Lecture 3: wedge Product and Representations of elements in Grassmanniann

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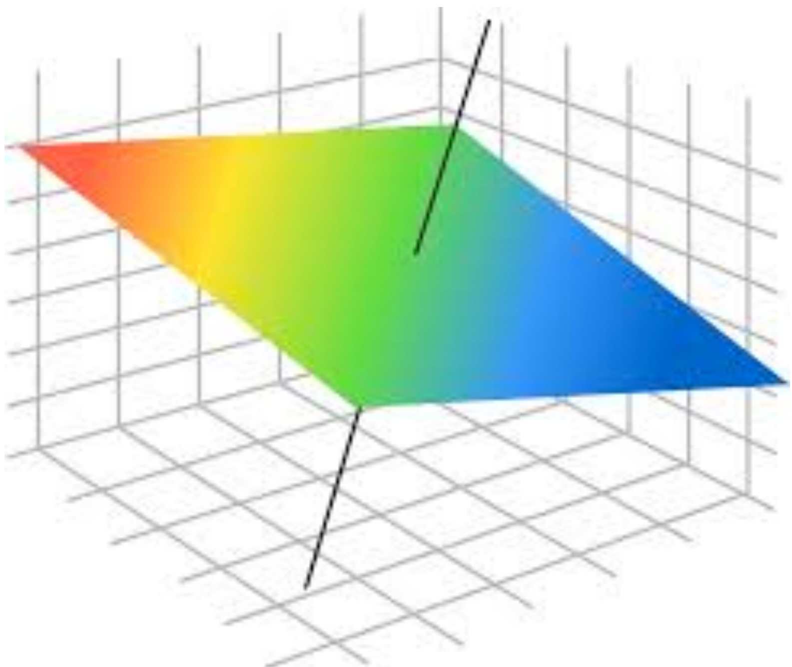
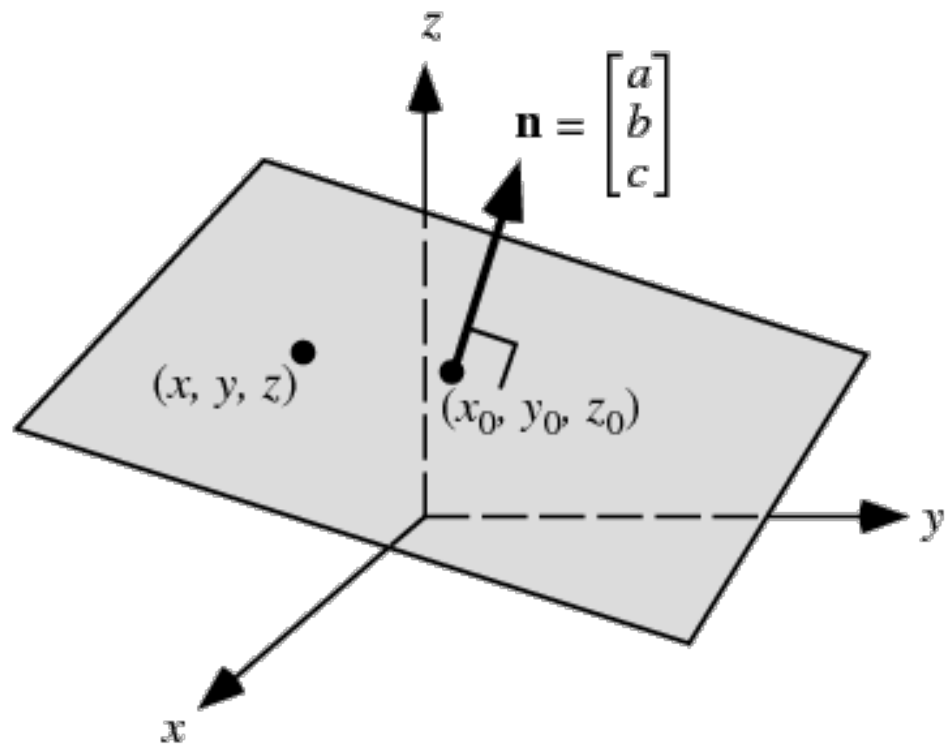
Recall: A plane and a line is in 1-1 correspondence.

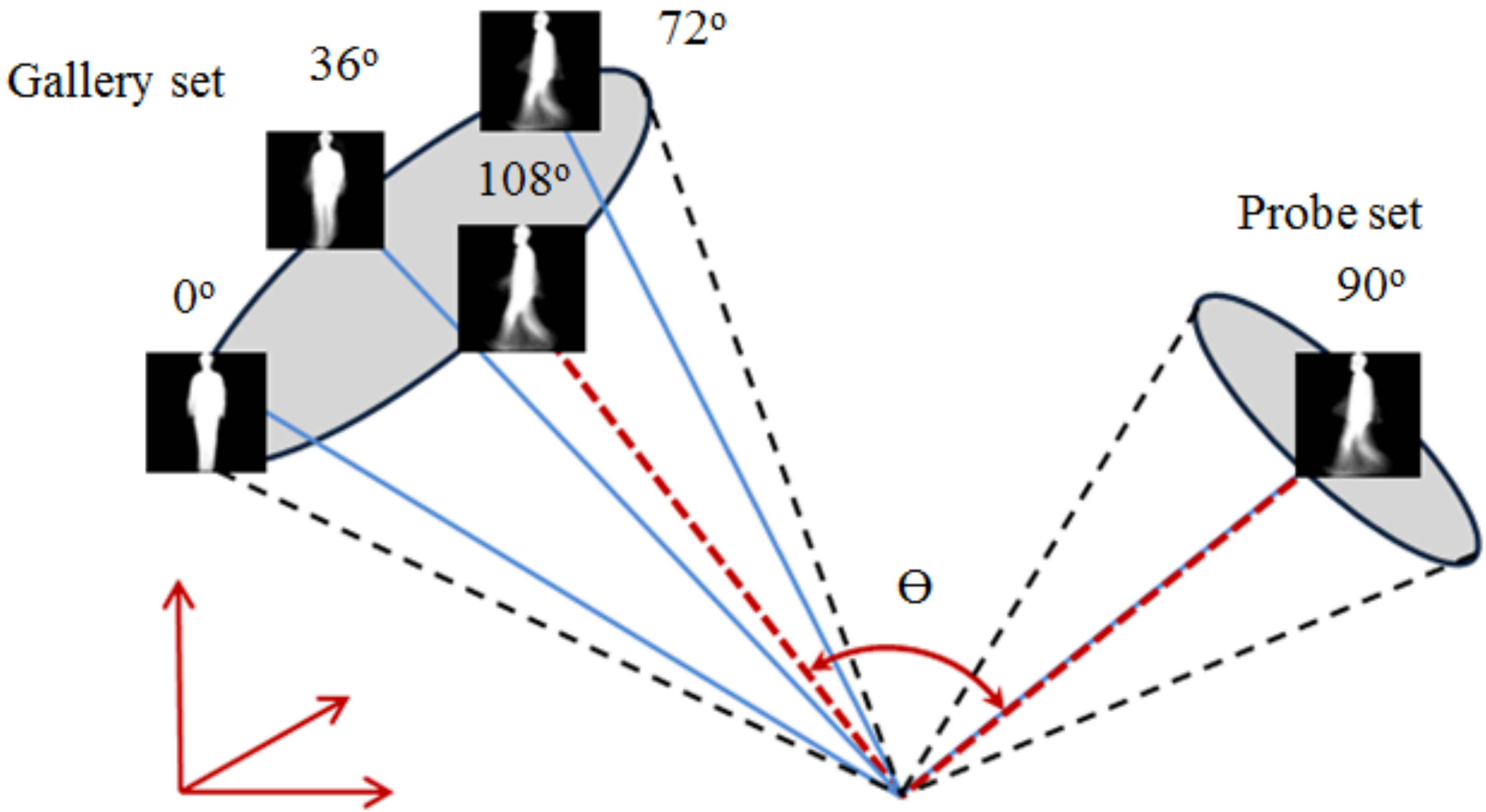




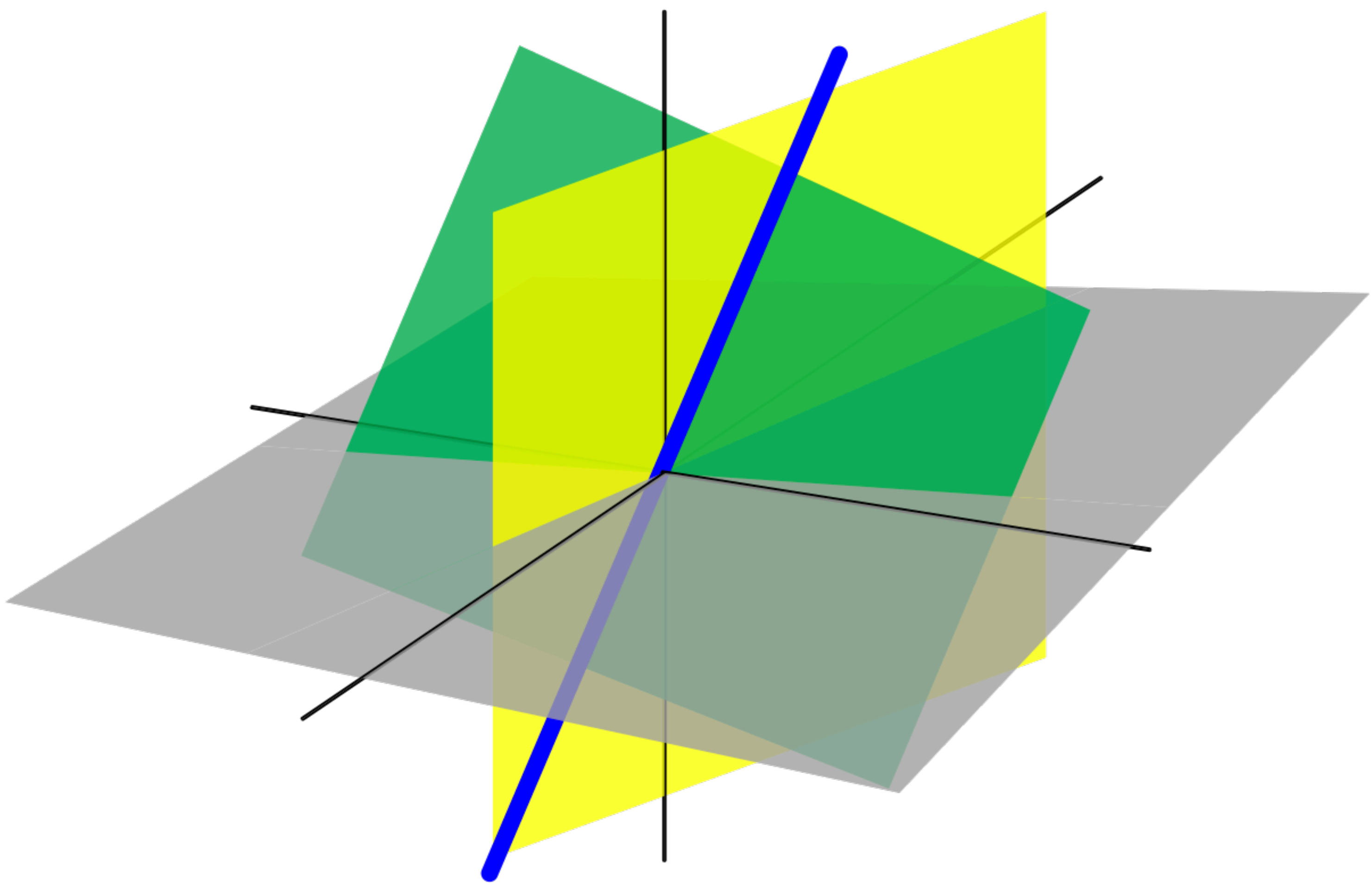
We can view n corresponding the face-up plane and (-n) corresponds to the face-down plane.
We call them oriented planes

Motivation

For e.g. Organizing images projected to planes and designing distance between images

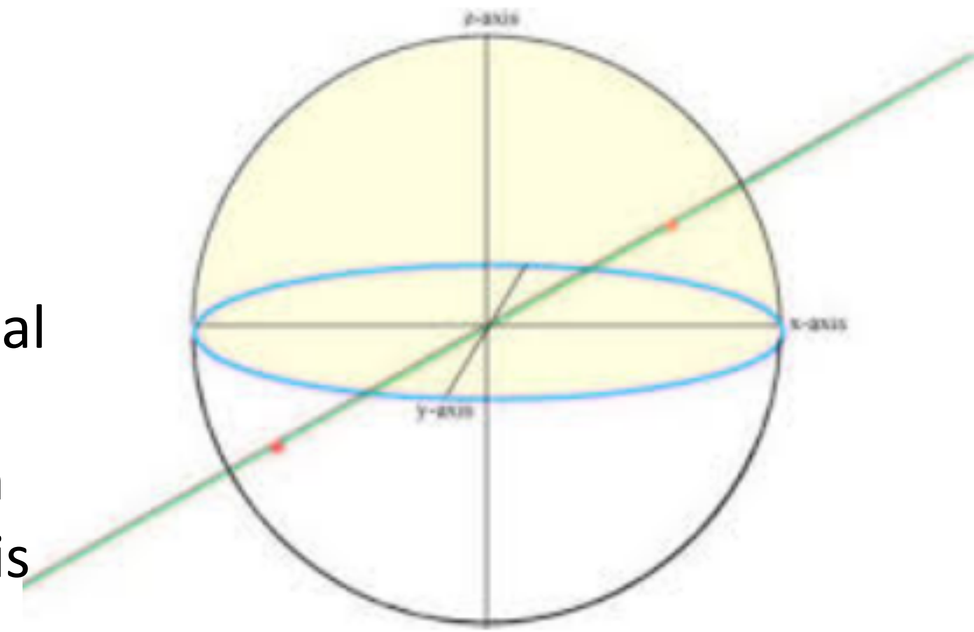


The set of planes in R^3 passing through the origin is hard to visualize

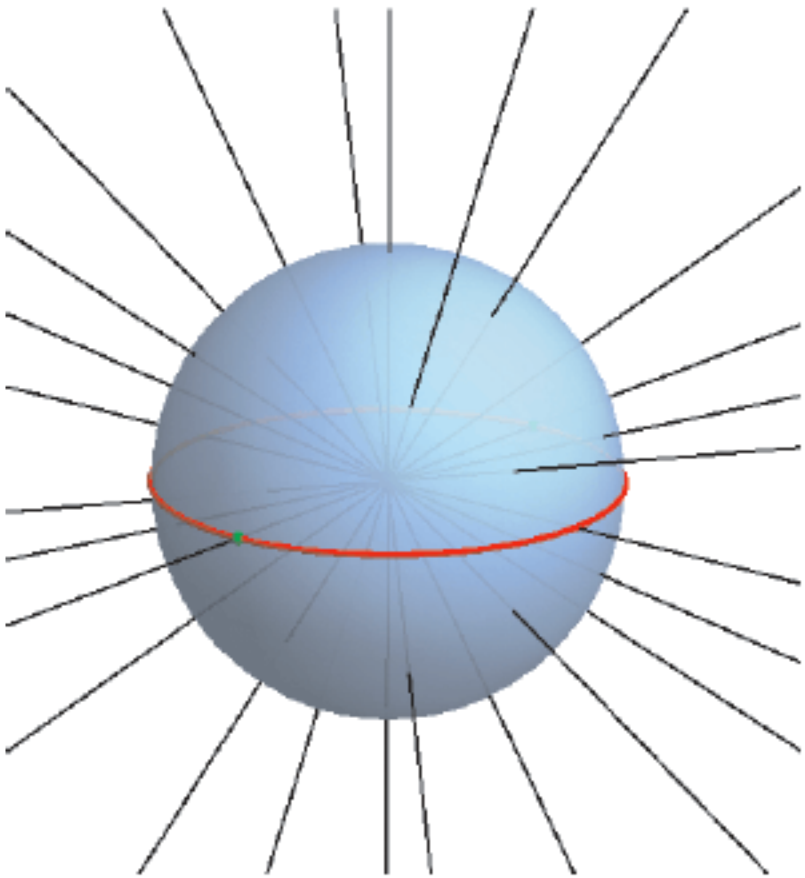


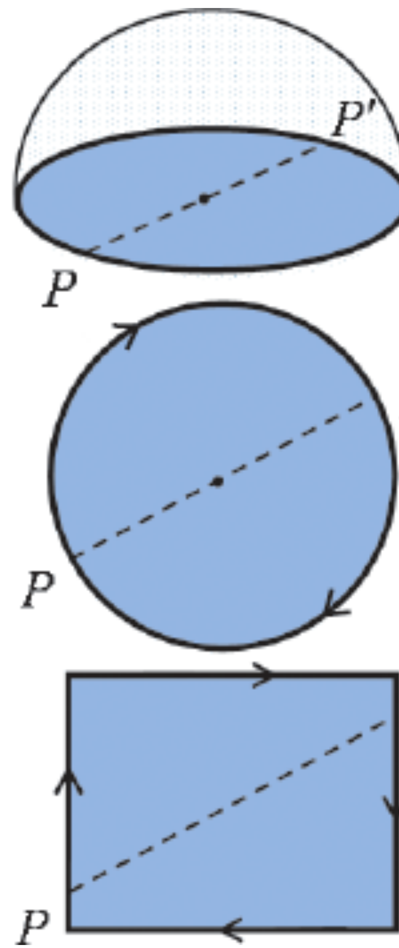
Note: Count the number of planes through the origin is equivalent to count the number of lines through the origin.

- Then in turn the line is captured by the two antipodal points on the unit sphere.
- Once these two antipodal points being identified, then the plane will be in 1-1 corresponding to this identified point.

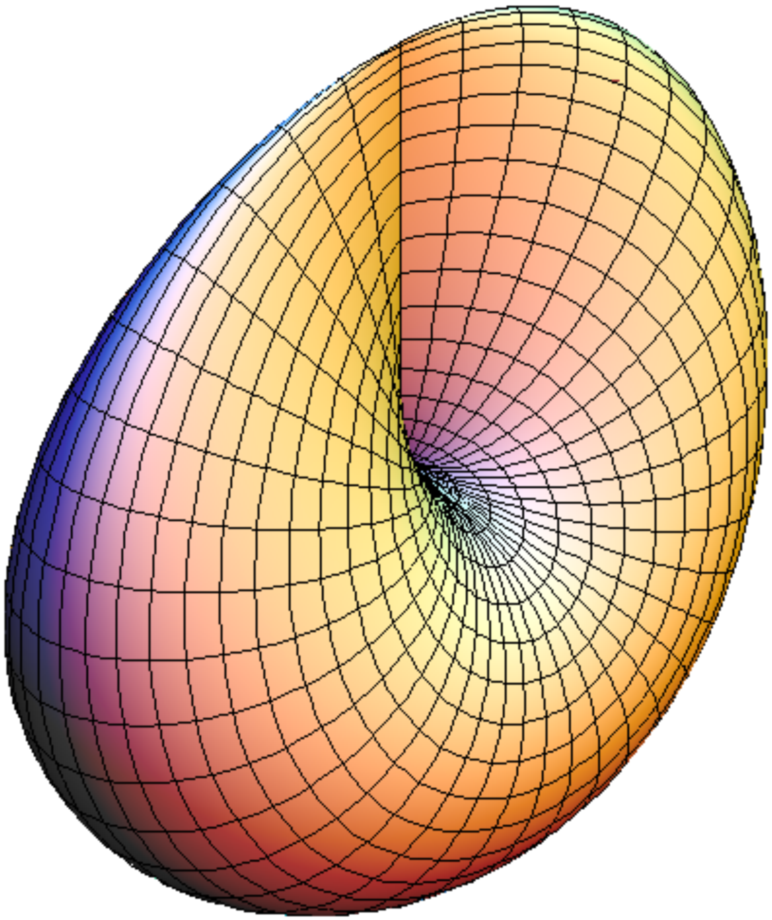


The set of all possible lines in R^3 through the origin is a manifold RP^2. How to get Real Projective Space RP^2?





RP^2



- Now each point on RP^2 is 1-1 corresponding to a plane.
- Visualizing and analyzing the set of points on RP^2 is much clearer and easier.
- For example, distance between two planes becomes distance between the two points on RP^2.
- Clustering a give set of planes (say corresponding to images), become clustering points on RP^2.
- Moreover, we can perform many of these analysis on the sphere S^2, then do the antipodal identification to map back to RP^2.

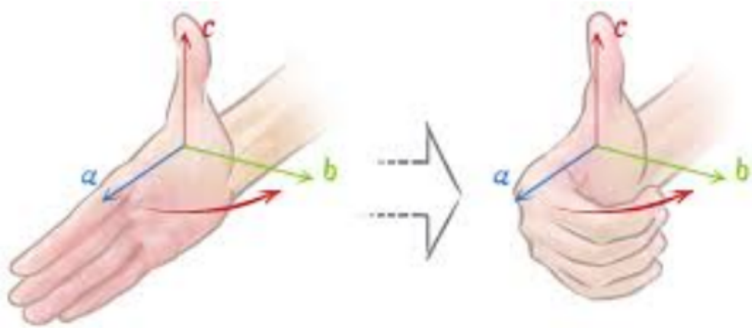
Note: Here we have used 1-1 corresponding between a 2-plane P in **R**^{3} and the normal vector **n**.

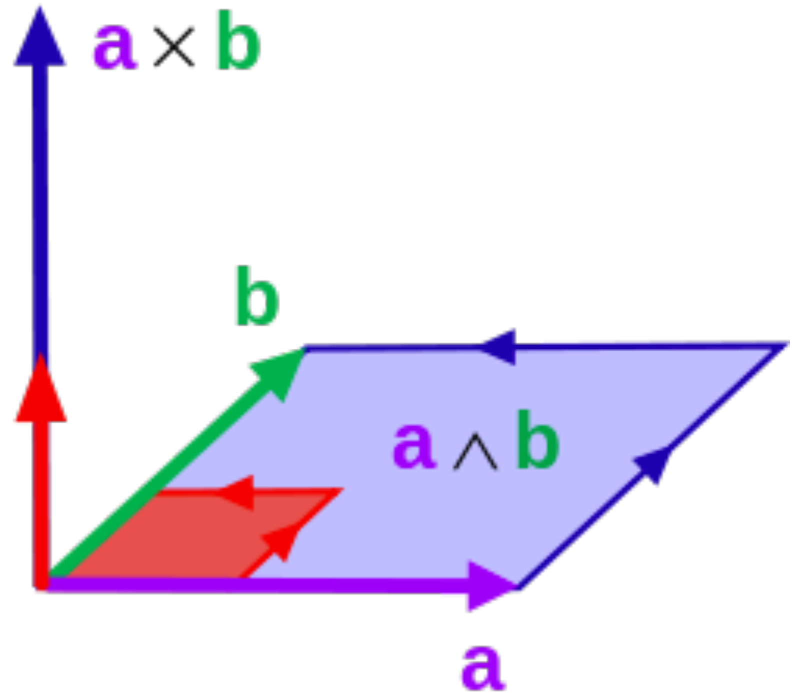
- This can be done so since R3 = P “+” **n**.

- What if we consider all 2 planes through the origin in **R**^{5}?
- It would not simply the problem if we use the normal vectors of each plane.
- We need to find different ways to represent 2-planes.

Another clever way to represent a 2-plane!

Oriented planes





Matrix representation:

- [a, b] represents the face-up plane.
- [b, a] represents the face-down plane.
- But if the frame {a, b} is rotated on the plane by an angle, then they still represent the same oriented plane!

We mimic ideas in linear algebra

- Just like A is equivalent to EA, where E is an elementary matrix, say for a homogenous linear system A**x** = **0**.

- Here a x b is equivalent to a’ x b’.

- Mathematicians developed clever new notation called wedge product, so that the representation will be unique no matter which oriented orthonormal basis one picks to represent the plane.
- Meaning The notation $\mathbf{a} \wedge \mathbf{b}$ will capture the change of basis!

What is a wedge product?

- Working out details with students on board.
- Consider R^n. See more details on

https://en.wikipedia.org/wiki/Exterior_algebra

- n=2:

$$\begin{aligned}\mathbf{v} \wedge \mathbf{w} &= (a\mathbf{e}_1 + b\mathbf{e}_2) \wedge (c\mathbf{e}_1 + d\mathbf{e}_2) \\ &= ac\mathbf{e}_1 \wedge \mathbf{e}_1 + ad\mathbf{e}_1 \wedge \mathbf{e}_2 + bc\mathbf{e}_2 \wedge \mathbf{e}_1 + bd\mathbf{e}_2 \wedge \mathbf{e}_2 \\ &= (ad - bc)\mathbf{e}_1 \wedge \mathbf{e}_2\end{aligned}$$

$$\text{Area} = \left|\det\begin{bmatrix}\mathbf{v} & \mathbf{w}\end{bmatrix}\right| = \left|\det\begin{bmatrix}a & c \\ b & d\end{bmatrix}\right| = |ad - bc|\,.$$

- n=3:

Work out the details with the students on the board.

- Please study details from

https://en.wikipedia.org/wiki/Exterior_algebra

Curves in Grassmannian G_kR^n

- Let's take k =2 for example.
- Note: We are considering a curve parametrized by t with each element of the curve is a 2-plane.
- Now we want to take the derivative of that curve.
- How?
- We need the proposition 5 on the following slide.

Properties of the Derivative of a Vector Valued Function

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

3. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

4. $\frac{d}{dt}[\mathbf{r}(t) \bullet \mathbf{u}(t)] = \mathbf{r}(t) \bullet \mathbf{u}'(t) + \mathbf{r}'(t) \bullet \mathbf{u}(t)$

5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

6. $\frac{d}{dt}[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$

Verification using an example

Verify Property 5 $\frac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{u}(t)\right]=\mathbf{r}(t)\times\mathbf{u}'(t)+\mathbf{r}'(t)\times\mathbf{u}(t)$

$$\mathbf{r}(t)=\langle t^2,2t^3,-t\rangle \quad , \quad \mathbf{u}(t)=\langle t,t^4,4\rangle$$

$$\vec{r}\times\vec{u}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ t^2&2t^3&-t\\ t&t^4&4\end{vmatrix}=\begin{vmatrix}2t^3&-t\\ t^4&4\end{vmatrix}\vec{i}-\begin{vmatrix}t^2&-t\\ t&4\end{vmatrix}\vec{j}+\begin{vmatrix}t^2&2t^3\\ t&t^4\end{vmatrix}\vec{k}$$

$$\vec{r}\times\vec{u}=\langle 8t^3+t^5,\ -5t^2,\ t^6-2t^4\rangle$$

$$\frac{d}{dt}\left[\vec{r}\times\vec{u}\right]=\langle 24t^2+5t^4,\ -10t,\ 6t^5-8t^3\rangle$$

$$\frac{d}{dt}\left[\vec{r}(t)\times\vec{u}(t)\right]=\vec{r}(t)\times\vec{u}'(t)+\vec{r}'(t)\times\vec{u}(t)$$

$$\vec{r}=\langle t^2,2t^3,-t\rangle \qquad \vec{r}'=\langle 2t,6t^2,-1\rangle$$

$$\vec{u}=\langle t,t^4,4\rangle \qquad \vec{u}'=\langle 1,4t^3,0\rangle$$

$$\vec{r}\times\vec{u}'=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ t^2&2t^3&-t\\ 1&4t^3&0\end{vmatrix}=\begin{vmatrix}2t^3&-t\\ 4t^3&0\end{vmatrix}\vec{i}-\begin{vmatrix}t^2&-t\\ 1&0\end{vmatrix}\vec{j}+\begin{vmatrix}t^2&2t^3\\ 1&4t^3\end{vmatrix}k$$

$$\vec{r}\times\vec{u}'=\langle 4t^4,\ -t,\ 4t^5-2t^3\rangle$$

$$\vec{r}'\times\vec{u}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ 2t&6t^2&-1\\ t&t^4&4\end{vmatrix}=\begin{vmatrix}6t^2&-1\\ t^4&4\end{vmatrix}\vec{i}-\begin{vmatrix}2t&-1\\ t&4\end{vmatrix}\vec{j}+\begin{vmatrix}2t&6t^2\\ t&t^4\end{vmatrix}k$$

$$\vec{r}'\times\vec{u}=\langle 24t^2+t^4,\ -9t,\ 2t^5-6t^3\rangle$$

$$\vec{r}\times\vec{u}'+\vec{r}'\times\vec{u}=\langle 24t^2+5t^4,\ -10t,\ 6t^5-8t^3\rangle$$

Decomposable wedge product

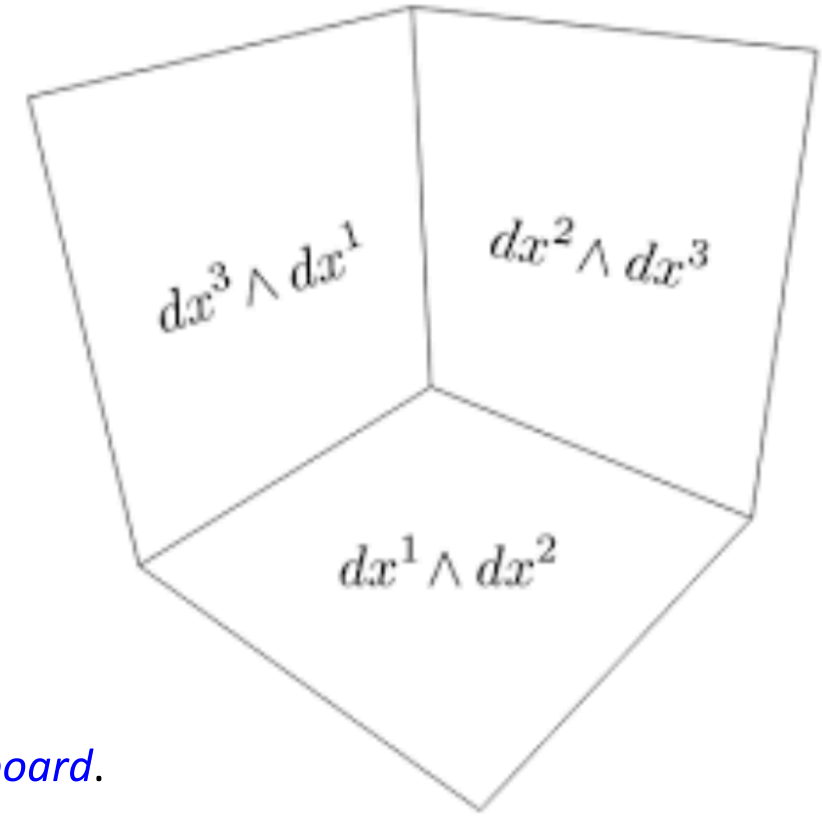
- Think a plane is an element on G_2R^3.
- Here are the "basis"(oriented) planes in R^3:

$e_1 x e_2$, $e_2 x e_3$, $e_3 x e_1$. Where $\{e_1, e_2, e_3\}$ is a basis of R^3.

- Any other plane a x b can be written as a linear combination of the these "basis" planes provide this linear combination is decomposable! Meaning:

axb $= c^{12}e_1xe_2 + c^{23}e_2xe_3 + c^{31}e_3xe_1 =$
$(a^1e_1+a^2e_2+a^3e_3)\, x(b^1e_1+b^2e_2+b^3e_3)$.
This idea can be generated to Grassmanian just extend the cross product to a wedge product.

- This idea also can be generated to the dual space of a vector space.
- Then can be generated to co-tangent space of a manifold.



Working out details with the students on board.

What is a dual space?

- https://en.wikipedia.org/wiki/Dual_space