
Math 173
Problem Set 2
September 24, 2018

Note: The Extra Credit (EC) problem is worth as much as a regular problem; even if you don't attempt it, you should read it thoroughly since these problems will hint at future directions in the course or other interesting topics.

1 (Lax 3.14) Let $\mathcal{V}$ be a vector space of arbitrary dimension and $T : \mathcal{V} \to \mathcal{V}$ be rank 1.

(a) Show there exists a unique number c such that $T^2 = cT$.

(b) Assuming $\dim \mathcal{V} < \infty$, show that $c \neq 1$ implies $I - T$ has an inverse.

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2 Let $\mathcal{V}$ be a vector space, and recall that two linear spaces are isomorphic if there exists an invertible linear map from one to the other.

(a) If $\mathcal{V}$ is infinite dimensional, prove that $\dim \mathcal{V} < \dim \mathcal{V}'$, or in other words that there exists a linear injection from $\mathcal{V}$ to $\mathcal{V}'$ that is not a surjection.

(b) On the other hand, if $\mathcal{V}$ is finite dimensional prove that $\mathcal{V}$ is isomorphic to $\mathcal{V}'$.

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3 Let $A \in \mathbb{C}^{m \times n}$ and decompose $A = U\Sigma V^*$ into its singular value decomposition where Σ is the diagonal matrix composed of the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Write $\Sigma_k = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ for $k = 1, 2, \ldots, \max\{m, n\}$ and define the truncated singular value decomposition $A_k = U\Sigma_k V^*$. Prove[a] that the largest singular value of $A - A_k$ is σ_{k+1}.

[a]As we'll see later in the course, this says that the distance between A and A_k in the so-called *spectral norm* is σ_{k+1}. It turns out that this is the *best* rank-k approximation to A in both the spectral and Frobenius norms.

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4 Load the image located at `https://math173.github.io/img/dog.jpg` into a computational language of your choice. We can think of this grayscale image as a matrix $A \in \mathbb{R}^{m \times n}$. Where each element A_{ij} is the intensity of the pixel A_{ij}.

(a) Plot[a] (in decreasing order) the singular values of A.

(b) Plot the truncated SVD A_k as an image for $k = 5, 10, 15, 20$ and compare it to A.

(c) How many entries do we need to store to represent A_k for general $A \in \mathbb{R}^{m \times n}$? Using this, interpret A_k as a lossy compression scheme for a matrix A.

[a]You can use an SVD function. No need to implement it.

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