LARSON—MATH 511—CLASSROOM WORKSHEET 09
Gilbert Strang Lecture 6.

More on Strang's Lectures

1. What is a *positive definite matrix*?

2. Is $S = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ positive definite?

3. One equivalent condition is that a symmetric matrix S is positive definite if, for every vector $\hat{x}$, the *energy* $\hat{x}^T S \hat{x} > 0$. Show that S has positive energy for every vector $\hat{x}$.

4. It is true in general for a symmetric matrix S that, if the energy $\hat{x}^T S \hat{x}$ is positive for every vector $\hat{x}$, then S is positive definite. Why?

5. We showed that $A^T A$ is symmetric. It is a **key fact** that if A has linearly independent columns then $A^T A$ is positive definite (and this had positive eigenvalues). Why?

SVD Algorithm

Suppose A is any $m \times n$ matrix—with linearly independent columns.

- Find $A^T A$ (its $n \times n$)
  This matrix is symmetric and positive definite.

- Find the eigenvalues and corresponding (unit) eigenvectors λ, $\hat{v}$ (with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$). Let Λ be the diagonal matrix with the λs on the diagonal.

  Vectors corresponding to different eigenvalues are orthogonal. For each eigenspace with dimension greater than 1, Gram-Schmidt can be used to find an orthogonal basis.

- Let $V = [\hat{v}_1 ... \hat{v}_n]$.
  V is orthogonal.

- Let $\sigma_i = \sqrt{\lambda_i}$ and let Σ be the diagonal matrix with diagonal entries $\sigma_1, \ldots, \sigma_n$

- Let $\hat{u}_i = \frac{1}{\sigma_i} A\hat{v}_i$ (that is, $A\hat{v}_i = \sigma_i \hat{u}_i$).
  $\hat{u}_i$'s are orthogonal.

- Let $U = [\hat{u}_1 \ldots \hat{u}_n]$.
  U is orthogonal.

- $AV = U\Sigma$ and $A = U\Sigma V^T$.

Sage/CoCalc

(a) Start the Chrome browser.

(b) Go to `http://cocalc.com`

(c) Login (likely using **your VCU email address**).

(d) You should see an existing Project for our class. Click on that.

(e) Click "New", then "Sage Worksheet", then call it **c09**.

6. Input A= $\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ and find A^T.

7. Find A^TA. Check that A^TA is symmetric and positive definite.

8. Find the eigenvalue-eigenvectors for A^TA and form Σ and V.

9. Check that the $\hat{v}$'s are orthogonal (and hence V is orthogonal).

10. Find the vectors $\hat{u}_i = \frac{1}{\sigma_i} A\hat{v}_i$ and form U.

11. Check that the $\hat{u}$'s are orthogonal (and hence U is orthogonal).

12. Check that $A = U\Sigma V^T$.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c09 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!