LARSON—MATH 511—CLASSROOM WORKSHEET 25
Low-Rank and Compressed Sensing

Return to Least Squares

Suppose we have m data points (each with k *features*):

$$(A_{1,1}, A_{1,2}, \ldots, A_{1,k}, b_1)$$

$$(A_{2,1}, A_{2,2}, \ldots, A_{2,k}, b_2)$$

$$\ldots$$

$$(A_{m,1}, A_{m,2}, \ldots, A_{m,k}, b_m)$$

and we have *already* found the least-squares solution $\hat{c}$ to $f(x) = c_1x_1 + c_2x_2 + \ldots + c_kx_k$, where:

$$f(A_{i,1}, A_{i,2}, \ldots, A_{i,k}) = b_i$$

for $i = 1, 2, \ldots, m$, where $\hat{c} = (A^TA)^{-1}A^T\hat{b}$.

Suppose we then get a new data point:

$$(A_{m+1,1}, A_{m+1,2}, \ldots, A_{m+1,k}, b_{m+1})$$

How can we use the Sherman-Morrison-Woodbury formula to efficiently update $(A^TA)^{-1}$ (for the *updated* data matrix A, given that we already know the $(A^TA)^{-1}$ from the original data points)?

We let $\hat{r}$ be the column vector with entries, $A_{m+1,1}, A_{m+1,2}, \ldots, A_{m+1,k}$, the "new A" be $\begin{bmatrix} A \\ \hat{r}^T \end{bmatrix}$ and the "new $\hat{b}$" be $\begin{bmatrix} \hat{b} \\ b_{m+1} \end{bmatrix}$ and found that the new (updated) least-squares solution could be computed by finding:

$$(\begin{bmatrix} A^T \hat{r} \end{bmatrix} \begin{bmatrix} A \\ \hat{r}^T \end{bmatrix})^{-1} = (\begin{bmatrix} A^TA + \hat{r}\hat{r}^T \end{bmatrix})^{-1}.$$

1. Now apply the **Sherman-Morrison-Woodbury formula**:

$$(A - UV^T)^{-1} = A^{-1} + A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}$$

What will you use for U and V?

The Derivative of A^{-1}

2. Let $A(t) = \begin{bmatrix} t & \frac{1}{t} \\ t^2 & t^2+1 \end{bmatrix}$. Find $A(1)$, $A(2)$.

3. Find $\frac{dA}{dt}$.

4. How would we find A^{-1} (shorthand for $A(t)^{-1}$) and $\frac{dA^{-1}}{dt}$?

5. Let $A = A(1)$ and $B = A(2)$. Are they invertible?

7. (**A Very Useful Formula**). Check: $B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}$.

8. Use this to find $\frac{\Delta A^{-1}}{\Delta t}$ and a formula for $\frac{dA^{-1}}{dt}$.

9. Now find $\frac{dA^{-1}}{dt}(1)$.

Sage/CoCalc

10. (a) Start the Chrome browser.
    (b) Go to `http://cocalc.com`
    (c) Login (likely using **your VCU email address**).
    (d) You should see an existing Project for our class. Click on that.
    (e) Click "New", then "Sage Worksheet", then call it **c25**.

11. Let $A(t) = \begin{bmatrix} t & \frac{1}{t} \\ t^2 & t^2+1 \end{bmatrix}$.

12. Find $A(1)$, $A(2)$.

13. Find $\frac{dA}{dt}$. (Sage can find the derivative of a symbolic matrix!)

14. Find A^{-1} (Sage can find the inverse of a symbolic matrix!)

15. Find $\frac{dA^{-1}}{dt}$.

16. Check that $\frac{dA^{-1}}{dt}(-1)$ agrees with what we found earlier.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c25 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!