Lecture 6

Math 178

Nonlinear Data Analytics

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Today

- Cell phone data in quaternion representations
- Gyroscope measurement models
- Accelerometer measurement models
  Choosing the state and modeling its dynamics
- Model for the prior and probabilistic models (only time permits)

Cell phone data in quaternion representations

Recently, microelectromechanical system (MEMS such as with cell phones) inertial sensors (3D accelerometers and 3D gyroscopes) have become widely available due to their small size and low cost.

Inertial sensor measurements are obtained at high sampling rates and can be integrated to obtain position and orientation information. These estimates are accurate on a short time scale, but suffer from integration drift over longer time scales.

(a) Left bottom: an Xsens MTx IMU [156]. Left top: a Trivisio Colibri Wireless IMU [148]. Right: a Samsung Galaxy S4 mini smartphone.

(b) A Samsung gear VR.[1]

(c) A Wii controller containing an accelerometer and a MotionPlus expansion device containing a gyroscope.[2]

Figure 1.1: Examples of devices containing inertial sensors.

Microelectromechanical System (MEMS)

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Figure 1.1: Examples of devices containing inertial sensors.

(a) Back pain therapy using serious gaming. IMUs are placed on the chest-bone and on the pelvis to estimate the movement of the upper body and pelvis. This movement is used to control a robot in the game and promotes movements to reduce back pain.

(b) Actor Seth MacFarlane wearing 17 IMUs to capture his motion and animate the teddy bear Ted. The IMUs are placed on different body segments and provide information about the relative position and orientation of each of these segments.

Figure 1.2: Examples illustrating the use of multiple IMUs placed on the human body to estimate its pose. Courtesy of Xsens Technologies.

(a) Inertial sensors are used in combination with GNSS measurements to estimate the position of the cars in a challenge on cooperative and autonomous driving.

(b) Due to their small size and low weight, IMUs can be used to estimate the orientation for control of an unmanned helicopter.

Figure 1.3: Examples illustrating the use of a single IMU placed on a moving object to estimate its pose. Courtesy of Xsens Technologies.

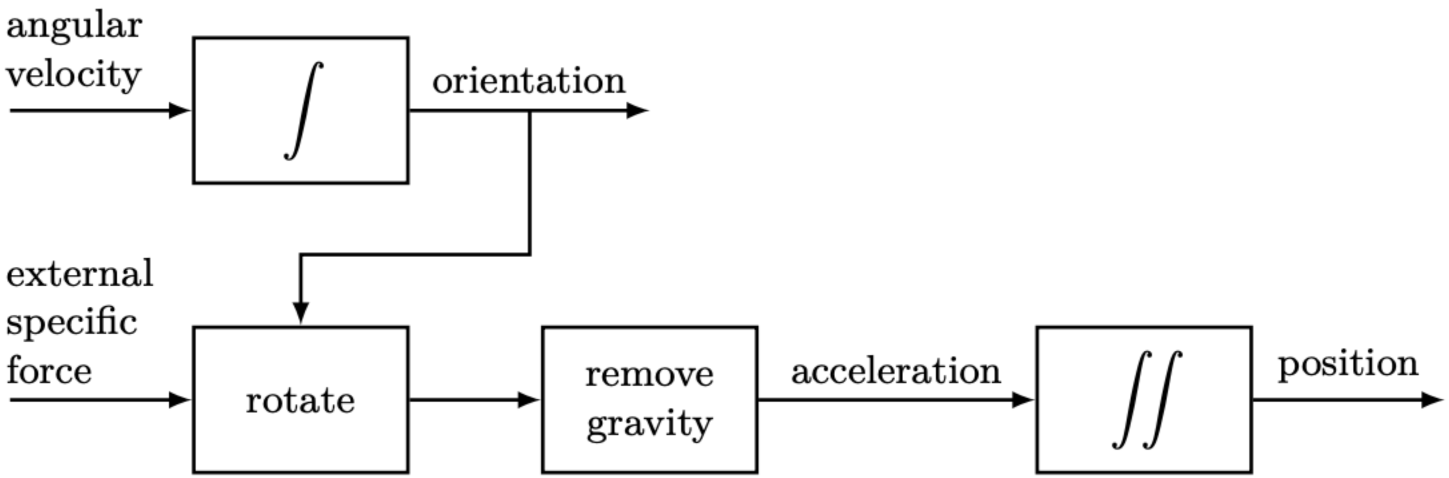


Figure 1.4: Schematic illustration of dead-reckoning, where the accelerometer measurements (external specific force) and the gyroscope measurements (angular velocity) are integrated to position and orientation.

Recall: A unit quaternion represent a rotation How to find out this rotation’s matrix representation?

- Work out details with the students.

Recall: Given a unit quaternion q. Recall we can define a rotation Rq. Now we want to find the matrix representation of Rq. So we let Rq acts on the basis 1, i, j, k.
Rq(p) = qpq*. (recall: qp not equal to pq. E.g. ij = k, and ji = -k, so ij is not equal to ji.)

- Rq(1) = q1q* = 1
- Rq(i) = qiq* = (a+bi+cj+dk)i(a-bi-cj-dk) = work out by students
- Rq(j) = qjq* = (a+bi+cj+dk)j(a-bi-cj-dk) = work out by students
- Rq(k) = qkq* = (a+bi+cj+dk)k(a-bi-cj-dk) = work out by students

Say q = a+bi+cj+dk = q_1+ q_ii+q_jj+q_kk

$$\mathbf{R} = \begin{bmatrix} 1-2s(q_j^2+q_k^2) & 2s(q_iq_j-q_kq_r) & 2s(q_iq_k+q_jq_r) \\ 2s(q_iq_j+q_kq_r) & 1-2s(q_i^2+q_k^2) & 2s(q_jq_k-q_iq_r) \\ 2s(q_iq_k-q_jq_r) & 2s(q_jq_k+q_iq_r) & 1-2s(q_i^2+q_j^2) \end{bmatrix}$$

Here $s = ||q||^{-2}$ and if q is a unit quaternion, $s = 1$.

Recovering the axis-angle representation

The axis a and angle θ corresponding to a quaternion $\mathbf{q} = q_r + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$ can be extracted via:

$$(a_x, a_y, a_z) = \frac{(q_i, q_j, q_k)}{\sqrt{q_i^2 + q_j^2 + q_k^2}}$$

$$\theta = 2\,\mathrm{atan2}\left(\sqrt{q_i^2 + q_j^2 + q_k^2}, q_r\right),$$

where $\mathrm{atan2}$ is the two-argument arctangent. Care should be taken when the quaternion approaches a real quaternion, since due to degeneracy the axis of an identity rotation is not well-defined.

Extension of Euler's formula

A Euclidean vector such as $(2, 3, 4)$ or (a_x, a_y, a_z) can be rewritten as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ or $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors representing the three Cartesian axes. A rotation through an angle of θ around the axis defined by a unit vector

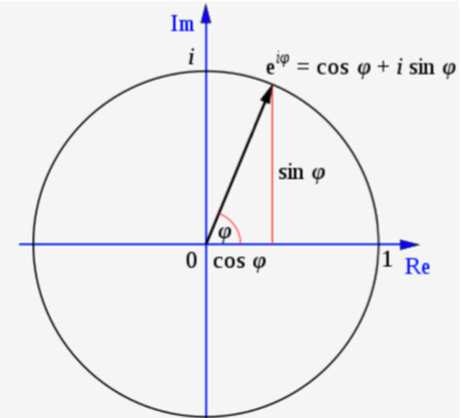
$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Recall: Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Differentiation with respect to the rotation quaternion

The rotated quaternion $\mathbf{p}'=\mathbf{q}\,\mathbf{p}\,\mathbf{q}^*$ needs to be differentiated with respect to the rotating quaternion $\mathbf{q}$, when the rotation is estimated from numerical optimization. The estimation of rotation angle is an essential procedure in 3D object registration or camera calibration. The derivative can be represented using the Matrix Calculus notation.

$$\frac{\partial \mathbf{p}'}{\partial \mathbf{q}} \equiv \left[\frac{\partial \mathbf{p}'}{\partial q_0}, \frac{\partial \mathbf{p}'}{\partial q_x}, \frac{\partial \mathbf{p}'}{\partial q_y}, \frac{\partial \mathbf{p}'}{\partial q_z}\right]$$

$$= \left[\mathbf{pq} - (\mathbf{pq})^*, (\mathbf{pqi})^* - \mathbf{pqi}, (\mathbf{pqj})^* - \mathbf{pqj}, (\mathbf{pqk})^* - \mathbf{pqk}\right].$$

Types of Matrix Derivatives

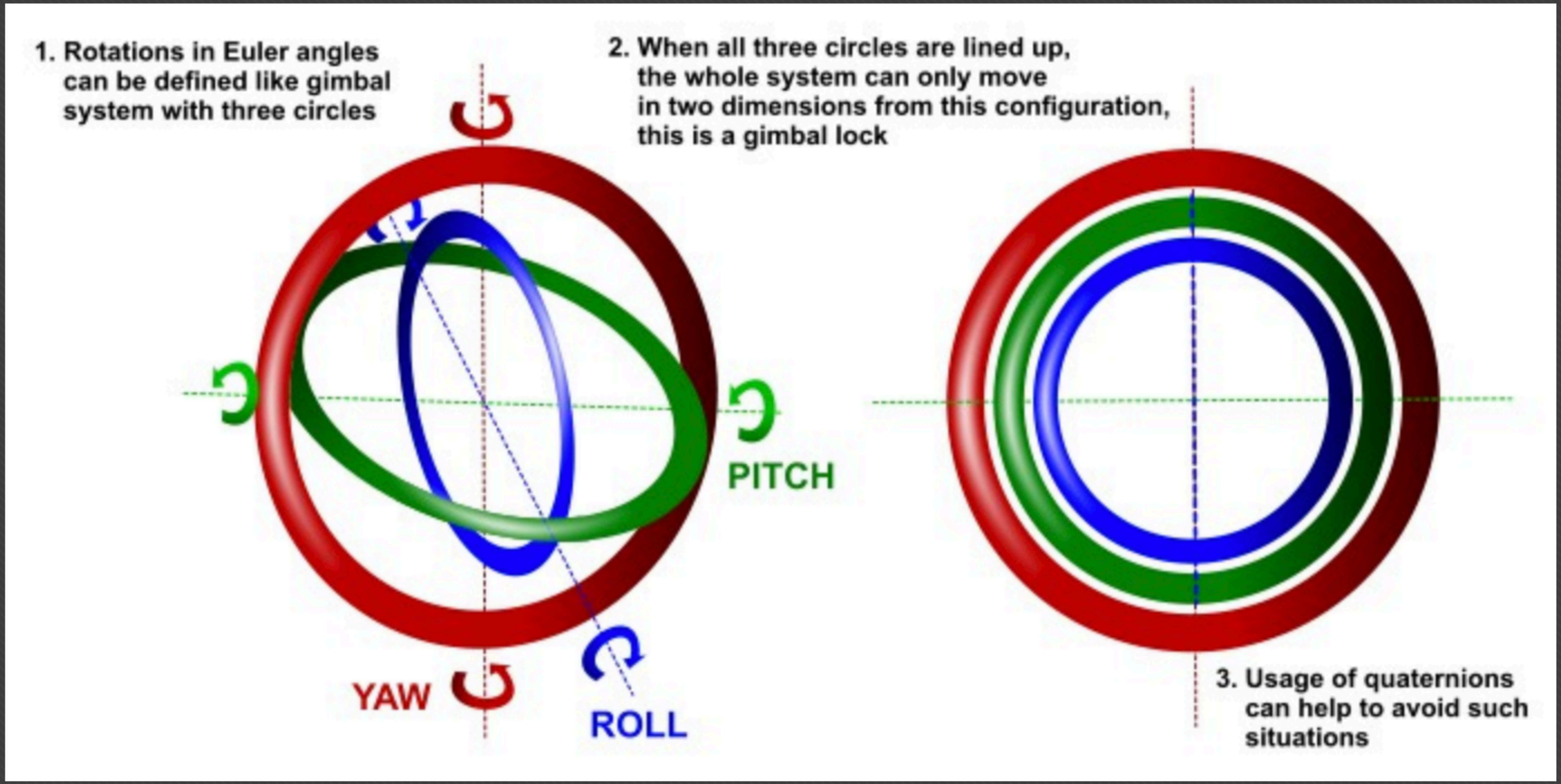
Types	Scalar	Vector	Matrix
Scalar	$\frac{\partial y}{\partial x}$	$\frac{\partial \mathbf{y}}{\partial x}$	$\frac{\partial \mathbf{Y}}{\partial x}$
Vector	$\frac{\partial y}{\partial \mathbf{x}}$	$\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ work out by students	
Matrix	$\frac{\partial y}{\partial \mathbf{X}}$		

Will be covered in Math 173, Advanced linear Algebra.

Quaternions for Quest3D

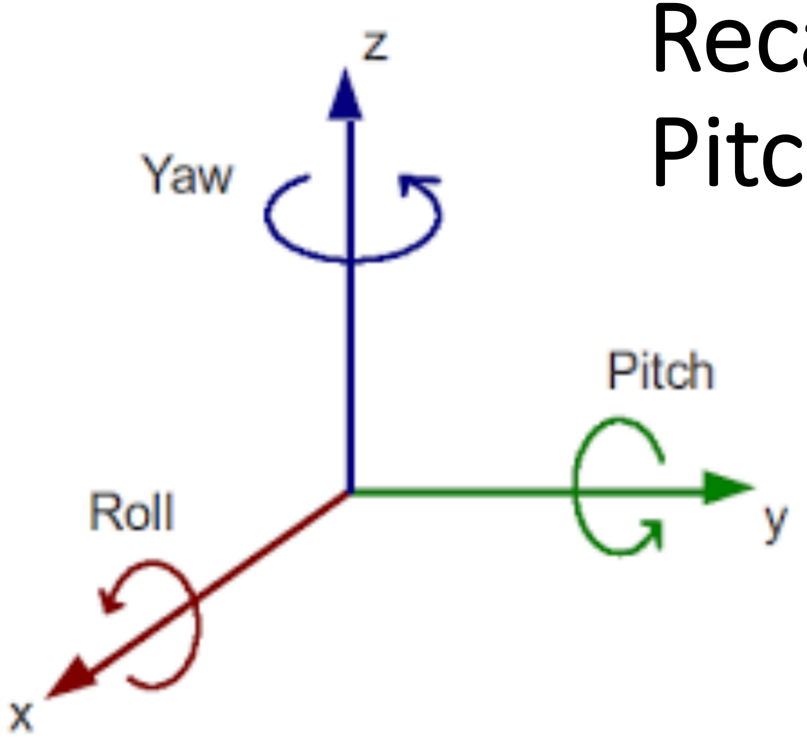
Quaternion is a data type suitable for defining object orientation and rotations. Quaternions are easier to work with than matrices and using quaternions helps to avoid gimbal lock problem like in case of Euler angles usage.

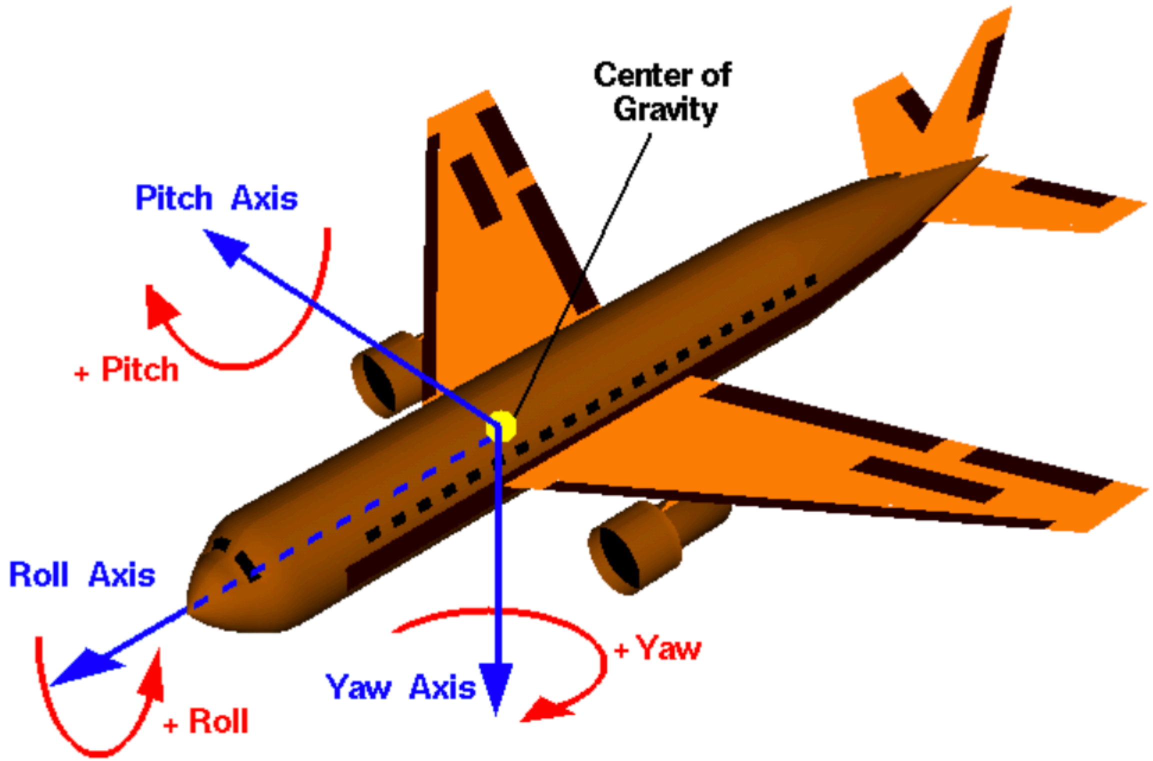
Tasks like smooth interpolation between three-dimensional rotations and building rotation by vector are fairly simpler to solve with quaternions than with Euler angles or matrices. Industrial grade inertial trackers and many other orientation sensors can return rotational data in quaternion form, also to avoid gimbal lock problem, and make such values easier to filter by interpolation.



Gimbal lock problem

Recall:
Pitch, Roll, and Yaw





Unit Quaternion and Euler Angles

- Each unit quaternion can be associated to a rotation around an axis.

$$\mathbf{q}_0 = \mathbf{q}_w = \cos(\alpha/2)$$
$$\mathbf{q}_1 = \mathbf{q}_x = \sin(\alpha/2)\cos(\beta_x)$$
$$\mathbf{q}_2 = \mathbf{q}_y = \sin(\alpha/2)\cos(\beta_y)$$
$$\mathbf{q}_3 = \mathbf{q}_z = \sin(\alpha/2)\cos(\beta_z)$$

$$\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} q_w & q_x & q_y & q_z \end{bmatrix}^T$$
$$|\mathbf{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_w^2 + q_x^2 + q_y^2 + q_z^2 = 1$$

We also can add a scalar to a vector and find inverse of a vector!

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + \vec{v}.$$

$$a + \vec{v} = (a, \vec{0}) + (0, \vec{v}).$$

$$(s + \vec{v})^{-1} = \frac{(s + \vec{v})^*}{\|s + \vec{v}\|^2} = \frac{s - \vec{v}}{s^2 + \|\vec{v}\|^2}$$

Now we can multiply two vectors in R^3 and in R^4 !
First define it in R^3
by viewing them as pure imaginary quaternions

We can express quaternion multiplication in the modern language of vector cross and dot products (which were actually inspired by the quaternions in the first place [6]). When multiplying the vector/imaginary parts, in place of the rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ we have the quaternion multiplication rule:

$$\vec{v}\vec{w} = \vec{v} \times \vec{w} - \vec{v} \cdot \vec{w},$$

where:

- $\vec{v}\vec{w}$ is the resulting quaternion,
- $\vec{v} \times \vec{w}$ is vector cross product (a vector),
- $\vec{v} \cdot \vec{w}$ is vector scalar product (a scalar).

Quaternion multiplication is noncommutative (because of the cross product, which anti-commutes), while scalar–scalar and scalar–vector multiplications commute. From these rules it follows immediately that (see details):

$$(s+\vec{v})(t+\vec{w}) = (st - \vec{v}\cdot\vec{w}) + (s\vec{w} + t\vec{v} + \vec{v}\times\vec{w}).$$

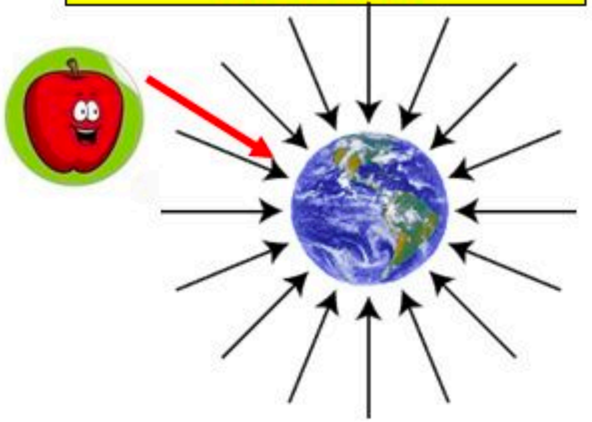
Complication: There are several different Fields, Poles, and Frames

- **Gravitational fields**
- **Electric fields**
- **Magnetic fields**
- **Magnetic pole**
- **Geographic pole**
- **Heliocentric frame**
- **Geocentric frame**

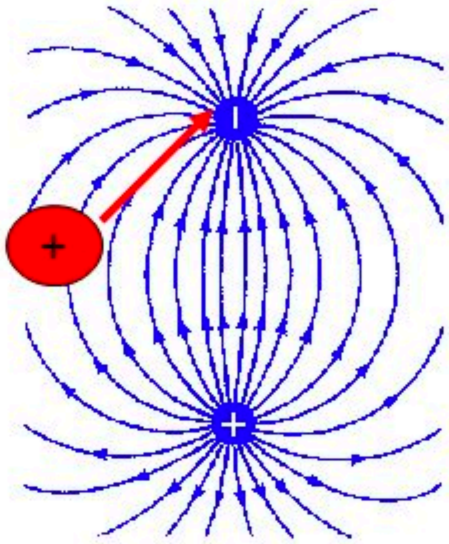
Examples of **gravitational fields**:

- Things falling to Earth
- The Earth orbiting the sun

Test object: Anything with mass



Test object: charged object (+)



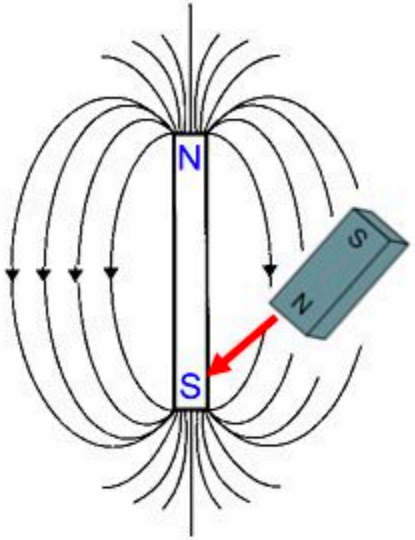
Examples of **electric fields**:

- Static electricity
- Lightning

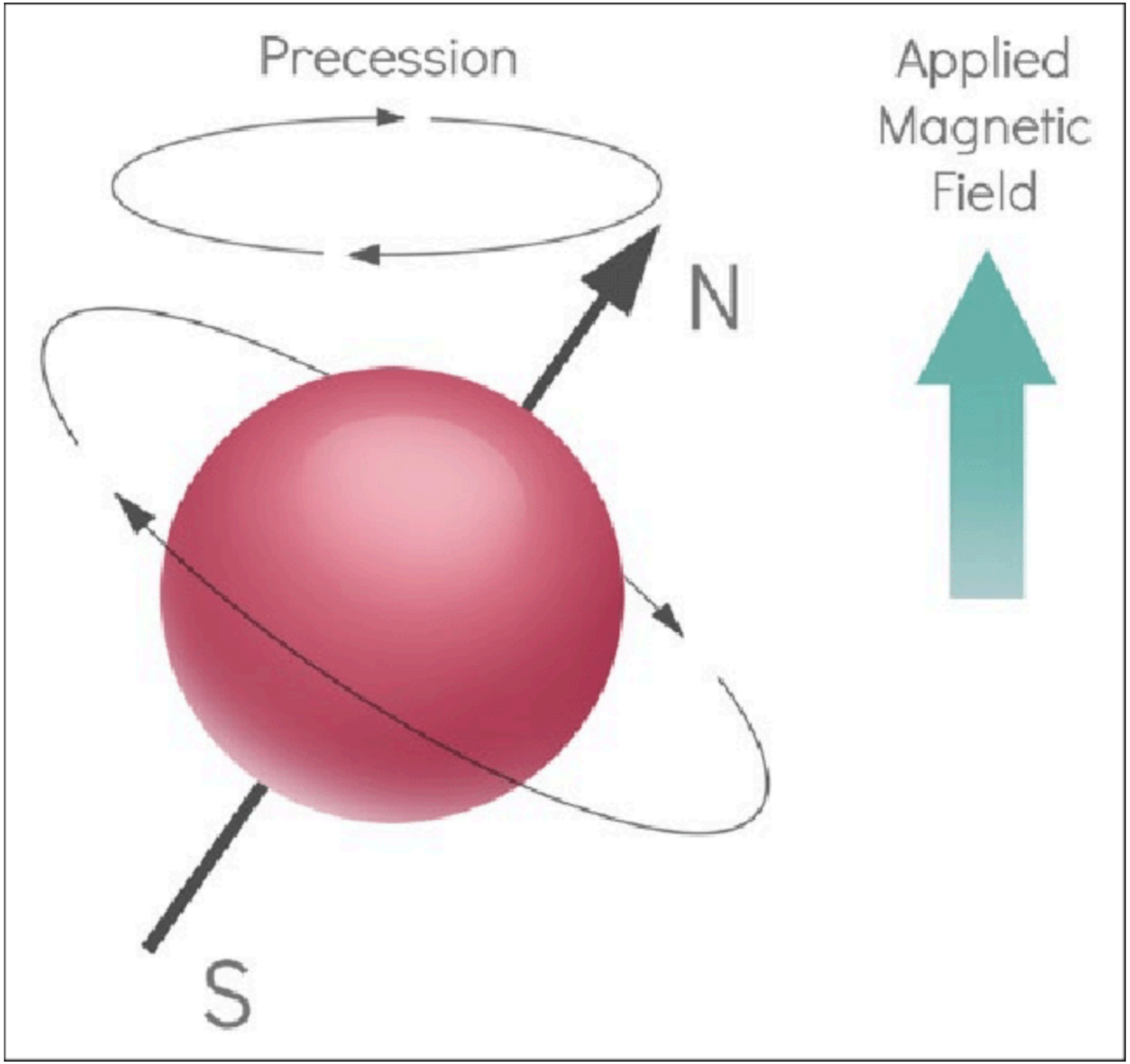
Examples of **magnetic fields**:

- Magnets
- Using a compass

Test object: magnet

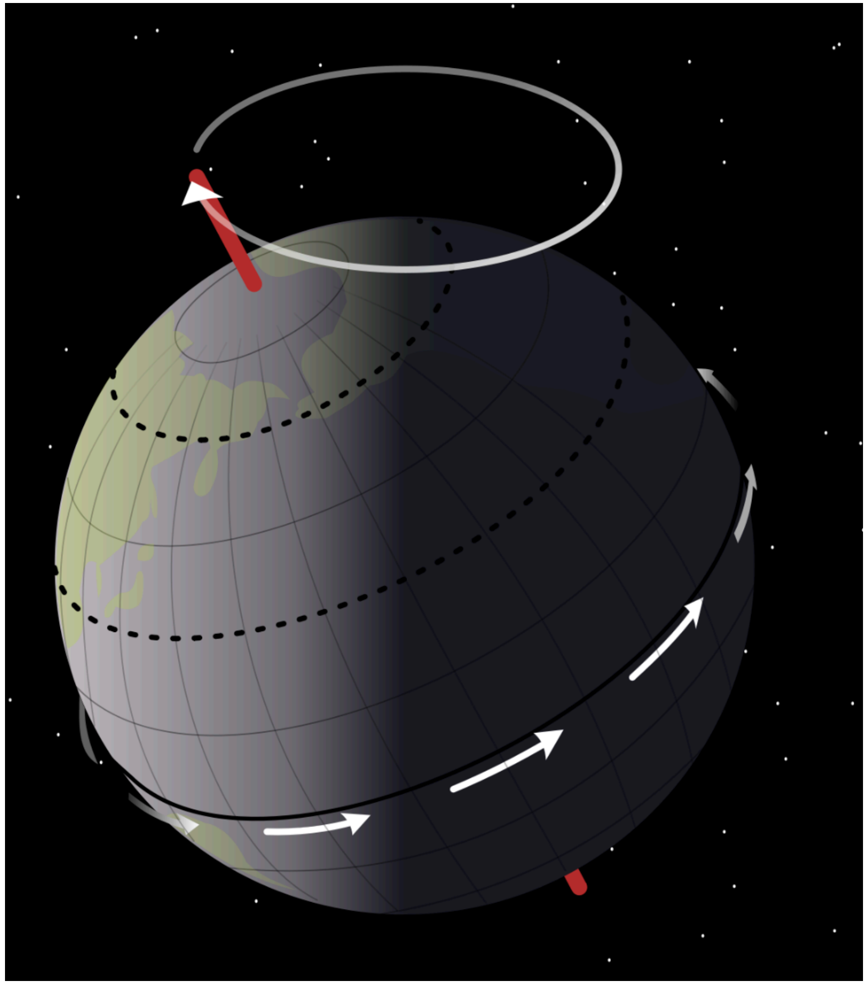


- A field line (or vector diagrams) tells us the direction and strength of a field
  - The direction of a field is determined by the direction a test object will move

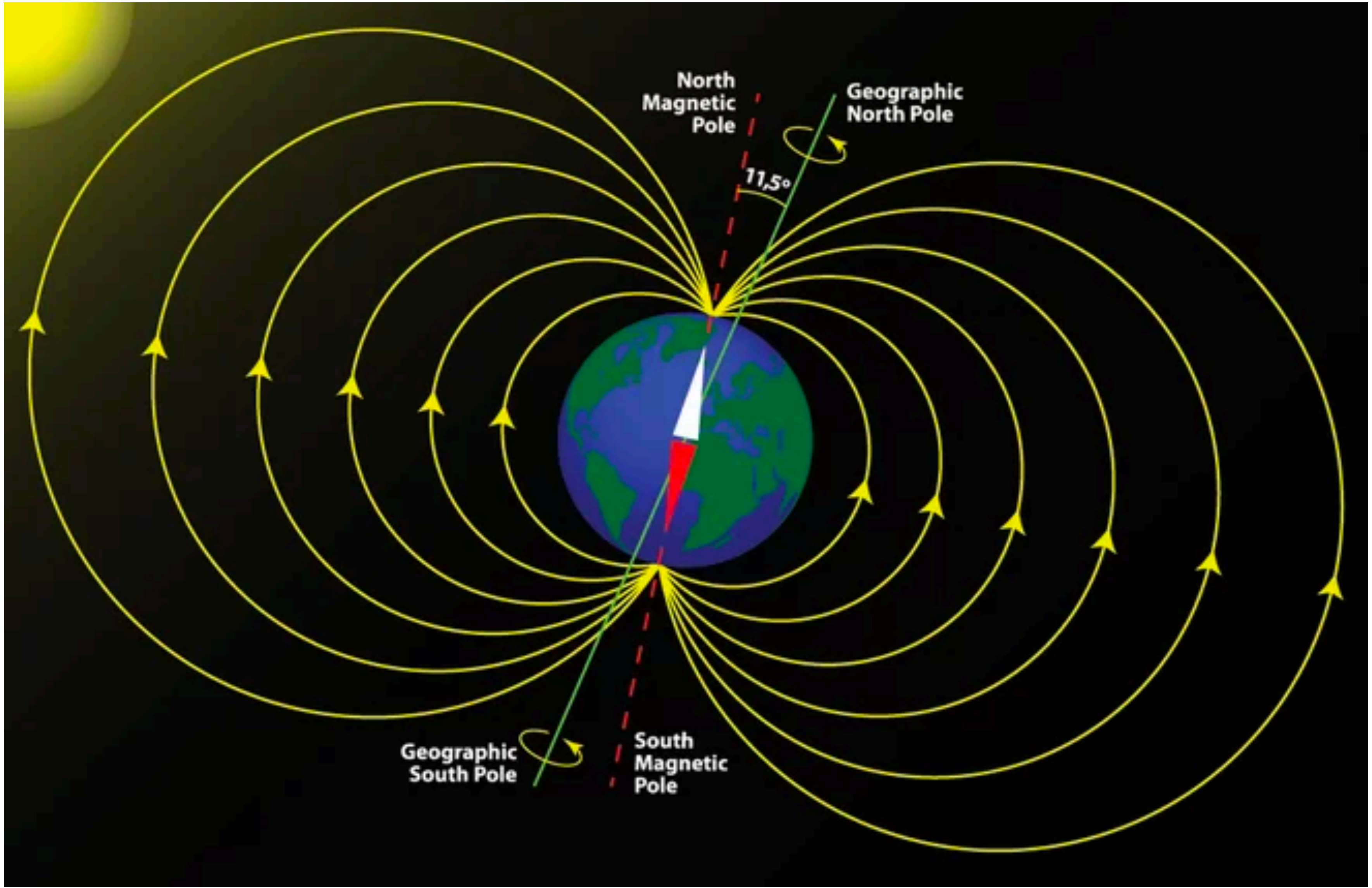

Precession
Applied
Magnetic
Field
N
S

Precession

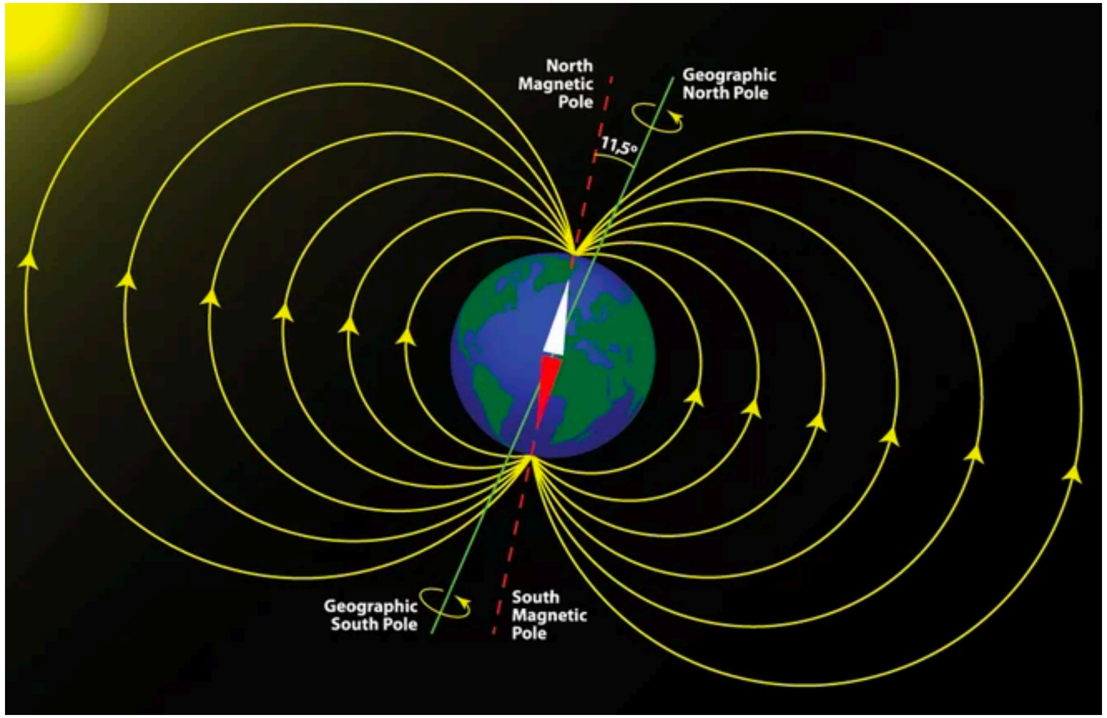
- The slow movement of the axis of a spinning body around another axis due to a torque (such as gravitational influence) acting to change the direction of the first axis. It is seen in the circle slowly traced out by the pole of a spinning gyroscope.



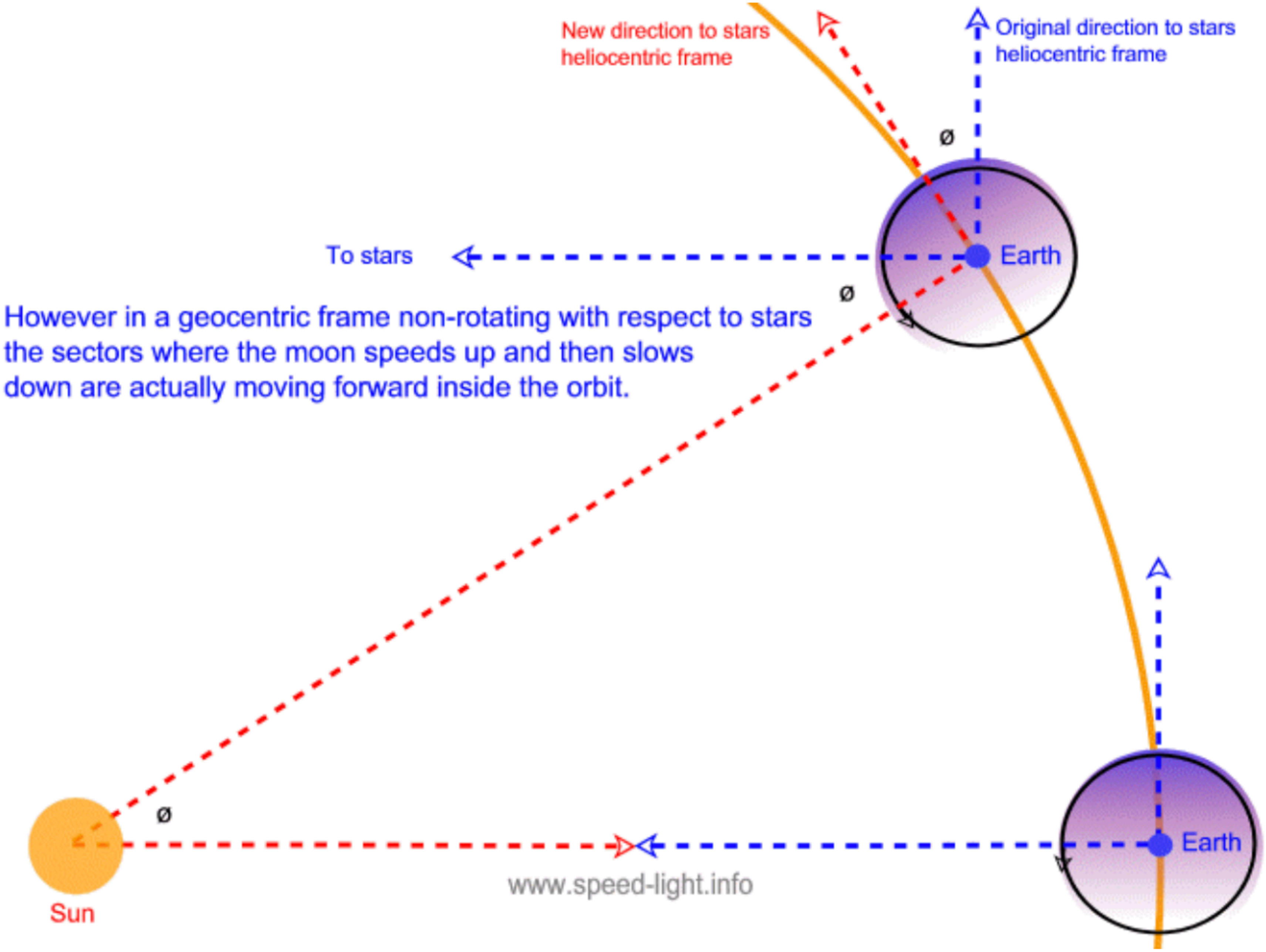
North Magnetic pole and Geographic North pole



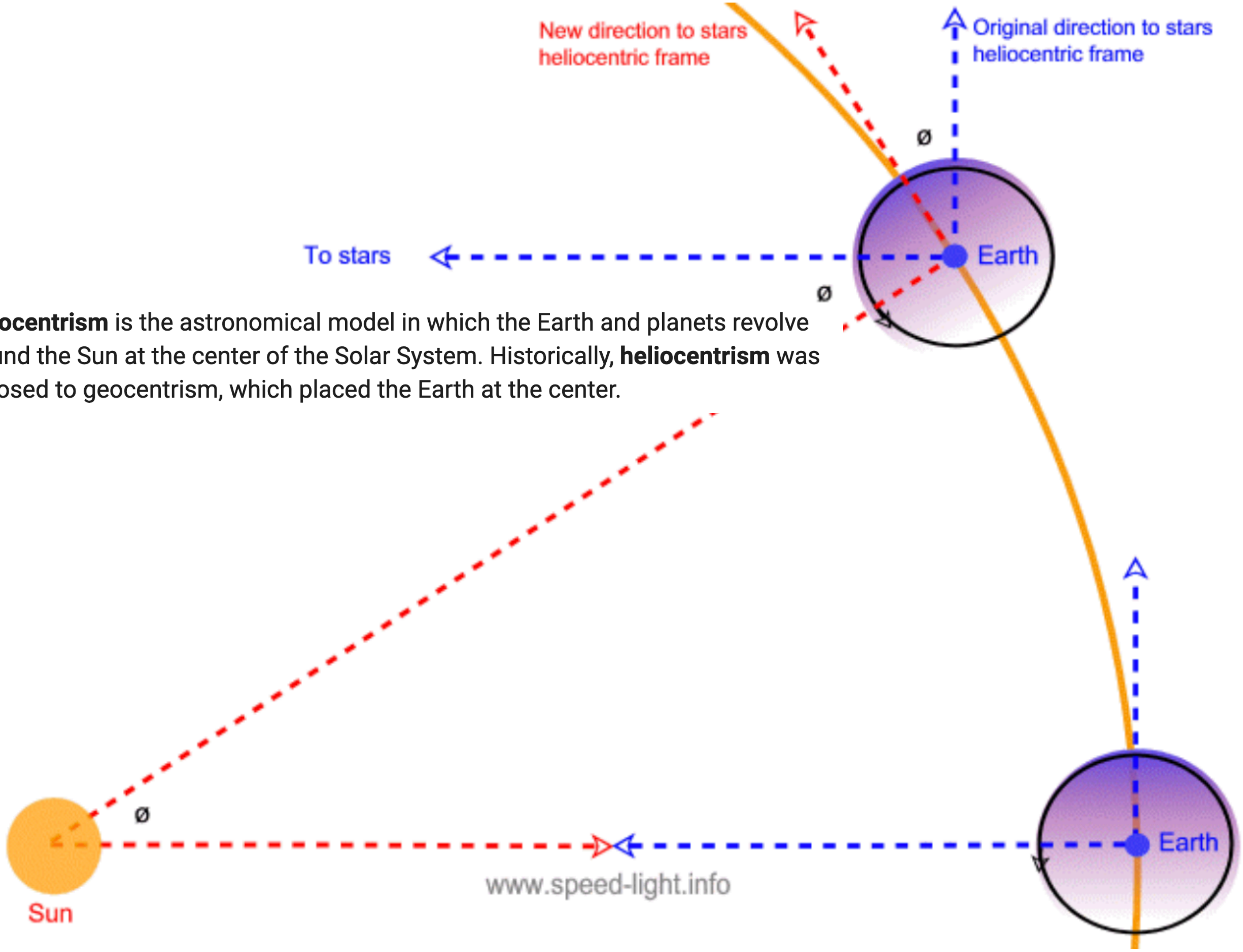
North Magnetic pole
and
Geographic North pole

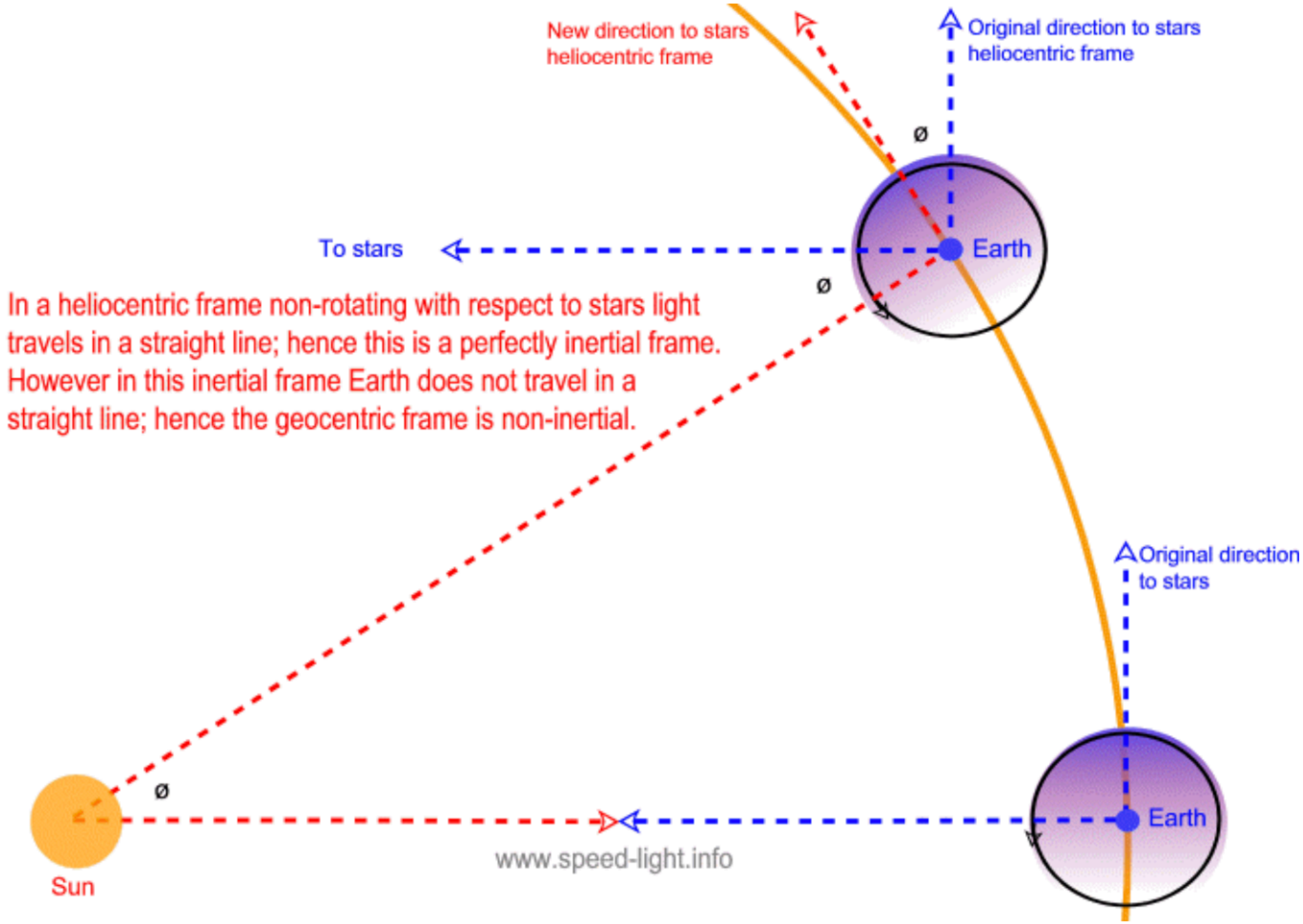


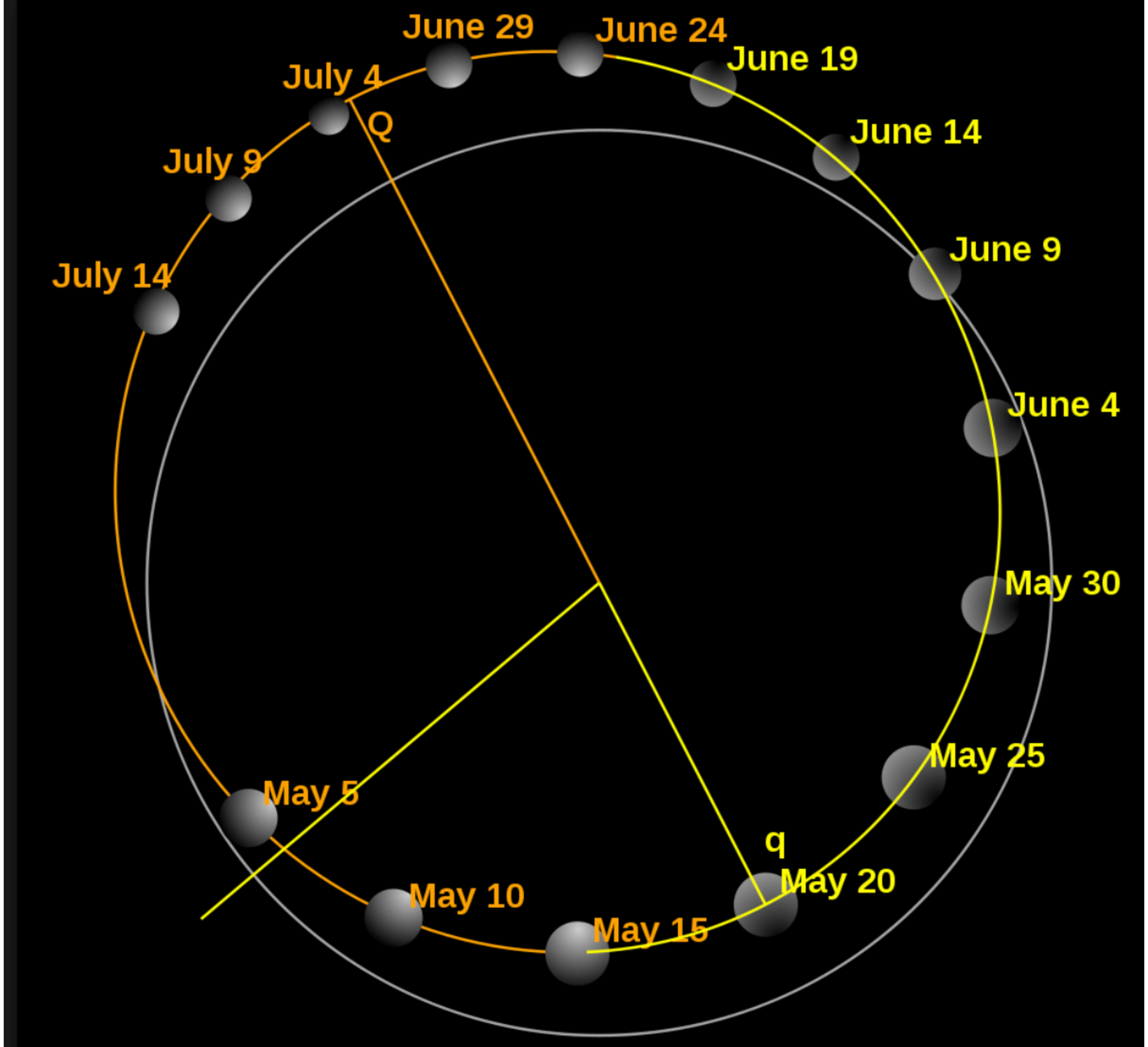
Geographic north and **south poles** are determined by the earth's spin. They are the locations on earth through which the axis **of** the earth's spin passes. **Magnetic north is** determined by the direction a compass points. **Magnetic** variance, or declination, **is** the **difference between geographic north** and **magnetic north**.


New direction to stars
heliocentric frame
Original direction to stars
heliocentric frame
ø
To stars
Earth
ø
However in a geocentric frame non-rotating with respect to stars
the sectors where the moon speeds up and then slows
down are actually moving forward inside the orbit.
ø
Sun
www.speed-light.info
Earth

Heliocentrism is the astronomical model in which the Earth and planets revolve around the Sun at the center of the Solar System. Historically, **heliocentrism** was opposed to geocentrism, which placed the Earth at the center.



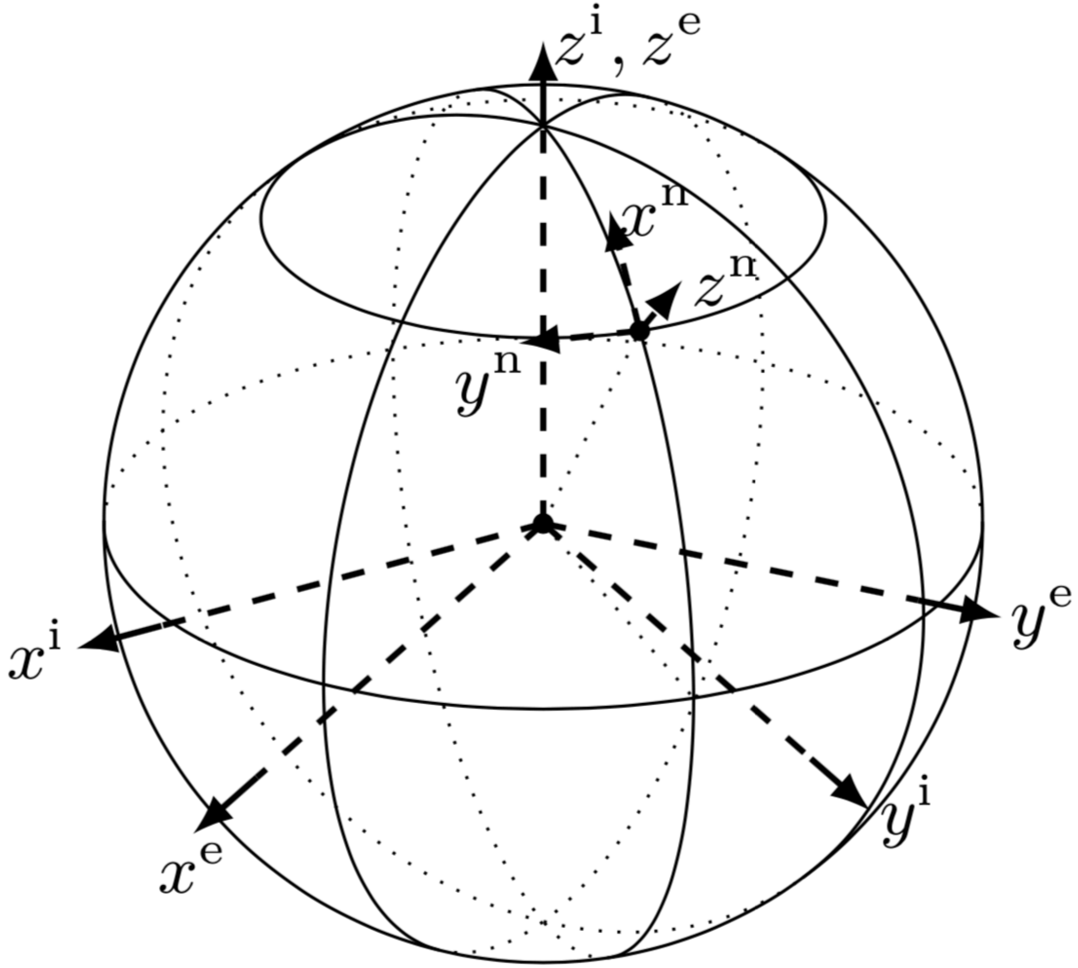

New direction to stars heliocentric frame
Original direction to stars heliocentric frame
ø
To stars
Earth
ø
In a heliocentric frame non-rotating with respect to stars light travels in a straight line; hence this is a perfectly inertial frame. However in this inertial frame Earth does not travel in a straight line; hence the geocentric frame is non-inertial.
Original direction to stars
ø
Earth
www.speed-light.info
Sun



June 29
June 24
June 19
July 4
Q
June 14
July 9
June 9
July 14
June 4
May 30
May 25
May 5
q
May 20
May 10
May 15

Using subscripts b, e, n, I to denote the four different frames

- the n-frame at a certain location on the earth,
- the e-frame rotating with the earth and
- the i-frame.



How to rotate vectors from one frame to another?

Example 2.1 (Rotation of vectors to different coordinate frames) *Consider a vector x expressed in the body frame b. We denote this vector x^b. The rotation matrix R^{nb} rotates a vector from the body frame b to the navigation frame n. Conversely, the rotation from navigation frame n to body frame b is denoted $R^{bn} = (R^{nb})^\mathsf{T}$. Hence, the vector x expressed in the body frame (x^b) and expressed in the navigation frame (x^n) are related according to*

$$x^n = R^{nb}x^b, \qquad x^b = (R^{nb})^\mathsf{T}x^n = R^{bn}x^n. \tag{2.1}$$

$$R^{nb}$$

Rotate from from b to n

$$X^n = R^{nb}X^b$$

As if b cancelled, left n

What does a gyroscope exactly measure?

The gyroscope measures the angular velocity of the (cell phone) body frame with respect to the inertial frame, expressed in the body frame,

denoted by $\omega_{\mathrm{ib}}^{\mathrm{b}}$. This angular velocity can be expressed as

$$\omega_{\mathrm{ib}}^{\mathrm{b}} = R^{\mathrm{bn}}\left(\omega_{\mathrm{ie}}^{\mathrm{n}} + \omega_{\mathrm{en}}^{\mathrm{n}}\right) + \omega_{\mathrm{nb}}^{\mathrm{b}},$$

where R^{bn} is the rotation matrix from the navigation frame to the body frame. The *earth rate*, *i.e.* the angular velocity of the earth frame with respect to the inertial frame is denoted by ω_{ie}. The earth rotates around its own z-axis in 23.9345 hours with respect to the stars [101]. Hence, the earth rate is approximately $7.29 \cdot 10^{-5}$ rad/s.

In case the navigation frame is not defined stationary with respect to the earth, the angular velocity ω_{en}, *i.e.* the *transport rate* is non-zero. The angular velocity required for navigation purposes — in which we are interested when determining the orientation of the body frame with respect to the navigation frame — is denoted by ω_{nb}.

What does a accelerometer exactly measure?

- **The accelerometer measures the specific force f in the body frame b.** This can be expressed as

$$f^{\mathrm{b}} = R^{\mathrm{bn}}(a_{\mathrm{ii}}^{\mathrm{n}} - g^{\mathrm{n}}),$$

where g denotes the gravity vector and an $a_{\mathrm{ii}}^{\mathrm{n}}$ denotes the linear acceleration of the sensor expressed in the navigation frame, which is

$$a_{\mathrm{ii}}^{\mathrm{n}} = R^{\mathrm{ne}} R^{\mathrm{ei}} a_{\mathrm{ii}}^{\mathrm{i}}.$$

The subscripts are used to indicate in which frame the differentiation is performed.

Ask yourself: In which frame the derivative was taken?

For example:

For navigation purposes, we are interested in the position of the sensor in the navigation frame p^n and its derivatives as performed in the navigation frame:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}}\right|_{\mathrm{n}} = v_{\mathrm{n}}^{\mathrm{n}}, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{n}}\right|_{\mathrm{n}} = a_{\mathrm{nn}}^{\mathrm{n}}.$$

How are a_{ii} and a_{nn} are exactly related?

- A relation between a_{ii} and a_{nn} can be derived by using the relation between two rotating coordinate frames. Given a vector x in a coordinate frame u,

Like a product rule, but be caution

$$\frac{\mathrm{d}}{\mathrm{d}t}x^{\mathrm{u}}\Big|_{\mathrm{u}} = \frac{\mathrm{d}}{\mathrm{d}t}R^{\mathrm{uv}}x^{\mathrm{v}}\Big|_{\mathrm{u}} = R^{\mathrm{uv}}\ \frac{\mathrm{d}}{\mathrm{d}t}x^{\mathrm{v}}\Big|_{\mathrm{v}} + \omega^{\mathrm{u}}_{\mathrm{uv}} \times x^{\mathrm{u}},$$

where $\omega^{\mathrm{u}}_{\mathrm{uv}}$ is the angular velocity of the v-frame with respect to the u-frame, expressed in the u-frame.

where we have use the two equations on previous 2 slides and use the fact that the angular velocity of the earth is constant,

$$i.e.\ \frac{\mathrm{d}}{\mathrm{d}t}\omega^{\mathrm{i}}_{\mathrm{ie}} = 0.$$

The students get their hands dirty

$$\begin{matrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{matrix} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

=

$$\begin{matrix} c \\ b \\ a \end{matrix} \quad \text{cross product with} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

We often want to view v_i and a_{ii} in the inertial frame. How?

- Using the fact that

$$p^{\mathrm{i}} = R^{\mathrm{ie}} p^{\mathrm{e}},$$

the velocity v_{i} and acceleration a_{ii} can be expressed as

$$v_{\mathrm{i}}^{\mathrm{i}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{i}}\right|_{\mathrm{i}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{ie}} p^{\mathrm{e}}\right|_{\mathrm{i}} = R^{\mathrm{ie}} \left.\frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\right|_{\mathrm{e}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}} = v_{\mathrm{e}}^{\mathrm{i}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}, \tag{2.8a}$$

$$a_{\mathrm{ii}}^{\mathrm{i}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{i}}^{\mathrm{i}}\right|_{\mathrm{i}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} v_{\mathrm{e}}^{\mathrm{i}}\right|_{\mathrm{i}} + \left.\frac{\mathrm{d}}{\mathrm{d}t} \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}\right|_{\mathrm{i}} = a_{\mathrm{ee}}^{\mathrm{i}} + 2\omega_{\mathrm{ie}}^{\mathrm{i}} \times v_{\mathrm{e}}^{\mathrm{i}} + \omega_{\mathrm{ie}}^{\mathrm{i}} \times \omega_{\mathrm{ie}}^{\mathrm{i}} \times p^{\mathrm{i}}, \tag{2.8b}$$

Similarly we can express velocity v and acceleration a in earth coordinates

Using the relation between the earth and navigation frames,

$$p^{\mathrm{e}} = R^{\mathrm{en}} p^{\mathrm{n}} + n^{\mathrm{e}}_{\mathrm{ne}},$$

where n_{ne} is the distance from the origin of the earth coordinate frame to the origin of the navigation coordinate frame, expressions similar to (2.8) can be derived. Note that in general it can not be assumed that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_{\mathrm{en}} = 0$. Inserting the obtained expressions into (2.8), it is possible to derive the relation between a_{ii} and a_{nn}. Instead of deriving these relations, we will assume that the navigation frame is fixed to the earth frame, and hence R^{en} and $n^{\mathrm{e}}_{\mathrm{ne}}$ are constant and

$$v^{\mathrm{e}}_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{e}}\Big|_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} R^{\mathrm{en}} p^{\mathrm{n}}\Big|_{\mathrm{e}} = R^{\mathrm{en}} \frac{\mathrm{d}}{\mathrm{d}t} p^{\mathrm{n}}\Big|_{\mathrm{n}} = v^{\mathrm{e}}_{\mathrm{n}}, \tag{2.10a}$$

$$a^{\mathrm{e}}_{\mathrm{ee}} = \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{e}}_{\mathrm{e}}\Big|_{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{d}t} v^{\mathrm{e}}_{\mathrm{n}}\Big|_{\mathrm{n}} = a^{\mathrm{e}}_{\mathrm{nn}}. \tag{2.10b}$$

- This is a reasonable assumption as long as the sensor does not travel over significant distances as compared to the size of the earth and it will be one of the model assumptions that we will use in this course.

Bonus slides

Now we can derive the relation of accelerations in different frames.

Inserting (2.10) into (2.8) and rotating the result, it is possible to express a_{ii}^{n} in terms of a_{nn}^{n} as

$$a_{\text{ii}}^{\text{n}} = a_{\text{nn}}^{\text{n}} + 2\omega_{\text{ie}}^{\text{n}} \times v_{\text{n}}^{\text{n}} + \omega_{\text{ie}}^{\text{n}} \times \omega_{\text{ie}}^{\text{n}} \times p^{\text{n}},$$

Coriolis acceleration *centrifugal acceleration*

This is a reasonable assumption as long as the sensor does not travel over significant distances as compared to the size of the earth and it will be one of the model assumptions.

Because this model assumption, it is not a good idea to put all the data collected from everywhere in this world into on data set. Keeping them apart has its advantages.

Example 2.2 (Magnitude of centrifugal and Coriolis acceleration) *The centrifugal acceleration depends on the location on the earth. It is possible to get a feeling for its magnitude by considering the property of the cross product stating that*

$$\|\omega_{ie}^n \times \omega_{ie}^n \times p^n\|_2 \leq \|\omega_{ie}^n\|_2 \|\omega_{ie}^n\|_2 \|p^n\|_2. \tag{2.12}$$

Since the magnitude of ω_{ie} *is approximately* $7.29 \cdot 10^{-5}$ rad/s *and the average radius of the earth is* 6371 km *[101], the magnitude of the centrifugal acceleration is less than or equal to* $3.39 \cdot 10^{-2}$ m/s^2.

The Coriolis acceleration depends on the speed of the sensor. Let us consider a person walking at a speed of 5 km/h. *In that case the magnitude of the Coriolis acceleration is approximately* $2.03 \cdot 10^{-4}$ m/s^2. *For a car traveling at* 120 km/h, *the magnitude of the Coriolis acceleration is instead* $4.86 \cdot 10^{-3}$ m/s^2.

We can use them to detect whether a person is in car or not.