Lecture 2

Math 178

Nonlinear Data Analytics

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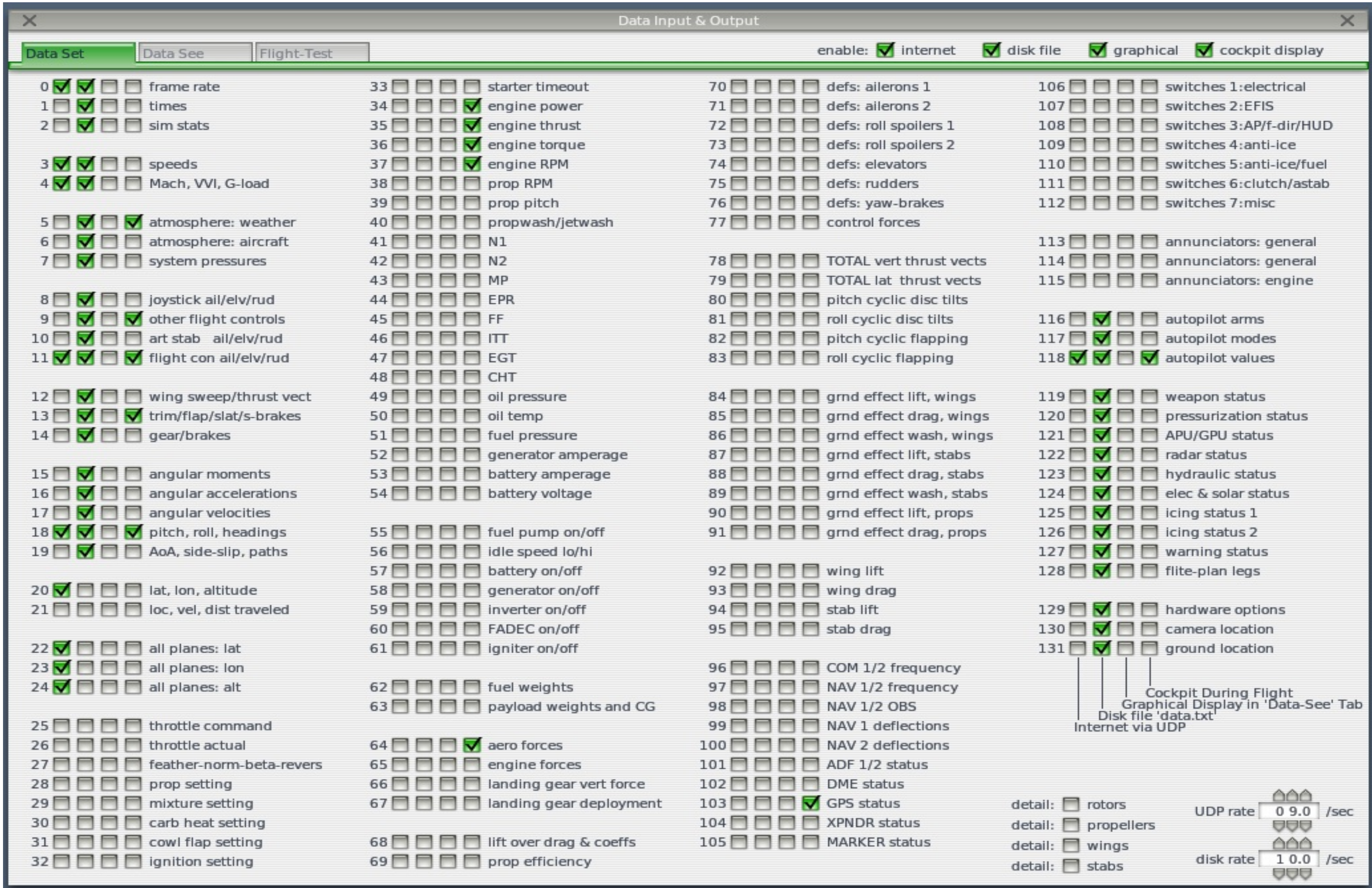
Overview of Lecture 2

- Intro to UAV's data, cell phone's data, and other robotic data
- Their common mathematical representations
- The set of rotations and translations and the set of rigid motions
- Euler angles
- Unit Quaternions

Examples of UAV data



Often you need to remove the metadata before your algorithms access the data

Metadata is "data [information] that provides information about other data".

```
baro_,_inHG | edens,_part | vacum,ratio | vacum,ratio | _elec,ratio | _elec,ratio | _AHRS,ratio | _AHRS,ratio |
_elev,yoke1 | ailrn,yoke1 | ruddr,yoke1 | _vect,_rqst | sweep,_rqst | incid,_rqst | dihed,_rqst | retra,_rqst |
water,jetts | _elev,astab | ailrn,astab | ruddr,astab | _elev,_surf | ailrn,_surf | ruddr,_surf | nwhel,steer |
sweep,1,deg | sweep,2,deg | sweep,h,deg | _vect,ratio | sweep,ratio | incid,ratio | dihed,ratio | retra,ratio |
_trim,_elev | _trim,ailrn | _trim,ruddr | _flap,handl | _flap,postn | _slat,ratio | sbrak,handl | sbrak,postn |
_gear,_0/1 | wbrak,_part | lbrak,_part | rbrak,_part | ____M,_ftlb | ____L,_ftlb | ____N,_ftlb | ____Q,rad/s |
____P,rad/s | ____R,rad/s | pitch,__deg | _roll,__deg | hding,_true | hding,__mag | alpha,__deg | _beta,__deg |
hpath,__deg | vpath,__deg | _slip,__deg | __mag,_comp | mavar,__deg | __lat,__deg | __lon,__deg | __alt,ftmsl |
__alt,ftagl | ___on,runwy | __alt,__ind | __lat,south | __lon,_west | ____X,____m | ____Y,____m | ____Z,____m |
___vX,__m/s | ___vY,__m/s | ___vZ,__m/s | _dist,___ft | _dist,___nm | lat_1,__deg | lat_2,__deg | lat_3,__deg |
lat_4,__deg | lat_5,__deg | lat_6,__deg | lat_7,__deg | lat_8,__deg | lon_1,__deg | lon_2,__deg | lon_3,__deg |
lon_4,__deg | lon_5,__deg | lon_6,__deg | lon_7,__deg | lon_8,__deg | alt_1,ftmsl | alt_2,ftmsl | alt_3,ftmsl |
alt_4,ftmsl | alt_5,ftmsl | alt_6,ftmsl | alt_7,ftmsl | alt_8,ftmsl | throt,_part | throt,_part | mode1,_0123 |
prop1,__set | mixt1,ratio | heat1,ratio | cowl1,__set | igni1,__set | star1,__sec | power,_1,hp | thrst,_1,lb |
trq_1,_ftlb | rpm_1,engin | rpm_1,_prop | ptch1,__deg | pwash,___kt | N1__1,_pcnt | N2__1,_pcnt | MP__1,_inhg |
FF__1,_lb/h | ITT_1,__deg | EGT_1,__deg | CHT_1,__deg | OILP1,__psi | OILT1,__deg | FUEP1,__psi | genr1,__amp |
batt1,__amp | batt1,_volt | pump1,__0/1 | idle1,__0/1 | batt1,__0/1 | genr1,__0/1 | invr1,__0/1 | fadec,__0/1 |
igni1,__0/1 | _fuel,_1_lb | _fuel,_2_lb | _fuel,_3_lb | _fuel,_4_lb | _fuel,_5_lb | _fuel,_6_lb | _fuel,_7_lb |
_fuel,_8_lb | empty,___lb | payld,___lb | _fuel,totlb | jetti,___lb | curnt,___lb | maxim,___lb | ___cg,ftref |
_lift,___lb | _drag,___lb | _side,___lb | norml,___lb | axial,___lb | _side,___lb | _gear,___lb | _gear,___lb |
_gear,___lb | _gear,__rat | _gear,__rat | _gear,__rat | __L/D,ratio | ___cl,total | ___cd,total | __L/D,*etaP |
Peff1,ratio |
   54.90465 |    75.11517 |     0.01821 |     0.00000 |     1.00000 |     1.00000 |     0.14571 |     0.01331
|   0.01331 |     0.00000 |     1.19308 |    17.19308 |   399.54227 |     0.00000 |     0.00000 |
0.00000 |     0.00000 |     0.00000 |     0.00000 | 1643516.00000 |     0.00000 |   -49.53402 |     0.36138 |
0.37173 |     0.36138 |   -57.00274 |     0.42778 |     0.41587 |     0.00055 |     0.00000 |     2.41929 |
-0.00000 |    0.00000 |    30.15301 |    25.20321 |     0.08707 |    36.05242 |     0.00041 |     0.00000 |
0.00000 |    29.07655 |    23.21409 |    23.21408 |     0.94505 |   670.77869 |     0.00044 |    32.14350 |
29.92000 |    0.94505 |     0.01343 |     0.01343 |     0.00000 |     0.00000 |     1.00000 |     1.00000 |
0.00000 |     0.00000 |     0.00000 |     0.00000 |     0.00000 |     0.00000 |     0.00000 |     0.00000 |
```

You may describe the data in a separate file, as in the H-MOG data.

Auto car data and other similar data

Auto car or auto vehicle data also contains accelerometer data and gyroscope data. They may name differently.

Vehicle Data

Deprecated, see explanation

Final Business Group Report 24 November 2014

Editors:

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Abstract

This specification defines a standard for Vehicle Data which might be available in a vehicle. It is designed to be used in conjunction with the Vehicle API Specification.

https://www.w3.org/2014/automotive/data_spec.html

This link contains codes for analyzing auto vehicle data.

Examples of Cell phone data:

Left: one user's accelerometer data
Right: the same user's gyroscope data

	x	y	z	timestamp
0	-1.8520508	3.9649048	9.772293	1.5076E+15
1	-0.8470612	4.6779633	7.757538	1.5076E+15
2	-1.2873383	4.821533	5.6781616	1.5076E+15
3	-2.280365	3.7495575	6.2285156	1.5076E+15
4	-3.0149536	3.4695892	9.791428	1.5076E+15
5	-1.9214325	3.6969147	12.045471	1.5076E+15
6	-0.8566284	3.5437775	12.196213	1.5076E+15
7	-1.3184509	3.728012	10.87059	1.5076E+15
8	-1.9405823	3.620346	8.992233	1.5076E+15
9	-2.0626068	3.373886	7.1186523	1.5076E+15
10	-1.9788666	3.653839	7.154541	1.5076E+15
11	-1.3184509	4.2113647	8.571091	1.5076E+15
12	0.11245728	4.450653	9.085556	1.5076E+15
13	2.7924194	2.1248322	8.140381	1.5076E+15
14	0.4355011	3.8045807	7.7192535	1.5076E+15
15	0.08613586	3.952942	7.8556366	1.5076E+15
16	0.11724854	3.912262	8.444275	1.5076E+15
17	-0.0191498	3.9792633	8.528015	1.5076E+15
18	-0.8949127	3.8811646	9.114258	1.5076E+15
19	-1.524231	3.969696	8.951553	1.5076E+15
20	-1.6725769	4.3980103	8.2456665	1.5076E+15
21	-1.5649109	4.8430786	7.6905365	1.5076E+15
22	-1.0887299	5.027313	7.9369965	1.5076E+15
23	-1.7324066	4.8717804	9.114258	1.5076E+15
24	-1.6462555	4.689926	9.511475	1.5076E+15
25	-1.1868439	4.7401886	9.109482	1.5076E+15

x	y	z	timestamp
0.11390603	-0.1478937	-0.941989	1.5089E+12
0.11016844	-0.186758	-0.955114	1.5089E+12
0.08951247	-0.1771261	-0.9596861	1.5089E+12
0.06113922	-0.1548789	-0.9596196	1.5089E+12
0.03615296	-0.1389114	-0.956291	1.5089E+12
0.0223047	-0.1385345	-0.9565526	1.5089E+12
0.01718072	-0.1503342	-0.9618695	1.5089E+12
0.01438952	-0.1642485	-0.9647654	1.5089E+12
0.01408425	-0.1700877	-0.9647263	1.5089E+12
0.0183536	-0.1697293	-0.965854	1.5089E+12
0.02105704	-0.1659252	-0.9672645	1.5089E+12
0.01959361	-0.1644334	-0.9668481	1.5089E+12
0.01783217	-0.1635982	-0.9665674	1.5089E+12
0.01781886	-0.1601314	-0.967697	1.5089E+12
0.0195245	-0.1543056	-0.9690824	1.5089E+12
0.02081427	-0.1493792	-0.9692894	1.5089E+12
0.01884452	-0.1496343	-0.9684128	1.5089E+12
0.01422125	-0.1514588	-0.9677539	1.5089E+12
0.00634989	-0.1518705	-0.9686434	1.5089E+12
-0.0004003	-0.1499035	-0.9693387	1.5089E+12
-0.0052467	-0.1497015	-0.9687631	1.5089E+12
-0.0053686	-0.1510176	-0.9682288	1.5089E+12
-0.0012975	-0.1540584	-0.9681965	1.5089E+12
0.00321152	-0.1593155	-0.968185	1.5089E+12
0.00429164	-0.1659338	-0.9681739	1.5089E+12
0.00188042	-0.171566	-0.9689082	1.5089E+12
-0.0005783	-0.1726523	-0.9708517	1.5089E+12

Key: How to find data characteristics for each user?

New geometric techniques to find characteristics of Multi-V time series

- First kind of characteristics:
- **Curvature and Torsion**
  - The trick here is that theoretic differential geometry using continuous and differential curves parametrized by arclength to derive curvature and torsion.
  - One needs to cleverly apply them to discrete data.
  - One way to do it is to do a curve fitting. But the curves one obtained may not be parametrized by arclength.
  - Reparamtrizing the curves to arclength usually is too expensive to do.

- How to over come the challenges?

Powerful Multi-V Time Series Features: **Curvature and Torsion Chararistics**

- What if a curve is not parametrized by arclength, how to find curvature and torsion?

The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

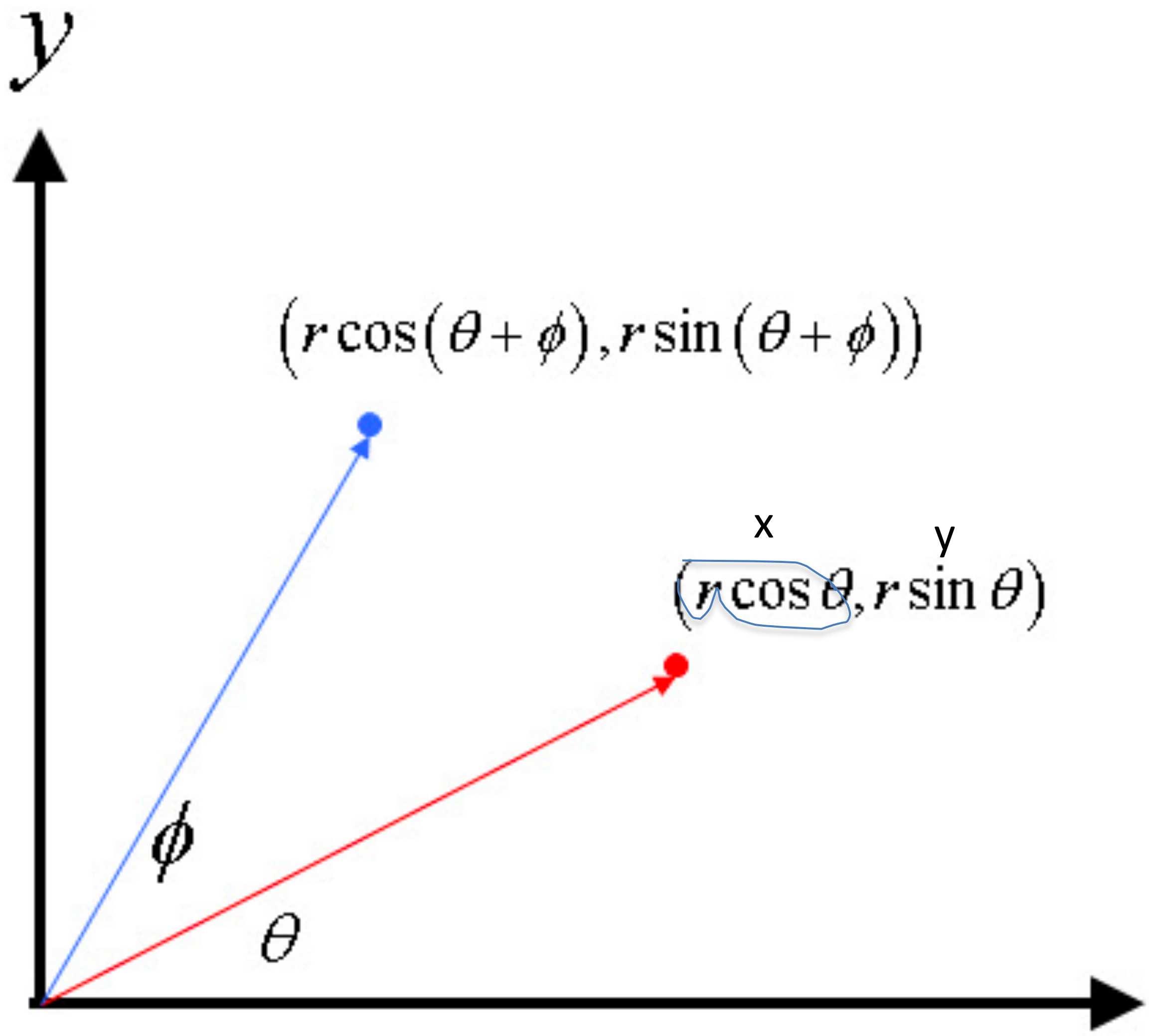
Recall: Rotation is a linear map.
What is the matrix representation of this map?
Why?

2D Rotations

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Could you imagine the math expression of the following figure someone described to you?

2D Rotations



What are some of the properties of a rotation matrix?

- 1.
- 2.
- …

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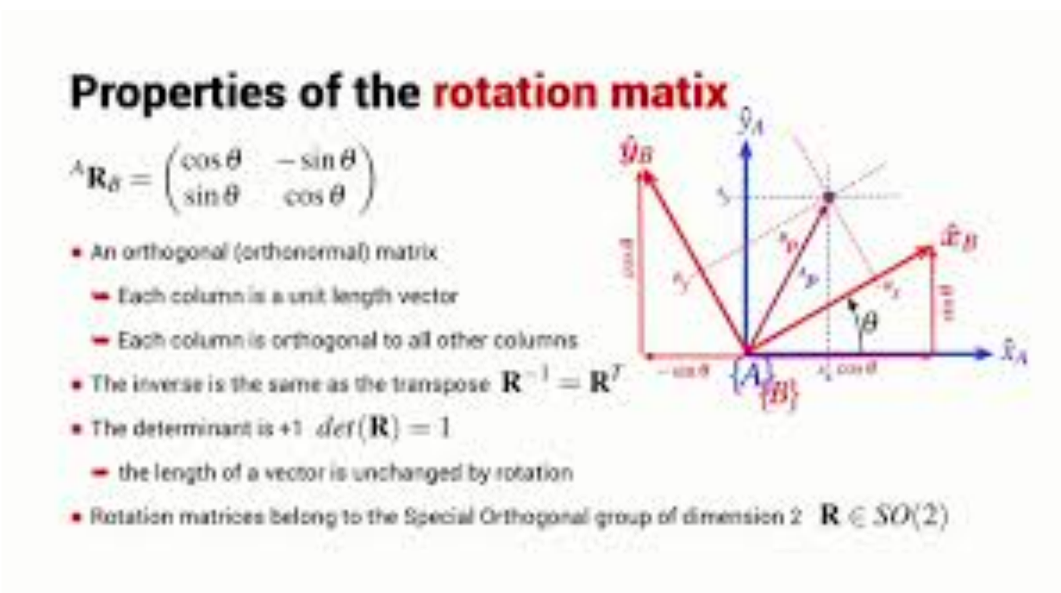
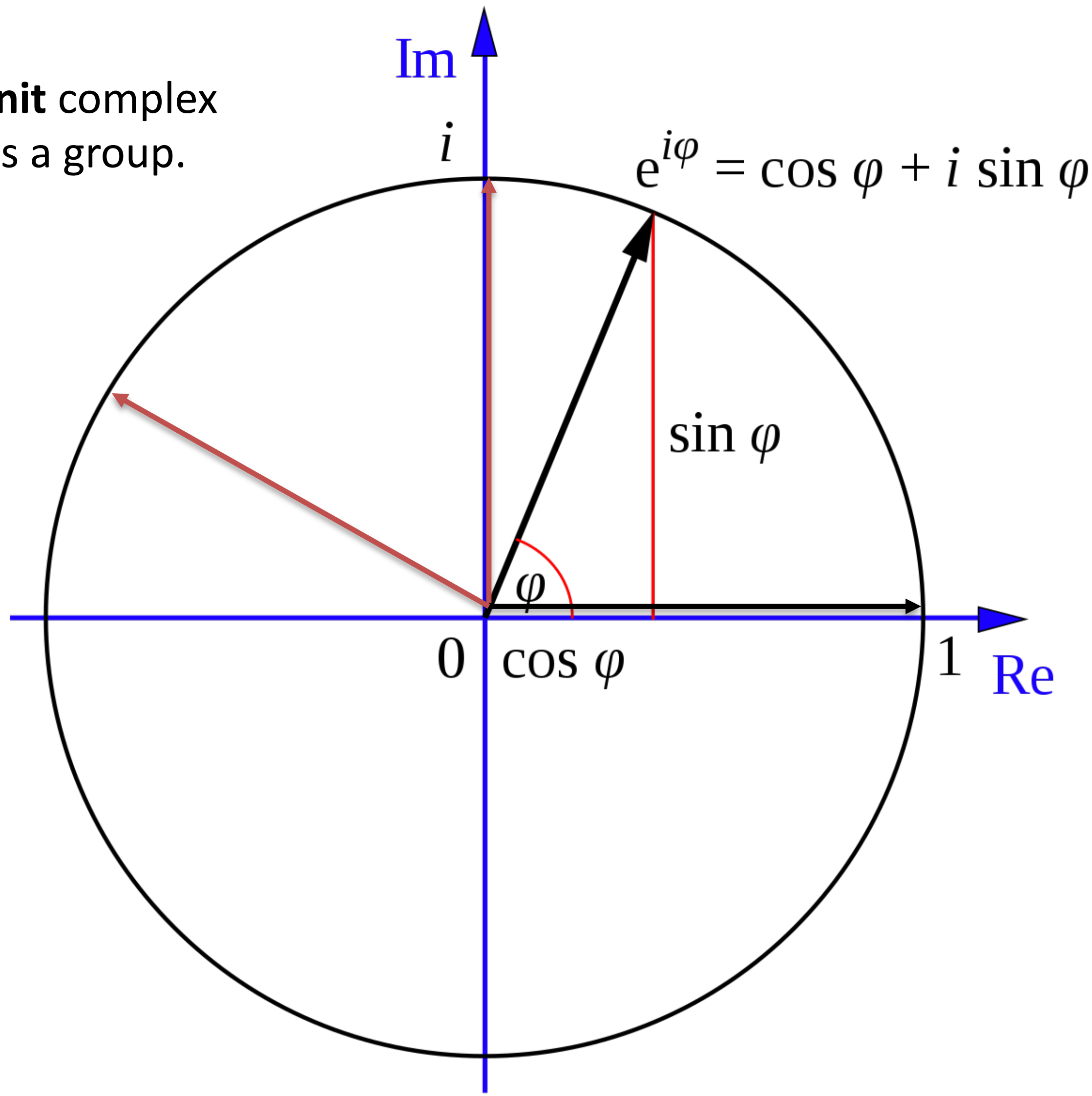
- 1.
- 2.
- …



Euler's Formula

Claim: the set of **unit** complex Numbers (S^1) forms a group.



What is a group?

Definition [edit]

A group is a set, G, together with an operation $\cdot$ (called the *group law* of G) that combines any two elements a and b to form another element, denoted $a \cdot b$ or ab. To qualify as a group, the set and operation, $(G, \cdot)$, must satisfy four requirements known as the *group axioms*:[5]

Closure

For all a, b in G, the result of the operation, $a \cdot b$, is also in G.[b]

Associativity

For all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity element

There exists an element e in G such that, for every element a in G, the equation $e \cdot a = a \cdot e = a$ holds. Such an element is unique (see below), and thus one speaks of *the* identity element.

Inverse element

For each a in G, there exists an element b in G, commonly denoted a^{-1} (or $-a$, if the operation is denoted "+"), such that $a \cdot b = b \cdot a = e$, where e is the identity element.

> The axioms for a group are short and natural... Yet somehow hidden behind these axioms is the monster simple group, a huge and extraordinary mathematical object, which appears to rely on numerous bizarre coincidences to exist. The axioms for groups give no obvious hint that anything like this exists.
>
> Richard Borcherds in *Mathematicians: An Outer View of the Inner World* [4]

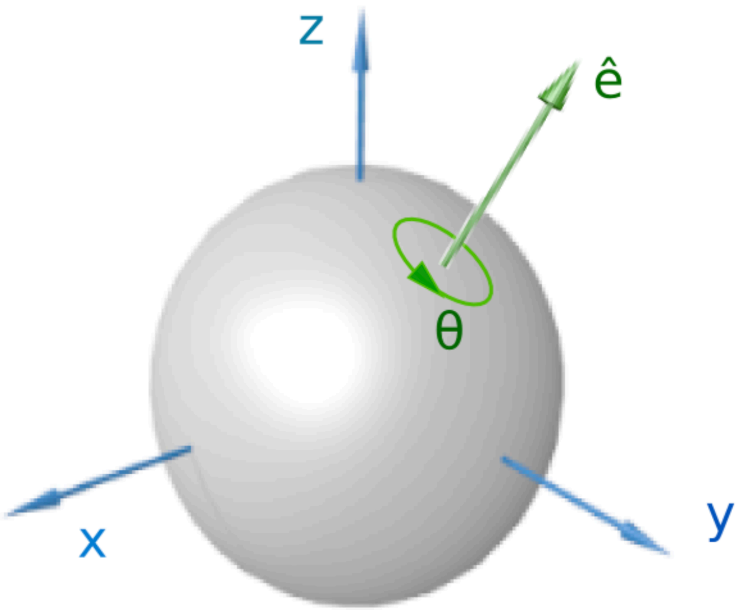
3D Rotations

General rotations in three-dimensional space can be decomposed into three rotations about the three axes.

$$
\begin{aligned}
R &= R_z(\alpha)\, R_y(\beta)\, R_x(\gamma) \\
&= \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{yaw}} \underbrace{\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}}_{\text{pitch}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}}_{\text{roll}} \\
&= \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}
\end{aligned}
$$

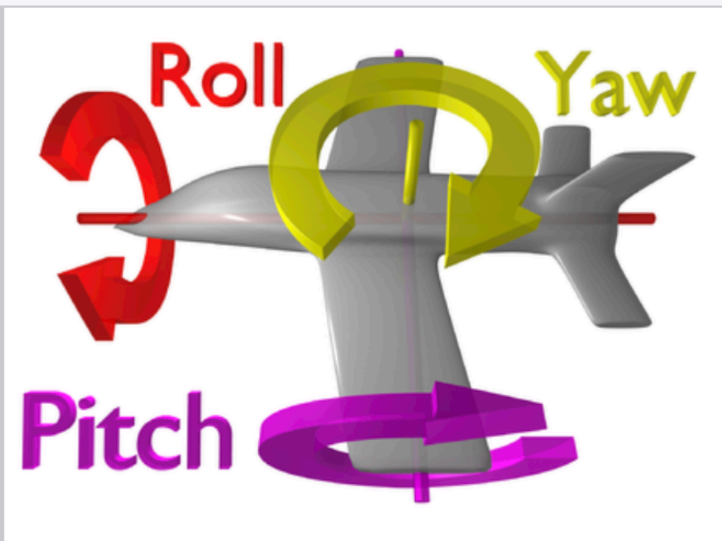
Key: Euler's Rotation Theorem

Euler's rotation theorem states that, in three-dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.

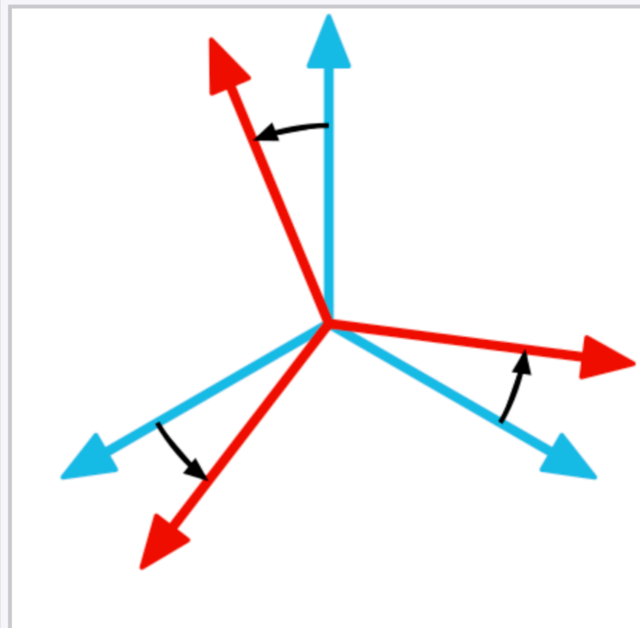


A rotation represented by an Euler axis and angle.

A **rigid body** (also known as a **rigid object** [2]) is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.



The position of a rigid body is determined by the position of its center of mass and by its attitude (at least six parameters in total).[1]



Changing orientation of a rigid body is the same as rotating the axes of a reference frame attached to it.

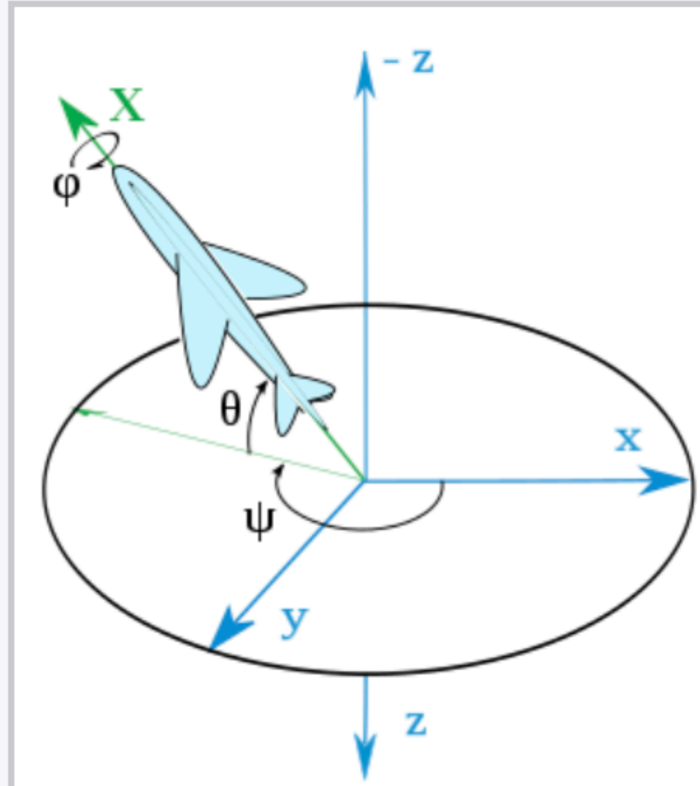
3D Rotations about Axis and Angle

The matrix of a proper rotation R by angle θ around the axis $\mathbf{u} = (u_x, u_y, u_z)$, a unit vector with $u_x^2 + u_y^2 + u_z^2 = 1$, is given by

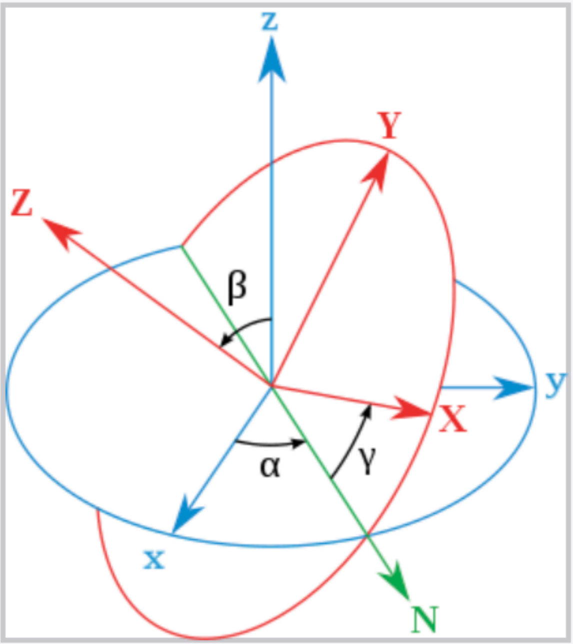
$$R = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

Attitude of a rigid body

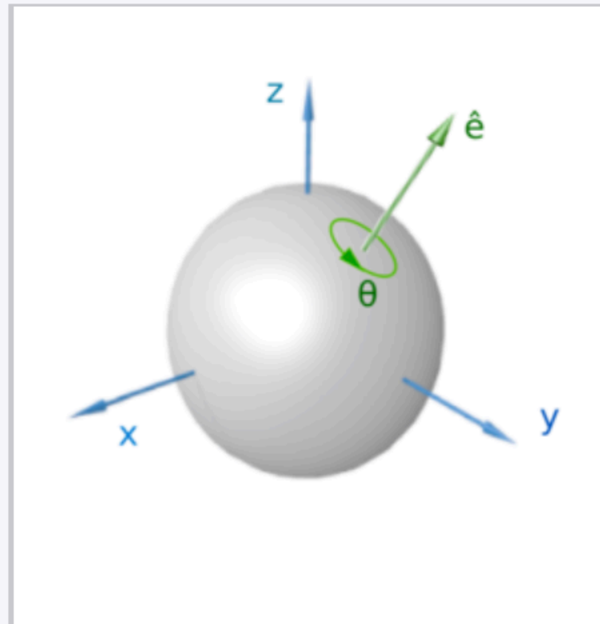
- The attitude of a rigid body is its orientation as described, for example, by the orientation of a frame fixed in the body relative to a fixed reference frame. The attitude is described by *attitude coordinates*, and consists of at least three coordinates.[6] One scheme for orienting a rigid body is based upon body-axes rotation; successive rotations three times about the axes of the body's fixed reference frame, thereby establishing the body's Euler angles. Another is based upon roll, pitch and yaw,[9] although these terms also refer to incremental deviations from the nominal attitude.



The orientation of a rigid body is determined by three angles



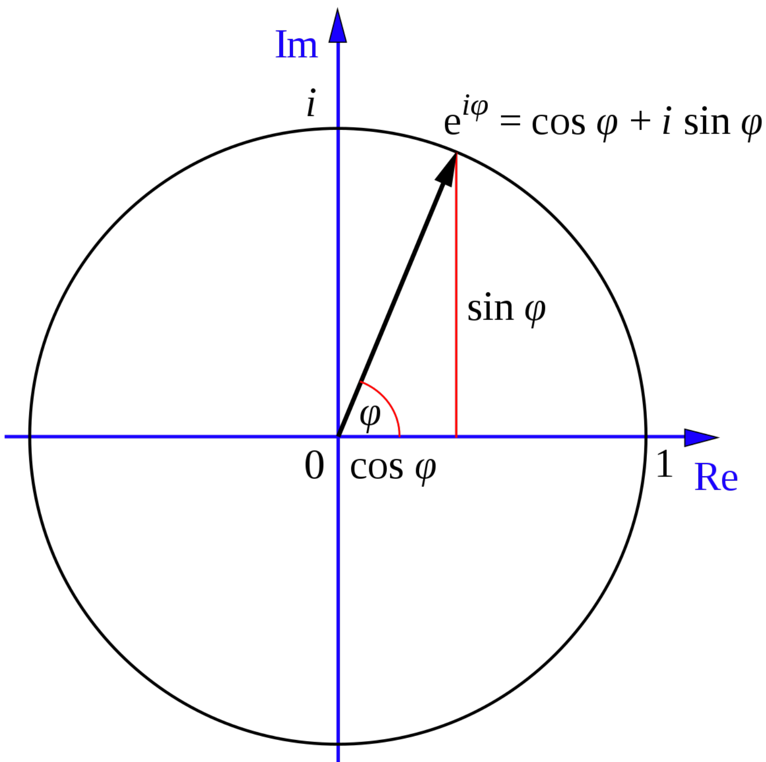
Euler angles, one of the possible ways to describe an orientation

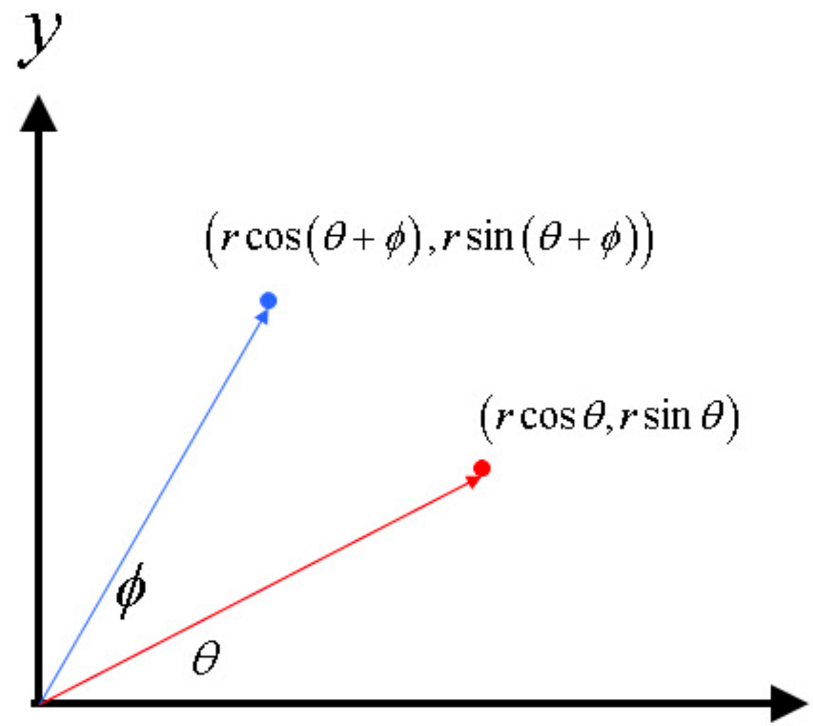


A rotation represented by an Euler axis and angle.

Use unit quaternions to represent a 3D rotation

- Recall, we can use a unit complex number to represent a 2D rotation. Similarly, we can use unit quaternions to represent a 3D rotation.





What are quaternion numbers?

- Double 2 real numbers → complex number
- Double 2 complex numbers → quaternion number
- Double 2 quaternion numbers → Cayley number
- …

What are quaternion numbers?

- Double 2 real numbers a and b: (a, b)→
- $z = a + bi$ complex number
- Double 2 complex numbers (z, w) where $z = a + bi$ and $w = c+di$ →
- $q = a + bi + cj + dk = (a + bi) + (c+di)j$ quaternion number
- Double 2 quaternion numbers→ Cayley number
- …

Unit Quaternions

Quaternions are generally represented in the form:

$$a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

where *a*, *b*, *c*, and *d* are real numbers, and ***i***, ***j***, and ***k*** are the fundamental *quaternion units*.

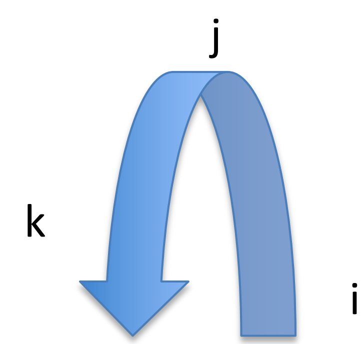
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\begin{aligned} ij &= k, & ji &= -k, \\ jk &= i, & kj &= -i, \\ ki &= j, & ik &= -j. \end{aligned}$$

Quaternion multiplication

×	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1

×	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1



Hamilton describes a quaternion $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, as consisting of a scalar part and a vector part. The quaternion $b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ is called the *vector part* (sometimes *imaginary part*) of q, and a is the *scalar part* (sometimes *real part*) of q. A quaternion that equals its real part (that is, its vector part is zero) is called a *scalar*, and is identified with the corresponding real number. That is, the real numbers are a subset of the quaternions. A quaternion that equals its vector part is called a *vector quaternion*.

Closed under Sum and Scalar Multiplication

The set of quaternions is made a 4 dimensional vector space over the real numbers, with $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as a basis, by the componentwise addition

$$
\begin{aligned}
&(a_1 + b_1\,\mathbf{i} + c_1\,\mathbf{j} + d_1\,\mathbf{k}) + (a_2 + b_2\,\mathbf{i} + c_2\,\mathbf{j} + d_2\,\mathbf{k}) \\
&= (a_1 + a_2) + (b_1 + b_2)\,\mathbf{i} + (c_1 + c_2)\,\mathbf{j} + (d_1 + d_2)\,\mathbf{k}
\end{aligned}
$$

and the componentwise scalar multiplication

$$\lambda(a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}) = \lambda a + (\lambda b)\,\mathbf{i} + (\lambda c)\,\mathbf{j} + (\lambda d)\,\mathbf{k}\,.$$

Hamilton Product

For two elements $a_1 + b_1 i + c_1 j + d_1 k$ and $a_2 + b_2 i + c_2 j + d_2 k$, their product, called the **Hamilton product** $(a_1 + b_1 i + c_1 j + d_1 k)$ $(a_2 + b_2 i + c_2 j + d_2 k)$, is determined by the products of the basis elements and the distributive law. The distributive law makes it possible to expand the product so that it is a sum of products of basis elements. This gives the following expression:

$$\begin{aligned}
&a_1 a_2 + a_1 b_2 i + a_1 c_2 j + a_1 d_2 k \\
&+ b_1 a_2 i + b_1 b_2 i^2 + b_1 c_2 ij + b_1 d_2 ik \\
&+ c_1 a_2 j + c_1 b_2 ji + c_1 c_2 j^2 + c_1 d_2 jk \\
&+ d_1 a_2 k + d_1 b_2 ki + d_1 c_2 kj + d_1 d_2 k^2 .
\end{aligned}$$

Now the basis elements can be multiplied using the rules given above to get:[7]

$$\begin{aligned}
&a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\
&+ (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i \\
&+ (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) j \\
&+ (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) k.
\end{aligned}$$

So closed under the product!

The product of two rotation quaternions[24] will be equivalent to the rotation $a_2 + b_2 i + c_2 j + d_2 k$ followed by the rotation $a_1 + b_1 i + c_1 j + d_1 k$.

Key theorem:
The quaternions form a division algebra.

- The real quaternion 1 is the identity element.
- The real quaternions commute with all other quaternions, that is $aq = qa$ for every quaternion q and every real quaternion a. In algebraic terminology this is to say that the field of real quaternions are the center of this quaternion algebra.
- The product is first given for the basis elements (see next subsection), and then extended to all quaternions by using the distributive property and the center property of the real quaternions. The Hamilton product is not commutative, but associative, thus the quaternions form an associative algebra over the reals.
- Additionally, every nonzero quaternion has an inverse with respect to the Hamilton product:

$$(a + b\,\mathbf{i} + c\,\mathbf{j} + d\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}\,(a - b\,\mathbf{i} - c\,\mathbf{j} - d\mathbf{k}).$$

Scalar and Vector parts of quaternions

If a quaternion is divided up into a scalar part and a vector part, i.e.

$$q = (r,\ \vec{v}),\ q \in \mathbf{H},\ r \in \mathbf{R},\ \vec{v} \in \mathbf{R}^3$$

then the formulas for addition and multiplication are:

$$(r_1,\ \vec{v}_1) + (r_2,\ \vec{v}_2) = (r_1 + r_2,\ \vec{v}_1 + \vec{v}_2)$$
$$(r_1,\ \vec{v}_1)(r_2,\ \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2,\ r_1 \vec{v}_2 + r_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$

where "·" is the dot product and "×" is the cross product.

Homework: Check the above holds!

Conjugation and the norm

Conjugation of quaternions is analogous to conjugation of complex numbers and to transposition (also known as reversal) of elements of Clifford algebras. To define it, let $q = a + bi + cj + dk$ be a quaternion. The **conjugate** of q is the quaternion $q^* = a - bi - cj - dk$. It is denoted by q^*, $\overline{q}$,[7] q^t, or $\tilde{q}$. Conjugation is an involution, meaning that it is its own inverse, so conjugating an element twice returns the original element. The conjugate of a product of two quaternions is the product of the conjugates *in the reverse order*. That is, if p and q are quaternions, then $(pq)^* = q^* p^*$, not $p^* q^*$.

The square root of the product of a quaternion with its conjugate is called its **norm**

$$\|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

This is always a non-negative real number, and it is the same as the Euclidean norm on **H** considered as the vector space $\mathbf{R}^4$.

$$\|\alpha q\| = \|\alpha\|\|q\|.$$

This is a special case of the fact that the norm is *multiplicative*, meaning that

$$\|pq\| = \|p\|\|q\|.$$

for any two quaternions *p* and *q*. Multiplicativity is a consequence of the formula for the conjugate of a product. Alternatively it follows from the identity

$$\det\begin{pmatrix} a+ib & id+c \\ id-c & a-ib \end{pmatrix} = a^2+b^2+c^2+d^2,$$

(where *i* denotes the usual imaginary unit) and hence from the multiplicative property of determinants of square matrices.

This norm makes it possible to define the **distance** *d*(*p*, *q*) between *p* and *q* as the norm of their difference:

$$d(p,q) = \|p-q\|.$$

This makes **H** a metric space. Addition and multiplication are continuous in the metric topology. Indeed, for any scalar, positive *a* it holds

$$\|(p+ap_1+q+aq_1)-(p+q)\| = a\|p_1+q_1\|.$$

Continuity follows from taking *a* to zero in the limit. Continuity for multiplication holds similarly.

Unit quaternion and Reciprocal

A **unit quaternion** is a quaternion of norm one. Dividing a non-zero quaternion q by its norm produces a unit quaternion $\mathbf{U}q$ called the **versor** of q:

$$\mathbf{U}q = \frac{q}{\|q\|}.$$

Every quaternion has a polar decomposition $q = \|q\| \cdot \mathbf{U}q$.

Using conjugation and the norm makes it possible to define the reciprocal of a non-zero quaternion. The product of a quaternion with its reciprocal should equal 1, and the considerations above imply that the product of q and $q^* / \|q\|^2$ (in either order) is 1. So the **reciprocal** of q is defined to be

$$q^{-1} = \frac{q^*}{\|q\|^2}.$$

This makes it possible to divide two quaternions p and q in two different ways (when q is non-zero). That is, their quotient can be either pq^{-1} or $q^{-1}p$. The notation $\frac{p}{q}$ is ambiguous because it does not specify whether q divides on the left or the right.

Division Algebra

The set **H** of all quaternions is a vector space over the real numbers with dimension 4. (In comparison, the real numbers have dimension 1, the complex numbers have dimension 2, and the octonions have dimension 8.) Multiplication of quaternions is associative and distributes over vector addition, but it is not commutative. Therefore, the quaternions **H** are a non-commutative associative algebra over the real numbers. Even though **H** contains copies of the complex numbers, it is not an associative algebra over the complex numbers.

Because it is possible to divide quaternions,
they form a division algebra. This is a structure similar to a field except for the non-commutativity of multiplication. Finite-dimensional associative division algebras over the real numbers are very rare. The Frobenius theorem states that there are exactly three: **R**, **C**, and **H**. The norm makes the quaternions into a normed algebra, and normed division algebras over the reals are also very rare: Hurwitz's theorem says that there are only four: **R**, **C**, **H**, and **O** (the octonions). The quaternions are also an example of a composition algebra and of a unital Banach algebra.

Unit quaternion matrix representation

Euler-Rodrigues Formula

A rotation about the origin is represented by four real numbers, a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = 1$. When the rotation is applied, a point at position $\mathbf{x}$ rotates to its new position

$$\mathbf{x}' = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & a^2 + c^2 - b^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & a^2 + d^2 - b^2 - c^2 \end{bmatrix} \mathbf{x}$$