
Math 173
Problem Set 2
September 24, 2018

1 (Lax 3.14) Let $\mathcal{V}$ be a vector space of arbitrary dimension and $T : \mathcal{V} \to \mathcal{V}$ be rank 1.

(a) Show there exists a unique number c such that $T^2 = cT$.

(b) Assuming $\dim \mathcal{V} < \infty$, show that $c \neq 1$ implies $I - T$ has an inverse.

■

2 Let $\mathcal{V}$ be a vector space, and recall that two linear spaces are isomorphic if there exists an invertible linear map from one to the other.

(a) If $\mathcal{V}$ is infinite dimensional, prove that $\dim \mathcal{V} < \dim \mathcal{V}'$, or in other words that there exists a linear injection from $\mathcal{V}$ to $\mathcal{V}'$ that is not a surjection.

(b) On the other hand, if $\mathcal{V}$ is finite dimensional prove that $\mathcal{V}$ is isomorphic to $\mathcal{V}'$.

■

3

(a) Prove that the set of diagonalizable matrices is dense in the set of all matrices under the metric $d(A, B) = \max_{ij} |A_{ij} - B_{ij}|$ on $\mathsf{M}_n(\mathbb{C})$.

(b) Using this, prove that the trace of a matrix is the sum of its eigenvalues, and that the determinant of a matrix is the product of its eigenvalues.

■

4 Let $A, B : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ be linear operators on the space of square summable sequences. If $\sum_{i,j} |A_{ij}|^2 < \infty$ we call A a Hilbert-Schmidt operator. Prove that the functional

$$\langle A, B \rangle = \sum_{ij} A_{ij} \overline{B}_{ij}$$

is an inner product. That is, prove that $\langle \cdot, \cdot \rangle$

(a) is well-defined between Hilbert-Schmidt operators.

(b) is linear in the first coordinate.

(c) satisfies $\langle A, B \rangle = \overline{\langle B, A \rangle}$

(d) requires $A = 0$ if $\langle A, A \rangle = 0$.

■