Last name ______________________

First name ______________________

LARSON—MATH 550—HOMEWORK WORKSHEET h06
Test 1 Review

1. What is a *recurrence* relation? Give an example.

2. (**Towers of Hanoi**) Let T_n be the minimum number of moves to solve the n disk Towers of Hanoi problem. Find T_1 and T_2. Explain.

3. Explain why $T_n \leq 2T_{n-1} + 1$.

4. Explain why $T_n \geq 2T_{n-1} + 1$.

5. What is the *recurrence* for T_n?

6. Use the recurrence for T_n to find T_4, T_5 and T_6.

7. *Solve* the recurrence for T_n.

8. *Prove* the closed formula for T_n.

9. (**Lines in the Plane**) What is the maximum number of regions defined by n lines in the plane? Try the methodology developed in the Towers of Hanoi problem

   (a) *Name* the quantity you want to count/investigate.

   (b) Find some values of that quantity.

   (c) Find a recurrence relation for that quantity.

   (d) Use the recurrence to find more values of that quantity.

   (e) Use these values to *guess* a (non-recurrence closed-form) formula for that quantity.

   (f) *Prove* your formula.

10. (**Quicksort**). Find C_2, C_3 where $C_n (n \geq 0)$ is defined as follows:

$$C_0 = C_1 = 0$$

$$C_n = (n+1) + \frac{2}{n}\sum_{k=0}^{n-1} C_k \text{ (for } n > 1)$$

11. We found

$$\frac{2}{n+1}C_n = \frac{2}{n}C_{n-1} + \frac{4}{n+1}, (n > 1).$$

12. Find C_2, C_3 using this formula.

13. Let

$$U_n = \frac{2}{n+1}C_n, (n > 1).$$

and find a recurrence for U_n.

14. Solve the recurrence for U_n to get a formula for C_n.

15. Use the *perturbation method* and sum rules to find a formula for the *geometric series*

$$S_n = \sum_{k=0}^{n} ax^n.$$

16. Use the perturbation method to find a formula for the

$$S_n = \sum_{k=0}^{n} k2^k.$$

17. What does $\sum_{1 \leq i,j \leq 3} a_i b_j$ *mean.*

18. Find the sum of the elements in the matrix:

$$\begin{bmatrix} a_1a_1 & a_1a_2 & \dots & a_1a_{n-1} & a_1a_n \\ a_2a_1 & a_2a_2 & \dots & a_2a_{n-1} & a_2a_n \\ \dots & \dots & & & \dots \\ a_{n-1}a_1 & a_{n-1}a_2 & \dots & a_{n-1}a_{n-1} & a_{n-1}a_n \\ a_na_1 & a_na_2 & \dots & a_na_{n-1} & a_na_n \end{bmatrix}$$

19. Explain why this identity is true (for $j, k, n \in \mathbb{Z}$):

$$[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n].$$

20. Expand and simplify:

$$\sum_{1 \leq i,j \leq 3} a_i b_i.$$

21. Use this to prove the following Chebyshev Monotonic Inequality:

$$(\sum_{k=1}^{n} a_k)(\sum_{k=1}^{n} b_k) \geq n(\sum_{k=1}^{n} a_k b_k) \text{ if } a_1 \leq \dots a_n, b_1 \geq \dots \geq b_n.$$

22. Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as $\sum_{1 \leq j \leq k \leq n} 2^k$.

23. Define $x^{\underline{m}}$ (for $x \in \mathbb{R}$)..

24. Define $x^{\overline{m}}$ (for $x \in \mathbb{R}$).

25. Define $\lceil x \rceil$ and $\lfloor x \rfloor$ (for $x \in \mathbb{R}$). Give examples.

26. Argue: $\lceil x \rceil = x \Leftrightarrow x$ is an integer (for $x \in \mathbb{R}$).

27. Argue: $\lceil x \rceil - \lfloor x \rfloor = [x \text{ is not an integer }]$ (for $x \in \mathbb{R}$).

28. Argue: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ (for $x \in \mathbb{R}$).

29. Argue: $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (for $n \in \mathbb{Z}, x \in \mathbb{R}$) .

30. Explain: $(n!)^2 = (1 \cdot 2 \dots n)(1 \cdot 2 \dots n) = \prod_{k=1}^{n} k(n + 1 - k)$