Homework 4

J.E. Paguyo
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Exercise 1. Let $f : G \to \mathbb{C}$ be a continuous function on an open set $G \subset \mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve in G.

a) Find a counterexample demonstrating that the inequality

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, dz$$

no longer makes sense for integrals along a curve γ.

Proof. □

b) Show that

$$\left| \int_\gamma f(z) \right| \leq \int_\gamma |f(z)|\, |dz|$$

where the latter is defined by

$$\int_\gamma |f(z)|\, |dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt$$

Proof. □

Exercise 2. Deduce from Exercise 1 that

$$\left| \int_\gamma f \right| \leq Ml(\gamma)$$

where $M \geq 0$ is a real constant such that $|f(z)| \leq M$ for all points z on γ and

$$l(y) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

is the length of the curve.

Proof. We have that

$$\left|\int_\gamma f(z)\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt = \int_a^b M|\gamma'(t)|\,dt$$
$$= M\int_a^b |\gamma'(t)|\,dt = Ml(\gamma)$$

Where the inequality is from Exercise 1 □

Exercise 3. Let γ be the arc of the circle $|z| = 2$ in the first quadrant $(x, y > 0)$. Establish the inequality

$$\left|\int_\gamma \frac{dz}{1+z^2}\right| \leq \frac{\pi}{3}$$

without performing the integral explicitly.

Proof. □

Exercise 4. Compute $\int_\gamma f(z)\,dz$ for the following

a) $f(z) = -y^2 + x^2 - 2ixy$ and γ the straight line from 0 to $-1-i$.

Proof. □

b) $f(z) = (2+z)/z$ and γ the semi-circle $z = e^{i\theta}$, $0 \leq \theta \leq \pi$.

Proof. □

c) $f(z) = 1/z$ and γ any path in the right half plane $\mathrm{Re}(z) \geq 0$ beginning at $-i$, ending at i, avoiding the origin.

Proof. □

Exercise 5. Let f, g be continuous functions, c_1, c_2 complex constants, and $\gamma, \gamma_1, \gamma_2$ piecewise smooth curves. Show that

a) $\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g$

Proof.

$$\int_\gamma (c_1 f + c_2 g) = \int_a^b (c_1 f(\gamma(t)) + c_2 g(\gamma(t)))\gamma'(t)\,dt = \int_a^b (c_1 f(\gamma(t))\gamma'(t) + c_2 g(\gamma(t))\gamma'(t))\,dt$$
$$= \int_a^b c_1 f(\gamma(t))\gamma'(t)\,dt + \int_a^b c_2 g(\gamma(t))\gamma'(t)\,dt$$
$$= c_1 \int_a^b f(\gamma(t))\gamma'(t)\,dt + c_2 \int_a^b g(\gamma(t))\gamma'(t)\,dt$$
$$= c_1 \int_\gamma f + c_2 \int_\gamma g$$

□

b) $\int_{-\gamma} f = -\int_{\gamma} f$

Proof.

$$\int_{-\gamma} f = \int_a^b f(\gamma(t))(-\gamma'(t))\,dt = -\int_a^b f(\gamma(t))\gamma'(t)\,dt = -\int_{\gamma} f$$

□

c) $\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

Proof. □