Last name ______________________

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LARSON—MATH 550—CLASSROOM WORKSHEET 30
The Sorting Example, Inversion & Derangements.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Review

1. $\binom{n}{k}$ is the number of k-subsets of an n-set (for $n, k \in \mathbb{Z}^{\geq 0}$).
2. We proved the *absorption identity* $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$ and the *absorption identity* variation $k\binom{r}{k} = r\binom{r-1}{k-1}$ and $(r-k)\binom{r}{k} = r\binom{r-1}{k}$.
3. (**Negating the Upper index**). We proved: $\binom{r}{k} = (-1)^k\binom{k-r-1}{k}$.
4. (**Summing on the Upper Index**) Prove (for $n, n \in \mathbb{Z}$):

$$\sum_{0\leq k\leq n}\binom{k}{m} = \binom{n+1}{m+1}.$$

1. (**The Sorting Problem**) Simplify:

$$T = \sum_{k=0}^{n} k\binom{m-k-1}{m-n-1}\Big/\binom{m}{n}.$$

Steps:

(a) Let

$$S = \sum_{k=0}^{n} k\binom{m-k-1}{m-n-1}.$$

(b) Rewrite k as $m-(m-k)$,

(c) Use absorption to get:

$$S = mA - (m-n)B, \text{ where :}$$

$$A = \sum_{k=0}^{n} \binom{m-k-1}{m-n-1}, \text{ and } B = \sum_{k=0}^{n} \binom{m-k}{m-n}.$$

(d) Sum on the Upper Index to get:

$$B = \binom{m+1}{m-n+1}, \text{ and } A = \binom{m}{m-n}.$$

(e) Show:

$$S = \frac{n}{m-n+1} \cdot \binom{m}{m-n}, \text{ and } T = \frac{n}{m-n+1}.$$

The Duplication Formula

(f) Show:

$$r^{\underline{k}}(r - \frac{1}{2})^{\underline{k}} = (2r)^{\underline{2k}}/2^{2k} \text{ for } k \in \mathbb{Z}^{\geq}.$$

Inversion

Let $f(n)$ be a function defined on $\mathbb{Z}^{\geq 0}$, and define:

$$g(n) = \sum_{k} \binom{n}{k} (-1)^k f(k).$$

The **claim** is that:

$$f(n) = \sum_{k} \binom{n}{k} (-1)^k g(k).$$

One **question** is what can we *use* this for? And of course is (or why) is it true? And of course the first thing we should do is try (or test) this formula...