Last name ______________________

First name ______________________

LARSON—MATH 550—CLASSROOM WORKSHEET 38
Generating Functions & Convolutions.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Generating Functions

A sequence $\langle a_n \rangle = \langle a_0, a_1, a_2, \ldots \rangle$ can be represented as the coefficients of the formal power series,

$$A(z) = a_0 + a_1 z^1 + a_2 z^2 + \ldots = \sum_{k \geq 0} a_k z^k,$$

called the *generating function* of the sequence, with notation $a_n = [z^n]A(z)$.

1. Let $B(z)$ be the generating function of the sequence $\langle b_n \rangle = \langle b_0, b_1, b_2, \ldots \rangle$. Find the coefficient of z^n in $A(z)B(z)$, that is, find c_n where $c_n = [z^n]A(z)B(z)$.

   The sequence $\langle c_n \rangle = \langle c_0, c_1, c_2, \ldots \rangle$ is the *convolution* of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

   Examples

   We found $\frac{1}{1-z}$ is the generating function for the sequence $\langle 1, 1, 1 \ldots \rangle$.

   We found e^z is the generating function for the sequence $\langle \frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{0!}, \ldots \rangle$.

   We found the recurrence $n! = \sum_k \binom{n}{k} \text{¡}(n-k)$, divided both sides by $n!$ and found: $1 = \sum_k \frac{1}{k!} \frac{\text{¡}(n-k)}{(n-k)!}$.

2. Let $D(z) = \sum_{k \geq 0} \frac{\text{¡}k}{k!} z^k$, and explain why $\frac{1}{1-z} = e^z D(z)$.

3. Solve for $D(z)$ and equate coefficients of z^n to find a formula for $\frac{¡n}{n!}$.

Fibonacci Numbers

We defined $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.

4. (**Bee Trees**). A male bee has a single female parent. A female bee has one male parent and one female parent. Draw a tree representing the "ancestors" of a male bee.

5. (**Bee Trees**). Let $B_1 = 1$, representing a male bee. Then that bee has one (female) ancestor one generation back; and represent this by $B_2 = 1$. This bee has two parents, so our original bee has 2 ancestors two generations ago; represent this by $B_3 = 2$. How many ancestors B_n does our original bee have after $n-1$ generations? Explain.

6. (**Kepler/Cassini**). Check that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ holds for small values of n.

Goals: We'd like to find the generating function $F(z)$ for $\langle F_n \rangle$ and use this to find a *formula* for the Fibonacci numbers F_n.

7. Find a relationship between $F(z)$, $zF(z)$ and $z^2F(z)$, and solve to get a formula for $F(z)$.