Last name _____

First name _____

LARSON—MATH 550—HOMEWORK WORKSHEET h06
Test 1 Review

1. What is a *recurrence* relation? Give an example.

2. (**Towers of Hanoi**) Let T_n be the minimum number of moves to solve the n disk Towers of Hanoi problem. Find T_1 and T_2. Explain.

3. Explain why $T_n \leq 2T_{n-1} + 1$.

4. Explain why $T_n \geq 2T_{n-1} + 1$.

5. What is the *recurrence* for T_n?

6. Use the recurrence for T_n to find T_4, T_5 and T_6.

7. *Solve* the recurrence for T_n.

8. *Prove* the closed formula for T_n.

9. (**Lines in the Plane**) What is the maximum number of regions defined by n lines in the plane? Try the methodology developed in the Towers of Hanoi problem

   (a) *Name* the quantity you want to count/investigate.
   (b) Find some values of that quantity.
   (c) Find a recurrence relation for that quantity.
   (d) Use the recurrence to find more values of that quantity.
   (e) Use these values to *guess* a (non-recurrence closed-form) formula for that quantity.
   (f) *Prove* your formula.

10. (**Quicksort**). Find C_2, C_3 where $C_n(n \geq 0)$ is defined as follows:

$$C_0 = C_1 = 0$$

$$C_n = (n + 1) + \frac{2}{n} \sum_{k=0}^{n-1} C_k \text{ (for } n > 1)$$

11. We found
$$\frac{2}{n+1} C_n = \frac{2}{n} C_{n-1} + \frac{4}{n+1}, (n > 1).$$

12. Find C_2, C_3 using this formula.

13. Let
$$U_n = \frac{2}{n+1} C_n, (n > 1).$$

and find a recurrence for U_n.

14. Solve the recurrence for U_n to get a formula for C_n.

15. Use the *perturbation method* and sum rules to find a formula for the *geometric series*

$$S_n = \sum_{k=0}^{n} ax^n.$$

16. Use the perturbation method to find a formula for the

$$S_n = \sum_{k=0}^{n} k2^k.$$

17. What does $\displaystyle\sum_{1 \leq i,j \leq 3} a_i b_j$ *mean*.

18. Find the sum of the elements in the matrix:

$$\begin{bmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_{n-1} & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \cdots & a_2 a_{n-1} & a_2 a_n \\ \cdots & \cdots & & & \cdots \\ a_{n-1} a_1 & a_{n-1} a_2 & \cdots & a_{n-1} a_{n-1} & a_{n-1} a_n \\ a_n a_1 & a_n a_2 & \cdots & a_n a_{n-1} & a_n a_n \end{bmatrix}$$

19. Explain why this identity is true (for $j, k, n \in \mathbb{Z}$):

$$[1 \leq j < k \leq n] + [1 \leq k < j \leq n] = [1 \leq j, k \leq n] - [1 \leq j = k \leq n].$$

20. Expand and simplify: $\sum_{1 \leq i,j \leq 3} a_i b_i$.

21. Use the formula we found for $S = \displaystyle\sum_{1 \leq j < k \leq n} (a_k - a_j)(b_k - b_j)$ to prove the following Chebyshev Monotonic Inequality:

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \geq n\left(\sum_{k=1}^{n} a_k b_k\right) \text{ if } a_1 \leq \ldots a_n, b_1 \geq \ldots \geq b_n.$$

22. Evaluate $\sum_{k=1}^{n} k2^k$ by rewriting it as $\sum_{1 \leq j \leq k \leq n} 2^k$.

23. Define $x^{\underline{m}}$ (for $x \in \mathbb{R}$)..

24. Define $x^{\overline{m}}$ (for $x \in \mathbb{R}$).

25. Define $\lceil x \rceil$ and $\lfloor x \rfloor$ (for $x \in \mathbb{R}$). Give examples.

26. Argue: $\lceil x \rceil = x \Leftrightarrow x$ is an integer (for $x \in \mathbb{R}$).

27. Argue: $\lceil x \rceil - \lfloor x \rfloor = [x$ is not an integer $]$ (for $x \in \mathbb{R}$).

28. Argue: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ (for $x \in \mathbb{R}$).

29. Argue: $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (for $n \in \mathbb{Z}, x \in \mathbb{R}$) .

30. Explain: $(n!)^2 = (1 \cdot 2 \ldots n)(1 \cdot 2 \ldots n) = \prod_{k=1}^{n} k(n + 1 - k)$