LARSON—MATH 610–CLASSROOM WORKSHEET 06
Pivot Decomposition Theorem.

Chp. 1 of Garcia & Horn, Matrix Mathematics

1. (**Motivating Example**) Find a maximal set of linearly independent columns by greedily choosing the first non-zero column vector, adding the next available column vector, and iterating (until no column remain).

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(**Theorem 1.7.1**) Let $\beta = \hat{v}_1, \ldots, \hat{v}_p$ be a nonzero list of vectors in an $\mathbb{F}$-vector space. There is an $s \in \{1, \ldots, p\}$ and unique indices $j_1, \ldots, j_s$ such that:

(a) $1 \leq j_1 < j_2 < j_s \leq p$.

(b) $\gamma = \hat{v}_{j_1}, \hat{v}_{j_2}, \ldots, \hat{v}_{j_s}$ is linearly independent.

(c) $span(\gamma) = span(\beta)$.

(d) If $j < j_1$ then $\hat{v}_j = \hat{0}$.

(e) If $s > 1$, $2 \leq k \leq s$ and $j_{k-1} < j < j_k$, then $\hat{v}_j \in span\{v_{j_1}, \hat{v}_{j_2}, \ldots, \hat{v}_{j_{k-1}}\}$.

2. What is this theorem about, and why is it true?

(**Theorem 1.7.5, Pivot Column Decomposition**). Let $A = [\hat{a}_1 \dots \hat{a}_n] \in \mathbb{M}_{m\times n}(\mathbb{F})$ be non-zero, let $j_1 < j_2 < \dots < j_s$ be its pivot indices, and let $P = [\hat{a}_{j_1}\hat{a}_{j_2} \dots \hat{a}_{j_s}] \in \mathbb{M}_{m\times s}(\mathbb{F})$.

(a) The s columns of P are linearly independent, $1 \leq s \leq n$ and $col(P) = col(A)$.

(b) There is a unique $R = [\hat{r}_1 \dots \hat{r}_n] \in \mathbb{M}_{s\times n}(\mathbb{F})$ such that $A = PR$.

(c) $\hat{r}_{j_k} = \hat{e}_k \in \mathbb{F}^s$, for each $k = 1, \dots, s$.

(d) If $s > 1$, $2 \leq k \leq s$ and $j_{k-1} < k < j_k$ then $\hat{r}_j \in span\{\hat{e}_1 \dots \hat{e}_{j_{k-1}}\}$.

(e) The rows of R are linearly independent and $null(A) = null(R)$.

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(**Lemma 2.1.4. Replacement Lemma**). Let $\mathcal{V}$ be a non-zero $\mathbb{F}$-vector space and let r be a positive integer. Suppose that $\beta = \hat{u}_1, \hat{u}_2, \dots, \hat{u}_r$ spans $\mathcal{V}$. Let $v \in \mathcal{V}$ be non-zero and let

$$\hat{v} = \sum_{i=1}^{r} c_i \hat{u}_i$$

(a) $c_j \neq 0$ for some $j \in \{1, \dots, r\}$,
(b) If $c_j \neq 0$ then the list $\hat{v}, \hat{u}_1, \dots, \widehat{\hat{u}_j}, \dots, \hat{u}_r$ spans $\mathcal{V}$,
(c) If β is a basis for $\mathcal{V}$ and $c_j \neq 0$ then the list in (b) is a basis for $\mathcal{V}$,
(d) If $r \geq 2$, β is a basis for $\mathcal{V}$, $\hat{v} \notin span\{\hat{u}_1, \dots, \hat{u}_k\}$ for some $k \in \{1, 2, \dots, r-1\}$, then there is an index $j \in \{k+1, k+2, \dots, r\}$ such that

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