LARSON—MATH 511—CLASSROOM WORKSHEET 20
QR, Gram-Schmidt & Randomized Matrix Multiplication

Pseudo-inverses

1. Suppose A is a matrix with linearly independent columns. One "solution" of $A\hat{x} = \hat{b}$ (even when $\hat{b}$ isn't in the column space of A) is $\hat{x} = (A^TA)^{-1}A^T\hat{b}$. Check that when A has linearly independent columns that $A^+ = (A^TA)^{-1}A^T$ (and so $\hat{x} = A^+\hat{b}$).

Gram-Schmidt

Idea: Given linearly independent vectors $\hat{a}_1$, $\hat{a}_2$,...,$\hat{a}_n$, let $\hat{q}_1 = \frac{1}{||\hat{a}_1||}\hat{a}_1$, and at each step i ($i = 2, ..i = n$):

- Let $\hat{a}_i'$ be $\hat{a}_i$ minus the projection of $\hat{a}_i$ on each of the previously found $\hat{q}_1$,...,$\hat{q}_{i-1}$.
- Let $\hat{q}_i = \frac{1}{||\hat{a}_i'||}\hat{a}_i'$.

2. Use Gram-Schmidt to find an orthogonal basis, $\hat{q}_1, \hat{q}_2, \hat{q}_3$, of the columns of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Let Q be the matrix whose columns are $\hat{q}_1$, $\hat{q}_2$, and $\hat{q}_3$. Write $A = QR$ (for some matrix R).

4. What can we say about R?

5. How will this QR decomposition of A help us solve $A\hat{x} = \hat{b}$?

Improved Gram-Schmidt

Idea: Given linearly independent vectors $\hat{a}_1$, $\hat{a}_2$,...,$\hat{a}_n$, let $\hat{q}_j = \frac{1}{||\hat{a}_j||}\hat{a}_j$, where $||\hat{a}_i||$ is a maximum, and at each step i ($i = 2, ..i = n$):

- For remaining (not-yet-processed) $\hat{a}_i$'s, let new $\hat{a}_i$ be current $\hat{a}_i$ minus the projection of $\hat{a}_i$ on $\hat{q}_{i-1}$.
- Find the largest-norm remaining $\hat{a}_i$.
- Let $\hat{q}_i = \frac{1}{||\hat{a}_i'||}\hat{a}_i'$.

6. Use Gram-Schmidt to find an orthogonal basis, $\hat{q}_1, \hat{q}_2, \hat{q}_3$, of the columns of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

7. Let Q be the matrix whose columns are $\hat{q}_1$, $\hat{q}_2$, and $\hat{q}_3$. Is it still the case that $A = QR$ (for some matrix R)?

Sage/CoCalc

8. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c20**.

Randomized Matrix Multiplication

Idea: To get a matrix that approximates the product AB, we can take a selection of s columns of A, dot them with the corresponding column of B and add them up.

We'll take a weighted selection, favoring index choices where the products of the A-column and corresponding B-row are largest.

9.
```
P = [0.5, 0.1, 0.4]
X = GeneralDiscreteDistribution(P)
X.get_random_element()

def randomized_matrix_multiplication(A,B,s):
    rows_columns_A = A.dimensions()
    rowsA = rows_columns_A[0]
    columnsA = rows_columns_A[1]
    rows_columns_B = B.dimensions()
    rowsB = rows_columns_B[0]
    columnsB = rows_columns_B[1]

    weights = []
    for i in range(columnsA):
        weight = (A.column(i).norm())*(B.row(i).norm())
        weights.append(weight)
    total_weight = sum(weights)
    distribution = [weight/total_weight for weight in weights]
    Randomized_product = matrix(RDF, rowsA, columnsB, [0]*(rowsA*columnsB))

    X = GeneralDiscreteDistribution(distribution)
    for i in range(s):
        index = X.get_random_element()
        Randomized_product =  Randomized_product +
        matrix(RDF, rowsA,1,A.column(index))*matrix(RDF, 1,columnsB,B.row(ind
    return Randomized_product
```

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c20 worksheet attached" (so that it will be properly recorded).