LARSON—MATH 511—HOMEWORK WORKSHEET 13
Low-rank Approximation

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **h13**.

   Annotate your Sage Worksheet verbosely. Answer any questions by writing *comments* in your worksheet.

Low Rank Approximation & Eckart-Young Theorem

We showed $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$

We let $A_k = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_k \hat{u}_k \hat{v}_k^T$ (for $k \leq r$). We showed that A_k is the "best" (with respect to the spectral norm) rank-k approximation to A.

2. We'll start with a rank-3 matrix A simple enough where we might see what's going on. Let: `A=matrix(RDF,4,4,[1,1,0,0,2,2,0,0,0,0,3,0,0,0,0,4])`. Evaluate A to check.

3. Find the rank of A: `A.rank()`. Explain.

4. Find the SVD of A: `U,S,V=A.SVD()`. Evaluate U, S and V to see what they look like. Find USV^T.

5. To find the rank-1 approximation $A0$ (counting Python-ically, starting at 0), we need the corresponding columns of U and V. After we get those columns we will make proper matrices from them:

   ```
   u0=U.column(0)
   u0=matrix(RDF,4,1,u0)
   ```

   ```
   u1=U.column(1)
   u1=matrix(RDF,4,1,u1)
   ```

   ```
   v0=V.column(0)
   v0=matrix(RDF,4,1,v0)
   ```

   ```
   v1=V.column(1)
   v1=matrix(RDF,4,1,v1)
   ```

   Evaluate u0, u1, v0 and v1 to see what you have.

6. We'll also need the first 2 singular values (which are on the diagonal of the S matrix):

   ```
   sigma0=S[0,0]
   ```

   ```
   sigma1=S[1,1]
   ```

   Evaluate sigma0 and sigma1 to check they are what you expect.

7. Now we'll find the best rank-1 approximation (according to the Eckart-Young Theorem):

   ```
   A0=sigma0*u0*v0.transpose()
   ```

   Evaluate A0 to see what you have.

8. How good is A0 as an approximation? Evaluate the norm of the difference of A and the A0 approximation: `(A-A0).norm()`

9. Now we'll pick a rank-1 matrix B (the 4×4 all 1's matrix) just for comparison's sake. We proved $||A - A0|| \leq ||A - B||$. Evaluate:

   ```
   B=matrix(RDF,4,4,[1]*16)
   ```

10. Now check:

    ```
    (A-B).norm()
    ```

11. Choose your own rank-1 matrix B and check that $||A - A0|| \leq ||A - B||$.

12. Now we'll find the best rank-2 approximation:

    ```
    A1=A0+sigma1*u1*v1.transpose()
    ```

13. How good is A1 as an approximation? Evaluate the norm of the difference of A and the A1 approximation: `(A-A1).norm()`

14. Now we'll pick a rank-2 matrix B just for comparison's sake. We proved $||A - A1|| \leq ||A - B||$. Evaluate: `B=matrix(RDF,4,4,[0,0,0,0,0,0,0,0,0,0,3,0,0,0,0,4])`

15. Now check:

    ```
    (A-B).norm()
    ```

16. Choose your own rank-2 matrix B and check that $||A - A1|| \leq ||A - B||$.

17. Find the matrix $A2$ and the best rank-3 approximation of A (with respect to the spectral norm).

Getting your homework recorded

When you are done, ...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—h13 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!