LARSON—MATH 610–CLASSROOM WORKSHEET 07
Pivot Decomposition Theorem.

Concepts (Chp. 1): field, vector space, $\mathcal{P}$, $\mathbb{F}^n$, $\mathbb{M}_{m\times n}(\mathbb{F})$, subspace, null space, row(A), col(A), list of vectors, span of a list of vectors, linear independence, linear dependence, pivot column decomposition, direct sum $\mathcal{U} \oplus \mathcal{V}$, *orthogonal* matrix, *unitary* matrix.
Review:

(**Theorem 1.7.5, Pivot Column Decomposition**). Let $A = [\hat{a}_1 \dots \hat{a}_n] \in \mathbb{M}_{m\times n}(\mathbb{F})$ be non-zero, let $j_1 < j_2 < \dots < j_s$ be its pivot indices, and let
$P = [\hat{a}_{j_1}\hat{a}_{j_2} \dots \hat{a}_{j_s}] \in \mathbb{M}_{m\times s}(\mathbb{F})$.

(a) The s columns of P are linearly independent, $1 \le s \le n$ and $col(P) = col(A)$.

(b) There is a unique $R = [\hat{r}_1 \dots \hat{r}_n] \in \mathbb{M}_{s\times n}(\mathbb{F})$ such that $A = PR$.

(c) $\hat{r}_{j_k} = \hat{e}_k \in \mathbb{F}^s$, for each $k = 1, \dots, s$.

(d) If $s > 1$, $2 \le k \le s$ and $j_{k-1} < k < j_k$ then $\hat{r}_j \in span\{\hat{e}_1 \dots \hat{e}_{j_{k-1}}\}$.

(e) The rows of R are linearly independent and $null(A) = null(R)$.

Chp. 2 of Garcia & Horn, Matrix Mathematics

(**Lemma 2.1.4. Replacement Lemma**). Let $\mathcal{V}$ be a non-zero $\mathbb{F}$-vector space and let r be a positive integer. Suppose that $\beta = \hat{u}_1, \hat{u}_2, \dots, \hat{u}_r$ spans $\mathcal{V}$. Let $v \in \mathcal{V}$ be non-zero and let

$$\hat{v} = \sum_{i=1}^{r} c_i \hat{u}_i$$

(a) $c_j \neq 0$ for some $j \in \{1, \dots, r\}$,
(b) If $c_j \neq 0$ then the list $\hat{v}, \hat{u}_1, \dots, \widehat{\hat{u}_j}, \dots, \hat{u}_r$ spans $\mathcal{V}$,
(c) If β is a basis for $\mathcal{V}$ and $c_j \neq 0$ then the list in (b) is a basis for $\mathcal{V}$,
(d) If $r \ge 2$, β is a basis for $\mathcal{V}$, $\hat{v} \notin span\{\hat{u}_1, \dots, \hat{u}_k\}$ for some $k \in \{1, 2, \dots, r-1\}$, then there is an index $j \in \{k+1, k+2, \dots, r\}$ such that

$$\hat{v}, \hat{u}_1, \dots, \hat{u}_k, \hat{u}_{k+1}, \dots, \widehat{\hat{u}_j}, \dots, \hat{u}_r$$

is a basis for $\mathcal{V}$.

1. What is this theorem about? Consider the following example: $\beta = \hat{e}_1, \hat{e}_2, \hat{e}_3 \in \mathbb{R}^3$, and $\hat{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

2. Suppose $\beta = \hat{v}_1, \hat{v}_2, \hat{v}_3$ is a basis for a vector space $\mathcal{V}$, and $\hat{v} \in \mathcal{V}$. What does the theorem say. Why is it true?

Claim: If $\hat{v}_1, \ldots, \hat{v}_k, \hat{v}_{k+1}$ are linearly independent, and $\hat{u}_{k+1}, \ldots, \hat{u}_n$ are linearly independent and $\hat{v}_{k+1} \in Span(\hat{v}_1, \ldots, \hat{v}_k) + Span(\hat{u}_{k+1}, \ldots, \hat{u}_n)$ then $\hat{v}_{k+1} \in Span(\hat{u}_{k+1}, \ldots, \hat{u}_n)$.

3. Why is it true?

4. (**Theorem 2.1.10**). Let $\mathcal{V}$ be an $\mathbb{F}$-vector space and let r and n be positive integers. Suppose that $\beta = \hat{u}_1, \ldots, \hat{u}_n$ is a basis for $\mathcal{V}$ and $\gamma = \hat{v}_1, \ldots, \hat{v}_r$ is linearly independent.

   (a) $r \leq n$.

   (b) If $r = n$ then γ is a basis for $\mathcal{V}$.

5. What is this theorem about?

6. (**Corollary 2.1.11, All Bases have Same Cardinality**). If r and n are positive integers and $\hat{v}_1, \ldots, \hat{v}_r$ and $\hat{w}_1, \ldots, \hat{w}_n$ are bases of an $\mathbb{F}$-vector space $\mathcal{V}$ then $r = n$.

7. Why is this true?