Lecture 10: Isometries

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Math 143:
Topics in Geometry

Isometries

Motivation

Although the cylinder and the plane are distinct surfaces, their first fundamental forms are "equal" (at least in the coordinate neighborhoods that we have considered). This means that insofar as intrinsic metric questions are concerned (length, angle, area), the plane and the cylinder behave *locally* in the same way. (This is intuitively clear, since by cutting a cylinder along a generator we may unroll it into a part of a plane.) In this lecture, we shall see that many other important concepts associated to a regular surface depend only on the first fundamental form and should be included in the category of intrinsic concepts. It is therefore convenient that we formulate in a precise way what is meant by two regular surfaces having equal first fundamental forms.

Isometries

Definition

A diffeomorphism $\varphi : S \to \overline{S}$ is an *isometry* if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have

$$\langle w_1, w_2 \rangle_p = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}.$$

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Note

φ is an isometry if and only if (i) φ is a diffeomorphism and (ii) φ preserves the first fundamental form.

Isometries

Definition

A map $\varphi : V \to \overline{S}$ of a neighborhood V of $p \in S$ is a *local isometry* at p if there exists a neighborhood $\overline{V}$ of $\varphi(p) \in \overline{S}$ such that $\varphi : V \to \overline{V}$ is an isometry. If there exists a local isometry into $\overline{S}$ at every $p \in S$, the surface S is said to be *locally isometric* to $\overline{S}$. S and $\overline{S}$ are *locally isometric* if S is locally isometric to $\overline{S}$ and $\overline{S}$ is locally isometric to S.

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It is clear that if $\varphi : S \to \overline{S}$ is a diffeomorphism and a local isometry for every $p \in S$, then φ is an isometry (globally). It may, however, happen that two surfaces are locally isometric without being (globally) isometric, as shown in the following example.

Example:

Isometries

Proposition

Assume the existence of parametrizations $\mathbf{x} : U \to S$ *and* $\overline{\mathbf{x}} : U \to \overline{S}$ *such that* $E = \overline{E}$, $F = \overline{F}$, $G = \overline{G}$ *in* U. *Then the map* $\varphi = \overline{\mathbf{x}} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \overline{S}$ *is a local isometry.*

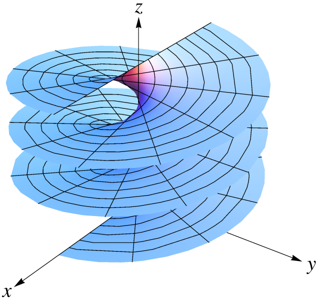
Proof.

Example: Showing that the catenoid is locally isometric to the helicoid.

Helicoid

Recall that the helicoid is generated from the helix $(\cos \overline{u}, \sin \overline{u}, a\overline{u})$ by drawing through each point a line parallel to the xy plane and intersecting the z axis. The helicoid admits the following parametrization:

$$\mathbf{x}(\overline{u}, \overline{v}) = (\overline{v} \cos \overline{u}, \overline{v} \sin \overline{u}, a\overline{u}), \quad 0 < \overline{u} < 2\pi, \quad -\infty < \overline{v} < \infty,$$
$$\overline{E}(\overline{u}, \overline{v}) = \overline{v}^2 + a^2, \qquad \overline{F}(\overline{u}, \overline{v}) = 0, \qquad \overline{G}(\overline{u}, \overline{v}) = 1.$$



Example (cont'd)

Catenoid

The surface of revolution of the *catenary*,

$$x = a\cosh v, \quad z = av, \quad -\infty < v < \infty,$$

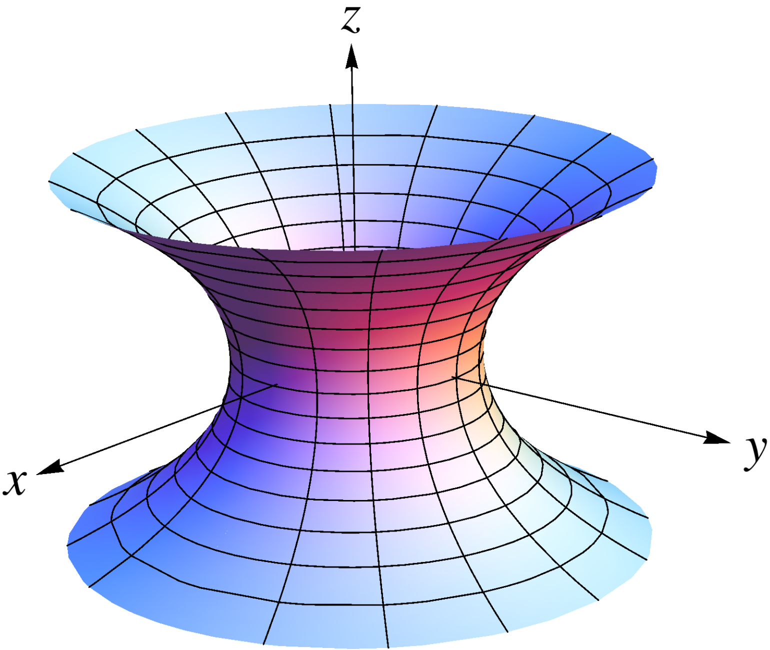
admits the following parametrization:

$$\mathbf{x}(u, v) = (a\cosh v\cos u, a\cosh v\sin u, av),$$
$$0 < u < 2\pi, \quad -\infty < v < \infty,$$

relative to which the coefficients of the first fundamental form are

$$\begin{aligned} E(u, v) &= a^2\cosh^2 v, \\ F(u, v) &= 0, \\ G(u, v) &= a^2(1 + \sinh^2 v) = a^2\cosh^2 v. \end{aligned}$$

This surface of revolution is called the *catenoid*.


z
x
y

Example (cont'd)

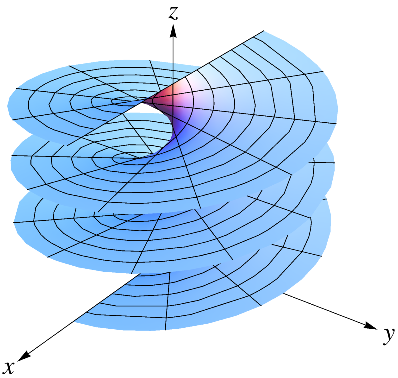
Recall:

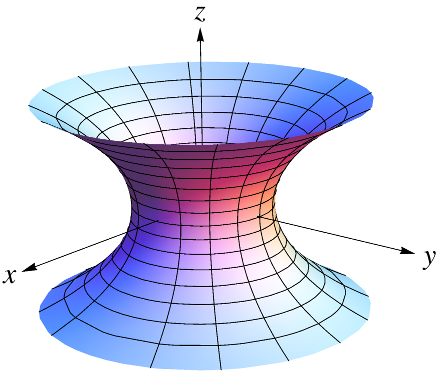
Let S be a surface of revolution and let

$$\mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$
$$a < v < b, \quad 0 < u < 2\pi, \quad f(v) > 0,$$

be a parametrization of S. The coefficients of the first fundamental form of S in the parametrization $\mathbf{x}$ are given by

$$E = (f(v))^2, \quad F = 0, \quad G = (f'(v))^2 + (g'(v))^2.$$





Animation:

`http://upload.wikimedia.org/wikipedia/commons/c/ce/Helicatenoid.gif`

Example

Plane and Cone

We shall prove that the one-sheeted cone (minus the vertex)

$$z = +k\sqrt{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

is locally isometric to a plane. The idea is to show that a cone minus a generator can be "rolled" onto a piece of a plane.