

1 If $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is a (real or complex) inner product space, prove

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x - iy\|^2 - i\|x + iy\|^2)$$

for all $x, y \in \mathcal{V}$. This is known as the *polarization identity*.

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2 If $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is a (real or complex) inner product space, prove

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$$

for any $x, y \in \mathcal{V}$. This is known as the *parallelogram law*.

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3 Von Neumann proved that the parallelogram law actually has a converse: a normed space $(\mathcal{V}, \|\cdot\|)$ is an inner product space (with inner product given by the polarization identity) if and only if $(\mathcal{V}, \|\cdot\|)$ satisfies the parallelogram law. Prove that the sequence space ℓ^p , $1 \leq p \leq \infty$, is a Hilbert space if and only if $p = 2$.

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4 Prove that a real or complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is separable (has a countable dense subset) if and only if \mathcal{H} has a countable orthonormal basis.

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