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LARSON—MATH 550—CLASSROOM WORKSHEET 37 Generating Functions & Convolutions.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Generating Functions

A sequence $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ can be represented as the coefficients of the formal power series,

$$A(z) = a_0 + a_1 z^1 + a_2 z^2 + \dots = \sum_{k>0} a_k z^k,$$

called the generating function of the sequence, with notation $a_n = [z^n]A(z)$.

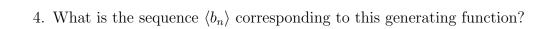
1. Let B(z) be the generating function of the sequence $\langle b_n \rangle = \langle b_0, b_1, b_2, \ldots \rangle$. Find the coefficient of z^n in A(z)B(z), that is, find c_n where $c_n = [z^n]A(z)B(z)$.

The sequence $\langle c_n \rangle = \langle c_0, c_1, c_2, \ldots \rangle$ is the *convolution* of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

Examples

2. Express $\frac{1}{(1-z)^{n+1}}$ as a power series.

3. Let n = 0. What do you notice?



5. Let
$$\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$$
 be any sequence. Find the convolution of $\langle b_n \rangle$ and $\langle a_n \rangle$.

6. Express
$$\frac{z^n}{(1-z)^{n+1}}$$
 as a power series. What do you notice?

7. Earlier we found the recurrence
$$n! = \sum_{k} \binom{n}{k}$$
 $\mathfrak{f}(n-k)$. Divide both sides by $n!$ and simplify.

8. What sequence does e^z generate?

9. Let
$$D(z) = \sum_{k \ge 0} \frac{\mathrm{j}k}{k!} z^k$$
, and explain why $\frac{1}{1-z} = e^z D(z)$.

10. Solve for D(z) and equate coefficients of z^n . What do you find?