Last name ______________________

First name ______________________

LARSON—MATH 610–CLASSROOM WORKSHEET 10
Linear Transformations.

Concepts (Chp. 1): field, vector space, $\mathcal{P}$, $\mathbb{F}^n$, $\mathbb{M}_{m\times n}(\mathbb{F})$, subspace, null space, row(A), col(A), list of vectors, span of a list of vectors, linear independence, linear dependence, pivot column decomposition, direct sum $\mathcal{U}\oplus\mathcal{V}$, *orthogonal* matrix, *unitary* matrix, *basis*, *dimension*.

Review:

1. (**Theorem 2.3.1**) Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. Then
$$dim(col(A)) = dim(col(A^T)) = dim(row(A)) \leq \min\{m, n\}.$$
2. (**Theorem 2.3.7. Full Rank Factorization**). Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$ be non-zero, let $r = rank(A)$, and let the columns of $X \in \mathbb{M}_{m\times r}(\mathbb{F})$ be a basis for $col(A)$. Then there is a unique $Y \in \mathbb{M}_{r\times n}(\mathbb{F})$ such that $A = XY$. Moreover, $rank(Y) = r$, the rows of Y are a basis for $row(A)$ and $null(A) = null(Y)$.
3. What is the β-*basis representation function*? What are the *coordinates* of a vector with respect to a basis?
4. What is a *linear transformation*?

Chp. 2 of Garcia & Horn, Matrix Mathematics

1. How does Full Rank Factorization show that, if $A \in \mathbb{M}_{n\times n}$ is invertible, then the columns of A are linealy independent?
2. How does the algorithm in our All Bases have the Same Cardinality argument show that *any* linearly independent set of vertices in a finite-dimensional vector space can be extended to a basis?
3. What is $\mathfrak{L}(\mathcal{V}, \mathcal{W})$?

4. (**Rank-Nullity Theorem**). Let $\mathcal{V}, \mathcal{W}$ be vector spaces with $dim(\mathcal{V}) = n$ and $dim(\mathcal{W}) = m$. Let $T \in \mathfrak{L}(\mathcal{V}, \mathcal{W})$. And let $\vec{v}_1, \ldots, \vec{v}_k$ be a basis for the $null(T)$. Argue:
$$rank(T) + nullity(T) = dim(\mathcal{V}).$$

5. How does any matrix $A \in \mathbb{M}_{m\times n}$ define a linear transformation?

6. How does any linear transformation $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and bases $\beta = \hat{v}_1, \ldots, \hat{v}_n$ of $\mathcal{V}$ and $\gamma = \hat{w}_1, \ldots, \hat{w}_m$ of $\mathcal{W}$ define a matrix $A \in \mathbb{M}_{m\times n}$?

7. What is ${}_\gamma[T]_\beta$?

8. What is the β-γ *change-of-basis* matrix (notation: ${}_\gamma[I]_\beta$)?

9. What is an example?