Lecture 12-PartA: Normal, Principal, Gaussian, and Mean Curvature

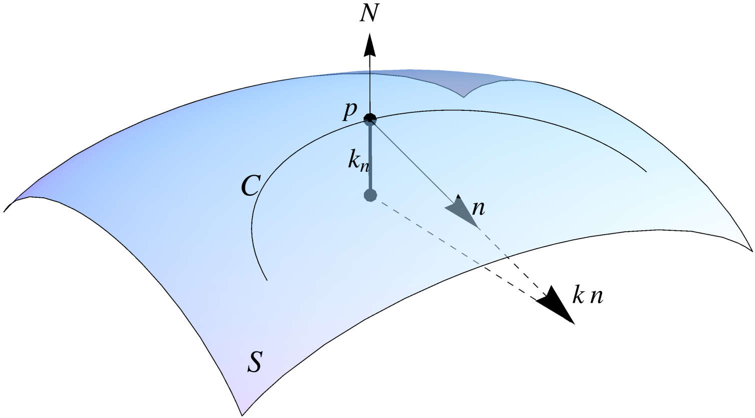
Prof. Weiqing Gu

Math 143:
Topics in Geometry

Normal Curvature

Definition

Let C be a regular curve in S passing through $p \in S$, k the curvature of C at p, and $\cos\theta = \langle n, N\rangle$, where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = k\cos\theta$ is then called the *normal curvature* of $C \subset S$ at p.



Normal Curvature

Proposition (Meusnier)

All curves lying on a surface S *and having at a given point* $p \in S$ *the same tangent line have at this point the same normal curvatures.*

Normal Curvature

Proposition (Meusnier)

All curves lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures.

Definition

This proposition allows us to speak of the *normal curvature along a given direction at* p. It is convenient to use the following terminology. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the *normal section* of S at p along v.

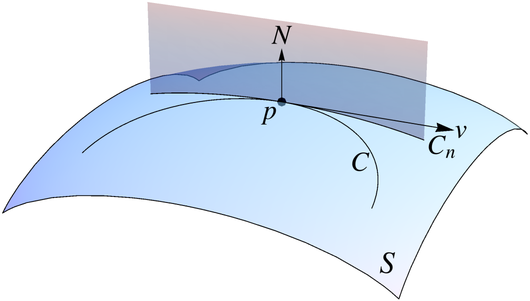
Normal Curvature

Proposition (Meusnier)

All curves lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvatures.

Definition

This proposition allows us to speak of the *normal curvature along a given direction at* p. It is convenient to use the following terminology. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the *normal section* of S at p along v.

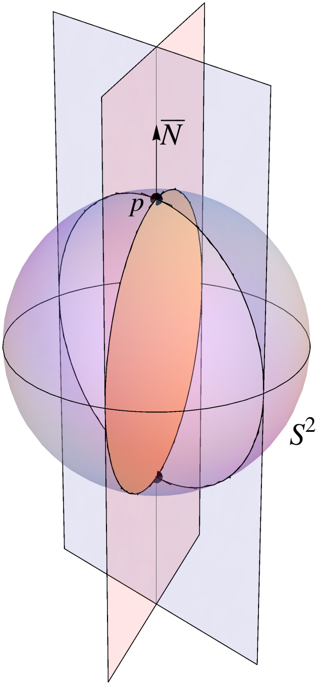


Examples

- In the plane, all normal sections are straight lines; hence, all normal curvatures are zero. thus, the second fundamental form is identically zero at all points. This agrees with the fact that $dN \equiv 0$.

Examples

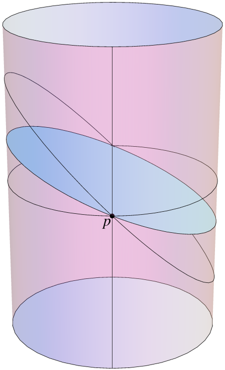
- In the plane, all normal sections are straight lines; hence, all normal curvatures are zero. thus, the second fundamental form is identically zero at all points. This agrees with the fact that $dN \equiv 0$.
- In the sphere $S^2(1)$, with outward orientation $\overline{N}$, the normal sections through a point $p \in S^2$ are circles with radius 1. Thus, all normal curvatures are equal to 1, and the second fundamental form is $II_p(v) = 1$ for all $p \in S^2$ and all $v \in T_p(S^2)$ with $\|v\| = 1$.


$\overline{N}$
p
S^2

Examples

Example

In a cylinder with radius 1, the normal sections at a point p vary from a circle perpendicular to the axis of the cylinder to a straight line parallel to the axis of the cylinder, passing through a family of ellipses. Thus, the normal curvatures vary from 1 to 0. It is not hard to see geometrically that 1 is the maximum and 0 is the minimum of the normal curvature at p.



Principal Curvature

Remark

Let us come back to the linear map dN_p. The theorem of the appendix to Ch. 3 of Do Carmo shows that for each $p \in S$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$. Moreover, k_1 and k_2, $(k_1 \geq k_2)$ are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

Principal Curvature

Remark

Let us come back to the linear map dN_p. The theorem of the appendix to Ch. 3 of Do Carmo shows that for each $p \in S$ there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$. Moreover, k_1 and k_2, $(k_1 \geq k_2)$ are the maximum and minimum of the second fundamental form II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

Definition

The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the *principal curvatures* at p; the corresponding directions, that is, the directions given by the eigenvectors e_1, e_2, are called *principal directions* at p.

The Euler Formula

The knowledge of the principal curvatures at p allows us to compute easily the normal curvature along a given direction of $T_p(S)$.

$$k_n = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

The Euler Formula

The knowledge of the principal curvatures at p allows us to compute easily the normal curvature along a given direction of $T_p(S)$.

$$k_n = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

The last expression is known classically as the *Euler formula*; actually, it is just the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

Examples

Example

In the plane, all directions at all points are principal directions. The same happens with a sphere. In both cases, this comes from the fact that the second fundamental form at each point, restricted to the unit vectors, is constant; thus, all directions are extremals for the normal curvature.

Examples

Example

In the plane, all directions at all points are principal directions. The same happens with a sphere. In both cases, this comes from the fact that the second fundamental form at each point, restricted to the unit vectors, is constant; thus, all directions are extremals for the normal curvature.

Example

In the cylinder above, the vectors v and w give the principal directions at p, corresponding to the principal curvatures 0 and 1, respectively. In the hyperbolic paraboloid, the x and y axes are along the principal directions with principal curvatures -2 and 2, respectively.

Gaussian and Mean Curvature

Definition

Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the *Gaussian curvature* K of S at p. The negative half of the trace of dN_p is called the *mean curvature* H of S at p.

In terms of the principal curvatures, we can write

$$K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}.$$

Special local shapes correspond to special values of curvatures and the Second Fundamental Form

Definition

A point of a surface S is called

1. Elliptic if $\det dN_p > 0$.
2. Hyperbolic if $\det dN_p < 0$.
3. Parabolic if $\det dN_p = 0$, with $dN_p \neq 0$.
4. Planar if $dN_p = 0$.

Examples

Example

- At an elliptic point the Gaussian curvature is positive. Both principal curvatures have the same sign, and therefore all curves passing through this point have their normal vectors pointing toward the same side of the tangent plane. The points of a sphere are elliptic points.

Examples

Example

- At an elliptic point the Gaussian curvature is positive. Both principal curvatures have the same sign, and therefore all curves passing through this point have their normal vectors pointing toward the same side of the tangent plane. The points of a sphere are elliptic points.
- At a hyperbolic point, the Gaussian curvature is negative. The principal curvatures have opposite signs, and therefore there are curves through p whose normal vectors at p point toward any of the sides of the tangent plane at p. The point $(0,0,0)$ of the hyperbolic paraboloid $z = y^2 - x^2$ is a hyperbolic point.

Examples

Example

- At a parabolic point, the Gaussian curvature is zero, but one of the principal curvatures is not zero. The points of a cylinder are parabolic points.

Examples

Example

- At a parabolic point, the Gaussian curvature is zero, but one of the principal curvatures is not zero. The points of a cylinder are parabolic points.
- Finally, at a planar point, all principal curvatures are zero. The points of a plane trivially satisfy this condition. A nontrivial examples of a planar point was given in Example 6 of Do Carmo.

Reading Materials

Definition

If at $p \in S$, $k_1 = k_2$, then p is called an *umbilical point* of S; in particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.

Reading Materials

Definition

If at $p \in S$, $k_1 = k_2$, then p is called an *umbilical point* of S; in particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.

Remark

All the points of a sphere and a plane are umbilical points. We may also verify that the point $(0,0,0)$ of the paraboloid $z = x^2 + y^2$ is a (nonplanar) umbilical point.

Reading Materials

Definition

If at $p \in S$, $k_1 = k_2$, then p is called an *umbilical point* of S; in particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.

Remark

All the points of a sphere and a plane are umbilical points. We may also verify that the point $(0, 0, 0)$ of the paraboloid $z = x^2 + y^2$ is a (nonplanar) umbilical point.

Proposition

If all points of a connected surface S are umbilical points, then S is either contained in a sphere or in a plane.