Lecture 9: The First Fundamental Form and Riemannian Metric

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Math 143:
Topics in Geometry

The First Fundamental Form

An Inner Product on the Tangent Plane

The natural inner product of $\mathbb{R}^3 \supset S$ induces on each tangent plane $T_p(S)$ of a regular surface S an inner product, to be denoted by $\langle \ , \ \rangle_p$: If $w_1, w_2 \in T_p(S) \subset \mathbb{R}^3$, then $\langle w_1, w_2 \rangle$ is equal to the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$.

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$$I_p(w) = \langle w, w \rangle_p = \|w\|^2 \geq 0. \tag{1}$$

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Definition

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Geometrically, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space $\mathbb{R}^3$ where the surface lies.

The First Fundamental Form

Expression in Local Coordinates

We shall now express the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parametrization $\mathbf{x}(u, v)$ at p.

$$\begin{aligned}
w = \alpha'(0) &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{x} \circ \tilde{\alpha}(t) = \left.\frac{d}{dt}\right|_{t=0} \mathbf{x}(u(t), v(t)) \\
&= \mathbf{x}_u u'(0) + \mathbf{x}_v v'(0) = \begin{pmatrix} \mathbf{x}_u & \mathbf{x}_v \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}
\end{aligned}$$

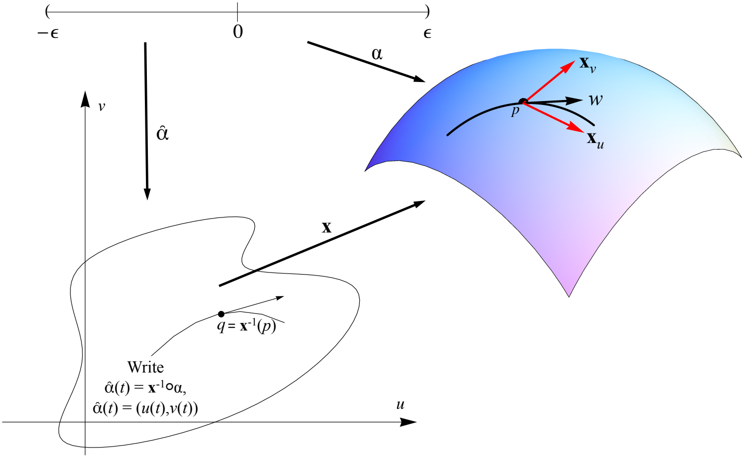
$$\begin{aligned}
I_p(w) &= I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p \\
&= \langle u'\mathbf{x}_u + v'\mathbf{x}_v, u'\mathbf{x}_u + v'\mathbf{x}_v \rangle \\
&= \|\mathbf{x}_u\|^2 (u')^2 + 2u'v' \langle \mathbf{x}_u, \mathbf{x}_v \rangle + \|\mathbf{x}_v\|^2 (v')^2 \\
&= E(u')^2 + 2Fu'v' + G(v')^2 \\
&= \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}
\end{aligned}$$

$$E(u, v) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \qquad F(u, v) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \qquad G(u, v) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

The First Fundamental Form

Remark

This says that the value of the first fundamental form on an arbitrary vector w is determined by the values of the inner product of the basis vectors.



Examples: Computing the First Fundamental Form

Example

A coordinate system for a plane $P \subset \mathbb{R}^3$ passing through $p_0 = (x_0, y_0, z_0)$ and containing the *orthonormal* vectors $w_1 = (a_1, a_2, a_3)$ and $w_2 = (b_1, b_2, b_3)$ is given as follows:

$$\mathbf{x}(u, v) = p_0 + uw_1 + vw_2, \quad (u, v) \in \mathbb{R}^2.$$

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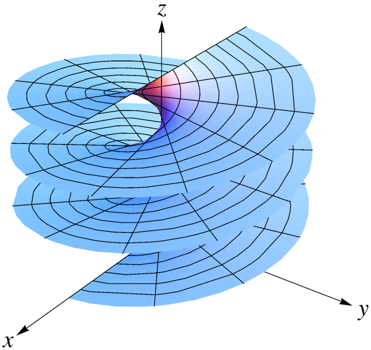
$$\begin{aligned} \Rightarrow E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle w_1, w_1 \rangle = 1 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = 1 \end{aligned}$$

Examples

Example

Consider a helix that is given by $(\cos u, \sin u, au)$. Through each point of the helix, draw a line parallel to the xy plane and intersecting the z axis. The surface generated by these lines is called a *helicoid* and admits the following parametrization:

$$\begin{aligned} \mathbf{x}(u,v) = (v\cos u, v\sin u, au),\\ 0 < u < 2\pi,\\ -\infty < v < \infty.\\ E = v^2 + a^2,\\ F = 0,\\ G = 1 \end{aligned}$$



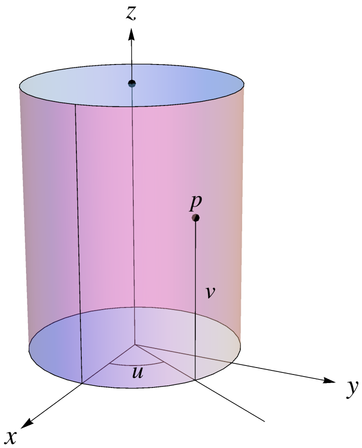
Examples

Example

The right cylinder over the circle $x^2 + y^2 = 1$ admits the parametrization $\mathbf{x} : U \to \mathbb{R}^3$, where

$$\mathbf{x}(u, v) = (\cos u, \sin u, v),$$
$$U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi,$$
$$-\infty < v < \infty\}.$$
$$E = 1,$$
$$F = 0,$$
$$G = 1$$

(Compare with first example)



Homework

Homework Hint: Page 95, Exmple 4, Do Carmo Diff. Geo. of Curves and Surfaces

You shall compute the first fundamental form of a sphere at a point of the coordinate neighborhood given by the parametrization

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Measurements

Arc Length

The arc length s of a parametrized curve $\alpha : I \to S$ is given by

$$s(t) = \int_0^t \|\alpha'(t)\| \, dt = \int_0^t \sqrt{I(\alpha'(t))} \, dt.$$

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In particular, if $\alpha(t) = \mathbf{x}(u(t), v(t))$ is contained in a coordinate neighborhood corresponding to the parametrization $\mathbf{x}(u, v)$, we can compute the arc length of α between, say, 0 and t by

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, dt. \tag{2}$$

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Remark

Because of Eq. **??**, many mathematicians talk about the "element" of arc length, ds, of S and write

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

Measurements

Angle

The angle θ under which two parametrized regular curves $\alpha : I \to S$, $\beta : I \to S$ intersect at $t = t_0$ is given by

$$\cos \theta = \frac{\langle \alpha'(t_0), \beta'(t_0) \rangle}{\|\alpha'(t_0)\| \|\beta'(t_0)\|}.$$

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In particular, the angle φ of the coordinate curves of a parametrization $\mathbf{x}(u, v)$ is

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it follows that *the coordinate curves of a parametrization are orthogonal if and only if* $F(u, v) = 0$ *for all* (u, v). Such a parametrization is called an *orthogonal parametrization*.

Example

As an application, let us determine the curves in this coordinate neighborhood of the sphere which make a constant angle β with the meridians $\varphi = \text{const}$. These curves are called *loxodromes* (rhumb lines) of the sphere.

Area

Definition

Let $R \subset S$ be a *bounded region* of a regular surface contained in the coordinate neighborhood of the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$. The positive number

$$\iint_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, du \, dv = A(R), \qquad Q = \mathbf{x}^{-1}(R),$$

is called the *area* of R. Note that $\|\mathbf{x}_u \wedge \mathbf{x}_v\| = \sqrt{EG - F^2}$.

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Recall

A (regular) *domain* of S is an open and connected subset of S such that its boundary is the image of a circle by a differentiable homeomorphism which is regular (that is, its differential is nonzero) except at a finite number of points. A *region* of S is the union of a domain with its boundary. A region of $S \subset \mathbb{R}^3$ is *bounded* if it is contained in some ball of $\mathbb{R}^3$.

Area

Why is $A(R)$ well-defined?

Let us show that the integral

$$\iint_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, du \, dv$$

does not depend on the parametrization $\mathbf{x}$.

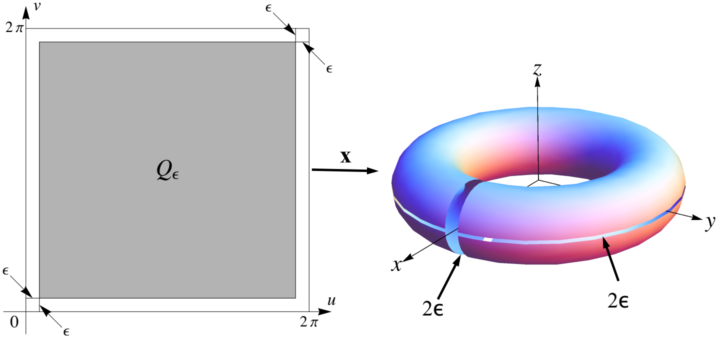
Examples

Example

Let us compute the area of the torus. For that, we consider the coordinate neighborhood corresponding to the parametrization

$$\mathbf{x}(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u),$$
$$0 < u < 2\pi, \qquad 0 < v < 2\pi,$$

which covers the torus, except for a meridian and a parallel.



Examples

Example (Surfaces of Revolution)

Let $S \subset \mathbb{R}^3$ be the set obtained by rotating a regular plane curve C about an axis in the plane which does not meet the curve; we shall take the xz plane as the plane for the curve and the z axis as the rotation axis.

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$$x = f(v), \qquad z = g(v), \qquad a < v < b, \qquad f(v) > 0,$$

be a parametrization for C and denote by u the rotation angle about the z axis.

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$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

from the open set $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, a < v < b\}$ into S.

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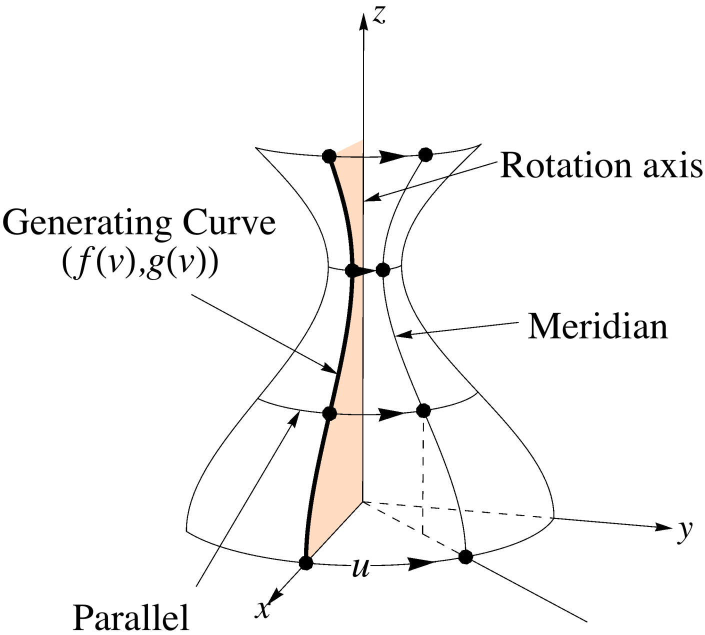
Claim

S is a regular surface which is called a *surface of revolution*.



z
Rotation axis
Generating Curve
$(f(v), g(v))$
Meridian
y
u
x
Parallel

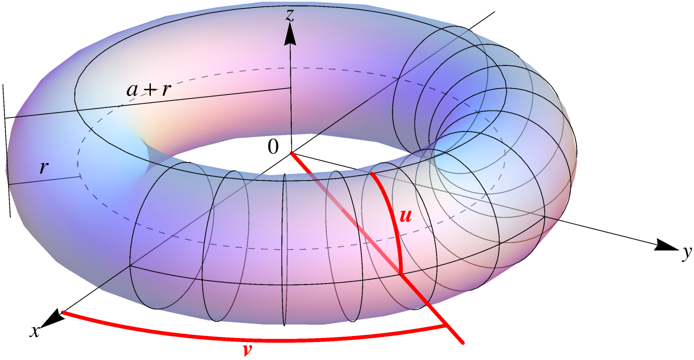
Examples

Example

A parametrization for the torus T can be given by

$$\mathbf{x}(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.



Extended Surfaces of Revolution

Remark

There is a slight problem with our definition of surface of revolution. If $C \subset \mathbb{R}^2$ is a closed regular plane curve which is symmetric relative to an axis r of $\mathbb{R}^3$, then, by rotating C about r, we obtain a surface which can be proved to be regular and should also be called a surface of revolution (when C is a circle and r contains a diameter of C, the surface is a sphere). To fit it in our definition, we would have to exclude two of its points, namely, the points where r meets C. For technical reasons, we want to maintain the previous terminology and shall call the latter surfaces *extended surfaces of revolution*.