Math 173
Problem Set 3
October 1, 2018

1 If $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a (real or complex) Hilbert space, prove

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2\right)$$

for all $x, y \in \mathcal{H}$. This is known as the *polarization identity.*

$\blacksquare$

2 If $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is a (real or complex) inner product space, prove

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

for any $x, y \in \mathcal{V}$. This is known as the *parallelogram law.*

$\blacksquare$

3 Von Neumann proved that the parallelogram law actually has a converse: a normed space $(\mathcal{V}, \|\cdot\|)$ is an inner product space (with inner product given by the polarization identity) if and only if $(\mathcal{V}, \|\cdot\|)$ satisfies the parallelogram law. Prove that the sequence space ℓ^p, $1 \le p \le \infty$, is a Hilbert space if and only if $p = 2$.

$\blacksquare$

4 Prove that a real or complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is separable (has a countable dense subset) if and only if $\mathcal{H}$ has a countable orthonormal basis.

$\blacksquare$