# Lecture 9 part 2

Key characteristics of manifold using moving frame method

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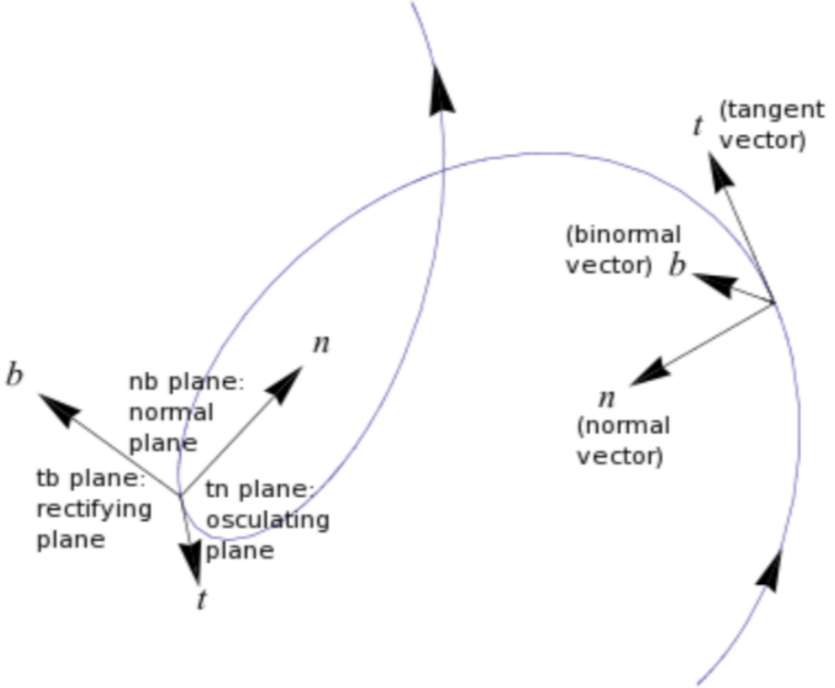
# Recall: Key Characteristics by Using Moving Frames

- For curves: Frenet frame and formul

$$\alpha'(s) \triangleq t(s)$$
$$\alpha''(s) = k(s)n(s)$$
$$t(s) \wedge n(s) = b(s)$$

$$\begin{cases} t' = kn, \\ n' = -kt - \tau b, \\ b' = \tau n \end{cases}$$




**Key: Express the rate change of the frame in the same frame!**
**The coefficients involved are the important characteristics.**

# Fundamental Theorem of the Local Theory of Curves

## Theorem

*Given differentiable functions* $k(s) > 0$ *and* $\tau(s), s \in I$, *there exists a regular parametrized curve* $\alpha : I \to \mathbb{R}^3$ *such that* $s$ *is the arc length,* $k(s)$ *is the curvature, and* $\tau(s)$ *is the torsion of* $\alpha$ *Moreover, any other curve* $\overline{\alpha}$ *satisfying the same conditions differs from* $\alpha$ *by a rigid motion; that is, there exists an orthogonal map* $\rho$ *of* $\mathbb{R}^3$*, with positive determinant, and a vector* $c$ *such that* $\overline{\alpha} = \rho \circ \alpha + c$.

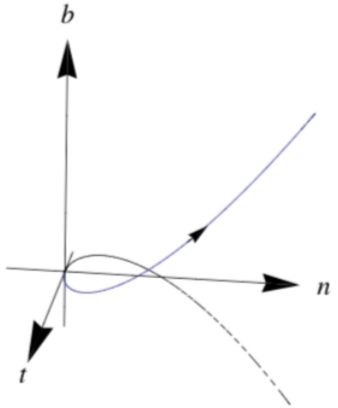
# Curvature and Torsion Locally totally determine a curve: Local Canonical Form

Let us now take the system $0xyz$ in such a way that the origin $0$ agrees with $\alpha(0)$ and that $t = (1,0,0)$, $n = (0,1,0)$, and $b = (0,0,1)$. Under these conditions, $\alpha(s) = (x(s), y(s), z(s))$ is given by

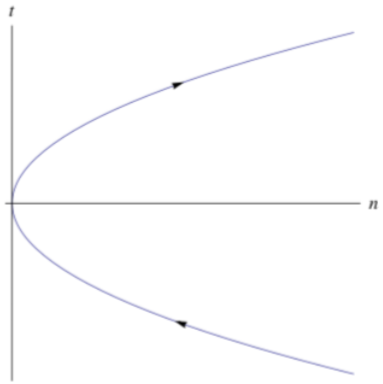
$$\begin{cases} x(s) = s - \dfrac{k^2 s^3}{6} + R_x, \\ y(s) = \dfrac{ks^2}{2} + \dfrac{k' s^3}{6} + R_y, \\ z(s) = -\dfrac{k\tau s^3}{6} + R_z, \end{cases} \tag{1}$$

where $R = (R_x, R_y, R_z)$. The representation (1) is called the *local canonical form* of $\alpha$, in a neighborhood of $s = 0$.

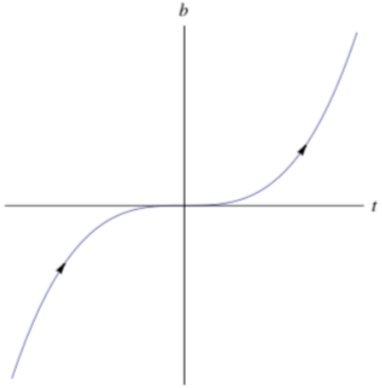
# A Sketch of projections of the trace of $\alpha$, for small $s$, in the $tn$, $tb$, and $nb$ planes:



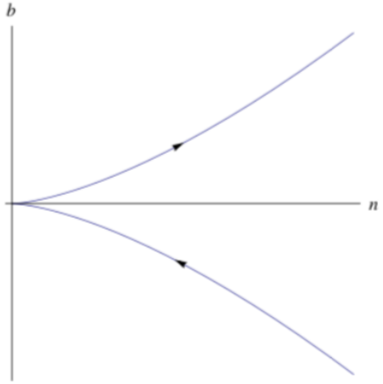

A Curve in $\mathbb{R}^3$




Projection over the plane *tn*




Projection over the plane *tb*




Projection over the plane *nb*

# For Surfaces: Christofel Symbols are basic characteristics!

- Recall:

## Trihedron at a Point of a Surface

$S$ will denote, as usual, a regular, orientable, and oriented surface. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization in the orientation of $S$. It is possible to assign to each point of $\mathbf{x}(U)$ a natural trihedron given by the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and $N$.

By expressing the derivatives of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$, and $N$ in the basis $\{\mathbf{x}_u, \mathbf{x}_v, N\}$, we obtain

$$
\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L_1 N, \\
\mathbf{x}_{uv} &= \Gamma^2_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v + L_2 N, \\
\mathbf{x}_{vu} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v + \overline{L}_2 N, \\
\mathbf{x}_{vv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v + L_3 N, \\
N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\
N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.
\end{aligned}
$$