Last name ______________________

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LARSON—MATH 610–CLASSROOM WORKSHEET 05
Pivot Decomposition Theorem.

Concepts (Chp. 1): field, vector space, $\mathcal{P}$, $\mathbb{F}^n$, $\mathbb{M}_{m\times n}(\mathbb{F})$, subspace, null space, row(A), col(A), list of vectors, span of a list of vectors, linear independence, linear dependence, pivot column decomposition, direct sum $\mathcal{U}\oplus\mathcal{V}$, *orthogonal* matrix, *unitary* matrix.

Review:

(**Theorem 1.5.9**). Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of an $\mathbb{F}$-vector space $\mathcal{V}$ and suppose that $\mathcal{U}\cap\mathcal{W}=\{0\}$. Then each vector in $\mathcal{U}\oplus\mathcal{W}$ is uniquely expressible as a sum of a vector in $\mathcal{U}$ and a vector in $\mathcal{W}$.

Chp. 1 of Garcia & Horn, Matrix Mathematics

1. (**Motivating Example**) Find a maximal set of linearly independent columns by greedily choosing the first non-zero column vector, adding the next available column vector, and iterating (until no column remain).

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(**Theorem 1.7.1**) Let $\beta=\hat{v}_1,\ldots,\hat{v}_p$ be a nonzero list of vectors in an $\mathbb{F}$-vector space. There is an $s\in\{1,\ldots,p\}$ and unique indices $j_1,\ldots,j_s$ such that:

(a) $1\le j_1<j_2<j_s\le p$.

(b) $\gamma=\hat{v}_{j_1},\hat{v}_{j_2},\ldots,\hat{v}_{j_s}$ is linearly independent.

(c) $span(\gamma)=span(\beta)$.

(d) If $j<j_1$ then $\hat{v}_j=\hat{0}$.

(e) If $s>1$, $2\le k\le s$ and $j_{k-1}<j<j_k$, then $\hat{v}_j\in span\{v_{j_1},\hat{v}_{j_2},\ldots,\hat{v}_{j_{k-1}}\}$.

2. What is this theorem about?

3. Why is this theorem true?

4. (**Theorem 1.7.5, Pivot Column Decomposition**). Let $A = [\hat{a}_1 \ldots \hat{a}_n] \in \mathbb{M}_{m\times n}(\mathbb{F})$ be non-zero, let $j_1 < j_2 < \ldots < j_s$ be its pivot indices, and let $P = [\hat{a}_{j_1} \hat{a}_{j_2} \ldots \hat{a}_{j_s}] \in \mathbb{M}_{m\times s}(\mathbb{F})$.

   (a) The s columns of P are linearly independent, $1 \leq s \leq n$ and $col(P) = col(A)$.

   (b) There is a unique $R = [\hat{r}_1 \ldots \hat{r}_n] \in \mathbb{M}_{s\times n}(\mathbb{F})$ such that $A = PR$.

   (c) $\hat{r}_{j_k} = \hat{e}_k \in \mathbb{F}^s$, for each $k = 1, \ldots, s$.

   (d) If $s > 1$, $2 \leq k \leq s$ and $j_{k-1} < k < j_k$ then $\hat{r}_j \in span\{\hat{e}_1 \ldots \hat{e}_{j_{k-1}}\}$.

   (e) The rows of R are linearly independent and $null(A) = null(R)$.

5. What is this theorem about?

6. Why is this theorem true?