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LARSON—MATH 511—HOMEWORK WORKSHEET 09 Test 1 Review

Write up a careful, complete test review and turn it in **before** our Test 1 on Thursday, Oct. 6. **Explain** everything.

1. Let
$$\theta = 30^{\circ}$$
 and $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- (a) Show Q is orthogonal.
- (b) Find $||\hat{x}||$.
- (c) Find $Q\hat{x}$ and $||Q\hat{x}||$.
- (d) Check that $||Q\hat{x}|| = ||\hat{x}||$.
- (e) Explain why $Q\hat{x}$ is a 30° rotation of \hat{x} .

2. Find the (column) rank of
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$$
.

3. Find a CR decomposition of
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$$
.

- 4. Explain why the column rank of A equals the number of columns of C.
- 5. Explain why the rows of R are necessarily linearly independent in the CR decomposition.
- 6. Explain why the row rank of A equals the number of rows of R.
- 7. Find an example of a positive definite matrix.
- 8. Use the equivalent energy definition to show that $A = \begin{bmatrix} 3 & 1 \\ 1 & -7 \end{bmatrix}$ is not positive definite.
- 9. Explain why A^TA is symmetric for any matrix A.
- 10. Show that the eigenvalues of a real symmetric matrix are real.
- 11. Show that, if S is a symmetric matrix and λ_1 and λ_2 are distinct eigenvalues with corresponding eigenvectors \hat{v}_1 , \hat{v}_2 then $A\hat{v}_1$ and $A\hat{v}_2$ are orthogonal.

SVD

12. Let
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
.

- 13. What is the rank r of A?
- 14. Find AA^T .
- 15. Find the r positive eigenvalues of AA^T (you should do this by hand—as if it were the test—but also check them with Sage/CoCalc). The remaining eigenvalues must be 0.
- 16. Find corresponding eigenvectors $\hat{u}_1, \dots, \hat{u}_r$, and normalize them.
- 17. Find an orthogonal basis for the null space of AA^T and call these vectors $\hat{u}_{r+1}, \ldots, \hat{u}_m$ (where m is the number of rows of A), and normalize them. (These are also eigenvectors of AA^T corresponding to 0).
- 18. Let $U = [\hat{u}_1 \dots \hat{u}_r \dots \hat{u}_m]$.
- 19. Find the r positive singular values $\sigma_1 \geq \ldots \geq \sigma_r$ of A.
- 20. Find $A^T A$.
- 21. The eigenvalues of A^TA must be the same as the ones you found for AA^T . Find corresponding eigenvectors $\hat{v}_1, \ldots, \hat{v}_r, \ldots, \hat{v}_n$ (where n is the number of rows of A), and normalize them.
- 22. Let $V = [\hat{v}_1 \dots \hat{v}_r \dots \hat{v}_n]$.
- 23. Let Σ be the $m \times n$ "diagonal" matrix with $\Sigma_{i,i} = \sigma_i$.
- 24. Check that $A = U\Sigma V^T$.
- 25. Check that $A = \sigma_1 \hat{u}_1 \hat{v}^T + \dots \sigma_r \hat{u}_r \hat{v}^T$.
- 26. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$. Follow all the steps to find the SVD of A.

(You can either start as above with AA^T and find the \hat{u} 's first or with A^TA and find the \hat{v} 's first; we proved the positive eigenvalues of A^TA and AA^T must be the same).