Last name ______________________

First name ______________________

LARSON—OPER 731—CLASSROOM WORKSHEET 26
Matroid Polytopes & Matchings

1. What is the *rank* of a matroid?

2. (**Scheduling**) Jobs $1, 2, \ldots, 10$ are to be processed by a single machine. Each job requires one hour to process, and has a profit c_j. The problem is to find an ordering of the jobs that maximizes total profit. Let the independent sets of a matroid M be sets of jobs that can mutually be completed by the deadline d_j.

Job j	c_j	d_j
1	20	3:00 PM
2	15	1:00 PM
3	10	2:00 PM
4	10	1:00 PM
5	6	2:00 PM
6	4	5:00 PM
7	3	5:00 PM
8	2	4:00 PM
9	2	2:00 PM
10	1	6:00 PM

Matroid Polytopes

Let M be a matroid. The convex hull of the characteristic vectors of the independent sets in M defines a polytope $\mathcal{P}_{\mathcal{I}(M)}$.

3. Let G be a path graph with 3 vertices and 2 edges. Let M be the corresponding graphic matroid. Find $\mathcal{P}_{\mathcal{I}(M)}$.

We know from Minkowski's Theorem that there is a representation of this polytope as an intersection of half-spaces (defined by linear inequalities). In fact there is an attractive theorem that gives us the inequalities in terms of the rank $r = r(M)$.

Matroid Polytope Theorem. For any matroid M,

$$\mathcal{P}_{\mathcal{I}(M)} = \{x \in \mathbb{R}^{E(M)} : \sum_{e \in T} x_e \leq r_M(T) \ \forall T \subseteq E(M)\}$$

4. Check that this the convex hull description of the polytope $\mathcal{P}_{\mathcal{I}(M)}$ of the graphic matroid in the last example is in fact the polytope defined half-space description of $\mathcal{P}_{\mathcal{I}(M)}$ given in the Matroid Polytope Theorem.

Matchings, Matching Matroid & the Matching Polytope

5. What is a *matching*? in a graph? What is a *maximum matching*?

6. What is $\delta(v)$? If M is a matching, what vertices are *covered* by M? What vertices are *exposed* by M?

7. If M is a matching in a graph, what is an *alternating path*? What is an *augmenting path*?

8. What is *Berge's Theorem*? Why is it true?

9. (**Matching Matroid Theorem**) Let G be a graph and $W \subseteq V(G)$. $M = (E(M), \mathcal{I}(M))$ with $E(M) = W$ and

$$\mathcal{I}(M) = \{X \subseteq W : G \text{ has a matching that covers } X\}.$$

is a matroid.