LARSON—MATH 511—CLASSROOM WORKSHEET 24
Low-Rank and Compressed Sensing

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c24**.

2. (**Rank-k changes**). Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Let $UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (rank-2).
   How does subtraction of UV^T change the inverse of A? (We know the inverse of A. What is the inverse of $M = A - UV^T$?)

3. (**Sherman-Morrison-Woodbury formula**) We checked the formula:

$$M^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1} U)^{-1} V^T A^{-1}$$

   with $M = A - UV^T$ and U, V^T from the previous example.

   Compute M^{-1} using this formula and compare it with the previous computation.

   (**Vandermonde Matrices**) We showed: Given $n+1$ points you can find a unique degree-n polynomial that fits them; that is, given points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, you can find a unique function $f(x) = c_0 + c_1 x^1 + \ldots + c_n x^n$ such that $f(x_0) = y_0, f(x_1) = y_1, \ldots, f(x_n) = y_n$ (assuming all x_i's are different of course).

   Find it by solving $V\hat{c} = \hat{y}$, where $\hat{c} = \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_n \end{bmatrix}$, $\hat{y} = \begin{bmatrix} y_0 \\ y_1 \\ \ldots \\ y_n \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 & x_0^1 & x_0^2 & \ldots & x_0^n \\ 1 & x_1^1 & x_1^2 & \ldots & x_1^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n^1 & x_n^2 & \ldots & x_n^n \end{bmatrix}$$

   We proved: V is invertible iff the x_is are all distinct.

4. (**Philosophical Implications**). What is the next term in the sequence $1, 2, 3, 4 \ldots$? Choose any number for the next term. Find the Vandermonde matrix V for your sequence, find $V^{-1}\hat{y}$ where $\hat{y}$ contains the sequence entries, form a degree-4 polynomial that fits these sequence terms.

Return to Least Squares

(**Sherman-Morrison-Woodbury formula**) If $M = A - UV^T$ and U, V^T then:

$$M^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1}U)^{-1}V^T A^{-1}$$

Suppose we have m data points (each with k *features*):

$$(A_{1,1}, A_{1,2}, \ldots, A_{1,k}, b_1)$$
$$(A_{2,1}, A_{2,2}, \ldots, A_{2,k}, b_2)$$
$$\ldots$$
$$(A_{m,1}, A_{m,2}, \ldots, A_{m,k}, b_m)$$

and we have *already* found the least-squares solution $\hat{c}$ to $f(x) = c_1x_1 + c_2x_2 + \ldots + c_kx_k$, where:

$$f(A_{i,1}, A_{i,2}, \ldots, A_{i,k}) = b_i$$

for $i = 1, 2, \ldots, m$, where $\hat{c} = (A^T A)^{-1}A^T\hat{b}$.

Suppose we then get a new data point:

$$(A_{m+1,1}, A_{m+1,2}, \ldots, A_{m+1,k}, b_{m+1})$$

How can we use the Sherman-Morrison-Woodbury formula to efficiently update $(A^T A)^{-1}$ (for the *updated* data matrix A, given that we already know the $(A^T A)^{-1}$ from the original data points)?

The Derivative of A^{-1}

5. Let $A(t) = \begin{bmatrix} t & \frac{1}{t} \\ t^2 & t^2+1 \end{bmatrix}$. Find $A(1)$, $A(2)$.

6. Find $\frac{dA}{dt}$.

7. Let $A = A(1)$ and $B = A(2)$. Are they invertible?

8. Let $\Delta A = B - A$. Find ΔA.

9. (**A Very Useful Formula**). Check: $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$.

10. Use this to find $\frac{\Delta A^{-1}}{\Delta t}$ and $\frac{dA^{-1}}{dt}$.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c24 worksheet attached" (so that it will be properly recorded).