Homework 4

February 6, 2016

1 Question 1

Let $f : G \to \mathbb{C}$ be a continuous function on an open set $G \subset \mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve in G.

(a) Find a counterexample demonstrating that the inequality

$$\left| \int_\gamma f(z)dz \right| \leq \int_\gamma |f(z)|dz$$

no longer makes sense for integrals along a curve γ.

Proof. Let $f(z) = z$ and γ be a straight line from $0 \to 1 + i$. Then I have that

$$\gamma(t) = (1+i)t, \quad t \in [0,1]$$

$$\gamma'(t) = 1 + i.$$

And then I have that $f(\gamma(t)) = t + it$. Taking the integral I have that,

$$\begin{aligned}\left| \int_\gamma f(z)dz \right| &= \left| (i+1) \int_0^1 (t+it)dt \right| \\ &= |i| \\ &= 1.\end{aligned}$$

Now taking the $|f(\gamma(t))|$ I have that $|f(\gamma(t))| = \sqrt{(2)}|t|$. Now by this into the right side of the inequality I obtain

$$\begin{aligned}\int_\gamma |f(z)|dz &= \int_0^1 \sqrt{2}|t|(1+i)dt \\ &= \sqrt{(2)}(1+1) \int_0^1 tdt \\ &= \frac{1+i}{\sqrt{2}}.\end{aligned}$$

if you calculate the right hand side you are going to get an approximation of $0.707107 + 0.707107i$, and therefore,

$$\left|\int_\gamma f(z)dz\right| \nleq \int_\gamma |f(z)|dz$$

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(b) Show that

$$\left|\int_\gamma f(z)\right| \leq \int_\gamma |f(z)||dz|$$

where the latter is defined by

$$\int_\gamma |f(z)||dz| = \int_a^b |f(\gamma(t))||\gamma'(t)|dt.$$

Proof. For a complex-valued function $g(t)$ on $[a,b]$, I have

$$\Re \int_a^b g(t)dt = \int_a^b \Re g(t)dt,$$

since $\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$ if $g(t) = u(t) + iv(t)$. Then we may use this fact to prove question (b). In Calculus we learned how to prove this inequality for g that are real-valued, but here g is complex-valued. Therefore, for our proof, I shall let $\int_a^b g(t)dt = re^{i\theta}$ for fixed r and θ, where $r \geq 0$, so that $r = e^{-i\theta}\int_a^b g(t)dt = \int_a^b e^{-i\theta}g(t)dt$. Thus,

$$r = \Re r = \Re \int_a^b e^{-i\theta}g(t)dt = \int_a^b \Re(e^{-i\theta}g(t))dt.$$

Then,

$$\Re(e^{-i\theta}g(t)) \leq |e^{-i\theta}g(t)| = |g(t)|, \quad \text{since } |e^{-i\theta}| = 1.$$

Therefore, $\int_a^b \Re(e^{-i\theta}g(t))dt \leq \int_a^b |g(t)|dt$, so I have that

$$\left|\int_a^b g(t)dt\right| = r \leq \int_a^b |g(t)|dt.$$

Using this fact and $|zz'| = |z||z'|$, I have

$$\left|\int_\gamma f\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt = \int_a^b |f(\gamma(t))||\gamma'(t)|dt.$$

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2 Question 2

Deduce from Question 1 that

$$\left|\int_\gamma f\right| \leq M\ell(y)$$

Where $M \geq 0$ is a real constant such that $|f(z)| \leq M$ for all points z on γ and

$$\ell(\gamma) = \int_a^b |\gamma'(t)| = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

is the length of the curve.

Proof. from Question 1 we have that

$$\begin{aligned}
\left| \int_\gamma f \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \\
&\leq \int_a^b |f(\gamma(t))||\gamma'(t)| dt \\
&= \int_a^b M|\gamma'(t)| dt \\
&= M \int_a^b |\gamma'(t)| dt \\
&= M \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt \\
&= M\ell(\gamma).
\end{aligned}$$

I may pull at the M because it is a $\Re(z)$. ■

3 Question 3

Let γ be that arc of the circle $|z| = 2$ in the first quadrant $(x, y > 0)$.
Establish the inequality

$$\left| \int_\gamma \frac{dz}{1 + z^2} \right| \leq \frac{\pi}{3}$$

without performing the integral explicitly.

Proof. Since I am in quadrant one, the $\arg \theta$ is $0 \leq \theta \leq \frac{\pi}{2}$ with $r = 2$. Therefore I have that,

$$\begin{aligned}
\left| \int_\gamma \frac{dz}{1 + z^2} \right| &\leq \int_\gamma \left| \frac{dz}{1 + z^2} \right| \\
' &\leq \int_\gamma \left| \frac{1}{1 + z^2} \right| |dz| \\
&\leq \int_\gamma \frac{1}{|1 + z^2|} 2 \cdot \frac{\pi}{2} \\
&\leq \int_\gamma \frac{1}{|1| + |z|^2} \cdot \pi \\
&\leq \frac{\pi}{5} \leq \frac{\pi}{3}.
\end{aligned}$$

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4 Question 4

compute $\int_\gamma f(z)dz$ for the following

(a) $f(z) = -y^2 + x^2 - 2ixy$ and γ the straight line from 0 to $-1-i$.

(b) $f(z) = (2+z)/z$ and γ the semi-circle $z = \exp(i\theta)$, $0 \leq \theta \leq \pi$.

(c) $f(z) = 1/z$ and γ any path in the right half plane $\Re(z) \geq 0$ beginning at $-i$, ending at i avoiding the orgin.

Proof. For part (a), I have that the complex function is defined as
$f(z) = -y^2 + x^2 - 2ixy$, and that γ is a straight line ranging from $0 \to -1-i$. I would encourage our readers to draw this on the number line and indicate what this line looks like. Now, I have the function γ with respect to t defined as,

$$\begin{aligned}\gamma(t) &= 0 + (\text{difference of starting point to ending point})t \\ &= 0 + (-1-i-0)t \\ &= 0 + (-1-i)t \quad t \in [0,1].\end{aligned}$$

Let's now check that $0 \leq t \leq 1$ is our correct bounds,

$$\gamma(0) = (-1-i)(0) = 0 \quad \text{and} \quad \gamma(1) = (-1-i)(1) = (-1-i).$$

Therefore, these bounds check out because we have remained in our function $\gamma(t)$. Now I may plug $f(\gamma(t))$ and $\gamma'(t)$ to obtain my function with respect to dt.

$$f(\gamma(t)) = -(-t)^2 + (-t)^2 - 2i(-t)(-t) = -t^2 + t^2 - 2it^2 = -2it^2,$$

$$\gamma'(t)) = -1-i.$$

And therefore I have that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 -2it^2(-1-i)dt \\ &= -2i(-1-i)\int_0^1 t^2 dt \\ &= -\frac{2i(-1-i)}{3}t^2\Big|_0^1 \\ &= -\frac{2}{3} + \frac{2}{3}i\end{aligned}$$

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Proof. For part (b), I have that the complex function is defined as
$f(z) = (2+z)/z$, and that γ is the semi-circle ranging from $0 \to \pi$. Now, I have the function γ with respect to t defined as

$$\gamma(t) = 1 \cdot e^{it} \quad t \in [0, \pi].$$

Now I may plug $f(\gamma(t))$ and $\gamma'(t)$ to obtain my function with respect to dt.

$$f(\gamma(t)) = \frac{2+e^{it}}{e^{it}},$$

$$\gamma'(t) = ie^{it}.$$

And therefore I have that

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^\pi \frac{2+e^{it}}{e^{it}} ie^{it} dt \\
&= \int_0^\pi (2i + ie^{it})dt \\
&= \int_0^\pi 2idt + \int_0^\pi ie^{it}dt \\
&= 2it\Big|_0^\pi + \frac{i}{i}e^{it}\Big|_0^\pi \\
&= -2 + 2\pi i.
\end{aligned}$$

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Proof. For part (c), I have that the complex function is defined as $f(z) = 1/z$. Let's choose γ to be a semi-circle ranging from $-\pi/2 \to \pi/2$. I have the function γ with respect to t and its derivative defined as

$$\gamma(t) = 1 \cdot e^{it} \quad t \in [-\pi/2, \pi/2].$$

Now I may plug $f(\gamma(t))$ and $\gamma'(t)$ to obtain my function with respect to dt.

$$\begin{aligned}
\int_\gamma \frac{1}{z} dz &= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{it}} e^{it} dt \\
&= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dt \\
&= it\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \pi i.
\end{aligned}$$

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5 Question 5

Let f, g be a continuous functions, c_1, c_2 complex constants and $\gamma, \gamma_1, \gamma_2$ piecewise smooth curves. Show that

(a) $\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g$

(b) $\int_{-\gamma} f = -\int_\gamma f$

(c) $\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$,

where $\gamma_1 + \gamma_2$ denotes concatenation of curves.

Proof. for (a) I want to show that,

$$\int_\gamma (c_1 f + c_2 g) = c_1 \int_\gamma f + c_2 \int_\gamma g.$$

Since f, g are piecewise continuous on γ; that is, the real and imaginary parts

$$u[x(t), y(t)] \quad \text{and} \quad v[x(t), y(t)]$$

of $f[z(t)], g[(z(t)]$ are piecewise continuous functions of t. I may define the linear integral of f, g along γ as

$$\begin{aligned}
\int_\gamma (c_1 f(z) + c_2 g(z))dz &= \int_a^b \left[c_1 f[z(t)]z'(t) + c_2 g[z(t)]z'(t)\right] dt \\
&= \int_a^b c_1 f[z(t)]z'(t)dt + \int_a^b c_2 g[z(t)]z'(t)dt \\
&= c_1 \int_a^b f[z(t)]z'(t)dt + c_2 \int_a^b g[z(t)]z'(t)dt \\
&= c_1 \int_\gamma f(z)dz + c_2 \int_\gamma g(z)dz.
\end{aligned}$$

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Proof. for (b) I want to show that,

$$\int_{-\gamma} f = -\int_\gamma f.$$

Since f is piecewise continuous on γ; that is, the real and imaginary parts

$$u[x(t), y(t)] \quad \text{and} \quad v[x(t), y(t)]$$

of $f[z(t)]$ is a piecewise continuous functions of t. I may define the linear integral of f along γ as

$$\int_{-\gamma} f(z)dz = \int_{-b}^{-a} f[z(-t)][-z'(-t)]dt,$$

and by changing the variable from $u = -t$ and rearranging the limits of a, b according to u I have that,

$$-\int_b^a f[z(u)][-z'(u)]du.$$

Therefore,

$$\begin{aligned}\int_{-\gamma} f(z)dz &= \int_{-b}^{-a} f[z(-t)][-z'(-t)]dt \\ &= -\int_{b}^{a} f[z(u)][-z'(u)]du \\ &= -\int_{a}^{b} f[z(u)[z'(u)]dz \\ &= -\int_{\gamma} f(z)dz.\end{aligned}$$

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Proof. for (c) I want to show that,

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Since f is piecewise continuous on γ; that is, the real and imaginary parts

$$u[x(t), y(t)] \quad \text{and} \quad v[x(t), y(t)]$$

of $f[z(t)]$ is piecewise continuous functions of t. If γ consists of a contour γ_1 from a_1 to b_1 and a contour γ_2 from a_2 to b_2 then it must be that $b_1 = a_2$ and therefore,

$$\begin{aligned}\int_{\gamma_1+\gamma_2} f(z)dz &= \int_{a_1+a_2}^{b_1+b_2} f[z(t)]z'(t)dt \\ &= \int_{a_1}^{b_1} f[z(t)]f'(z)dt + \int_{a_2=b_1}^{b_2} f[z(t)]z'(t)dt \\ &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.\end{aligned}$$

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