Last name ______________________

First name ______________________

LARSON—MATH 556–HOMEWORK WORKSHEET 09
Posets!

1. (**Divisibility poset**). Let $X = \{2, 3, \dots 20\}$ and define the divisibility relation "|": For $x, y \in X$, $x|y$ (that is, *x divides y*, or *y is divisible by x*)) if there is an integer k such that $kx = y$. We showed that $P = (X, |)$ is a poset.
   - Find a largest antichain in P.
   - Find a minimal chain decomposition in P.
   - Prove that your antichain is indeed maximal and your chain decomposition is indeed minimal.

2. Let $X = \{x_1, x_2, \dots, x_{101}\}$ be a sequence of 101 positive integers. Argue that X contains either an 11-term increasing subsequence or an 11-term decreasing subsequence. (**Hint:** Define an appropriate poset).

3. Let
$$A = \begin{pmatrix} 7/12 & 0 & 5/12 \\ 1/6 & 1/2 & 1/3 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$
   Use Birkhoff's algorithm to show that A is a convex combination of permutation matrices. (So for each iteration, you'll need to define a bipartite graph G_i, find a perfect matching M_i, a corresponding permutation matrix P_i and number m_i. If you get a 0 matrix after k iterations, you will have $A = m_1P_1 + \dots m_kP_k$ and $m_1 + \dots m_k = 1$.

Birkhoff's Algorithm (from class notes)

Let A_1 be a non-negative square matrix with constant (non-zero) row and column sums.

1. Let G_i be the associated bipartite graph (whose points represent the rows and columns of A_i and where ρ_j is adjacent to c_k if $(A_i)_{j,k}$ is non-zero).
2. Let M_i be a perfect matching in G_i.
3. Each line of M_i corresponds to an entry in A_i, each in a different row and different column. Let m_i be the minimum of these entries.
4. Let P_i be the permutation matrix with 1 entries in the coordinates corresponding to M_i.
5. Let $A_{i+1} = A_i - m_iP_i$.
6. If A_{i+1} is non-zero, repeat.
7. Else, if A_{i+1} is the zero matrix, then $A_i = m_1P_1 + m_2P_2 + \dots m_{i-1}P_{i-1}$.