Lecture 16: Parallel Transport and the Local Gauss-Bonnet Theorem

Prof. Weiqing Gu

Math 142:
Differential Geometry

Parallel Vector Fields

Definition

A vector field w along a parametrized curve $\alpha : I \to S$ is said to be *parallel* if $Dw/dt = 0$ for every $t \in I$.

Parallel Vector Fields

Definition

A vector field w along a parametrized curve $\alpha : I \to S$ is said to be *parallel* if $Dw/dt = 0$ for every $t \in I$.

Geometric Meaning

In the particular case of the plane, the notion of parallel field along a parametrized curve reduces to that of a constant field along the curve; that is, the length of the vector and its angle with a fixed direction are constant. Those properties are partially reobtained on any surface as the following proposition shows.

Fig. 4-10

Parallel Vector Fields

Proposition

Let w and v be parallel vector fields along $\alpha : I \to S$. Then $\langle w(t), v(t) \rangle$ is constant. In particular, $\|w(t)\|$ and $\|v(t)\|$ are constant, and the angle between $v(t)$ and $w(t)$ is constant.

Proof.

Parallel Vector Fields

Example

The tangent vector field of a meridian (parametrized by arc length) of a unit sphere S^2 is a parallel field on S^2. In fact, since the meridian is a great circle on S^2, the usual derivative of such a field is normal to S^2. Thus, its covariant derivative is zero.

Fig. 4-11

Parallel Transport

Proposition

Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.

Parallel Transport

Proposition

Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.

This proposition allows us to talk about parallel transport of a vector along a parametrized curve.

Parallel Transport

Proposition

Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.

This proposition allows us to talk about parallel transport of a vector along a parametrized curve.

Definition

Let $\alpha : I \to S$ be a parametrized curve and $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Let w be the parallel vector field along α, with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$, is called the *parallel transport* of w_0 along α at the point t_1.

Parallel Transport

Proposition

Let $\alpha : I \to S$ be a parametrized curve in S and let $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Then there exists a unique parallel vector field $w(t)$ along $\alpha(t)$, with $w(t_0) = w_0$.

This proposition allows us to talk about parallel transport of a vector along a parametrized curve.

Definition

Let $\alpha : I \to S$ be a parametrized curve and $w_0 \in T_{\alpha(t_0)}(S)$, $t_0 \in I$. Let w be the parallel vector field along α, with $w(t_0) = w_0$. The vector $w(t_1)$, $t_1 \in I$, is called the *parallel transport* of w_0 along α at the point t_1.

Remark

If $\alpha : I \to S$, $t \in I$, is regular, then the parallel transport does not depend on the parametrization of $\alpha(I)$.

Parallel Transport

Remark

Another interesting property of the parallel transport is that if two surfaces S and $\overline{S}$ are tangent along a parametrized curve α and w_0 is a vector of $T_{\alpha(t_0)}(S) = T_{\alpha(t_0)}(\overline{S})$, then $w(t)$ is the parallel transport of w_0 relative to the surface S if and only if $w(t)$ is the parallel transport of w_0 relative to $\overline{S}$. Indeed, the covariant derivative Dw/dt of w is the same for both surfaces. Since the parallel transport is unique, the assertion follows.

Parallel Transport

Remark

Another interesting property of the parallel transport is that if two surfaces S and $\overline{S}$ are tangent along a parametrized curve α and w_0 is a vector of $T_{\alpha(t_0)}(S) = T_{\alpha(t_0)}(\overline{S})$, then $w(t)$ is the parallel transport of w_0 relative to the surface S if and only if $w(t)$ is the parallel transport of w_0 relative to $\overline{S}$. Indeed, the covariant derivative Dw/dt of w is the same for both surfaces. Since the parallel transport is unique, the assertion follows.

The above property will allow us to give a simple and instructive example of parallel transport.

Parallel Transport

Example

Let C be a parallel of colatitude φ of an oriented unit sphere and let w_0 be a unit vector, tangent to C at some point p of C. Let us determine the parallel transport of w_0 along C, parametrized by arc length s, with $s = 0$ at p.

Parallel Transport

Example

Let C be a parallel of colatitude φ of an oriented unit sphere and let w_0 be a unit vector, tangent to C at some point p of C. Let us determine the parallel transport of w_0 along C, parametrized by arc length s, with $s = 0$ at p.

Fig. 4-12

Parallel Transport

Example

Let C be a parallel of colatitude φ of an oriented unit sphere and let w_0 be a unit vector, tangent to C at some point p of C. Let us determine the parallel transport of w_0 along C, parametrized by arc length s, with $s = 0$ at p.

Fig. 4-12

Consider the cone which is tangent to the sphere along C. The angle ψ at the vertex of this cone is given by $\psi = \frac{\pi}{2} - \varphi$. By the above property, the problem reduces to the determination of the parallel transport of w_0 along C, relative to the tangent cone.

Parallel Transport

Example

Let C be a parallel of colatitude φ of an oriented unit sphere and let w_0 be a unit vector, tangent to C at some point p of C. Let us determine the parallel transport of w_0 along C, parametrized by arc length s, with $s = 0$ at p.

Fig. 4-12 Fig 4-13

Consider the cone which is tangent to the sphere along C. The angle ψ at the vertex of this cone is given by $\psi = \frac{\pi}{2} - \varphi$. By the above property, the problem reduces to the determination of the parallel transport of w_0 along C, relative to the tangent cone.

Geodesic Curvature

Definition

Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$, $t \in I$, is a unit vector field, $(dw/dt)(t)$ is normal to $w(t)$, and therefore

$$\frac{Dw}{dt} = \lambda(N \wedge w(t)).$$

The real number $\lambda = \lambda(t)$, denoted by $[Dw/dt]$, is called the *algebraic value of the covariant derivative* of w at t.

Geodesic Curvature

Definition

Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S. Since $w(t)$, $t \in I$, is a unit vector field, $(dw/dt)(t)$ is normal to $w(t)$, and therefore

$$\frac{Dw}{dt} = \lambda(N \wedge w(t)).$$

The real number $\lambda = \lambda(t)$, denoted by $[Dw/dt]$, is called the *algebraic value of the covariant derivative* of w at t.

Definition

Let C be an oriented regular curve contained on an oriented surface S, and let $\alpha(s)$ be a parametrization of C, in a neighborhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative $[D\alpha'(s)/ds] = k_g$ of $\alpha'(s)$ at p is called the *geodesic curvature* of C at p.

Geodesic Curves

Definition

A nonconstant, parametrized curve $\gamma : I \longrightarrow S$ is said to be *geodesic* at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is,

$$\frac{D\gamma'(t)}{dt} = 0;$$

γ is a *parametrized geodesic* if it is geodesic for all $t \in I$.

Geodesic Curves

Definition

A nonconstant, parametrized curve $\gamma : I \longrightarrow S$ is said to be *geodesic* at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t; that is,

$$\frac{D\gamma'(t)}{dt} = 0;$$

γ is a *parametrized geodesic* if it is geodesic for all $t \in I$.

Note

1. $\|\gamma'(t)\|$ is constant.
2. A parametrized geodesic may admit self-intersections.

Geodesic Curves

Definition

A regular connected curve C in S is said to be *a geodesic* if, for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

Geodesic Curves

Definition

A regular connected curve C in S is said to be *a geodesic* if, for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

We note that, geometrically, a regular curve $C \subset S$ $(k \neq 0)$ is a geodesic if and only if its principal normal at each point $p \in C$ is parallel to the normal to S at p.

Geodesic Curves

Definition

A regular connected curve C in S is said to be *a geodesic* if, for every $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of p by the arc length s is a parametrized geodesic; that is, $\alpha'(s)$ is a parallel vector field along $\alpha(s)$.

We note that, geometrically, a regular curve $C \subset S$ $(k \neq 0)$ is a geodesic if and only if its principal normal at each point $p \in C$ is parallel to the normal to S at p.

Example

Observe that every straight line contained in a surface is a geodesic.

Geodesic Curves

Example

The great circles of a sphere S^2 are geodesics. Indeed, the great circles C are obtained by intersecting the sphere with a plane that passes through the center O of the sphere. The principal normal at a point $p \in C$ lies in the direction of the line that connects p to O because C is a circle of center O. Since S^2 is a sphere, the normal lies in the same direction, which verifies our assertion.

Geodesic Curves

Example

The great circles of a sphere S^2 are geodesics. Indeed, the great circles C are obtained by intersecting the sphere with a plane that passes through the center O of the sphere. The principal normal at a point $p \in C$ lies in the direction of the line that connects p to O because C is a circle of center O. Since S^2 is a sphere, the normal lies in the same direction, which verifies our assertion.

Later, we shall prove the general fact that for each point $p \in S$ and each direction in $T_p(S)$ there exists exactly one geodesic $C \subset S$ passing through p and tangent to this direction. For the case of the sphere, through each point and tangent to each direction there passes exactly one great circle, which, as we proved before, is a geodesic. Therefore, by uniqueness, the great circles are the only geodesics of a sphere.

Geodesic Curves

Example

For the right circular cylinder over the circle $x^2 + y^2 = 1$, it is clear that the circles obtained by the intersection of the cylinder with planes that are normal to the axis of the cylinder are geodesics. That is so because the principal normal to any of its points is parallel to the normal to the surface at this point.

Plane to cylinder isometry with geodesic

Geodesics in Local Coordinates

We shall now introduce the equations of a geodesic in a coordinate neighborhood. For that, let $\gamma : I \to S$ be a parametrized curve of S and let $\mathbf{x}(u, v)$ be a parametrization of S in a neighborhood V of $\gamma(t_0)$, $t_0 \in I$. Let $J \subset I$ be an open interval containing t_0 such that $\gamma(J) \subset V$. Let $\mathbf{x}(u(t), v(t))$, $t \in J$, be the expression of $\gamma : J \to S$ in the parametrization $\mathbf{x}$. Then, the tangent vector field $\gamma'(t)$, $t \in J$, is given by

$$w = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v.$$

Geodesics in Local Coordinates

We shall now introduce the equations of a geodesic in a coordinate neighborhood. For that, let $\gamma : I \to S$ be a parametrized curve of S and let $\mathbf{x}(u, v)$ be a parametrization of S in a neighborhood V of $\gamma(t_0)$, $t_0 \in I$. Let $J \subset I$ be an open interval containing t_0 such that $\gamma(J) \subset V$. Let $\mathbf{x}(u(t), v(t))$, $t \in J$, be the expression of $\gamma : J \to S$ in the parametrization $\mathbf{x}$. Then, the tangent vector field $\gamma'(t)$, $t \in J$, is given by

$$w = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v.$$

Therefore, the fact that w is parallel is equivalent to the the system of differential equations

$$\begin{aligned} u'' + \Gamma_{11}^1 (u')^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 (v')^2 = 0, \\ v'' + \Gamma_{11}^2 (u')^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 (v')^2 = 0, \end{aligned}$$

obtained by equating to zero the coefficients of $\mathbf{x}_u$ and $\mathbf{x}_v$.

Geodesics in Local Coordinates

In other words, $\gamma : I \to S$ is a geodesic if and only if the system on the previous slide is satisfied for every interval $J \subset I$ such that $\gamma(J)$ is contained in a coordinate neighborhood. The system on the previous slide is known as the *differential equations of the geodesics of S*.

Geodesics in Local Coordinates

In other words, $\gamma : I \to S$ is a geodesic if and only if the system on the previous slide is satisfied for every interval $J \subset I$ such that $\gamma(J)$ is contained in a coordinate neighborhood. The system on the previous slide is known as the *differential equations of the geodesics of* S.

The following proposition is an important consequence of the fact that geodesics are characterized by this system.

Geodesics in Local Coordinates

In other words, $\gamma : I \to S$ is a geodesic if and only if the system on the previous slide is satisfied for every interval $J \subset I$ such that $\gamma(J)$ is contained in a coordinate neighborhood. The system on the previous slide is known as the *differential equations of the geodesics of S*.

The following proposition is an important consequence of the fact that geodesics are characterized by this system.

Proposition

Given a point $p \in S$ and a vector $w \in T_p(S)$, $w \neq 0$, there exists an $\epsilon > 0$ and a unique parametrized geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p$, $\gamma'(0) = w$.

Geodesics in Local Coordinates

Example

Let us study locally the geodesics of a surface of revolution with the parametrization

$$x = f(v)\cos u, \qquad y = f(v)\sin u, \qquad z = g(v).$$

Geodesics in Local Coordinates

Example

Let us study locally the geodesics of a surface of revolution with the parametrization

$$x = f(v)\cos u, \qquad y = f(v)\sin u, \qquad z = g(v).$$

By Example 1 of Sec. 4-1 (Do Carmo), the Christoffel symbols are given by

$$\Gamma^1_{11} = 0, \quad \Gamma^2_{11} = -\frac{ff'}{(f')^2 + (g')^2}, \quad \Gamma^1_{12} = \frac{ff'}{f^2},$$
$$\Gamma^2_{12} = 0, \quad \Gamma^1_{22} = 0, \quad \Gamma^2_{22} = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Geodesics in Local Coordinates

Example (cont'd)

With these values, the system of differential equations for a geodesic becomes

$$\begin{aligned} u'' + \frac{2ff'}{f^2}u'v' &= 0, \\ v'' - \frac{ff'}{(f')^2+(g')^2}(u')^2 + \frac{f'f''+g'g''}{(f')^2+(g')^2}(v')^2 &= 0. \end{aligned}$$

Geodesics in Local Coordinates

Example (cont'd)

With these values, the system of differential equations for a geodesic becomes

$$\begin{aligned} u'' + \frac{2ff'}{f^2}u'v' &= 0, \\ v'' - \frac{ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 &= 0. \end{aligned}$$

First, as expected, the meridians $u = \text{const.}$ and $v = v(s)$, parametrized by arc length s, are geodesics.

Geodesics in Local Coordinates

Parallels

Second, we are going to determine which parallels $v =$ const., $u = u(s)$, parametrized by arc length, are geodesics. The first of the equations gives $u' =$ const. and the second becomes

$$\frac{ff'}{(f')^2 + (g')^2}(u')^2 = 0.$$

Geodesics in Local Coordinates

Parallels

Second, we are going to determine which parallels $v = \text{const.}$, $u = u(s)$, parametrized by arc length, are geodesics. The first of the equations gives $u' = \text{const.}$ and the second becomes

$$\frac{ff'}{(f')^2 + (g')^2}(u')^2 = 0.$$

In other words, a necessary condition for a parallel of a surface of revolution to be a geodesic is that such a parallel be generated by the rotation of a point of the generating curve where the tangent is parallel to the axis of revolution. The condition is clearly sufficient, since it implies that the normal line of the parallel agrees with the normal line to the surface.

Geodesics in Local Coordinates

Parallels

Second, we are going to determine which parallels $v =$ const., $u = u(s)$, parametrized by arc length, are geodesics. The first of the equations gives $u' =$ const. and the second becomes

$$\frac{ff'}{(f')^2 + (g')^2}(u')^2 = 0.$$

Fig. 4-19

In other words, a necessary condition for a parallel of a surface of revolution to be a geodesic is that such a parallel be generated by the rotation of a point of the generating curve where the tangent is parallel to the axis of revolution. The condition is clearly sufficient, since it implies that the normal line of the parallel agrees with the normal line to the surface.

Geodesics in Local Coordinates

Other Geodesics

Finally, we shall show that the system of differential equations for the geodesics of a surface of revolution may be integrated by means of primitives.

Geodesics in Local Coordinates

Other Geodesics

Finally, we shall show that the system of differential equations for the geodesics of a surface of revolution may be integrated by means of primitives.

Let $u = u(s)$, $v = v(s)$ be a geodesic parametrized by arc length, which we shall assume not to be a meridian or a parallel of the surface.

Geodesics in Local Coordinates

Other Geodesics

Finally, we shall show that the system of differential equations for the geodesics of a surface of revolution may be integrated by means of primitives.

Let $u = u(s)$, $v = v(s)$ be a geodesic parametrized by arc length, which we shall assume not to be a meridian or a parallel of the surface.

The first of the equations is then written as $f^2 u' = \text{const.} = c \neq 0$. Hence,

$$u = c \int \frac{1}{f} \sqrt{\frac{(f')^2 + (g')^2}{f^2 - c^2}}\, dv + \text{const.},$$

which is the equation of a segment of a geodesic of a surface of revolution which is neither a parallel nor a meridian.

The Gauss-Bonnet Theorem

The General Idea

The Gauss-Bonnet Theorem is probably the deepest theorem in the differential geometry of surfaces. A first version of this theorem was presented by Gauss in a famous paper and deals with geodesic triangles on surfaces (that is, triangles whose sides are geodesics).

The Gauss-Bonnet Theorem

The General Idea

The Gauss-Bonnet Theorem is probably the deepest theorem in the differential geometry of surfaces. A first version of this theorem was presented by Gauss in a famous paper and deals with geodesic triangles on surfaces (that is, triangles whose sides are geodesics).

Roughly speaking, it asserts that the excess over π of the sum of the interior angles $\varphi_1, \varphi_2, \varphi_3$ of a geodesic triangle T is equal to the integral of the Gaussian curvature K over T; that is,

$$\sum_{i=1}^{3} \varphi_i - \pi = \iint_T K \, d\sigma.$$

The Gauss-Bonnet Theorem

The General Idea

The Gauss-Bonnet Theorem is probably the deepest theorem in the differential geometry of surfaces. A first version of this theorem was presented by Gauss in a famous paper and deals with geodesic triangles on surfaces (that is, triangles whose sides are geodesics).

Roughly speaking, it asserts that the excess over π of the sum of the interior angles $\varphi_1, \varphi_2, \varphi_3$ of a geodesic triangle T is equal to the integral of the Gaussian curvature K over T; that is,

$$\sum_{i=1}^{3} \varphi_i - \pi = \iint_T K \, d\sigma.$$

Figure 4-23

The Gauss-Bonnet Theorem

Examples

1. If $K \equiv 0$, we ovtain that $\sum \varphi_i = \pi$, an extension of Thales' theorem of high school geometry to surfaces of zero curvature.
2. If $K \equiv 1$ (e.g., the sphere), then $\sum \varphi_i > \pi$.
3. If $K \equiv -1$ (e.g., the pseudosphere), then $\sum \varphi_i < \pi$.

Figures of Spherical and Hyperbolic Geometry

The Gauss-Bonnet Theorem

Theorem (GAUSS-BONNET THEOREM (Local).)

Let $\mathbf{x} : U \to S$ be an orthogonal parametrization (that is, $F = 0$) of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and $\mathbf{x}$ is compatible with the orientation of S. Let $R \subset \mathbf{x}(U)$ be a simple region of S and let $\alpha : I \to S$ be such that $\partial R = \alpha(I)$. Assume that α is positively oriented, parametrized by arc length s, and let $\alpha(s_0), \ldots, \alpha(s_k)$ and $\theta_0, \ldots, \theta_k$ be, respectively, the vertices and the external angles of α. Then

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+t}} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi,$$

where $k_g(s)$ is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S.