------
Math 173
Problem Set 1
September 17, 2018

**Note:** You may assume the Axiom of Choice in all these problems, but for this problem set in particular please point out where you use it. The Extra Credit (EC) problem is worth as much as a regular problem; even if you don't attempt it, you should read it thoroughly since these problems will hint at future directions in the course or other interesting topics.

**1** Under the Axiom of Choice, prove that every vector space, infinite dimensional or not, has a basis. *Hint:* Use Zorn's Lemma.

■

**2 (Lax 2.1)** Given a nonzero $x$ in a vector space $\mathcal{V}$ of arbitrary dimension, show that there exists a linear functional $f : \mathcal{V} \to \mathbb{F}$ such that $f(x) \neq 0$.

$\blacksquare$

**3** Let $C[0,1]$ be the vector space of continuous real functions on the interval $[0,1]$. Show that $C[0,1]$ has uncountable Hamel dimension. *Hint:* Consider $\{e^{\beta x} : \beta \in [0,1]\}$.

$\blacksquare$

**4**

(1) Prove that the real numbers form an infinite dimensional vector space over the rational numbers.

(2) Let $p \in \mathbb{Q}[x]$ be a rational coefficient polynomial. What is the minimal number of evaluations of $p$ needed to uniquely determine $p$ (a) if we are allowed to evaluate only at rational points and (b) if we are allowed to evaluate at real points.

$\blacksquare$

**EC** Let $x \in [0,1]$ and $X \sim \text{Binomial}(n,x)$ be a binomial random variable. Take $f \in C[0,1]$. Observe that

$$\mathbb{E}f(X/n) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

The right hand side of this, as a function of $x$, is called the $n$-th Bernstein polynomial of $f$, denoted $B_n(f)$. We will show that $B_n(f) \to f$ uniformly, which shows that $\{x^k(1-x)^{n-k} : 0 \le k \le n\}$ approximates a basis for $C[0,1]$ in some sense. This shows that, while $C[0,1]$ has uncountable Hamel dimension, it has 'almost-countable' dimension, a notion that will be made more precise later in the course. (**Remark:** This also proves of the Wierstrass Approximation Theorem, that says that the polynomials are dense in $C[0,1]$.)

(1) Prove that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{P}(|\tfrac{X}{n} - x| \ge \epsilon) = 0$$

for all $\epsilon > 0$. *Hint:* Recall Chebyshev's inequality[a] and use a diagonalization argument.

(2) Infer that

$$\lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{P}(|f(\tfrac{X}{n}) - f(x)| \ge \epsilon) = 0 \quad \text{to prove} \quad \lim_{n\to\infty} \sup_{x\in[0,1]} \mathbb{E}|f(\tfrac{X}{n}) - f(x)| = 0.$$

(3) Conclude that $\lim_{n\to\infty} \|B_n(f) - f\|_\infty = \lim_{n\to\infty} \sup_{x\in[0,1]} |B_n(f)(x) - f(x)| = 0.$

[a] Chebyshev's inequality says that $\mathbb{P}(|X - \mathbb{E}X| \ge \gamma) \le \gamma^{-2}\text{Var}(X)$.

∎