Homework 4

J.E. Paguyo
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Exercise 1. Let $f : G \to \mathbb{C}$ be a continuous function on an open set $G \subset \mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve in G.

a) Find a counterexample demonstrating that the inequality

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, dz$$

no longer makes sense for integrals along a curve γ.

Proof. $\square$

b) Show that

$$\left| \int_\gamma f(z) \right| \leq \int_\gamma |f(z)|\, |dz|$$

where the latter is defined by

$$\int_\gamma |f(z)|\, |dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt$$

Proof. $\square$

Exercise 2. Deduce from Exercise 1 that

$$\left| \int_\gamma f \right| \leq Ml(\gamma)$$

where $M \geq 0$ is a real constant such that $|f(z)| \leq M$ for all points z on γ and

$$l(y) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

is the length of the curve.

Proof. ∎

Exercise 3. Let γ be the arc of the circle $|z| = 2$ in the first quadrant $(x, y > 0)$. Establish the inequality

$$\left| \int_\gamma \frac{dz}{1+z^2} \right| \leq \frac{\pi}{3}$$

without performing the integral explicitly.

Proof. ∎

Exercise 4.

a) *Proof.* ∎

b) *Proof.* ∎

c) *Proof.* ∎

Exercise 5. Let f, g be continuous functions, c_1, c_2 complex constants, and $\gamma, \gamma_1, \gamma_2$ piece-wise smooth curves. Show that

a) $\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g$

Proof.

$$\begin{aligned}
\int_\gamma (c_1 f + c_2 g) &= \int_a^b (c_1 f(\gamma(t)) + c_2 g(\gamma(t)))\gamma'(t)\,dt = \int_a^b (c_1 f(\gamma(t))\gamma'(t) + c_2 g(\gamma(t))\gamma'(t))\,dt \\
&= \int_a^b c_1 f(\gamma(t))\gamma'(t)\,dt + \int_a^b c_2 g(\gamma(t))\gamma'(t)\,dt \\
&= c_1 \int_a^b f(\gamma(t))\gamma'(t)\,dt + c_2 \int_a^b g(\gamma(t))\gamma'(t)\,dt \\
&= c_1 \int_\gamma f + c_2 \int_\gamma g
\end{aligned}$$

∎

b) $\int_{-\gamma} f = -\int_\gamma f$

Proof. ∎

c) $\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

Proof. ∎