

Last name _____

First name _____

LARSON—MATH 511—HOMEWORK WORKSHEET 09

Test 1 Review

Write up a careful, complete test review and turn it in **before** our Test 1 on Thursday, Oct. 6. **Explain** everything.

1. Let $\theta = 30^\circ$ and $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
 - (a) Show Q is orthogonal.
 - (b) Find $\|\hat{x}\|$.
 - (c) Find $Q\hat{x}$ and $\|Q\hat{x}\|$.
 - (d) Check that $\|Q\hat{x}\| = \|\hat{x}\|$.
 - (e) Explain why $Q\hat{x}$ is a 30° rotation of \hat{x} .
2. Find the (column) rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$.
3. Find a CR decomposition of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$.
4. Explain why the column rank of A equals the number of columns of C .
5. Explain why the rows of R are necessarily linearly independent in the CR decomposition.
6. Explain why the row rank of A equals the number of rows of R .
7. Find an example of a positive definite matrix.
8. Use the equivalent *energy* definition to show that $A = \begin{bmatrix} 3 & 1 \\ 1 & -7 \end{bmatrix}$ is *not* positive definite.
9. Explain why $A^T A$ is symmetric for *any* matrix A .
10. Show that the eigenvalues of a real symmetric matrix are real.
11. Show that, if S is a symmetric matrix and λ_1 and λ_2 are distinct eigenvalues with corresponding eigenvectors \hat{v}_1 , \hat{v}_2 then \hat{v}_1 and \hat{v}_2 are orthogonal.

SVD

12. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
13. What is the rank r of A ?
14. Find AA^T .
15. Find the r positive eigenvalues of AA^T (you should do this by hand—as if it were the test—but also check them with Sage/CoCalc). The remaining eigenvalues must be 0.
16. Find corresponding eigenvectors $\hat{u}_1, \dots, \hat{u}_r$, and normalize them.
17. Find an orthogonal basis for the null space of AA^T and call these vectors $\hat{u}_{r+1}, \dots, \hat{u}_m$ (where m is the number of rows of A), and normalize them. (These are also eigenvectors of AA^T corresponding to 0).
18. Let $U = [\hat{u}_1 \dots \hat{u}_r \dots \hat{u}_m]$.
19. Find the r positive singular values $\sigma_1 \geq \dots \geq \sigma_r$ of A .
20. Find $A^T A$.
21. The eigenvalues of $A^T A$ must be the same as the ones you found for AA^T . Find corresponding eigenvectors $\hat{v}_1, \dots, \hat{v}_r, \dots, \hat{v}_n$ (where n is the number of columns of A), and normalize them.
22. Let $V = [\hat{v}_1 \dots \hat{v}_r \dots \hat{v}_n]$.
23. Let Σ be the $m \times n$ “diagonal” matrix with $\Sigma_{i,i} = \sigma_i$.
24. Check that $A = U\Sigma V^T$.
25. Check that $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \dots \sigma_r \hat{u}_r \hat{v}_r^T$.
26. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$. Follow all the steps to find the SVD of A .
(You can either start as above with AA^T and find the \hat{u} 's first or with $A^T A$ and find the \hat{v} 's first; we proved the positive eigenvalues of $A^T A$ and AA^T must be the same).