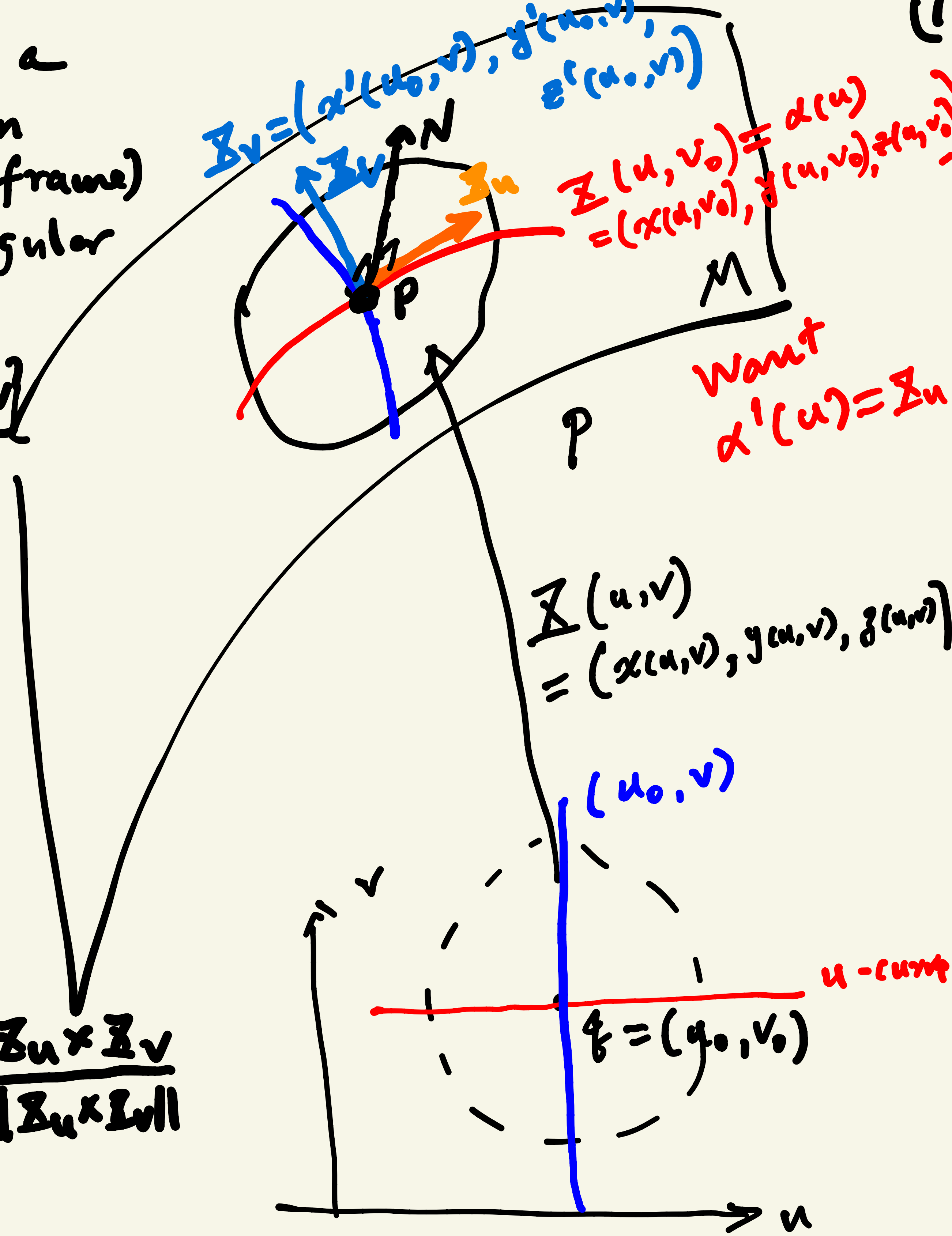
What is a trihedron (or a 3-frame) on a regular surface?

$\{\mathbf{X}_u, \mathbf{X}_v, N\}$



$$N = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$$

$$\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$\mathbf{X}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\mathbf{X}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\right)$$

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}$$

$$X_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$X_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\right)$$

Note: $N \perp X_u$, $N \perp X_v$

But $X_u \not\perp X_v$
may not true

$\{X_u, X_v, N\} \to$ tripot.

As p changes, it moves on the manifold. i.e. a 3-frame moves on the manifold.
may not O.N.

Note: The tangent plane at p has basis X_u & X_v.

<u>Key</u> : Understanding how this 3-frame moves tells us the how the underline mfld looks like.

Trick write X_{uu} etc on the below into a linear combination of the moving frame!

$$(X_u)_u = \boxed{\Gamma^1_{11}} X_u + \boxed{\Gamma^2_{11}} X_v + \boxed{L_1} N$$

$$(X_u)_v = \boxed{\Gamma^1_{12}} X_u + \boxed{\Gamma^2_{12}} X_v + \boxed{L_2} N$$

$$(X_v)_u = \boxed{\Gamma^1_{21}} X_u + \boxed{\Gamma^2_{21}} X_v + \boxed{L_2} N$$

$$(X_v)_v = \boxed{\Gamma^1_{22}} X_u + \boxed{\Gamma^2_{22}} X_v + \boxed{L_3} N$$

$$N_u = \cdots$$

$$N_v = \cdots$$

$\because$ derivative is linear, so it suffices to understand how X_u, X_v moves.

$$(\mathbf{x}_u)_u = \boxed{\Gamma_{11}^1}\,\mathbf{x}_u + \boxed{\Gamma_{11}^2}\,\mathbf{x}_v + \boxed{L_{11}}\,N \quad (*)$$

Want to find Γ_{11}^1 & Γ_{11}^2

Dot $\mathbf{x}_u$ bothsides of $(*)$

$\because N \perp \mathbf{x}_u$

$$\Rightarrow \underbrace{\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle}_{c} = \Gamma_{11}^1 \underbrace{\mathbf{x}_u \cdot \mathbf{x}_u}_{=E} + \Gamma_{11}^2 \underbrace{\mathbf{x}_u \cdot \mathbf{x}_v}_{=F} + 0$$

Dot $\mathbf{x}_v$ bothsides of $(*)$

$$\Rightarrow \underbrace{\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle}_{d} = \Gamma_{11}^1 \underbrace{\mathbf{x}_u \cdot \mathbf{x}_v}_{=F} + \Gamma_{11}^2 \underbrace{\mathbf{x}_v \cdot \mathbf{x}_v}_{=G} + 0$$

Let $E = \mathbf{x}_u \cdot \mathbf{x}_u = \|\mathbf{x}_u\|^2$

$F = \mathbf{x}_u \cdot \mathbf{x}_v = \mathbf{x}_v \cdot \mathbf{x}_u$

$G = \mathbf{x}_v \cdot \mathbf{x}_v = \|\mathbf{x}_v\|^2$

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{bmatrix}$$

$$> \underbrace{\langle X_{uu}, X_u \rangle}_{c} = \Gamma_{11}^{1} \overbrace{X_u \cdot X_u}^{=E} + \Gamma_{11}^{2} \overbrace{X_u \cdot X_v}^{=F}$$

$$> \underbrace{\langle X_{uu}, X_v \rangle}_{d} = \Gamma_{11}^{1} \overbrace{X_u \cdot X_v}^{=F} + \Gamma_{11}^{2} \overbrace{X_v \cdot X_v}^{=G}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{bmatrix}$$

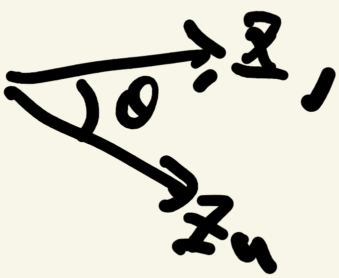
$$\Rightarrow \begin{bmatrix} \Gamma_{11}^{1} \\ \Gamma_{11}^{2} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\underbrace{\det\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{\det \neq 0 \;\leftarrow\; \text{Why ?}} = EG - F^2$$

<u>Claim</u> : $EG - F^2 = \|X_u \times X_v\|^2$

<u>Pf</u> : $\|X_u \times X_v\|^2 = \|X_u\|^2 \|X_v\|^2 \underbrace{\sin^2\theta}_{(1-\cos^2\theta)}$

since $\nparallel$ 0



$$= \|X_u\|^2 \|X_v\|^2 - \left[\|X_u\|\|X_v\|\cos\theta\right]^2$$

$$= \underbrace{\langle X_u, X_u\rangle}_{E} \underbrace{\langle X_v, X_v\rangle}_{F} - \langle X_u, X_v\rangle^2$$

$$= E \quad F \quad - G^2$$

$$= \det\begin{bmatrix} E & F \\ F & G \end{bmatrix} \neq 0$$

$\uparrow$ Therefore

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 \neq 0 \qquad \text{Why ?}$$

$\because$ The S is regular. $\mathbf{x}(u,v) = (\underline{x(u,v)}, \underline{y(u,v)}, \underline{z(u,v)})$

$d\mathbf{x}_q$ has rank 2

$$\| $$

$$\begin{array}{c} \nabla x \rightarrow \\ \nabla y \rightarrow \\ \nabla z \rightarrow \end{array} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

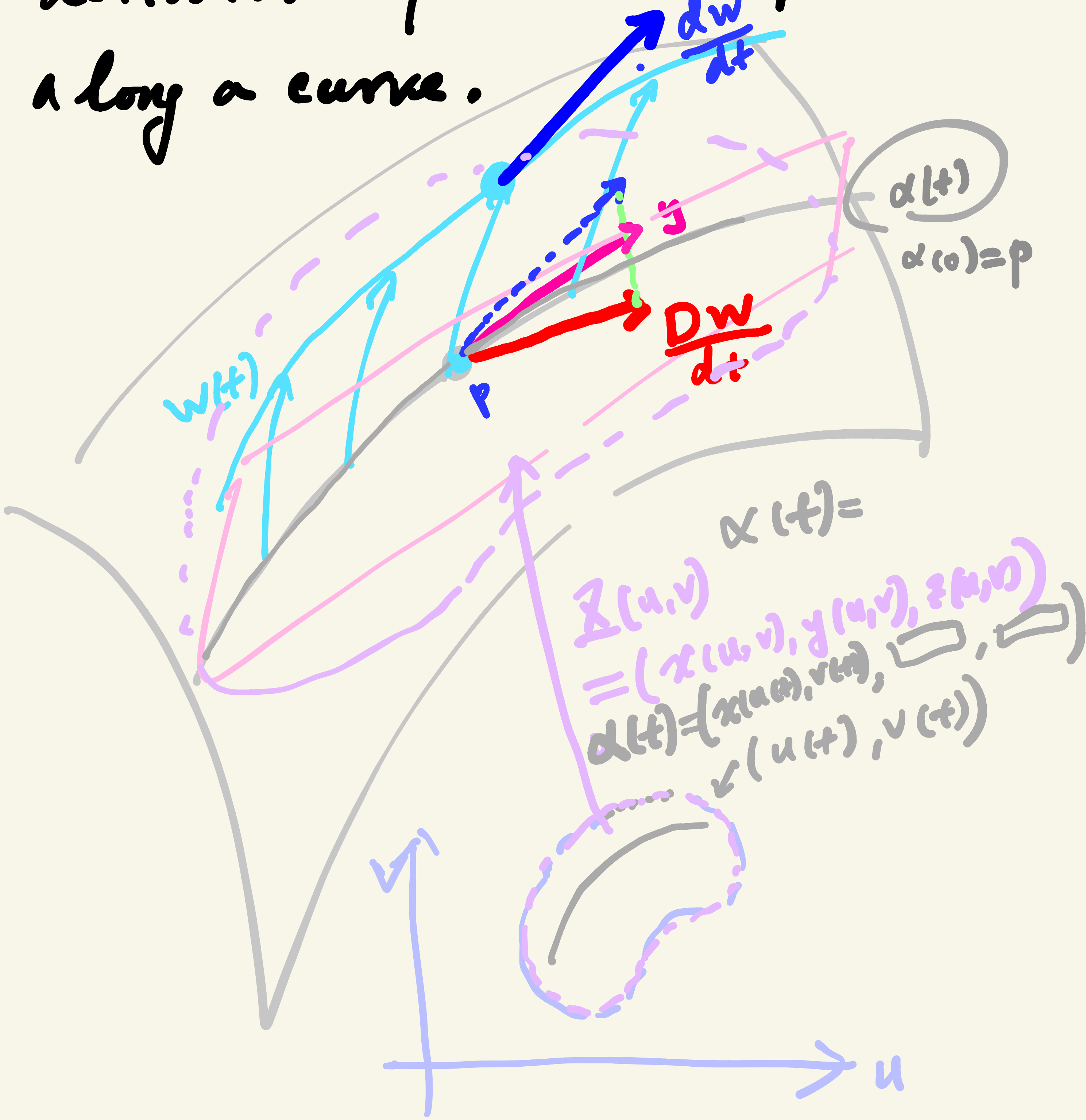
$$\mathbf{x}_u \qquad \mathbf{x}_v$$

$\because$ $\mathbf{x}_u$ and $\mathbf{x}_v$ is required to be linear independent.

$\Rightarrow \mathbf{x}_u \times \mathbf{x}_v \neq 0$

✓

Now Let's calculate the covariant derivative of a vector field along a curve.



$$\alpha(t) = \left(x(u(t), v(t)),\ y(u(t), v(t)),\ z(u(t), v(t))\right)$$

$$\alpha'(t) = \left(x_u u'(t) + x_v v'(t),\ y_u u'(t) + y_v v'(t),\ \cdots\right)$$

$$= u'(t)\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}^T + v'(t)\begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}^T$$

$$= \mathbf{X}_u u'(t) + \mathbf{X}_v v'(t)$$

$$\alpha(t) = \mathbf{X}(u(t), v(t))$$

$$\alpha'(t) = \mathbf{X}_u u'(t) + \mathbf{X}_v v'(t).$$

$$W = \underbrace{a(u,v)}\,\mathbf{X}_u + \underbrace{b(u,v)}\,\mathbf{X}_v$$

differentiable if a, b both differentible

Let W restrict to the curve $\alpha(t)$

$$\underbrace{w(u(t), v(t))}_{w(t)} = a(t)\, X_u(u(t), v(t)) + b(t)\, X_v(u(t), v(t))$$

$$\frac{dw}{dt} = a(t)\Big[X_{uu}\, u'(t) + X_{uv}\, v'(t) \Big] + a'(t)\, X_u + b(t)\Big[X_{vu}\, u'(t) + X_{vv}\, v'(t) \Big] + b'(t)\, X_v$$

$X_{uv} = \boxed{\Gamma^1_{12}} X_u + \boxed{\Gamma^2_{12}} X_v + \boxed{L_2} N$

$X_{uu} = \boxed{\Gamma^1_{11}} X_u + \boxed{\Gamma^2_{11}} X_v + \boxed{L_1} N$

$$= \boxed{a'(t) + \text{fcn of } \Gamma^k_{ij}}\, X_u + \boxed{b'(t) + \text{fcn of } \Gamma^k_{ij}}\, X_v \underbrace{+ \boxed{}\, N}$$

$$(X_u)_u = \boxed{\Gamma^1_{11}} X_u + \boxed{\Gamma^2_{11}} X_v + \boxed{L_1} N$$

$$(X_u)_v = \boxed{\Gamma^1_{12}} X_u + \boxed{\Gamma^2_{12}} X_v + \boxed{L_2} N$$

$$\| \qquad \| \qquad \|$$

$$(X_v)_u = \boxed{\Gamma^1_{21}} X_u + \boxed{\Gamma^2_{21}} X_v + \boxed{L_2} N$$

$$(X_v)_v = \boxed{\Gamma^1_{22}} X_u + \boxed{\Gamma^2_{22}} X_v + \boxed{L_3} N$$

$\frac{Dw}{dt}$ = only X_u & X_v parts, drop N part

If $\frac{DW}{dt} = 0$ along $\alpha(t)$,

then $W(t)$ is called parallel.

Now if you take a special $W(t) = \alpha'(t)$

Then
If $\frac{DW(t)}{dt} = \frac{D(\alpha'(t))}{dt} = 0$

Then $\alpha(t)$ is called geodesic at t.

Then this is true for all $t \in [a,b]$, $\alpha(t)$, $a \le t \le b$ is called a geodesic.