LARSON—MATH 511—CLASSROOM WORKSHEET 11
Gilbert Strang Lecture 7.

More on Strang's Lectures

1. We showed that A^TA is symmetric. It is a **key fact** that if A has linearly independent columns then A^TA is positive definite (and thus had positive eigenvalues). Why?

2. We showed that A^TA is symmetric. It is a **key fact** that A^TA is positive semi-definite (and thus has non-negative eigenvalues). Why?

SVD Algorithm (general case, A has rank r)

Suppose A is **any** $m \times n$ matrix.

- Find A^TA (its $n \times n$)
  This matrix has rank r, is symmetric and is positive semi-definite.

- Find the positive eigenvalues and corresponding (unit) eigenvectors λ, $\hat{v}$ (So $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$). The remaining eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$ are 0. The remaining (unit) eigenvectors $\hat{v}_{r+1}, \ldots, \hat{v}_n$ are a basis for the nullspace of A^TA. Let Λ be the diagonal matrix with the λs on the diagonal.

  Vectors corresponding to different eigenvalues are orthogonal. For each eigenspace with dimension greater than 1, Gram-Schmidt can be used to find an orthogonal basis.

- Let $V = [\hat{v}_1 \ldots \hat{v}_r \ldots \hat{v}_n]$.
  V is orthogonal.

- Let $\sigma_i = \sqrt{\lambda_i}$ and let Σ be the $m \times n$ matrix with diagonal entries $\sigma_1, \ldots, \sigma_r$

- Find AA^T (its $m \times m$)
  This matrix has rank r, is symmetric and is positive semi-definite.

- Check that $A\hat{v}_i$ is an eigenvector for AA^T with corresponding eigenvalue λ_i. This means that these are all the non-zero eigenvalues for AA^T. Let $u_i = \frac{1}{\sigma_i}A\hat{v}_i$ for $i = 1 \ldots r$. Check that these $\hat{u}_i$ are unit.
- The remaining eigenvalues of AA^T, $\lambda_{r+1}, \ldots, \lambda_m$ are 0. The remaining (unit) eigenvectors $\hat{u}_{r+1}, \ldots, \hat{u}_m$ are a basis for the nullspace of AA^T.
- Let $U = [\hat{u}_1 \ldots \hat{u}_r \ldots \hat{u}_m]$.
  U is orthogonal.

- $AV = U\Sigma$ and $A = U\Sigma V^T$.

3. **Low Rank Approximation**. Why does $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$?

Sage/CoCalc

(a) Start the Chrome browser.

(b) Go to `http://cocalc.com`

(c) Login (likely using **your VCU email address**).

(d) You should see an existing Project for our class. Click on that.

(e) Click "New", then "Sage Worksheet", then call it **c11**.

We will find the *singular values* $\sigma_1 \geq \ldots \sigma_r > 0$ of a matrix A with rank r in three ways: (1) via the eigenvalues of $A^T A$, (2) via the eigenvalues of AA^T, and (3) using Sage's built-in SVD method.

4. Input $A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ (remember to inform Sage you mean for the entries to be interpreted as elements of a Real Double Field (RDF).

5. Find the rank of A.

6. Find $A^T A$, its eigenvalues, and the singular values of A.

7. Find AA^T, its eigenvalues, and the singular values of A.

8. Try: `A.SVD()`. See the `help` in order to interpret the output.

9. Check that $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$.

10. Repeat for the matrix $A = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & 1 \end{bmatrix}$.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c11 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!