
Math 173
Problem Set 1
September 17, 2018

Note: You may assume the Axiom of Choice in all these problems, but for this problem set in particular please point out where you use it. The Extra Credit (EC) problem is worth as much as a regular problem; even if you don't attempt it, you should read it thoroughly since these problems will hint at future directions in the course or other interesting topics.

1 Under the Axiom of Choice, prove that every vector space, infinite dimensional or not, has a basis. *Hint:* Use Zorn's Lemma.

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2 (Lax 2.1) Given a nonzero x in a vector space $\mathcal{V}$ of arbitrary dimension, show that there exists a linear functional $f : \mathcal{V} \to \mathbb{F}$ such that $f(x) \neq 0$.

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3 Let $C[0,1]$ be the vector space of continuous real functions on the interval $[0,1]$. Show that $C[0,1]$ has uncountable Hamel dimension.

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(1) Prove that the real numbers form a vector space over the rational numbers of uncountable Hamel dimension.

(2) Let $p \in \mathbb{Q}[x]$ be a rational coefficient polynomial. If we are only allowed to query $p(x)$ for (a) rational x and (b) real x, what is the minimal number of queries needed to uniquely determine p?

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EC Let $x \in [0,1]$ and $X \sim \text{Binomial}(n,x)$ be a binomial random variable. Take $f \in C[0,1]$. Observe that

$$\mathbb{E}f(X/n) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

The right hand side of this, as a function of x, is called the n-th Bernstein polynomial of f, denoted $B_n(f)$. We will prove the Wierstrass Approximation Theorem: that the polynomials are dense in $C[0,1]$ under the sup-norm $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$, by showing that $B_n(f) \to f$ uniformly. This says that under a strong notion of distance, the vector space $C[0,1]$ of uncountable Hamel dimension is actually 'approximately' countable, in a way we will formalize later in the course.

(1) Prove that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{P}(|\tfrac{X}{n} - x| \geq \epsilon) = 0$$

for all $\epsilon > 0$. *Hint:* Recall Chebyshev's inequality[a].

(2) Infer that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{P}(|f(\tfrac{X}{n}) - f(x)| \geq \epsilon) = 0 \quad \text{to prove} \quad \lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{E}|f(\tfrac{X}{n}) - f(x)| = 0.$$

(3) Conclude that $\lim_{n\to\infty} \|B_n(f) - f\|_\infty = \lim_{n\to\infty} \sup_{x\in[0,1]} |B_n(f)(x) - f(x)| = 0$.

[a] Chebyshev's inequality says that $\mathbb{P}(|X - \mathbb{E}X| \geq \gamma) \leq \gamma^{-2}\text{Var}(X)$.

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