Last name ______________________

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LARSON—MATH 550—CLASSROOM WORKSHEET 27
The polynomial argument, & Vandermonde's convolution.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Review

1. $\binom{n}{k}$ is the number of k-subsets of an n-set (for $n, k \in \mathbb{Z}^{\geq 0}$).
2. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(0 \leq m \leq n,\ k, n \in \mathbb{Z})$.
3. We proved the *symmetry identity* $\binom{n}{k} = \binom{n}{n-k}$.
4. We proved the *addition formula*:
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$
5. The *binomial coefficients* $\binom{n}{k}$ $(n, k \in \mathbb{Z}^{\geq 0})$ can be generalized to $\binom{r}{k}$ $(r \in \mathbb{R}, k \in \mathbb{Z})$:
$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!} \text{ (if } k \geq 0)$$
$$\binom{r}{k} = 0 \text{ (if } k < 0).$$
6. We proved the *absorption identity* $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$ and the *absorption identity* variation $k\binom{r}{k} = r\binom{r-1}{k-1}$ and $(r-k)\binom{r}{k} = r\binom{r-1}{k}$.
7. (**Negating the Upper index**). We proved: $\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$.

New

1. What is the *Multiplication Principle*?

2. (**Vandermonde Convolution**) Prove ($n \in \mathbb{Z}$):

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}.$$

3. (**Summing on the Upper Index**) Prove (for $n, n \in \mathbb{Z}$):

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}.$$

4. (**The Sorting Problem**) Simplify:

$$T = \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1} \Big/ \binom{m}{n}.$$

Steps:

(a) Let

$$S = \sum_{k=0}^{n} k \binom{m-k-1}{m-n-1}.$$

(b) Rewrite k as $m - (m-k)$

(c) Use absorption to get:

$$S = mA - (m-n)B, \text{ where :}$$

$$A = \sum_{k=0}^{n} \binom{m-k-1}{m-n-1}, \text{ and } B = \sum_{k=0}^{n} \binom{m-k}{m-n}.$$

(d) Sum on the Upper Index to get:

$$B = \binom{m+1}{m-n+1}, \text{ and } A = \binom{m}{m-n}.$$

(e) Show:

$$S = \frac{n}{m-n+1} \cdot \binom{m}{n-m}, \text{ and } T = \frac{n}{m-n+1} \cdot \binom{m}{n-m}.$$