Last name _____

First name _____

LARSON—MATH 511—HOMEWORK WORKSHEET 09
Test 1 Review

Write up a careful, complete test review and turn it in **before** our Test 1 on Thursday, Oct. 6. **Explain** everything.

1. Let $\theta = 30^o$ and $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

    (a) Show Q is orthogonal.
    (b) Find $||\hat{x}||$.
    (c) Find $Q\hat{x}$ and $||Q\hat{x}||$.
    (d) Check that $||Q\hat{x}|| = ||\hat{x}||$.
    (e) Explain why $Q\hat{x}$ is a 30° rotation of $\hat{x}$.

2. Find the (column) rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$.

3. Find a CR decomposition of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$.

4. Explain why the column rank of A equals the number of columns of C.

5. Explain why the rows of R are necessarily linearly independent in the CR decomposition.

6. Explain why the row rank of A equals the number of rows of R.

7. Find an example of a positive definite matrix.

8. Use the equivalent *energy* definition to show that $A = \begin{bmatrix} 3 & 1 \\ 1 & -7 \end{bmatrix}$ is *not* positive definite.

9. Explain why $A^T A$ is symmetric for *any* matrix A.

10. Show that the eigenvalues of a real symmetric matrix are real.

11. Show that, if S is a symmetric matrix and λ_1 and λ_2 are distinct eigenvalues with corresponding eigenvectors $\hat{v}_1$, $\hat{v}_2$ then $A\hat{v}_1$ and $A\hat{v}_2$ are orthogonal.

SVD

12. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. What is the rank r of A?

14. Find AA^T.

15. Find the r positive eigenvalues of AA^T (you should do this by hand—as if it were the test—but also check them with Sage/CoCalc). The remaining eigenvalues must be 0.

16. Find corresponding eigenvectors $\hat{u}_1, \ldots, \hat{u}_r$, and normalize them.

17. Find an orthogonal basis for the null space of AA^T and call these vectors $\hat{u}_{r+1}, \ldots, \hat{u}_m$ (where m is the number of rows of A), and normalize them. (These are also eigenvectors of AA^T corresponding to 0).

18. Let $U = [\hat{u}_1 \ldots \hat{u}_r \ldots \hat{u}_m]$.

19. Find the r positive singular values $\sigma_1 \geq \ldots \geq \sigma_r$ of A.

20. Find $A^T A$.

21. The eigenvalues of $A^T A$ must be the same as the ones you found for AA^T. Find corresponding eigenvectors $\hat{v}_1, \ldots, \hat{v}_r, \ldots, \hat{v}_n$ (where n is the number of rows of A), and normalize them.

22. Let $V = [\hat{v}_1 \ldots \hat{v}_r \ldots \hat{v}_n]$.

23. Let Σ be the $m \times n$ "diagonal" matrix with $\Sigma_{i,i} = \sigma_i$.

24. Check that $A = U\Sigma V^T$.

25. Check that $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots \sigma_r \hat{u}_r \hat{v}_r^T$.

26. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$. Follow all the steps to find the SVD of A.

    (You can either start as above with AA^T and find the $\hat{u}$'s first or with $A^T A$ and find the $\hat{v}$'s first; we proved the positive eigenvalues of $A^T A$ and AA^T must be the same).