Last name ______________________

First name ______________________

LARSON—OPER 731—HOMEWORK WORKSHEET 14
Test 2 Review

1. What is a *face* of a polytope? What is a *facet* of a polytope? What is an *extreme point* of a polytope?

2. Find the extreme points in the polyhedron defined by the following linear inequalities ($x \in \mathbb{R}$, $x_i \geq 0$). Explain *how* to find *all* the extreme points systematically.

$$\begin{array}{ccccccc} x_0 & + & x_1 & & & \leq & 1 \\ x_0 & & & + & x_2 & \leq & 1 \\ & & x_1 & + & x_2 & \leq & 1 \end{array}$$

3. This polyhedron is a polytope. Find a convex hull description of this polytope.

4. What is the definition of the *dimension* of a polytope? What is the dimension of the above polytope? Explain.

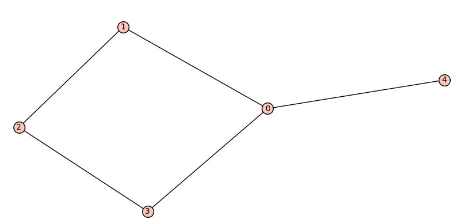
5. What is a *totally unimodular matrix*? Give a small example and show why it is totally unimodular from the definition.

6. Is the following matrix A is *totally unimodular*? Explain. $\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$.

7. Argue that if you multiply a row of a totally unimodular matrix A by -1 to get matrix A' then A' is totally unimodular. Give an example.

8. Show that the vertex-edge incidence matrix of a bipartite graph is totally unimodular. Give an example.

9. State the **Total Unimodularity implies Integrality** theorem. Explain its significance. Give an example.



10. What is a *matching* in a graph, find a maximum matching in the *tadpole graph* T (above) and explain why your matching is maximum.

11. What is a *vertex cover*, find a minimum vertex cover in the tadpole graph T, and explain why your vertex cover is minimum.

12. What is a *bipartite graph*, explain why the tadpole graph T is bipartite, and what König's Theorem says about T.

13. *Model* finding a maximum matching in the tadpole graph T as an integer programming (IP) problem. Explain how your IP in fact models this combinatorial problem: what do your variables represent, what do your constraints model, what will a solution corresponding to the IP optimum represent.

14. What is the constraint matrix A from your IP model? Explain why it is totally unimodular.

15. If the integer decision variables in your IP are allowed to be real (relaxed), why will the optimal value of the IP objective function equal the optimum value of this LP's objective function?

16. What is a *matroid*? Let $E(M) = \{a, b, c, d, e, f\}$. Let $\mathcal{I}(M)$ be all subsets of $E(M)$ with no more than 3 elements. Argue that $M = (E(M), \mathcal{I}(M))$ is a matroid.

17. What is the *rank* of a matroid? What is the rank of the matroid M of the last example? Explain.

18. What is a *base* of a matroid? Find all the bases of M. Explain.

19. What is a *tree*? What is a *spanning tree* in a graph? Find a spanning tree for the tadpole T.

20. What is a *forest* in a graph? List all the forests in the tadpole graph T.

21. What is a *graphic matroid*? Find the graphic matroid for the tadpole graph T. Explain.

22. What is the rank of this matroid? Explain.

23. Why, if the edges Y of a graph G induce a forest and the edges X of G induce a forest and $|Y| > |X|$, must it be the case that there is an edge $e \in Y$ such that $X \cup \{e\}$ induces a forest in G?

24. What is a *linear matroid*? Let $A = \begin{pmatrix} 0 & 1 & 2 & 4 & 6 \\ 1 & 0 & 3 & 5 & 7 \end{pmatrix}$. Describe the linear matroid M corresponding to A.

25. What is the rank of M? Explain.

26. What is the (cardinality) *greedy algorithm* for a matroid M? *Why* does the greedy algorithm *always* produce a largest cardinality independent set?

27. For the tadpole graph T define edge weights which are the sum of the edge's vertex labels. Define an appropriate matroid and use the *weighted greedy algorithm* to find a **minimum** weight spanning tree. Explain.

28. Find the **matching matroid** M for the tadpole graph T, with $E(M) = V(T)$..

29. Find the **matching polytope** $\mathcal{P}_{\mathcal{M}(T)}$ for the tadpole graph T.

30. Find the polyhedron $\mathcal{P}$ defined by the following constraints: one non-negative variable x_i for each edge e_i of the tadpole graph T and one "vertex" constraint for each vertex of T which models the requirement that the sum of the variables corresponding to the edges incident to any vertex is no more than 1. Does $\mathcal{P} = \mathcal{P}_{\mathcal{M}(T)}$?