LARSON—MATH 511—CLASSROOM WORKSHEET 26
Low-Rank and Other Matrix Updates

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c26**.

Least Squares Update Example

2. We will talk through the example in "least_squares_updated.sage" in your CoCalc project Handouts folder.

   Here is the mathematical background from last class:

   Suppose we have m data points (each with k *features*):

$$(A_{1,1}, A_{1,2}, \ldots, A_{1,k}, b_1)$$

$$(A_{2,1}, A_{2,2}, \ldots, A_{2,k}, b_2)$$

$$\ldots$$

$$(A_{m,1}, A_{m,2}, \ldots, A_{m,k}, b_m)$$

   and we have *already* found the least-squares solution $\hat{c}$ to $f(x) = c_1x_1 + c_2x_2 + \ldots + c_kx_k$, where:

$$f(A_{i,1}, A_{i,2}, \ldots, A_{i,k}) = b_i$$

   for $i = 1, 2, \ldots, m$, where $\hat{c} = (A^TA)^{-1}A^T\hat{b}$.

   Suppose we then get a new data point:

$$(A_{m+1,1}, A_{m+1,2}, \ldots, A_{m+1,k}, b_{m+1})$$

   How can we use the Sherman-Morrison-Woodbury formula to efficiently update $(A^TA)^{-1}$ (for the *updated* data matrix A, given that we already know the $(A^TA)^{-1}$ from the original data points)?

   We let $\hat{r}$ be the column vector with entries, $A_{m+1,1}, A_{m+1,2}, \ldots, A_{m+1,k}$, the "new A" be $\begin{bmatrix} A \\ \hat{r}^T \end{bmatrix}$ and the "new $\hat{b}$" be $\begin{bmatrix} \hat{b} \\ b_{m+1} \end{bmatrix}$ and found that the new (updated) least-squares solution could be computed by finding:

$$(\begin{bmatrix} A^T \hat{r} \end{bmatrix} \begin{bmatrix} A \\ \hat{r}^T \end{bmatrix})^{-1} = (\begin{bmatrix} A^TA + \hat{r}\hat{r}^T \end{bmatrix})^{-1}.$$

   Then we applied the **Sherman-Morrison-Woodbury formula**:

$$(A - UV^T)^{-1} = A^{-1} + A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}$$

   We let $\hat{r}^T$ be the new data row. Then $U = \hat{r}^T$ and $V = \hat{r}$.

The Derivative of A^{-1}

3. Let $A(t) = \begin{bmatrix} t & \frac{1}{t} \\ t^2 & t^2+1 \end{bmatrix}$. Find $A(1)$, $A(2)$.

4. Find $\frac{dA}{dt}$.

5. Let $A = A(1)$ and $B = A(2)$. Are they invertible?

6. Let $\Delta A = B - A$. Find ΔA.

7. (**A Very Useful Formula**). Check: $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$.

8. Use this to find $\frac{\Delta A^{-1}}{\Delta t}$ and $\frac{dA^{-1}}{dt}$.

Interlacing Eigenvalues for Symmetric Matrices

9. (A Rayleigh-Ritz-type formula for symmetric matrices). Let S be a symmetric $n \times n$ matrix with (real) eigenvalues $\lambda_1 \geq \ldots \geq \lambda_q \geq \ldots \lambda_p \geq \ldots \geq \lambda_n$, and corresponding eigenvectors $\hat{u}_1, \ldots, \hat{u}_n$. If $\hat{x}$ is a unit eigenvector in $Span(\{\hat{u}_p, \ldots, \hat{u}_q\})$ then $\lambda_p \leq \hat{u}^T S \hat{u} \leq \lambda_q$.

10. What are examples?

11. What is a *principal* submatrix of a A of a square matrix S?

12. (**Cauchy's Interlacing Theorem**) If A is a $(n-1) \times (n-1)$ principle submatrix of a symmetric matrix S with eigenvalues $\mu_1 \geq \ldots \geq \mu_{n-1}$ then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n$.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c26 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!