LARSON—MATH 511—CLASSROOM WORKSHEET 21
Gram-Schmidt and Random Matrix Multiplication Experiments

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c21**.

2. Open your CoCalc project Handouts folder, click on "random_matrix_multiplication.sage". We'll need code from this file. We will run the code here step-by-step in your c21 worksheet.

(Original) Gram-Schmidt

Idea: Given linearly independent vectors $\hat{a}_1$, $\hat{a}_2$,...,$\hat{a}_n$, let $\hat{q}_1 = \frac{1}{||\hat{a}_1||}\hat{a}_1$, and at each step i ($i = 2, ..i = n$):

- Let $\hat{a}'_i$ be $\hat{a}_i$ minus the projection of $\hat{a}_i$ on each of the previously found $\hat{q}_1$,...,$\hat{q}_{i-1}$.
- Let $\hat{q}_i = \frac{1}{||\hat{a}'_i||}\hat{a}'_i$.

We'll see that this mathematically correct idea can produce incorrect results. How can we test if the produced "orthogonal" matrix Q is indeed orthogonal? What other tests or measurements can we think up?

Improved Gram-Schmidt

Idea: Given linearly independent vectors $\hat{a}_1$, $\hat{a}_2$,...,$\hat{a}_n$, let $\hat{q}_j = \frac{1}{||\hat{a}_j||}\hat{a}_j$, where $||\hat{a}_i||$ is a maximum, and at each step i ($i = 2, ..i = n$):

- For remaining (not-yet-processed) $\hat{a}_i$'s, let new $\hat{a}_i$ be current $\hat{a}_i$ minus the projection of $\hat{a}_i$ on $\hat{q}_{i-1}$.
- Find the largest-norm remaining $\hat{a}_i$.
- Let $\hat{q}_i = \frac{1}{||\hat{a}'_i||}\hat{a}'_i$.

We'll see that this mathematically correct idea will produce better results than the original. How can we test if the produced "orthogonal" matrix Q is indeed orthogonal? What other tests or measurements can we think up?

Randomized Matrix Multiplication

Idea: To get a matrix that approximates the product AB, we can take a selection of s columns of A, dot them with the corresponding column of B and add them up.

We'll take a weighted selection, favoring index choices where the products of the A-column and corresponding B-row are largest.

Strang proved that with **any** probability distribution, a selected of s columns of A will produce a matrix whose *expected value* is the correct product AB.

He also proved that, if you take the probability distribution created by assigning products of vector-lengths to the possible selections, you will produce a matrix where the *variance* is minimized.

Outer-Product Expansion

Why is it true that, for an $m \times n$ matrix A with columns $\hat{a}_1, \ldots, \hat{a}_n$, and $n \times t$ matrix B, with rows $\hat{b}_1^T, \ldots, \hat{b}_n^T$, that:

$$AB = \hat{a}_1 \hat{b}_1^T + \hat{a}_2 \hat{b}_2^T + \ldots + \hat{a}_n \hat{b}_n^T.$$

Confirm with an example and *explain.*

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c21 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!