LARSON—MATH 511—CLASSROOM WORKSHEET 19
Pseudo-inverses & Gram-Schmidt

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c19**.

2. How is the *pseudo-inverse* (Moore-Penrose inverse) A^+ defined? Find the pseudo-inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3. Find $A * A^+$ and $A^+ * A$.

4. By definition, *every* matrix (square or not) has a pseudo-inverse. Find the pseudo-inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

5. Find $A * A^+$ and $A^+ * A$.

6. Define any symmetric 3×3 rank-3 matrix A. A must have an inverse. Find A^{-1} and A^+.

7. A QR decomposition of a $m \times n$ matrix A is an $m \times m$ orthogonal matrix Q and $m \times n$ upper-triangular matrix R where $A = QR$ (these are not necessarily unique). Use Sage to find a QR decomposition of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

   (The QR algorithm). The goal is to input a square matrix A and output a similar triangular matrix (and thus yielding the eigenvalues of A). We will repeatedly find a QR factorization of A, and then let $A = RQ$; this *new* A must be similar to the original A.

8. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. First use Sage to find the eigenvalues of A for reference. Then try several iterations of this QR algorithm and see if we get the claimed similar matrix.

9. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. First use Sage to find the eigenvalues of A for reference. Then try several iterations of this QR algorithm and see if we get the claimed similar matrix.

Pseudo-inverses

10. Check that $A^+ = A^{-1}$ when A is invertible and (thus) $\hat{x} = A^+\hat{b}$ is the solution of $A\hat{x} = \hat{b}$.

11. Suppose A is a matrix with linearly independent columns. One "solution" of $A\hat{x} = \hat{b}$ (even when $\hat{b}$ isn't in the column space of A) is $\hat{x} = (A^TA)^{-1}A^T\hat{b}$. Check that when A has linearly independent columns that $A^+ = (A^TA)^{-1}A^T$ (and so $\hat{x} = A^+\hat{b}$).

12. Solve $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Gram-Schmidt

13. Use Gram-Schmidt to find an orthogonal basis, $\hat{q}_1, \hat{q}_2, \hat{q}_3$, of the columns of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

14. Let Q be the matrix whose columns are $\hat{q}_1$, $\hat{q}_2$, and $\hat{q}_3$. Write $A = QR$ (for some matrix R).

15. What can we say about R? and how will this QR decomposition of A help us solve $A\hat{x} = \hat{b}$?

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c19 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!