Last name ________________________

First name ________________________

LARSON—MATH 550—CLASSROOM WORKSHEET 33
Inversion & Derangements.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Inversion

Let $f(n)$ be a function defined on $\mathbb{Z}^{\geq 0}$, and define:

$$g(n) = \sum_k \binom{n}{k}(-1)^k f(k).$$

We **proved**:

$$f(n) = \sum_k \binom{n}{k}(-1)^k g(k).$$

Derangements

A *derangement* of n objects is an ordering (listing, or permutation) of those objects so that none is in its original place; or, more formally, it is a bijection $\phi : [n] \to [n]$ such that $\phi(i) \neq i$, for every $i \in [n]$.

Let ¡n be the number of derangements of n objects.

Let $h(n, k)$ be the number of permutations of n objects where exactly k are fixed.

1. Find $h(4, 0)$, $h(4, 1)$, $h(4, 2)$, $h(4, 3)$, and $h(4, 4)$.

2. Why does
$$n! = \sum_k h(n,k)?$$

3. Why does
$$\sum_k h(n,k) = \sum_k \binom{n}{k} \text{¡}(n-k)?$$

4. Why does
$$\sum_k \binom{n}{k} \text{¡}(n-k) = \sum_k \binom{n}{k} \text{¡}(k)?$$

5. Use the Inversion Theorem to get a formula for ¡n.

6. Can this sum be simplified?

7. What is $\frac{1}{e}$?

8. KGP claim that ¡n converges to $\frac{n!}{e}$. What is their argument?