Last name ______________________

First name ______________________

LARSON—MATH 550—CLASSROOM WORKSHEET 37
Generating Functions & Convolutions.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Generating Functions

A sequence $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ can be represented as the coefficients of the formal power series,

$$A(z) = a_0 + a_1 z^1 + a_2 z^2 + \ldots = \sum_{k \geq 0} a_k z^k,$$

called the *generating function* of the sequence, with notation $a_n = [z^n]A(z)$.

1. Let $B(z)$ be the generating function of the sequence $\langle b_n \rangle = \langle b_0, b_1, b_2, \ldots \rangle$. Find the coefficient of z^n in $A(z)B(z)$, that is, find c_n where $c_n = [z^n]A(z)B(z)$.

   The sequence $\langle c_n \rangle = \langle c_0, c_1, c_2, \ldots \rangle$ is the *convolution* of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

   Examples

2. Express $\frac{1}{(1-z)^{n+1}}$ as a power series.

3. Let $n = 0$. What do you notice?

4. What is the sequence $\langle b_n \rangle$ corresponding to this generating function?

5. Let $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ be any sequence. Find the convolution of $\langle b_n \rangle$ and $\langle a_n \rangle$.

6. Express $\frac{z^n}{(1-z)^{n+1}}$ as a power series. What do you notice?

7. Earlier we found the recurrence $n! = \sum_k \binom{n}{k}$ ¡$(n-k)$. Divide both sides by $n!$ and simplify.

8. What sequence does e^z generate?

9. Let $D(z) = \sum_{k \geq 0} \frac{¡k}{k!} z^k$, and explain why $\frac{1}{1-z} = e^z D(z)$.

10. Solve for $D(z)$ and equate coefficients of z^n. What do you find?