LARSON—MATH 511—CLASSROOM WORKSHEET 16
Eckart-Young Theorem

Low Rank Approximation & Eckart-Young Theorem

We showed $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$

Now let $A_k = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_k \hat{u}_k \hat{v}_k^T$ (for $k \leq r$). We will show that A_k is the "best" low rank approximation to A.

For any $m \times n$ matrix A, let $||A|| = \max \frac{||A\hat{x}||}{||\hat{x}||}$ (for any $\hat{x} \in \mathbb{R}^n$).

1. Find $||A - A_k||$.

2. Let B be *any* $m \times n$ matrix with rank k. The dimension of the null space of B (the "nullity") is $n - k$. Explain why there must be a non-0 vector $\hat{x}$ in $N(B) \cap span(\{\hat{v}_1, \ldots, \hat{v}_{k+1}\}$.

3. (We can assume $\hat{x}$ is unit). Argue that $||(A - B)\hat{x}|| \geq \sigma_{k+1}$.

4. Argue that $||A - A_k|| \leq ||A - B||$.

5. Explain why A_k is the "best" rank-k approximation of A.

Sage/CoCalc

6. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c16**.

7. Input $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (remember to inform Sage you mean for the entries to be interpreted as elements of a Real Double Field (RDF).

8. What is the rank of A?

9. Find the U, S, V from the SVD by evaluating: `U,S,V = A.SVD()`. Check what you have for u, S, V. What are the singular values of A?

10. Find the approximation matrix A_1.

11. Find the norm of $A - A_1$.

12. Let B be *any* 2×2 rank-1 matrix. Find the norm of $A - B$ and check that $||A - A_1|| \geq ||A - B||$.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c16 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!