Linear Algebra Review

Due date: ______________

A1: For which of the following matrices are you *guaranteed* a real diagonal form or no real diagonal form at all without first determining the existence of an eigenbasis? Why?

$$A = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 3 & 3 \\ -1 & 3 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

A2: Let

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Is A similar to a diagonal matrix? If so, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Is P unique? Explain.

(c) Find the eigenvalues of A^{-1}.

(d) Find the eigenvalues and corresponding eigenvectors of A^2.

A3: Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by

$$L(a + bt + ct^2) = (2a - c) + (a + b - c)t + ct^2.$$

(a) Find the matrix A representing L with respect to the standard basis of $\mathcal{P}_2$.

(b) Find all the eigenvalues of A. For each eigenvalue, find all eigenvectors associated with that eigenvalue.

(c) Find a matrix P such that $P^{-1}AP$ is diagonal.

(d) Find A^n where n is an integer. What is L^{100}?

A4: Let A be an $n \times n$ real matrix.

(a) Prove that the coefficient of λ^{n-1} in the characteristic polynomial of A is given by $-\operatorname{tr} A$.

(b) Prove that $\operatorname{tr} A$ is the sum of the eigenvalues of A.

(c) Prove that the constant coefficient of the characteristic polynomial of A is $\pm$ the product of the eigenvalues of A.

A5: Let A be a 5×5 matrix. Suppose A has distinct eigenvalues $-1, 1, -10, 5, 2$.

(please turn over)

(a) What is $\det A$? What is $\operatorname{tr} A$?

(b) If A and B are similar, what is $\det B$? Why?

(c) Do you expect that all eigenvectors of A are mutually orthogonal? Why?

A6: *This is an extra credit-type problem.* Let $p_1(\lambda)$ be the characteristic polynomial of A_{11} and $p_2(\lambda)$ the characteristic polynomial of A_{22}. What is the characteristic polynomial of each of the following partitioned matrices?

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{pmatrix}$$

A7: (a) Prove that similar matrices have the same eigenvalues.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a matrix A with associated eigenvectors $x_1, x_2, \ldots, x_k$. Prove that $x_1, x_2, \ldots, x_k$ are linearly independent.

(c) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $L(X) = AX$. Let $V_\lambda = \{\xi \in \mathbb{R}^n \mid L(\xi) = \lambda\xi\}$. Prove V_λ is a subspace of $\mathbb{R}^n$. (This subspace is called the eigenspace associated with λ.)

(d) Let λ be an eigenvalue of A with multiplicity r. Let $\dim V_\lambda = s$. Prove $s \le r$. (That is, the dimension of the eigenspace associated with λ is at most the multiplicity of λ.)

A8: Let

$$u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 2 \\ 2 \\ -\sqrt{6} \end{pmatrix}.$$

(a) Find $\|u\|, \|v\|$. Find a unit vector in the direction of u.

(b) Find the distance between v and w.

(c) Find angle between u and v.

(d) Show that v and w are orthogonal.

A9: (a) Prove the Cauchy-Schwarz Inequality: If u and v are any vectors in an inner product space V, then $\langle u, v\rangle^2 \le \|u\|^2\|v\|^2$.

(b) Consider $\mathbb{R}^n$ with the standard inner product. Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Prove that

$$\left(\sum_{i=1}^n u_i v_i\right)^2 \le \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right).$$

(c) Let V be the vector space of all continuous real-valued functions on the unit interval $[0,1]$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Prove

$$\left(\int_0^1 f(t)g(t)\,dt\right)^2 \le \left(\int_0^1 f^2(t)\,dt\right)\left(\int_0^1 g^2(t)\,dt\right).$$

(please turn over)

A10: Let $C = [c_{ij}]$ be an $n \times n$ symmetric matrix and let V be an n-dimensional vector space with ordered basis $S = \{u_1, u_2, \ldots, u_n\}$. For $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$ and $w = b_1u_1 + b_2u_2 + \cdots + b_nu_n$ in V, define

$$(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j.$$

Prove that this defines an inner product on V if and only if C is a positive-definite matrix.

A11: Let V be the vector space of all continuous functions on the interval $[-\pi, \pi]$. For f and g in V, define $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$.

(a) Show that this defines an inner product on V.

(b) Show that the following set is an orthogonal set:

$$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt, \ldots\}.$$

(c) Convert the above set into an orthonormal set.

A12: A linear transformation $L : V \to V$, where V is an n-dimensional Euclidean space, is called *orthogonal* if $\langle Lv, Lw\rangle = \langle v, w\rangle$.

(a) Let A be an $n \times n$ matrix. Show that A is orthogonal if and only if the columns (and rows) of A form an orthonormal basis for $\mathbb{R}^n$.

(b) Let S be an orthonormal basis for V and let the matrix A represent the orthogonal linear transformation L with respect to S. Prove that A is an orthogonal matrix.

(c) Prove that for any vectors $u, v \in \mathbb{R}^n$, $\langle Lu, Lv\rangle = \langle u, v\rangle$ if and only if for any $u \in \mathbb{R}^n$, $\|Lu\| = \|u\|$.

(d) Let $L : V \to V$ be an orthogonal linear transformation. Show that if λ is an eigenvalue of L, then $|\lambda| = 1$.

A13: Let W be the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product with basis $S = \{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Transform S to an orthonormal basis $T = \{w_1, w_2, w_3\}$ using the Gram-Schmidt process.

A14: (a) Let

$$A = \begin{pmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{pmatrix}.$$

Find a 3×3 matrix P with $P^{-1} = P^T$ such that $P^TAP = D$, where D is a 3×3 diagonal matrix.

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(b) (Extra credit) Show that all the eigenvalues of a real symmetric matrix are real numbers.

(c) Show that if A is a symmetric real matrix, then eigenvectors that belong to distinct eigenvalues of A are orthogonal.

(d) Prove that a symmetric matrix A is positive-definite if and only if $A = P^T P$ for a nonsingular matrix P.

(e) Prove that if the matrix A is similar to a diagonal matrix, then A is similar to A^T.

A15: Consider two adjoining cells separated by a permeable membrane and suppose that a fluid flows from the first cell to the second one at a rate (in milliliters per minute) that is numerically equal to three times the volume (in milliliters) of the fluid in the first cell. It then flows out of the second cell at a rate (in milliliters per minute) that is numerically equal to twice the volume in the second cell. Let $x_1(t)$ and $x_2(t)$ denote the volumes of the fluid in the first and second cells at time t, respectively. Assume that initially the first cell has 40 milliliters of fluid, while the second one has 5 milliliters of fluid. Find the volume of fluid in each cell at time t.

A16: Consider a plant that can have red flowers (R), pink flowers (P), or white flowers (W), depending upon the genotypes RR, RW, and WW. When we cross each of these genotypes with a genotype RW, we obtain the transition matrix

$$M = \begin{pmatrix} 0.5 & 0.25 & 0.0 \\ 0.5 & 0.5 & 0.5 \\ 0.0 & 0.25 & 0.5 \end{pmatrix}.$$

Suppose that each successive generation is produced by crossing only with plants of RW genotype. When the process reaches equilibrium, what percentage of the plants will have red, pink, or white flowers?

A17: (a) Compute the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) Verify that if $A = PBP^{-1}$ and k is a positive integer, then $A^k = PB^kP^{-1}$.

(c) Using a hand calculator or MATLAB, compute f_8, f_{12}, and f_{20}, where f_n is the nth Fibonacci number, starting with $f_0 = f_1 = 1$.

A18: Determine which of the given quadratic forms in three variables are equivalent:

$$\begin{aligned} g_1(\mathbf{x}) &= x_1^2 + x_2^2 + x_3^3 + 2x_1x_2 \\ g_2(\mathbf{x}) &= 2x_2^2 + 2x_3^2 + 2x_2x_3 \\ g_3(\mathbf{x}) &= 3x_2^2 - 3x_3^2 + 8x_2x_3 \\ g_4(\mathbf{x}) &= 3x_2^2 + 3x_3^2 - 4x_2x_3. \end{aligned}$$

(please turn over)

A19: Which of the following matrices are positive-definite?

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}.$$

A20: Let $g(\mathbf{x}) = 3x_1^2 - 3x_2^2 - 3x_3^2 + 4x_2x_3$ be a quadratic form in three variables.

(a) Find a quadratic form in the type given in the Principal Axis Theorem that is equivalent to g. What is the rank of g? What is the signature of g?

(b) Identify the surface $g(\mathbf{x}) = 9$.