Topic: Curves and Their Applications

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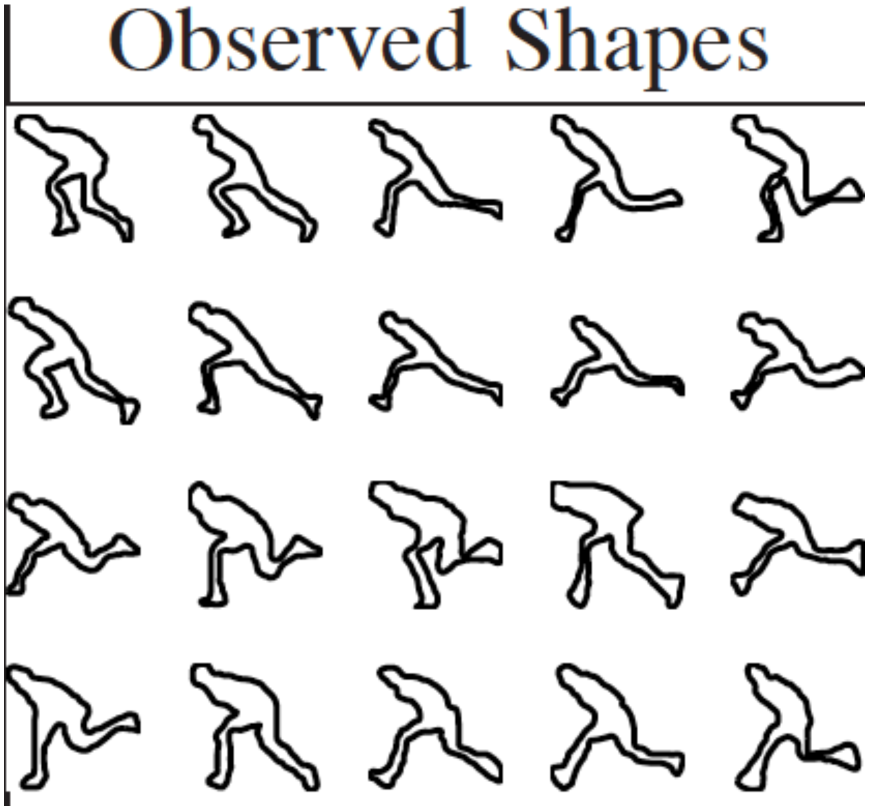
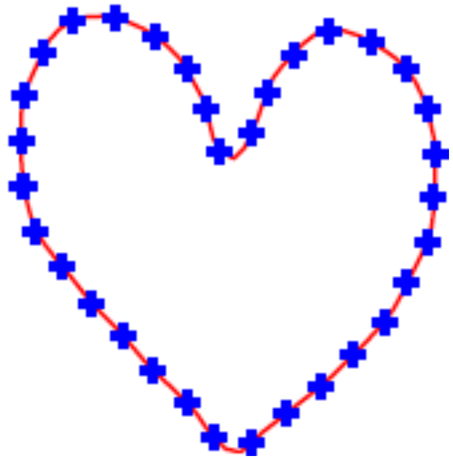
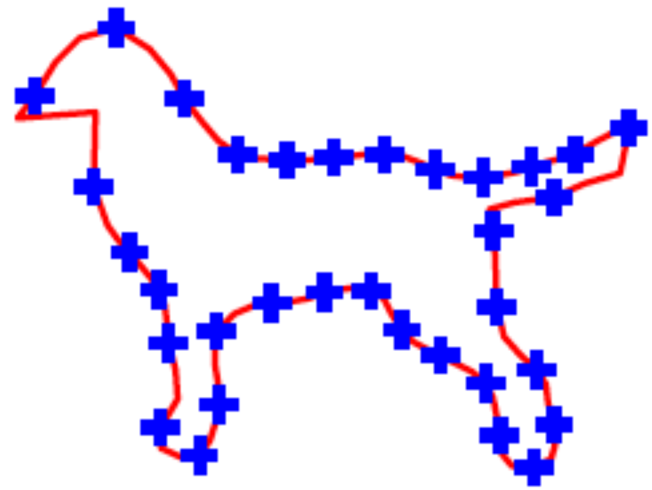
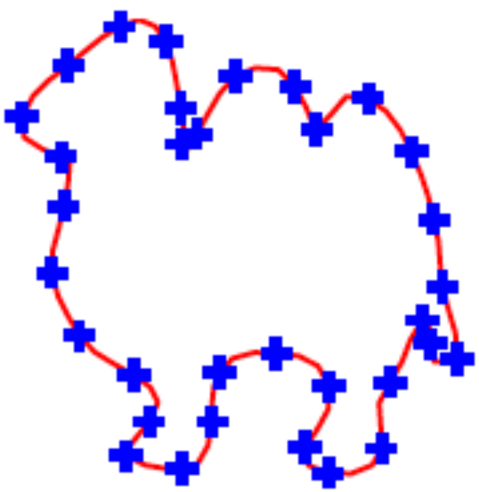
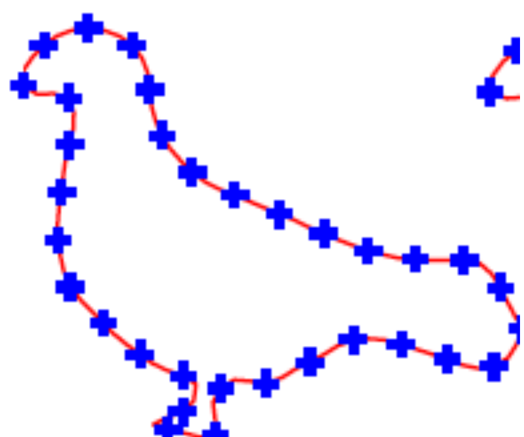
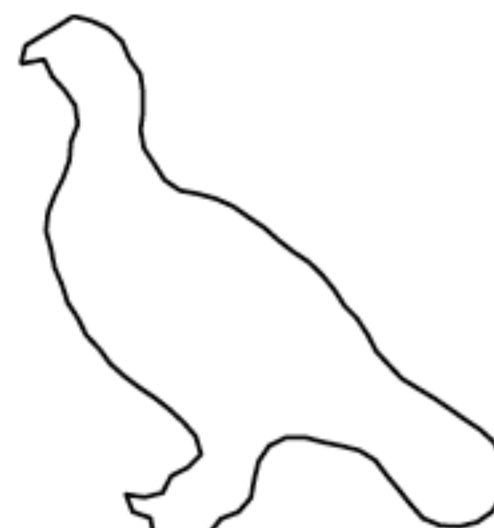
Curves have many applications

- For example: Pedestrian recognition. This depends on the shape analysis of curves.
- *Working out details with students on the board.*



The Shape Manifold

Different ways to represent curves

- Parameterized curves
  - Regular curve
  - Reparametrize a curve
  - Parametrize a curve by its arc-lengthen
- Represent a curve as a point on certain manifold
- Represent a curve using its tangent vectors
  - Elastic representation (or square-root velocity (SRV) representation.)
  - Elastic analysis of curves

Different ways to study the Shape Space

- Parametrized methods – use representations of parametrized curves.
- Un-paramatrized method – use representations of non-parametrized curves.

The shape Space SP^n is topologically CP^{n-2}.

Here CP^k is a complex projective space, which, we will show later, is a manifold.

For example, CP^1 is a sphere S^2.

Parametrized and Regular Curves

Definition

A *parametrized differentiable curve* is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Definition

A parametrized differentiable curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

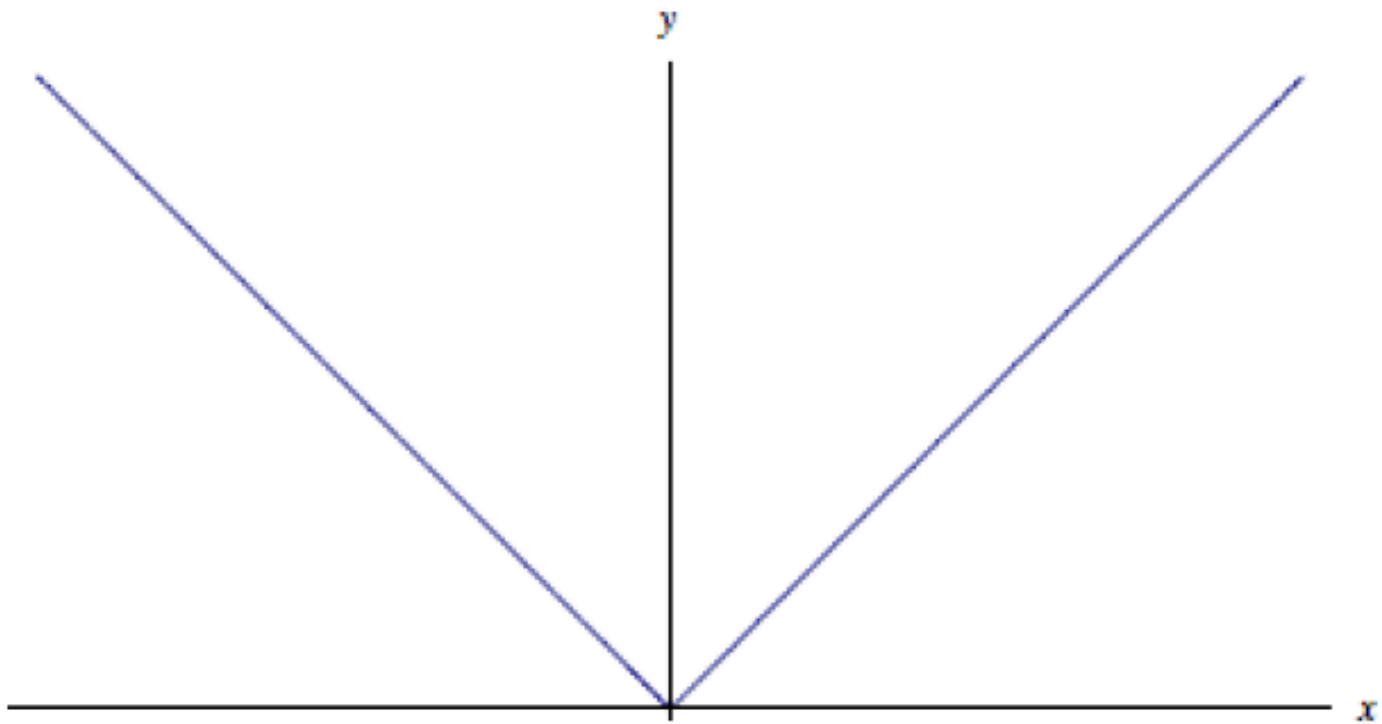
Example:

- We can parametrize a circle or a elliptic curve in different ways.
- While a shape of a closed curve is just a trace of different ways of its parametrization.
- ***Work out details with students on the board.***
- Note: Using arclength to paramatrize the curve is only a representative in the equivalent class of reparametrized the trace of the curve.

Examples

Example 1

The map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|), t \in \mathbb{R}$ (not differentiable).

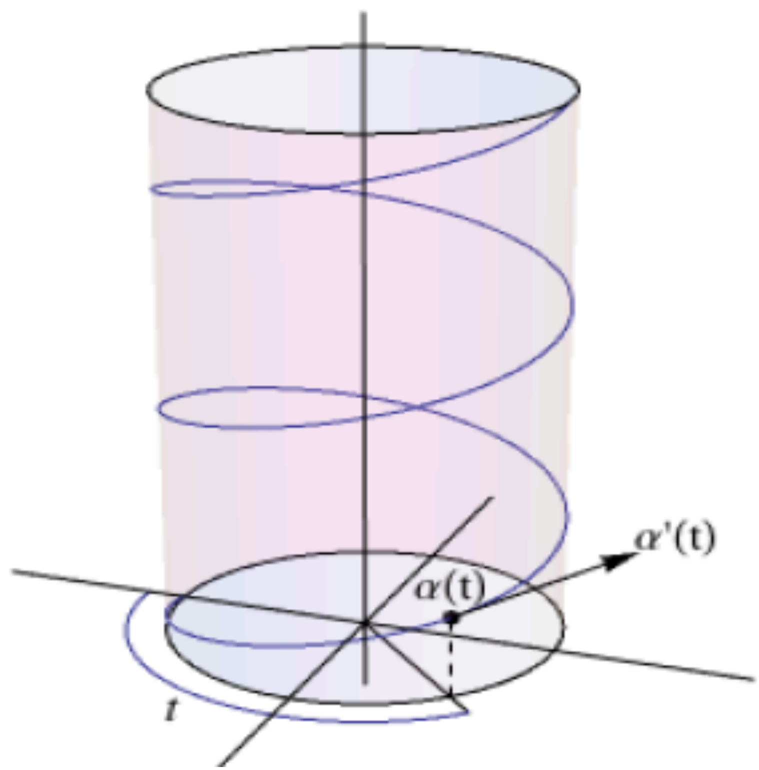


How to parametrize a curve on surface?

Example 2

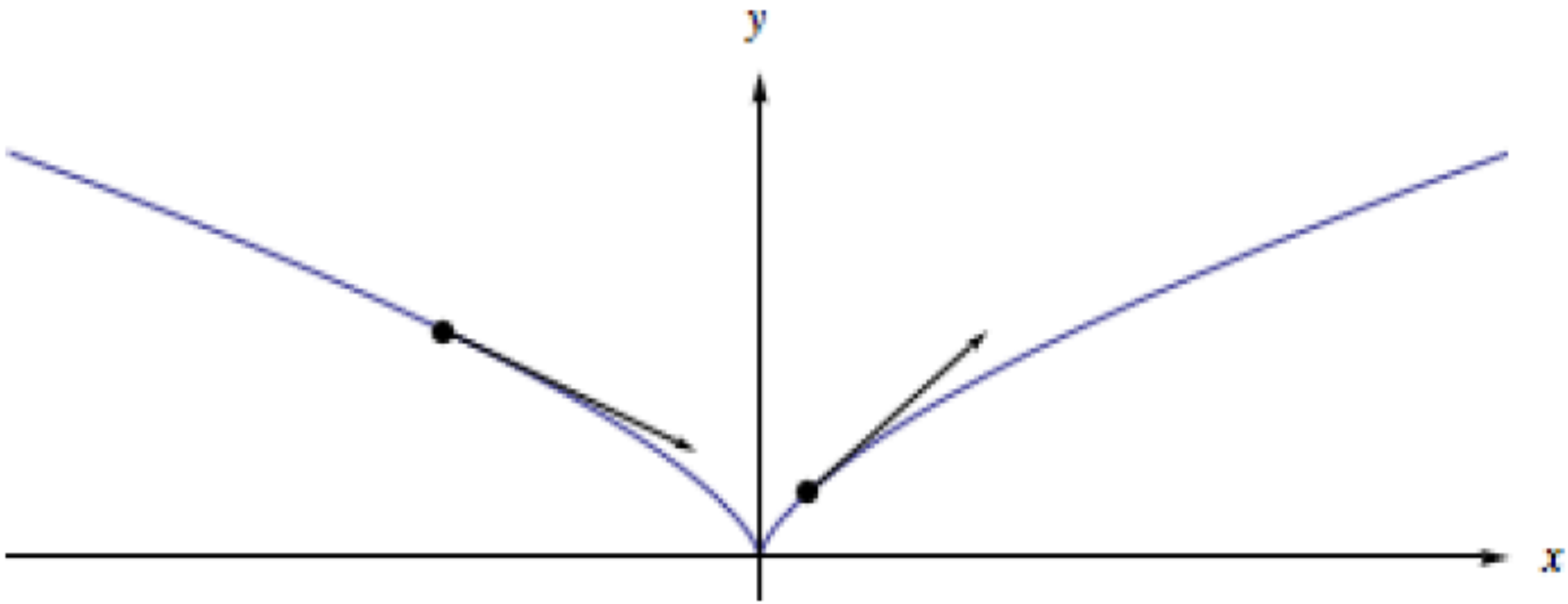
A helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.

Is helix regular?



Example 3

The map $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t^3, t^2), t \in \mathbb{R}$.



- We will deal with curves which are not regular later.

Arc Length of a Curve

Definition

Given $t \in I$, the *arc length* of a regular parametrized curve $\alpha : I \to \mathbb{R}^3$, from the point t_0, is by definition

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| \, dt,$$

where

$$\|\alpha'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$.

Definition

A parametrized curve $\alpha : I \to \mathbb{R}^3$ is said to be parametrized by arc length if $\|\alpha'(t)\| = 1$ (that is, if α has unit speed) for all $t \in I$.

Parametrization by Arc Length

Proposition (Geometric meaning of above definition)

A curve $\alpha : I \to \mathbb{R}^3$ is parametrized by arc length if and only if the parameter t is the arc length of α measured from some point.

Proof.

$\square$

Proposition (Advantages of $\|\alpha'(s)\| = 1$)

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length. Then $\alpha''(s)$ is orthogonal to $\alpha'(s)$ for all $s \in I$.

Proof.

$\square$

Reparametrization by Arc Length

Theorem

If α is a regular curve in $\mathbb{R}^3$, then there exists a reparametrization β of α such that β has unit speed.

Note: We can consider the set of all kinds of reparamatrizations of the same curve, they form an equivalent class of the same curve, the one parametrized by arclength can be viewed as a representation of them.
We can take other parametrization as another representation (Elastic representation for example, late).

Reparametrization by Arc Length

Example

Consider the helix $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by $\alpha(t) = (\cos t, \sin t, t)$.

Homework!

Curvature

Geometric Meaning

Let $\alpha : I = (a, b) \to \mathbb{R}^3$ be a curve parametrized by arc length s. Since the tangent vector $\alpha'(s)$ has unit length, the norm $\|\alpha''(s)\|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s. $\|\alpha''(s)\|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s.

Definition

Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $\|\alpha''(s)\| = k(s)$ is called the *curvature* of α at s.

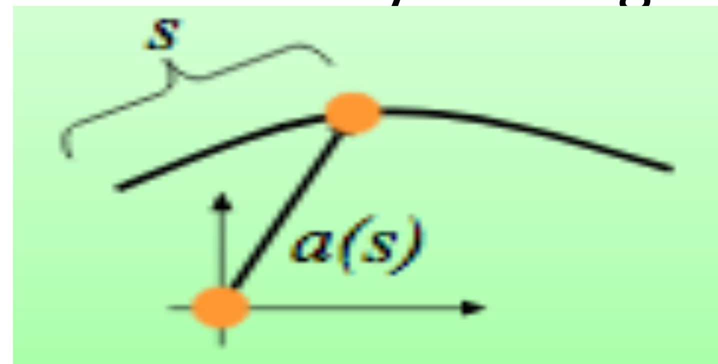
$k(s) = \|\alpha''(s)\| = 0$ for any $s \in \mathbb{R}$ if and only if α is a straight line.

Curvature of a curve

Parametrization of curve & then reparametrize it by arclength

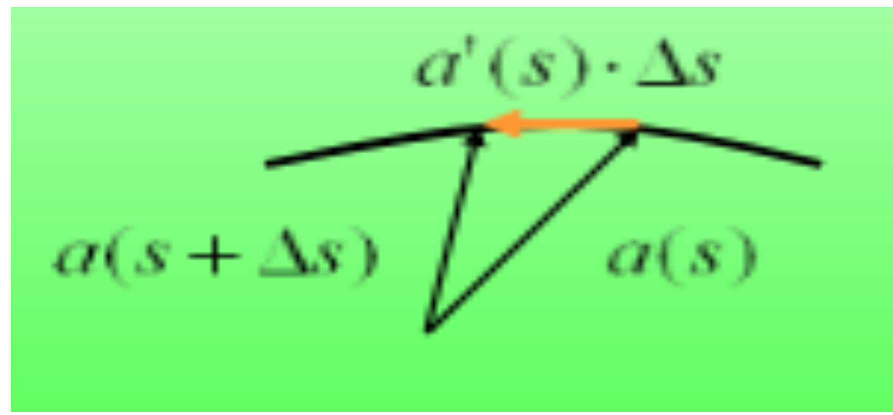
- Curve—s arc length

$$a(s) = (\,x(s), y(s)\,)$$



- Tangent of a curve

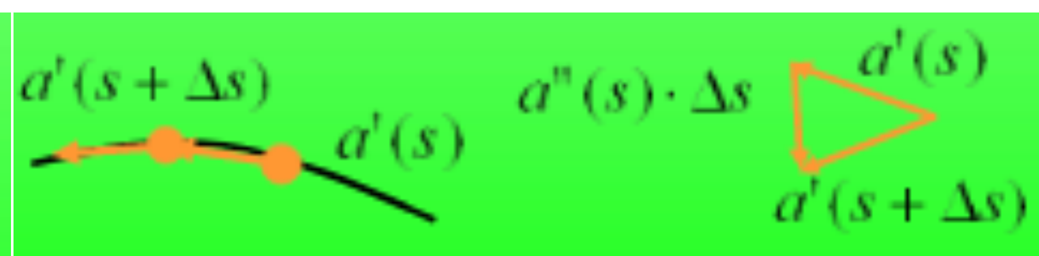
$$a'(s) = (\,x'(s), y'(s)\,)$$



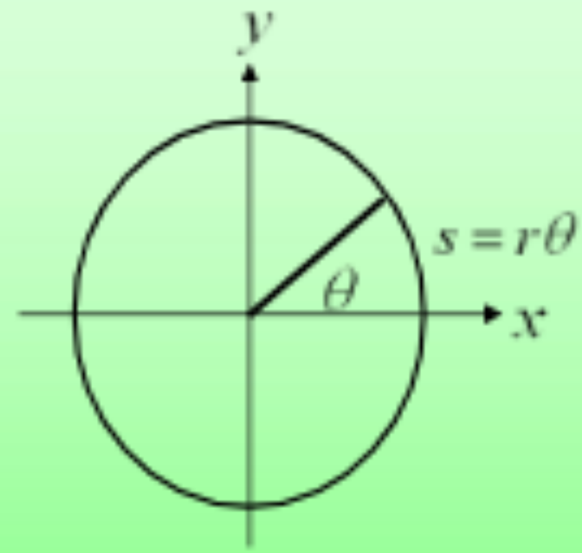
- Curvature of a curve

$$a''(s) = (\,x''(s), y''(s)\,)$$

$|a''(s)|$ -- curvature



Example: Circle



1. Arc length, s

$$s = r\theta$$

2. coordinates

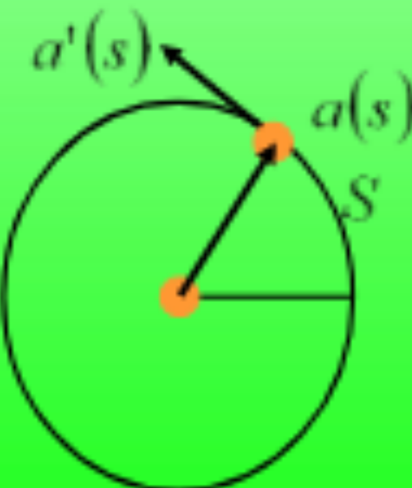
$$x = r\cos\theta = r\cos\left(s/r\right) \qquad y = r\sin\theta = r\sin\left(s/r\right)$$

$$a(s) = \left(x(s), y(s)\right) = \left(r\cos\left(s/r\right), r\sin\left(s/r\right)\right)$$

3. tangent

$$a'(s) = \left(-\sin\left(s/r\right), \cos\left(s/r\right)\right)$$

4. curvature

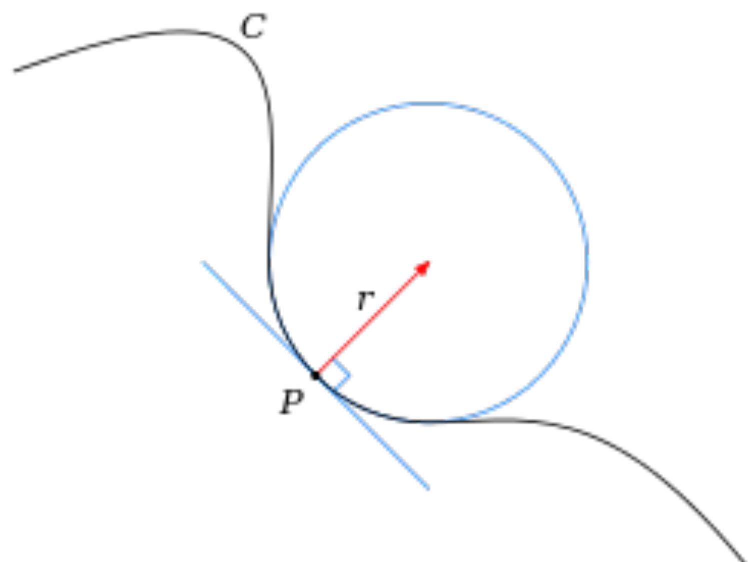


$$a''(s) = \left(-\cos(s/r)/r, -\sin(s/r)/r\right) = -1/r^2 a(s)$$

$$\left|a''(s)\right| = 1/r$$

Another meaning of Curvature of a curve

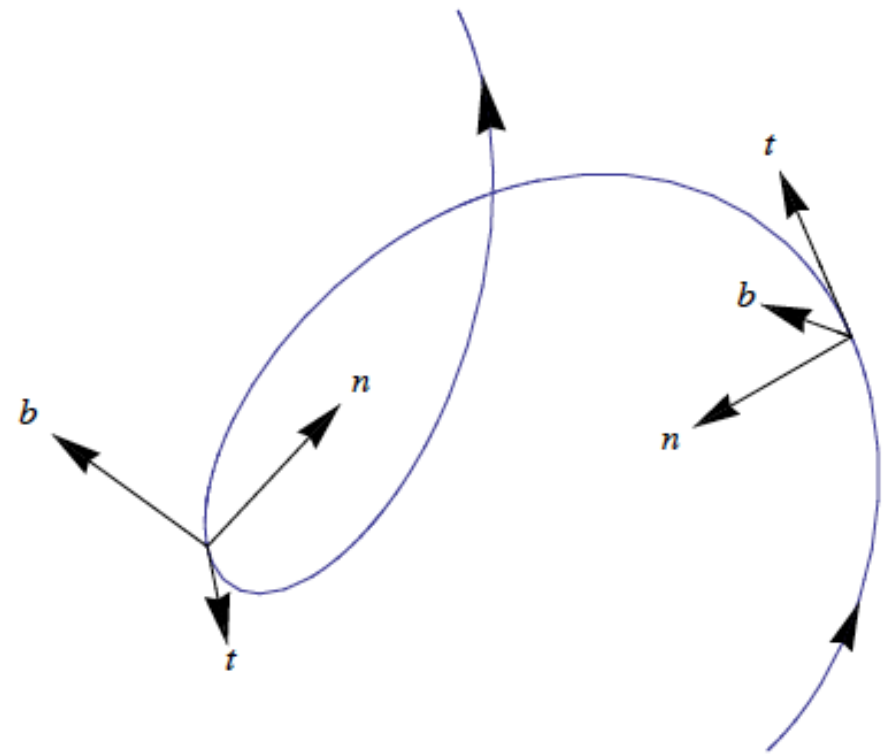
- Viewed as an inverse of the radius of the circle best tangent to the curve.



Torsion

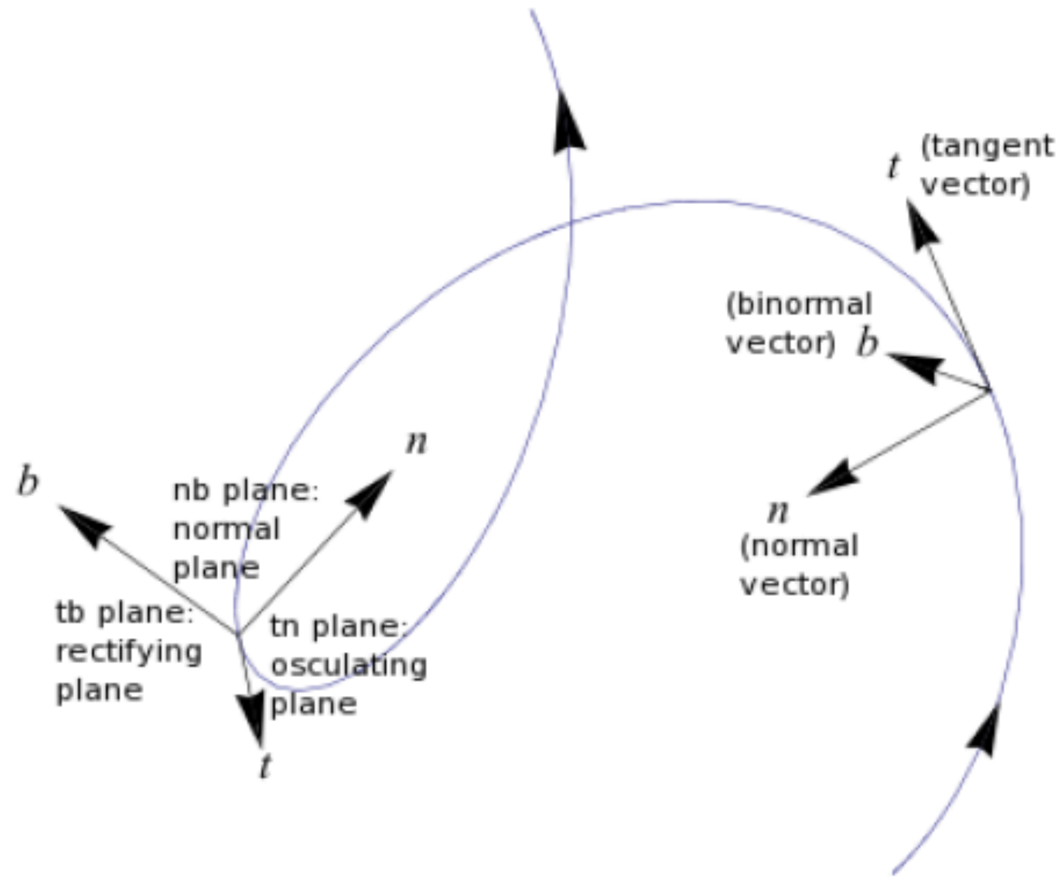
Geometric Meaning

Since $b(s)$ is a unit vector, the length $\|b'(s)\|$ measures the rate of change of the neighboring osculating planes with the osculating plane at s; that is $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.



Frenet Frame

$$\alpha'(s) \triangleq t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$



Frenet Formulas

$$\begin{cases} t' = kn, \\ n' = -kt - \tau b, \\ b' = \tau n \end{cases}$$

Fundamental Theorem of the Local Theory of Curves

Theorem

Given differentiable functions $k(s) > 0$ *and* $\tau(s), s \in I$, *there exists a regular parametrized curve* $\alpha : I \to \mathbb{R}^3$ *such that* s *is the arc length,* $k(s)$ *is the curvature, and* $\tau(s)$ *is the torsion of* α *Moreover, any other curve* $\overline{\alpha}$ *satisfying the same conditions differs from* α *by a rigid motion; that is, there exists an orthogonal map* ρ *of* $\mathbb{R}^3$, *with positive determinant, and a vector* c *such that* $\overline{\alpha} = \rho \circ \alpha + c$.

Idea of equivalent class on curve representations

- Note: We can consider the set of all kinds of reparamatrizations of the same curve,
- they form an equivalent class of the same curve,
- the one which is parametrized by arclength can be viewed as a standard representation of all those reparametrizations.
- We can pick a different kind of representation, for example, the square root velocity representation, which turned out that it is very useful in shape analysis of curves. (Details later).

We can represent curves in different ways!

- The spaces of such representations turn out to have a nonlinear geometry because of various constraints such as
  - rotation or
  - scale invariance
- on the definitions of shapes.
- Riemannian approaches are particularly well suited for the formulation and analysis of continuous-valued representations arising from such complex applications,
- since they can efficiently exploit the intrinsic nonlinearity of the representations and the geometry of the underlying spaces of those representations.

- **For example: Elastic Shape Analysis (Details later).**

There are many kinds of applications of Shapes Analysis

Curve shapes are geometric descriptions of the underlying morphological information of objects from images.

- Many kinds of applications including
  - statistical pattern recognition
  - machine vision
  - medical imaging
  - bio-signals
  - Intelligent informatics-based applications
    - human activity