Last name ______________________

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LARSON—MATH 601—HOMEWORK WORKSHEET h11
The "Big" Determinant Formula

For be an $n \times n$ matrix A over a *commutative ring* define:

$$\det A = \sum_{\sigma \in S_n} (sgn\ \sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

We'll investigate what this definition says.

Let $[n] = \{1, 2, \ldots, n\}$. Let S_n be the set of bijective functions $\sigma : [n] \to [n]$. (These functions are called *permutations* or *permutation functions* as they can be viewed as re-ordering—permuting—the elements of $[n]$.)

1. Write out all of the functions $\sigma : [2] \to [2]$, by writing explicitly what their values are. (There are $2! = 2$ of them).

2. Write out all of the functions $\sigma : [3] \to [3]$, by writing explicitly what their values are. (There are $3! = 6$ of them).

3. **Argue** that there are $4! = 24$ functions (bijections) in S_4.

4. Consider the function $\sigma \in S_4$ defined as follows:

$$\sigma(1) = 3$$
$$\sigma(2) = 1$$
$$\sigma(3) = 2$$
$$\sigma(4) = 4$$

This function can be represented compactly in cycle notation as $(1, 3, 2)(4)$. The first part $(1, 3, 2)$ says $1 \to 3 \to 2 \to 1$, while the second part (4) says $4 \to 4$. (It is also conventional to drop any "cycles" consisting of a single number mapping to itself—any missing numbers can be assumed to map to themselves.)

Write the function $\gamma \in S_4$ in cycle notation where:

$$\gamma(1) = 2$$
$$\gamma(2) = 1$$
$$\gamma(3) = 4$$
$$\gamma(4) = 3$$

5. Functions in S_n can be composed (or "multiplied"). For σ, γ above, **find** $\sigma \circ \gamma$ (where $\sigma \circ \gamma(n) = \sigma(\gamma(n))$

6. **Argue** that if $\sigma, \gamma \in S_n$ then $\sigma \circ \gamma$ is in S_n (that is, that $\sigma \circ \gamma$ is a bijection from $[n]$ to $[n]$).

   A function σ in S_n where $\sigma(a) = b$ and $\sigma(b) = a$ (with $a \neq b$) and is the identity for every other element is a *transposition*. So, for instance the function $\sigma \in S_4$ defined by:

$$\sigma(1) = 3$$
$$\sigma(2) = 2$$
$$\sigma(3) = 1$$
$$\sigma(4) = 4$$

   is a transposition. It can be written in cycle notation as: $(1, 3)$. (By definition and convention, every transposition can be written as a single cycle with two entries).

   Importantly, any function if S_n can be written as product of transpositions. Let $\sigma = (1, 3, 2, 4) \in S_4$. **Check** that $\sigma = (1, 3) \circ (3, 2) \circ (2, 4)$ (that is, as a composition of three transpositions, written more simply as $\sigma = (1, 3)(3, 2)(2, 4)$).

   For $\sigma \in S_n$, define $sgn\ \sigma$ to be 1 if the number of transpositions of σ is even when it is written as a product (composition) of transpositions, and -1 if σ is an odd number of transpositions.

   So for $\sigma = (1, 3, 2, 4) = (1, 3)(3, 2)(2, 4) \in S_4$, we have $sgn\ \sigma = -1$ as σ is a product of **three** transpositions, which is odd.

7. Write each function/permutation $\sigma \in S_2$ as a product of transpositions, and then **find** $sgn\ \sigma$.

8. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Use the following formula to **find** $\det A$:

$$\det A = \sum_{\sigma \in S_2} (sgn\ \sigma) \prod_{i=1}^{2} A_{i,\sigma(i)}.$$

9. Write each function/permutation $\sigma \in S_3$ as a product of transpositions, and then **find** $sgn\ \sigma$.

10. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Use the following formula to **find** $\det A$:

$$\det A = \sum_{\sigma \in S_3} (sgn\ \sigma) \prod_{i=1}^{3} A_{i,\sigma(i)}.$$