Last name ______________________

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LARSON—MATH 610—CLASSROOM WORKSHEET 14
Minimal Polynomials, Invariant Subspaces, and Upper-Triangular Matrices.

Concepts & Notation

- (Chp. 5) *eigenvalue*, *eigenvector*, *invariant subspace*, *minimal polynomial*,
- (Chp. 8) *generalized eigenvector*.

1. (**Claim**:) Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

Minimal Polynomials

2. (**Existence, uniqueness, and degree of minimal polynomial**). If V is a finite-dimensional vector space, and $T \in \mathcal{L}(V)$, then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree with $p(T) = 0$.

3. What is the *minimal polynomial* of $T \in \mathcal{L}(V)$ (for finite-dimensional V)?

Invariant Subspaces and Upper-Triangular Matrices

4. What is an *invariant subspace* of $T \in \mathcal{L}(V)$?

5. For $T \in \mathcal{L}(V)$ the *eigenspace* $U = \{v : T(v) = \lambda v\}$ corresponding to an eigenvalue λ of T is an invariant subspace of T.

6. (**Claim**:) Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. Then the following are equivalent:

   (a) the matrix of T with respect to $(v_1, \ldots, v_n)$ is upper-triangular;

   (b) $T(v_k) \in span(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$;

   (c) $span(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

7. (**Claim**:) Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Complex Vector Spaces

8. Let $T \in \mathcal{L}(\mathbb{F}^3)$ with $T(z_1, z_2, z_3) = (4z_1, 0, 5z_3)$. Check that $\mathbb{C}^3 = null\ T^{dim\ \mathbb{C}^3} \oplus range\ T^{dim\ \mathbb{C}^3}$.

9. (**Claim**:) Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then $V = null\ T^{dim\ V} \oplus range\ T^{dim\ V}$.