Lecture 16 Part B- High Dimensional Manifolds and Their Applications

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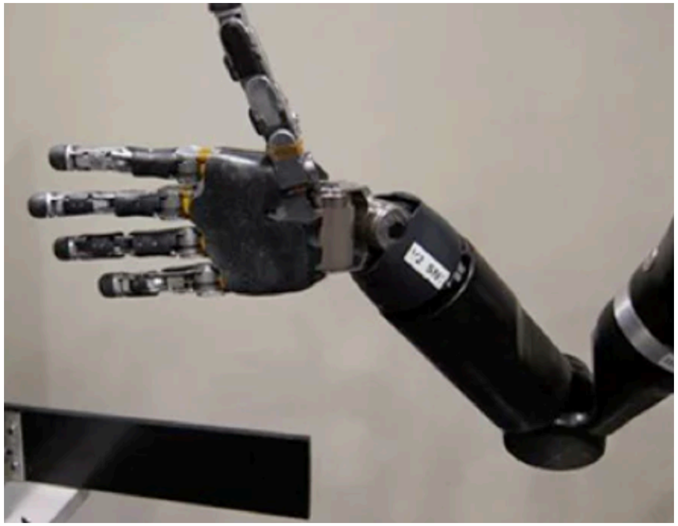
Q: Can we see some examples of concrete high dimensional manifolds?

Example: Mathematics behind of a robotic arm



- We can use high dimensional manifolds to describe a variety of situations. Above is just one example we have illustrated.

Manifold of planar robot arm

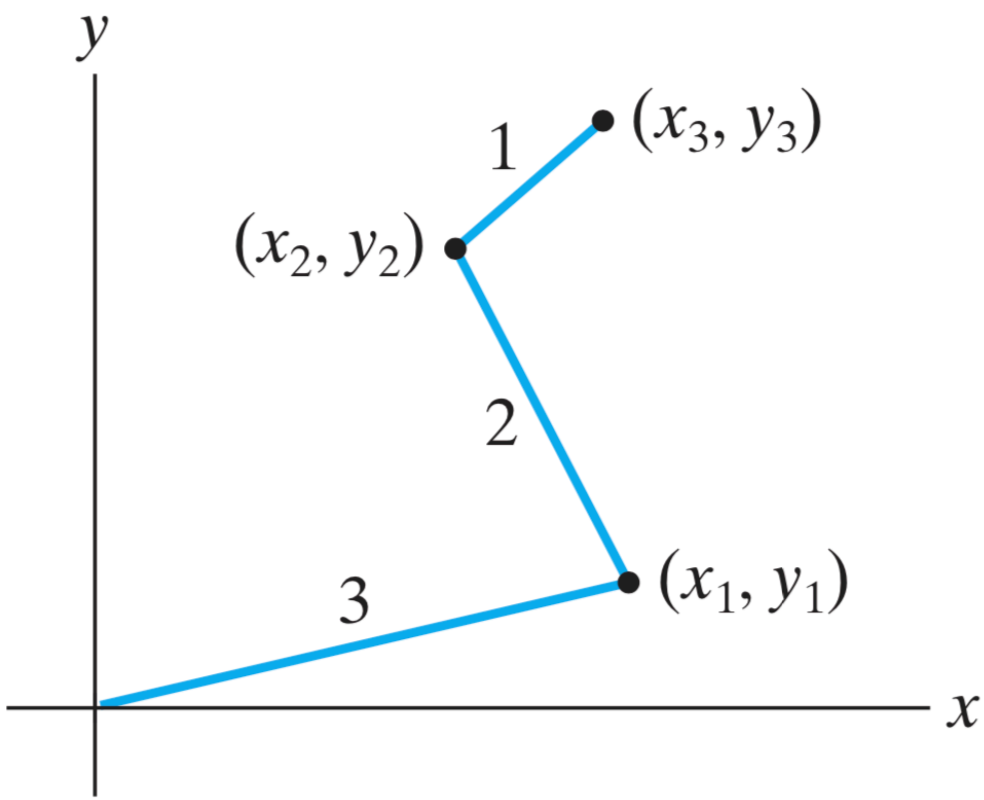
A planar robot arm is constructed consisting of three linked rods of lengths 1, 2, and 3.

The rod of length 3 is anchored at the origin of $\mathbf{R}^2$ but free to rotate about the origin.

The rod of length 2 is attached to the free end of the rod of length 3.

The rod of length 1 is, in turn, attached to the free end of the rod of length 2.

We claim the set of positions that the arm can take forms a manifold of dimension 3 in $\mathbf{R}^6$.



Manifold of planar robot arm

- Work out details with the students on the board.

Clearly, each state of the robot arm is determined by the coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3) of the linkage points, which we may consider to form a vector $\mathbf{x} = (x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbf{R}^6$. However, not all vectors in $\mathbf{R}^6$ represent a state of the robot arm. In particular, the point (x_1, y_1) must lie on the circle of radius 3, centered at the origin, the point (x_2, y_2) must lie on the circle of radius 2, centered at (x_1, y_1), and the point (x_3, y_3) must lie on the circle of radius 1, centered at (x_2, y_2). Thus, for $\mathbf{x} = (x_1, y_1, x_2, y_2, x_3, y_3)$ to represent a state of the robot arm, we require

$$\begin{cases} x_1^2 + y_1^2 = 9 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = 4 \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 = 1 \end{cases} . \tag{2}$$

Q: How many free variables here?

To answer this question, we can try to parametrize the configuration space.

Let's parametrize the manifold

$$\begin{aligned}(x_1, y_1) &= (3\cos\theta_1, 3\sin\theta_1),\\ (x_2, y_2) &= (x_1 + 2\cos\theta_2, y_1 + 2\sin\theta_2)\\ &= (3\cos\theta_1 + 2\cos\theta_2, 3\sin\theta_1 + 2\sin\theta_2),\end{aligned} \tag{3}$$

and

$$\begin{aligned}(x_3, y_3) &= (x_2 + \cos\theta_3, y_2 + \sin\theta_3)\\ &= (3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3, 3\sin\theta_1 + 2\sin\theta_2 + \sin\theta_3),\end{aligned}$$

where $0 \le \theta_1, \theta_2, \theta_3 < 2\pi$. Therefore, the map $\mathbf{X}\colon [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \to \mathbf{R}^6$ given by

$$\mathbf{X}(\theta_1, \theta_2, \theta_3) = (x_1, y_1, x_2, y_2, x_3, y_3),$$

where $(x_1, y_1, x_2, y_2, x_3, y_3)$ are given in terms of θ_1, θ_2, and θ_3 by means of the equations in (3), exhibits the set of states of the robot arm as a parametrized 3-manifold in $\mathbf{R}^6$. We leave it to you to check that $\mathbf{X}$ defines a smooth parametrized 3-manifold. ◆

Homework

- **Check that the parametrized 3-manifold in the Example is in fact a smooth parametrized 3-manifold.**

Hint: Show linear independence of $\mathbf{T}_{\theta_1}$, $\mathbf{T}_{\theta_2}$, $\mathbf{T}_{\theta_3}$ by solving the vector equation $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{\theta_2} + c_3\mathbf{T}_{\theta_3} = \mathbf{0}$ for c_1, c_2, c_3.

Q: How do we usually show that a set in R^n is a manifold?

EXAMPLE 2 Let $D = [0, 1] \times [1, 2] \times [-1, 1]$ and $\mathbf{X}\colon D \to \mathbf{R}^5$ be given by

$$\mathbf{X}(u_1, u_2, u_3) = (u_1 + u_2, 3u_2, u_2u_3^2, u_2 - u_3, 5u_3).$$

We show that $M = \mathbf{X}(D)$ is a smooth parametrized 3-manifold in $\mathbf{R}^5$.

Read only for this example! No time to cover in class.

Note first that $\mathbf{X}$ is continuous (in fact, of class C^∞) since its component functions are polynomials. To see that $\mathbf{X}$ is one-one, consider the equation

$$\mathbf{X}(\mathbf{u}) = \mathbf{X}(\tilde{\mathbf{u}}); \tag{1}$$

we show that $\mathbf{u} = \tilde{\mathbf{u}}$. Equation (1) is equivalent to a system of five equations:

$$\begin{cases} u_1 + u_2 = \tilde{u}_1 + \tilde{u}_2 \\ 3u_2 = 3\tilde{u}_2 \\ u_2 u_3^2 = \tilde{u}_2 \tilde{u}_3^2 \\ u_2 - u_3 = \tilde{u}_2 - \tilde{u}_3 \\ 5u_3 = 5\tilde{u}_3 \end{cases}.$$

The second equation implies $u_2 = \tilde{u}_2$, and the last equation implies $u_3 = \tilde{u}_3$. Hence, the first equation becomes

$$u_1 + u_2 = \tilde{u}_1 + u_2 \quad \Longleftrightarrow \quad u_1 = \tilde{u}_1.$$

Thus,

$$\mathbf{u} = (u_1, u_2, u_3) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \tilde{\mathbf{u}}.$$

To check the smoothness of M, note that the tangent vectors to the three coordinate curves are

$$\mathbf{T}_{u_1} = \frac{\partial \mathbf{X}}{\partial u_1} = (1, 0, 0, 0, 0);$$

$$\mathbf{T}_{u_2} = \frac{\partial \mathbf{X}}{\partial u_2} = (1, 3, u_3^2, 1, 0);$$

$$\mathbf{T}_{u_3} = \frac{\partial \mathbf{X}}{\partial u_3} = (0, 0, 2u_2u_3, -1, 5).$$

Therefore, to have $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 = \mathbf{0}$, we must have

$$(c_1 + c_2, 3c_2, u_3^2c_2 + 2u_2u_3c_3, c_2 - c_3, 5c_3) = (0, 0, 0, 0, 0).$$

It readily follows that $c_1 = c_2 = c_3 = 0$ is the only possibility for a solution. Hence, $\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}$ are linearly independent at all $\mathbf{u} \in D$ and so M is smooth at all points. ◆

Reference

- Colley S. J. Vector Calculus