**LARSON—MATH 310—Homework 13**
**SVD!**

**Show all your work.**

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

1. $A$ is an $m \times n$ matrix. Find $m$ and $n$ and the rank $r$ of $A$.
2. Find $A^T A$.
3. Check that the rank of $A^T A$ is the same as the rank $r$ of $A$.
4. Find the eigenvalues of $A^T A$. There will be $r$ positive eigenvalues: $\sigma_1^2, \ldots, \sigma_r^2$. (Arrange these so $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \sigma_r^2$.
5. Find the *singular* values $\sigma_1, \ldots, \sigma_r$.
6. Find the corresponding eigenvectors for the $r$ positive eigenvalues of $A^T A$ and normalize them. Call these: $\vec{v_1}, \ldots, \vec{v_r}$.
7. Check that your eigenvalue-eigenvector pairs work.
8. The eigenvectors corresponding to the 0-eigenvalues of $A^T A$ are a basis for the null space of $A^T A$. They might not be orthogonal. Use Gram-Schmidt to convert these to an orthogonal (actually, orthonormal) basis for the null space of $A^T A$. Call these $v_{r+1}, \ldots, v_n$.
9. What is an *orthogonal* matrix?
10. Let $V = \begin{bmatrix} v_1 \ldots v_r \ldots v_n \end{bmatrix}$, check that $V$ is $n \times n$, and show that $V$ is orthogonal.
11. For each $i \in \{1, \ldots, r\}$, find $\|A\vec{v_i}\|$.
12. Let $\Sigma$ be the $m \times n$ matrix with the $\sigma_i$'s on the diagonal for $i = 1, \ldots, r$, and 0 for every other entry. Find $\Sigma$.
13. Find $AA^T$.
14. Check that the rank of $AA^T$ is the same as the rank $r$ of $A$.
15. Find the eigenvalues of $AA^T$. Call the positive ones: $\sigma_1^2, \ldots, \sigma_r^2$.
16. Check that your eigenvalue-eigenvector pairs work.
17. Find the corresponding eigenvectors for the eigenvalues of $AA^T$ and normalize them. Call the eigenvectors corresponding to the $r$ positive eigenvalues: $\vec{u_1}, \ldots, \vec{u_r}$.
18. The eigenvectors corresponding to the 0-eigenvalues of $AA^T$ are the null space of $AA^T$. We want an orthogonal basis. Find any basis and use Gram-Schmidt to convert these to an orthogonal (actually, orthonormal) basis for the null space of $AA^T$. Call these $v_{r+1}, \ldots, v_n$.
19. Let $U = \begin{bmatrix} u_1 \ldots u_r \ldots u_m \end{bmatrix}$, check that $U$ is $m \times m$, and show that $U$ is orthogonal.
20. For each $i = 1, \ldots, r$, show that $A\vec{v_i} = \sigma_i \vec{u_i}$.