

Lecture 17: Derivatives on manifolds and their applications

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Math 143:
Topics in Differential Geometry

Directional Derivatives

Definition

Let f be a differentiable real-valued function on \mathbf{E}^3 , and let \mathbf{v}_p be a tangent vector to \mathbf{E}^3 . Then the number

$$\mathbf{v}_p[f] = \frac{d}{dt}(f(\mathbf{p} + t\mathbf{v}))|_{t=0}$$

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U_1, U_2, U_3 is the standard frame field on \mathbf{E}^3 , then $U_i[f] = \partial f / \partial x_i$.

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Lemma

If $\mathbf{v}_p = (v_1, v_2, v_3)_p$ is a tangent vector to \mathbf{E}^3 , then

$$\mathbf{v}_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(\mathbf{p}) = (\mathbf{v} \cdot \nabla f)(p).$$

Directional Derivatives

Theorem

Let f and g be functions on \mathbf{E}^3 , \mathbf{v}_p and \mathbf{w}_p tangent vectors, a and b numbers. Then

1. $(a\mathbf{v}_p + b\mathbf{w}_p)[f] = a\mathbf{v}_p[f] + b\mathbf{w}_p[f]$.
2. $\mathbf{v}_p[af + bg] = a\mathbf{v}_p[f] + b\mathbf{v}_p[g]$.
3. $\mathbf{v}_p[fg] = \mathbf{v}_p[f] \cdot g(\mathbf{p}) + f(\mathbf{p}) \cdot \mathbf{v}_p[g]$.

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Corollary

If V and W are vector fields on \mathbf{E}^3 and f, g, h are real-valued functions, then

1. $(fV + gW)[h] = fV[h] + gW[h]$.
2. $V[af + bg] = aV[f] + bV[g]$ for all real numbers a and b .
3. $V[fg] = V[f] \cdot g + f \cdot V[g]$.

Covariant Derivatives on \mathbf{E}^3

Definition

Let W be a vector field on \mathbf{E}^3 and let \mathbf{v} be a tangent vector to \mathbf{E}^3 at the point \mathbf{p} . Then the *covariant derivative* of W with respect to \mathbf{v} is the tangent vector

$$\nabla_{\mathbf{v}} W = W(\mathbf{p} + t\mathbf{v})'(0)$$

at the point \mathbf{p} .

Covariant Derivatives on \mathbf{E}^3

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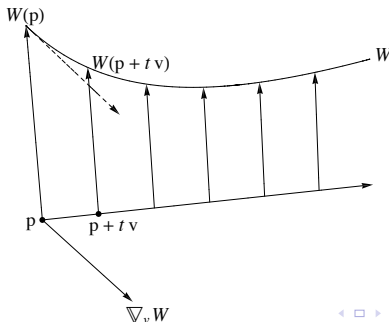
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Evidently

$\nabla_{\mathbf{v}} W$ measures the initial rate of change of $W(\mathbf{p})$ as \mathbf{p} moves in the \mathbf{v} direction.



Covariant Derivatives

Example

Suppose $W = x^2 U_1 + yz U_3$, and $\mathbf{v} = (-1, 0, 2)$ at $\mathbf{p} = (2, 1, 0)$.

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Lemma

If $W = \sum w_i U_i$ is a vector field on \mathbf{E}^3 , and \mathbf{v} is a tangent vector at \mathbf{p} , then

$$\nabla_{\mathbf{v}} W = \sum \mathbf{v}[w_i] U_i(\mathbf{p}).$$

In short, to apply $\nabla_{\mathbf{v}}$ to a vector field, apply \mathbf{v} to its Euclidean coordinates.

Covariant Derivatives

Theorem

Let \mathbf{v} and \mathbf{w} be tangent vectors to \mathbf{E}^3 at \mathbf{p} , and let Y and Z be vector fields on \mathbf{E}^3 . Then

1. $\nabla_{a\mathbf{v}+b\mathbf{w}}Y = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{w}}Y$ for all numbers a and b .
2. $\nabla_{\mathbf{v}}(aY + bZ) = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{v}}Z$ for all numbers a and b .
3. $\nabla_{\mathbf{v}}(fY) = \mathbf{v}[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_{\mathbf{v}}Y$ for all (differentiable) functions f .
4. $\mathbf{v}[Y \cdot Z] = \nabla_{\mathbf{v}}Y \cdot Z(\mathbf{p}) + Y(\mathbf{p}) \cdot \nabla_{\mathbf{v}}Z$.

Covariant Derivatives

Note

We can define $\nabla_V W$ naturally, where V is also a vector field.

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Corollary

Let V, W, Y , and Z be vector fields on \mathbf{E}^3 . Then

1. $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$ for all numbers a and b .
2. $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$ for all functions f and g .
3. $\nabla_V(fY) = V[f]Y + f\nabla_V Y$ for all functions f .
4. $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$.

Covariant Derivative on a Surface

Motivation

We want to systematically study the intrinsic geometry of a surface.

Key

We need to generalize Gauss's idea

$$dN_p : T_p(S) \rightarrow T_p(S),$$

where N is a normal vector field.

How

We must find a way of differentiating a vector field with respect to a direction, and this must be compatible with differentiation in a Euclidean space.

Covariant Derivative on a Surface

Definition

Let w be a differentiable vector field in an open set $U \subset S$ and $p \in U$. Let $y \in T_p(S)$. Consider a parametrized curve

$$\alpha : (-\epsilon, \epsilon) \rightarrow U,$$

with $\alpha(0) = p$ and $\alpha'(0) = y$, and let $w(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field w to the curve α . The vector field obtained by the normal projection of $w'(0)$ onto the plane $T_p(S)$ is called the *covariant derivative* at p of the vector field w relative to the vector y . This covariant derivative is denoted by $(Dw/dt)(0)$ or $(D_y w)(p)$.

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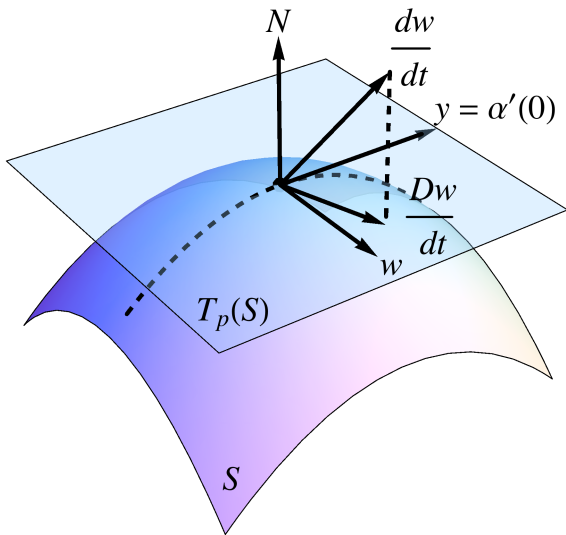
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Where:

A (*tangent*) vector field in an open set $U \subset S$ of a regular surface S is a correspondence w that assigns to each $p \in U$ a vector $w(p) \in T_p(S)$.

The vector field w is *differentiable* at p if, for some parametrization $\mathbf{x}(u, v)$ in p , the components a and b of $w = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ are differentiable functions at p . w is differentiable in U if it is differentiable for every $p \in U$.



The Covariant Derivative in Local Coordinates

The above definition makes use of the normal vector of S and of a particular curve α , tangent to y at p . To show that covariant differentiation is a concept of the intrinsic geometry and that it does not depend on the choice of the curve α , we shall obtain its expression in terms of a parametrization $\mathbf{x}(u, v)$ of S in p .

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$$\begin{aligned}\frac{Dw}{dt} = & (a' + \Gamma_{11}^1 au' + \Gamma_{12}^1 av' + \Gamma_{12}^1 bu' + \Gamma_{22}^1 bv')\mathbf{x}_u \\ & + (b' + \Gamma_{11}^2 au' + \Gamma_{12}^2 av' + \Gamma_{12}^2 bu' + \Gamma_{22}^2 bv')\mathbf{x}_v.\end{aligned}\quad (1)$$

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This expression shows that Dw/dt depends only on the vector $(u', v') = y$ and not on the curve α . Furthermore, the surface makes its appearance in Eq. ?? through the Christoffel symbols, that is, through the first fundamental form. Our assertions are, therefore, proved.

The Covariant Derivative in Local Coordinates

Note

The definition of covariant derivative of a vector field is the analogue for surfaces of the usual differentiation of vectors in the plane.

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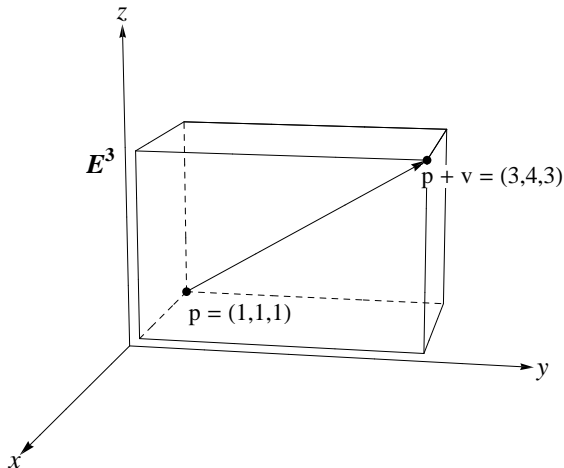
The definition of covariant derivative of a vector field is the analogue for surfaces of the usual differentiation of vectors in the plane.

From the local expression of Dw/dt , we see that if S is a plane, we know that it is possible to find a parametrization in such a way that $E = G = 1$ and $F = 0$. A quick inspection of the equations that give the Christoffel symbols shows that in this case the Γ_{ij}^k become zero. In this case, it follows from Eq. ?? that the covariant derivative agrees with the usual derivative of vectors in the plane (this can also be seen geometrically from the definition). The covariant derivative is, therefore, a generalization of the usual derivative of vectors in the plane.

Appendix: Vector Fields on \mathbf{E}^3

Definition

A *tangent vector* \mathbf{v}_p to \mathbf{E}^3 consists of two points of \mathbf{E}^3 : its *vector part* \mathbf{v} and its *point of application* \mathbf{p} .



Appendix: Vector Fields on \mathbf{E}^3

Definition

Let \mathbf{p} be a point of \mathbf{E}^3 . The set $T_p(\mathbf{E}^3)$ consisting of all tangent vectors that have \mathbf{p} as point of application is called the *tangent space* of \mathbf{E}^3 at \mathbf{p} .

Fig. 1.3

Note

We define $\mathbf{v}_p + \mathbf{w}_p$ to be $(\mathbf{v} + \mathbf{w})_p$, and if c is a number we define $c(\mathbf{v}_p)$ to be $(c\mathbf{v})_p$.

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A *vector field* V on \mathbf{E}^3 is a function that assigns to each point \mathbf{p} of \mathbf{E}^3 a tangent vector $V(\mathbf{p})$ to \mathbf{E}^3 at \mathbf{p} .

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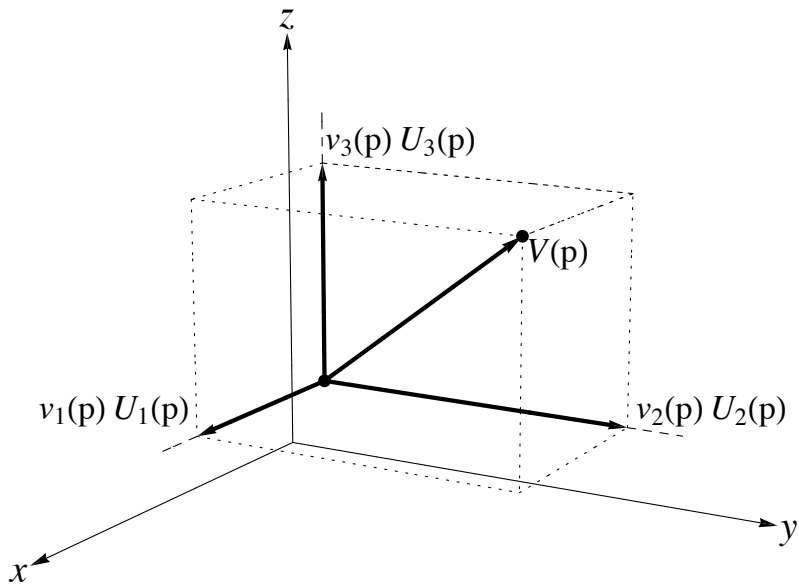
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Lemma

If V is a vector field on \mathbf{E}^3 , there are three uniquely determined real-valued functions v_1, v_2, v_3 on \mathbf{E}^3 such that

$$V = v_1 U_1 + v_2 U_2 + v_3 U_3.$$

The functions v_1, v_2, v_3 are called the Euclidean coordinate functions of V .



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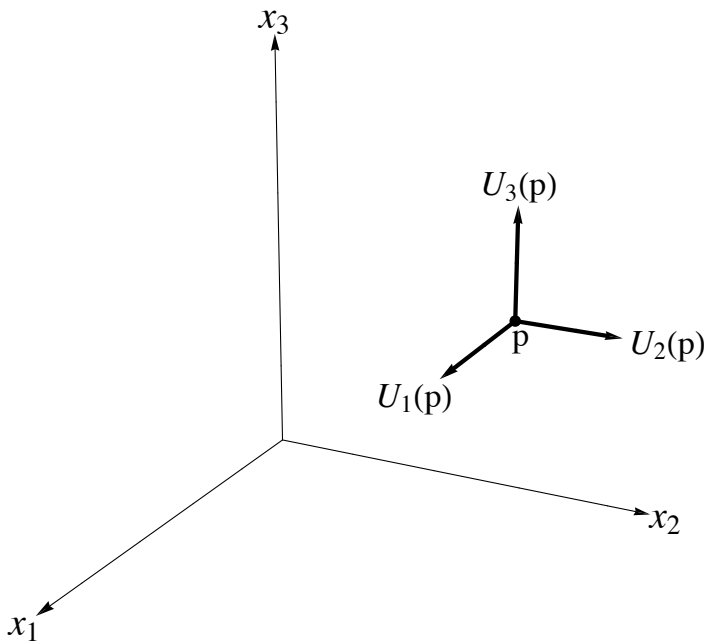
Let U_1 , U_2 , and U_3 be the vector fields on \mathbf{E}^3 such that

$$U_1(\mathbf{p}) = (1, 0, 0)_p$$

$$U_2(\mathbf{p}) = (0, 1, 0)_p$$

$$U_3(\mathbf{p}) = (0, 0, 1)_p$$

for each point \mathbf{p} of \mathbf{E}^3 . We call U_1 , U_2 , U_3 —collectively—the *natural frame field* on \mathbf{E}^3 .



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Definition

Vector fields E_1, E_2, E_3 on \mathbf{E}^3 constitute a *frame field* on \mathbf{E}^3 provided

$$E_i \cdot E_j = \delta_{ij} \quad (1 \leq i, j \leq 3)$$

where δ_{ij} is the Kronecker delta.

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The term *frame field* is justified by the fact that at each point \mathbf{p} the three vectors $E_1(\mathbf{p}), E_2(\mathbf{p}), E_3(\mathbf{p})$ form a frame at \mathbf{p} . We anticipated this by calling U_1, U_2, U_3 the natural frame field on \mathbf{E}^3 .

Appendix: Vector Fields on \mathbf{E}^3

Example: The Cylindrical Frame Field

Let r, θ, z be the usual cylindrical coordinate functions on \mathbf{E}^3 . We shall pick a unit vector field in the direction in which each coordinate increases (when the other two are held constant).

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$$E_1 = \cos \theta U_1 + \sin \theta U_2,$$

pointing straight out from the z axis.

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It is easy to check that $E_i \cdot E_j = \delta_{ij}$, so this is a frame field (defined on all of \mathbf{E}^3 except the z axis). We call it the *cylindrical frame field* on \mathbf{E}^3 .

Appendix: Vector Fields on \mathbf{E}^3

Example: The Spherical Frame Field

Let E_1, E_2, E_3 be the cylindrical frame field. For spherical coordinates, the unit vector field F_2 in the direction of increasing θ is the same as above, so $F_2 = E_2$.

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Thus the formulas for E_1, E_2, E_3 yield

$$F_1 = \sin \varphi (\cos \theta U_1 + \sin \theta U_2) + \cos \varphi U_3$$

$$F_2 = -\sin \theta U_1 + \cos \theta U_2$$

$$F_3 = -\cos \varphi (\cos \theta U_1 + \sin \theta U_2) + \sin \varphi U_3.$$

Appendix: Vector Fields on Surfaces

Definition

A *vector field* w in an open set $U \subset S$ of a regular surface S is a correspondence which assigns to each $p \in U$ a vector $w(p) \in T_p(S)$. The vector field w is differentiable at $p \in U$ if, for some parametrization $\mathbf{x}(u, v)$ at p , the functions $a(u, v)$ and $b(u, v)$ given by

$$w(p) = a(u, v)\mathbf{x}_u + b(u, v)\mathbf{x}_v$$

are differentiable functions at p ; it is clear that this definition does not depend on the choice of \mathbf{x} .

Appendix: Vector Fields on Surfaces

Compare

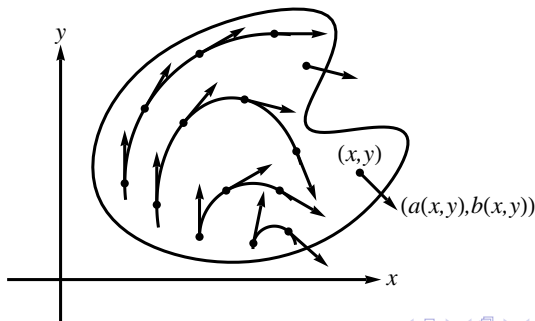
A *vector field* in an open set $U \subset \mathbb{R}^2$ is a map which assigns to each $q \in U$ a vector $w(q) \in \mathbb{R}^2$. The vector field w is said to be *differentiable* if writing $q = (x, y)$ and $w(q) = (a(x, y), b(x, y))$, the functions a and b are differentiable functions in U .

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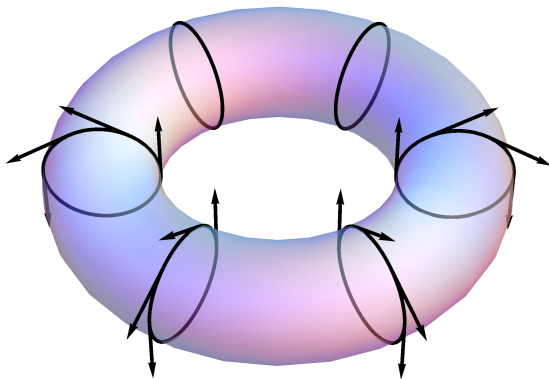
Geometrically, the definition corresponds to assigning to each point $(x, y) \in U$ a vector with coordinates $a(x, y)$ and $b(x, y)$ which vary differentiably with (x, y) .



Appendix: Vector Fields on Surfaces

Example

A vector field in the usual torus T is obtained by parametrizing the meridians of T by arc length and defining $w(p)$ as the velocity vector of the meridian through p . Notice that $\|w(p)\| = 1$ for all $p \in T$. It is left as an exercise to verify that w is differentiable.



Appendix: Vector Fields on Surfaces

Example

A similar procedure, this time on the sphere S^2 and using the semimeridians of S^2 , yields a vector field w defined in the sphere minus the two poles N and S . To obtain a vector field defined in the whole sphere, reparametrize all the semimeridians by the same parameter t , $-1 < t < 1$, and define $v(p) = (1 - t^2)w(p)$ for $p \in S^2 \setminus \{N, S\}$ and $v(N) = v(S) = 0$.

