

Last name _____

First name _____

LARSON—MATH 310—CLASSROOM WORKSHEET 27
SVD & Rank-1 Matrices

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The SVD here is $A = U\Sigma V^T$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. What is m , n and the rank r of the given matrix A ?
2. Do the sizes of U , Σ and V make sense?
3. How many positive singular values must there be?
4. Call the positive singular values $\sigma_1, \dots, \sigma_r$. What are the positive singular values $\sigma_1 \geq \dots \geq \sigma_r$? Find Σ .
5. What are the \vec{v}_i 's?
6. Find V .
7. Check that V is orthogonal.

8. What are the \vec{u}_i 's?

9. Check that U is orthogonal.

10. Check that $A\vec{v}_i = \sigma_i\vec{u}_i$ for $i = 1, \dots, r$. Check that $A\vec{v}_i = 0$ for $i > r$.

11. Find the matrices $\sigma_1\vec{u}_1\vec{v}_1^T, \dots, \sigma_r\vec{u}_r\vec{v}_r^T$.

12. Check that the rank of each of $\sigma_1\vec{u}_1\vec{v}_1^T, \dots, \sigma_r\vec{u}_r\vec{v}_r^T$ is 1.

13. Check that $A = \sigma_1\vec{u}_1\vec{v}_1^T + \dots + \sigma_r\vec{u}_r\vec{v}_r^T$ (that is, that A is a sum of r rank-1 matrices).