Last name ______________________

First name ______________________

LARSON—MATH 550—HOMEWORK WORKSHEET h14
Test 2 Review

1. Define Pascal's Triangle and draw several levels.
2. Define $\binom{n}{m}$ (for non-negative integers n, m).
3. State and prove the formula we found for $\binom{n}{m}$.
4. Find $\binom{3}{1}$, $\binom{3}{2}$, $\binom{n}{1}$, and $\binom{n}{n-1}$.
5. Draw several levels of the Binomial Coefficients Triangle.
6. What needs to be shown to prove that Pascal's Triangle is the same as the Binomial Coefficients Triangle?
7. Give a combinatorial (non-algebraic) explanation of why the *symmetry identity* is true:
$$\binom{n}{k} = \binom{n}{n-k}.$$
8. Give a combinatorial (non-algebraic) explanation of why the *addition formula* is true:
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$
9. State the Binomial Theorem for $(x+y)^n$ ($n \in \mathbf{Z}^{\geq 0}$), check it for $n = 0, 1, 2$ and then *explain* why it is true for any n.
10. Explain the *main ideas* of the *Polynomial Argument* we used to prove:
$$(r-k)\binom{r}{k} = r\binom{r-1}{k} \ (r \in \mathbb{R}, k \in \mathbb{Z}).$$
11. What is a *derangement*?
12. Let ¡n be the number of derangements of n objects. Find ¡3, ¡4. Explain.
13. Let $h(n,k)$ be the number of permutations of n objects where exactly k are fixed. Explain why:
$$n! = \sum_k h(n,k)?$$
14. If ¡n is the number of ways to arrange n hats so that the hat of the i^{th} person to check their hat doesn't match the i^{th} hat. What is the probability that *none* of the n people who come in get their hat back? What is this probability when $n = 4$?

15. (**Vandermonde Convolution**) Give a combinatorial proof that :

$$\sum_{k\geq 0}\binom{r}{k}\binom{s}{n-k}=\binom{r+s}{n}\ (r,s,n\in\mathbb{Z}^{\geq 0}).$$

16. Let $f(n)=n^2$ $(n\in\mathbb{Z}^{\geq 0})$, let:

$$g(n)=\sum_k\binom{n}{k}(-1)^k f(k).$$

Check that the following formula holds for $n=2$:

$$f(n)=\sum_k\binom{n}{k}(-1)^k g(k).$$

17. (**Inversion**) If $f(n)$ is a function defined on $\mathbb{Z}^{\geq 0}$, and we define:

$$g(n)=\sum_k\binom{n}{k}(-1)^k f(k).$$

Prove:

$$f(n)=\sum_k\binom{n}{k}(-1)^k g(k).$$

18. What is the power series definition of e^x? What is $\frac{1}{e}$?
19. What is a *generating function*?
20. Express $\frac{1}{1-z}$ as a power series. Explain.
21. $\frac{1}{1-z}$ is the generating function for the sequence $\langle b_n\rangle$. What is $\langle b_n\rangle$?
22. What sequence does e^z generate?
23. What sequence does $\frac{1}{e}$ generate?
24. What is the *convolution* of sequences $\langle a_n\rangle$ and $\langle b_n\rangle$?
25. Let $\langle a_n\rangle=\langle a_0,a_1,a_2,\ldots\rangle$ be any sequence. Find the convolution of $\langle b_n\rangle=\langle 1,1,1,\ldots\rangle$ and $\langle a_n\rangle$.
26. Define the Fibonacci numbers F_n and find the first few terms of the sequence $\langle F_n\rangle$.
27. Let $F(z)$ be the generating function for the Fibonacci numbers $\langle F_n\rangle$. Find a relationship between $F(z)$, $zF(z)$ and $z^2F(z)$, and solve to get a formula for $F(z)$.
28. (**Bee Trees**). A male bee has a single female parent. A female bee has one male parent and one female parent. Draw a tree representing the "ancestors" of a male bee. Let $M_1=1$, representing a male bee. Then that bee has one (female) ancestor one generation back, and zero male ancestors; represent this by $M_2=0$. This bee has two parents, so our original bee has 2 ancestors two generations ago; one is male so represent this by $M_3=1$. How many **male** ancestors M_n does our original bee have after $n-1$ generations? Explain.