Lecture 13: The Second Fundamental Form
(Note: First, some students will give midterm project presentation.)

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Definition of the 2nd Fundamental Form

$$\mathrm{II}_p(\vec{v}) = -\langle dN_p(\vec{v}), \vec{v} \rangle$$

Where dN_p is the differential of the Gauss map at p.

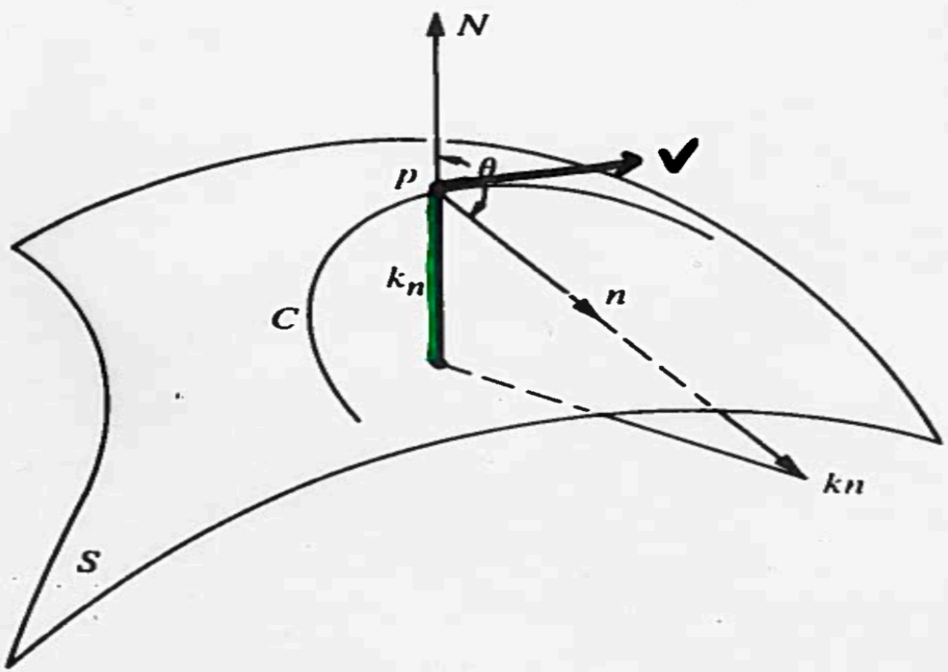
$$\vec{v} = \alpha'(0) \in T_pS$$

Here S is a regular surface (i.e. 2-dim'l mfld)

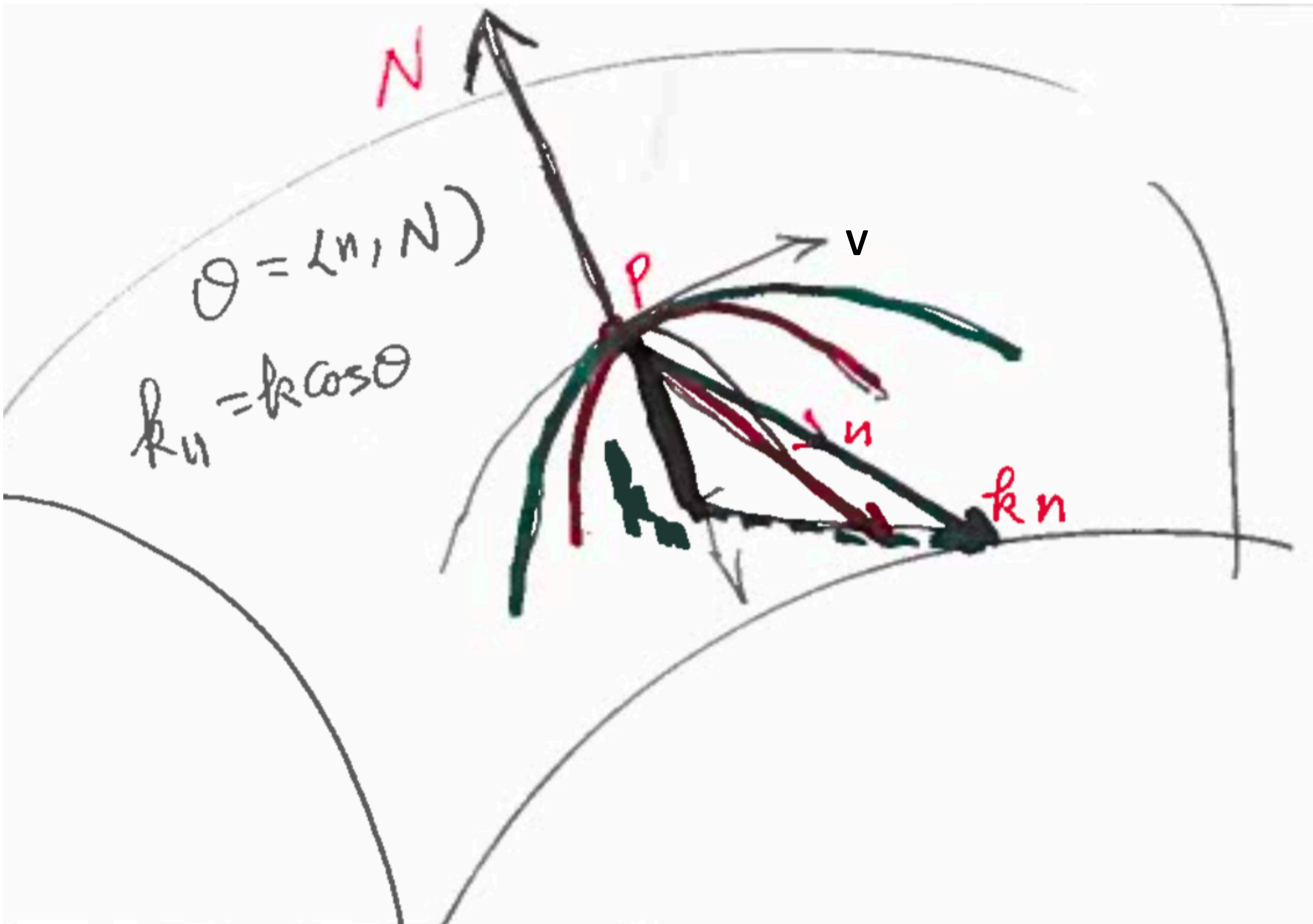
- Like the first fundamental form, the second fundamental form also "eats a vector spit out a number."

Geometric Meaning: the number spitted out by the 2rd fundamental Form is the normal curvature.

DEFINITION 3. *Let* C *be a regular curve in* S *passing through* $p \in S$, k *the curvature of* C *at* p, *and* $\cos\theta = \langle n, N \rangle$, *where* n *is the normal vector to* C *and* N *is the normal vector to* S *at* p. *The number* $k_n = k\cos\theta$ *is then called the* normal curvature *of* $C \subset S$ *at* p.



The 2nd fundamental form is well defined!



PROPOSITION 2 (Meusnier). *All curves lying on a surface* S *and having at a given point* p ∈ S *the same tangent line have at this point the same normal curvatures.*

The above proposition allows us to speak of the *normal curvature along a given direction at p*. It is convenient to use the following terminology. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the *normal section* of S at p along v (Fig. 3-9).

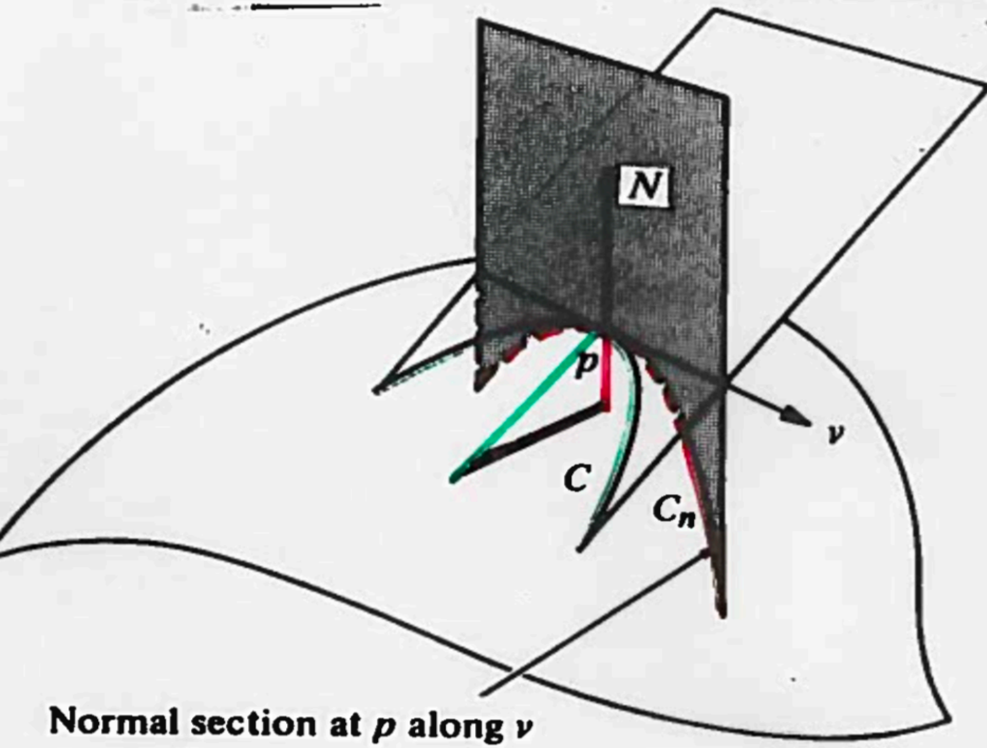


Figure 3-9. Meusnier theorem: C and C_n have the same normal curvature at p along v.

In the feature, we can extend these ideas to some submanifolds.

- The following slides are just for giving some heads-up only.

In Euclidean space, the curvature tensor of a submanifold can be described by the following formula:

$$\langle R(u,v)w,z\rangle = \langle \mathrm{I\!I}(u,z),\mathrm{I\!I}(v,w)\rangle - \langle \mathrm{I\!I}(u,w),\mathrm{I\!I}(v,z)\rangle.$$

This is called the **Gauss equation**, as it may be viewed as a generalization of Gauss's Theorema Egregium.

For general Riemannian manifolds one has to add the curvature of ambient space; if N is a manifold embedded in a Riemannian manifold (M, g) then the curvature tensor R_N of N with induced metric can be expressed using the second fundamental form and R_M, the curvature tensor of M:

$$\langle R_N(u,v)w,z\rangle = \langle R_M(u,v)w,z\rangle + \langle \mathrm{I\!I}(u,z), \mathrm{I\!I}(v,w)\rangle - \langle \mathrm{I\!I}(u,w), \mathrm{I\!I}(v,z)\rangle.$$