Last name ______________________

First name ______________________

LARSON—MATH 610–CLASSROOM WORKSHEET 18
Cauchy-Schwartz.

Concepts (Chp. 1): field, vector space, $\mathcal{P}$, $\mathbb{F}^n$, $\mathbb{M}_{m\times n}(\mathbb{F})$, subspace, null space, row(A), col(A), list of vectors, span of a list of vectors, linear independence, linear dependence.
(Chp. 2).pivot column decomposition, direct sum $\mathcal{U} \oplus \mathcal{V}$, *orthogonal* matrix, *unitary* matrix, *basis*, *dimension*, *linear transformation.*
(Chp. 3). *conformable* matrix addition and multiplication.
(Chp. 4). *nullity*, $A \oplus B$.
(Chp. 5). *inner product*, *inner product space*, $\langle\ \rangle$, *orthogonal* vectors, $\perp$, $||\cdot||$.

Review:

1. What is a *norm* induced by an inner product? What is the notation $||\cdot||$?
2. What is a *unit* vector in an inner product space?
3. What properties does an inner product in a $\mathbb{C}$-vector space have?
4. What is the **Pythagorean Theorem** in an inner product space?

   The *projection* of $\hat{v}$ onto $\hat{u}$ is

$$\hat{x} = \langle \hat{v}, \frac{\hat{u}}{||\hat{u}||}\rangle \frac{\hat{u}}{||\hat{u}||}.$$

Chp. 5 of Garcia & Horn, Matrix Mathematics

1. Check that, for the projection $\hat{x}$ of $\hat{v}$ on $\hat{u}$, $\hat{v}-\hat{x}$ is orthogonal to $\hat{u}$ (and also to $\hat{x}$).

   (**Theorem 5.4.14. Cauchy-Schwartz**). Let $\mathcal{V}$ be an inner product space with inner product $\langle\ ,\ \rangle$, and derived norm $||\ ||$. Then

$$|\langle \hat{u}, \hat{v}\rangle| \le ||\hat{u}|| \cdot ||\hat{v}||.$$

2. Use Cauchy-Schwartz to show that, for $x_1, \ldots, x_n \in \mathbb{R}$ that $(\sum_{i=1}^n x_i)^2 \le n \sum_{i=1}^n x_i^2$.

Chp. 6 of Garcia & Horn, Matrix Mathematics

3. What is an *orthonormal list*?

4. What is an *orthonormal basis*?

5. (**Theorem 6.2.5**). Let $\mathcal{V}$ be a finite-dimensional inner product space, let $\beta = \hat{u}_1, \ldots, \hat{u}_n$ be an orthonormal basis for $\mathcal{V}$ and let $\hat{v} \in \mathcal{V}$. Then

$$\hat{v} = \sum_{i=1}^{n} \langle \hat{v}, \hat{u}_i \rangle \hat{u}_i,$$

$$||\hat{v}||^2 = \sum_{i=1}^{n} |\langle \hat{v}, \hat{u}_i \rangle|^2, \text{ and}$$

$$[\hat{v}]_\beta = \begin{bmatrix} \langle \hat{v}, \hat{u}_1 \rangle \\ \langle \hat{v}, \hat{u}_2 \rangle \\ \vdots \\ \langle \hat{v}, \hat{u}_n \rangle \end{bmatrix}$$

6. What does this theorem say?

7. Why is this theorem true?

(**Theorem 6.3.5, Gram-Schmidt Process**). Let $\mathcal{V}$ be an inner product space, and suppose that $\hat{v}_1, \ldots, \hat{v}_n \in \mathcal{V}$ are linearly independent. There is an orthonormal list $\hat{u}_1, \ldots, \hat{u}_n$ such that

$$span\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_k\} = span\{\hat{u}_1, \hat{u}_2 \ldots, \hat{u}_k\} \text{ for } k = 1, 2, \ldots, n.$$

8. What does this theorem say?

9. Why is this theorem true?