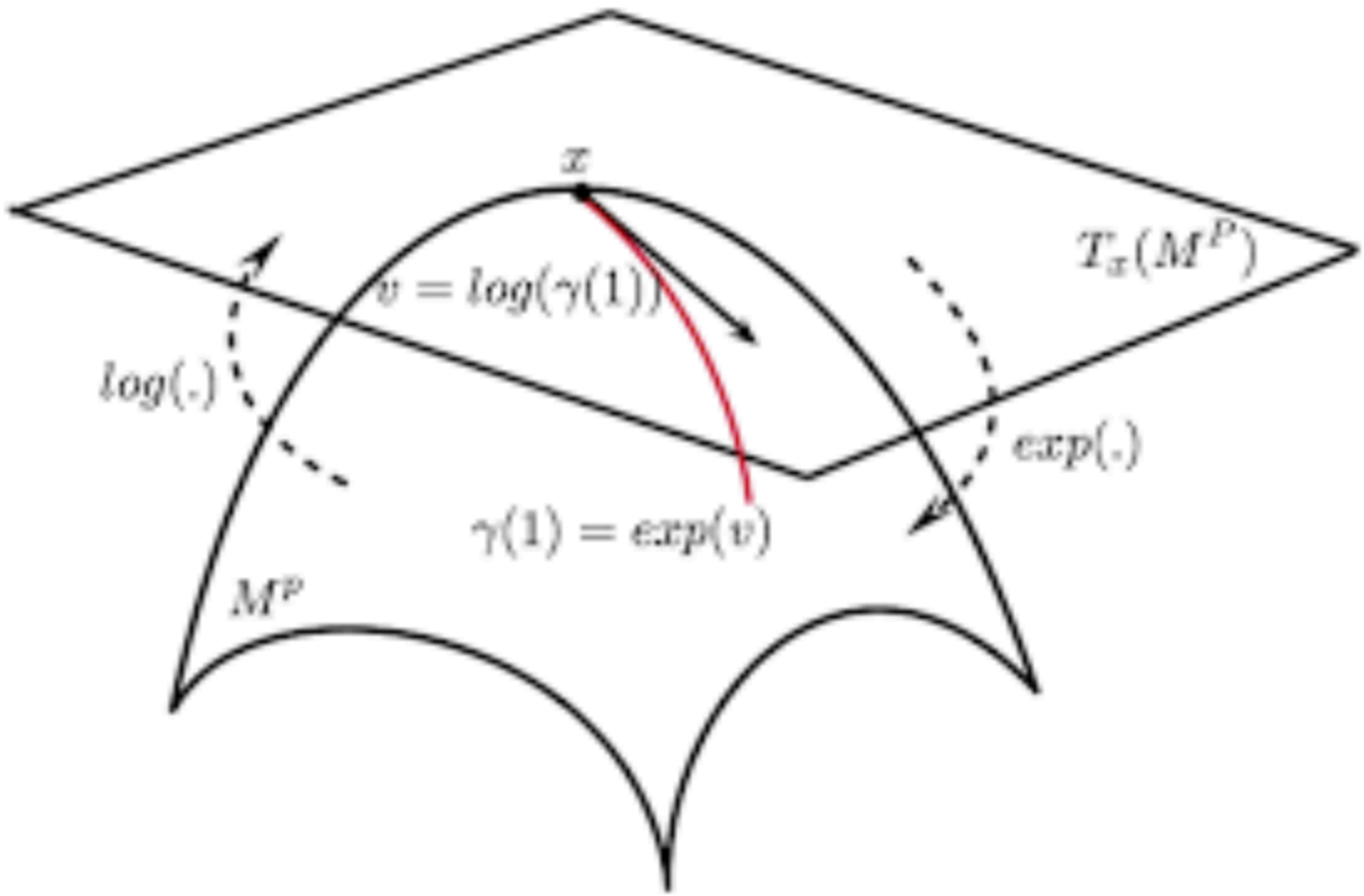
$$\frac{d^2 u_k}{dt^2} + \sum_{ij} \Gamma_{ij}^k \frac{du_i}{dt} \frac{du_j}{dt} = 0, \qquad k = 1, \ldots, n,$$

with $u_i = pr_i \circ \varphi \circ \gamma$ $(n = \dim(M))$.

- Just extend dim = 2 to dim = n.

Exponential map

- Also log map



Exponential Map on manifold

Definition 7.2.1 Let (M, g) be a Riemannian manifold. For every $p \in M$, let $\mathcal{D}(p)$ (or simply, $\mathcal{D}$) be the open subset of T_pM given by

$$\mathcal{D}(p) = \{v \in T_pM \mid \gamma_v(1) \quad \text{is defined}\},$$

where γ_v is the unique maximal geodesic with initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The *exponential map* is the map, $\exp_p\colon \mathcal{D}(p) \to M$, given by

$$\exp_p(v) = \gamma_v(1).$$

Geodesic Polar Coordinates

1. The *normal coordinates* which correspond to a system of rectangular coordinates in the tangent plane $T_p(S)$.
2. The *geodesic polar coordinates* which correspond to polar coordinates in the tangent plane $T_p(S)$ (Fig. 4-38).

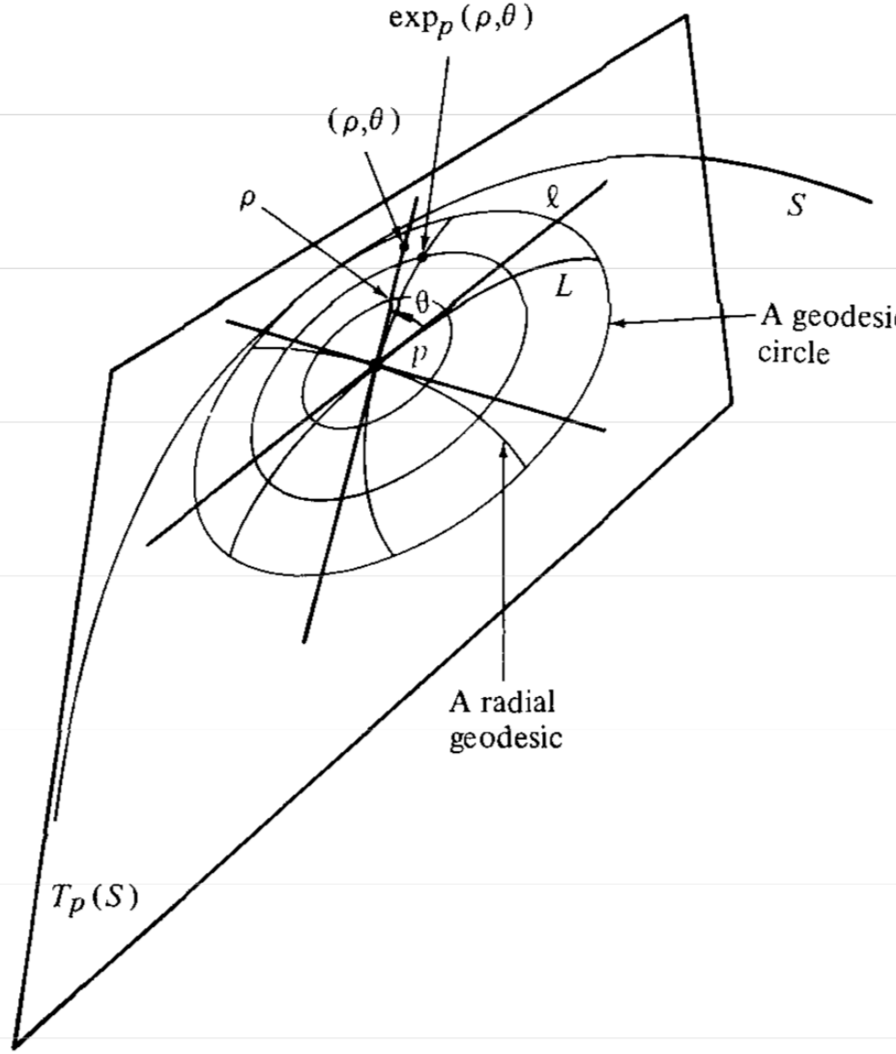


Figure 4-38 Polar coordinates.