NEW CONDITIONS FOR GRAPH HAMILTONICITY

N. BUSHAW, V. GUPTA, C. E. LARSON, S. LOEB[1],
M. NORGE, J. PARRISH, N. VAN CLEEMPUT[2], J. YIRKA, G. WU

Abstract. We present results on new sufficient or necessary conditions for the existence of a Hamilton cycle in a graph. We are especially interested in finding conditions which are not implied by any of a number of well-known theorems in the literature on graph hamiltonicity. We also report a number of unresolved conjectures.

1. Introduction

We present results of an experiment to find new sufficient or necessary conditions for the existence of a Hamilton cycle in a graph. We are especially interested in finding conditions which are not implied by any of a number of well-known theorems in the literature on graph hamiltonicity.

This research was done in the context of an annual summer research program designed to introduce undergraduates to substantive mathematical research. We used the open-source Conjecturing program towards this end. While the body of the paper is focused on our results, our research methodology may be of interest to some readers and is described in an appendix. This program and this approach can be used in any analogous context.

We use standard notation where possible (see, e.g., [8]); in particular, n is the *order* of the graph.

2. Hamiltonicity

We refer to a spanning cycle as a *Hamilton cycle*; a graph containing such a cycle is *Hamiltonian*. The *hamiltonian problem*, then, refers to determining the existence of a Hamilton cycle (or, equivalently, determining *hamiltonicity*). As the name suggests (an homage to Sir William Rowan Hamilton), the problem dates to at least the 1850s and is one of the most natural questions in all of graph theory – when is it possible to traverse across a graph, visiting every vertex once and returning where you started?

It should be noted that throughout this manuscript, we assume that our graphs are connected and contain at least three vertices. For disconnected graphs, many of these results fail in a trivial manner – the same is true of the connected but too-small graphs K_2 and K_1. In order to avoid clutter, we will avoid rewriting these conditions in the following theorems.

For a complete history, the interested reader should visit the three surveys of Ron Gould [15, 16, 17], as well as the survey on closure based techniques due to Broersma, Ryjáček, and Schiermeyer [10].

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3. Early Results

In this section, we collect a few of our early results. As our results build directly on these theorems and lemmas, seeing these short proofs in their barest form will help to set the stage for the later (more involved) proofs.

Observation 1. *If G is a 2-connected outerplanar graph, then it is Hamiltonian.*

Proof. A graph is *outerplanar*[1] if it has an embedding in the plane in which the boundary of the infinite face contains every vertex. This boundary may have cut vertices—and in this case are not Hamiltonian. The result is trivial, however, when the graph is 2-connected: there are no cut vertices and the boundary of the infinite face is a cycle (and is spanning). □

Observation 2. *If G is a Gallai tree and is 2-connected, then G is Hamiltonian.*

Proof. A *block* of a graph is a maximal connected subgraph with no cut-vertex. A graph is a *Gallai tree* if every block is either a complete graph or an odd cycle (such blocks are called *bricks*). Since any two blocks in a graph are separated by a cut vertex, a two-connected Gallai tree must be just a single brick. This is either an odd cycle or a complete graph, and is thus Hamiltonian. □

Observation 3. *If G is a bipartite line graph which has diameter equal to radius, then it is Hamiltonian.*

Proof. The *line graph of a graph H* is the graph $L(H)$ with $V(L(H)) = E(H)$ and $ef \in E(L(H))$ whenever edges e and f share an endpoint in H. A graph G is then *a line graph* if it is the line graph of another graph H. Krausz [21] showed that a graph is a line graph if and only if it can be partitioned into edge disjoint cliques such that every vertex lies in exactly two cliques. This is not difficult to prove; if G is the line graph of H, then the edges incident to a vertex of H form a complete subgraph of G, and every edge of G lies in exactly one of these. Since each edge of H has two vertices, the corresponding vertex of G is in at most two of these subgraphs. Since our graph G is bipartite, it is triangle-free, so such cliques are singletons or edges. Every vertex is in exactly two such cliques, and so every vertex has degree at most two. Since G is connected, our graph is either a path or a cycle. Paths consisting of more than a single edge do not have radius equal to diameter, so our graph is a spanning cycle. □

Observation 4. *If G is a line graph whose complement is chordal and which has radius equal to diameter, then G is Hamiltonian.*

Proof. A further characterization of line graphs was given by Beineke [6], where he showed that line graphs are characterized by nine forbidden induced subgraphs. Among these is the claw $K_{1,3}$, and so every line graph is claw free. A graph is *chordal* if every induced cycle is length three. Since the complement of P_6 contains an induced C_4, every co-chordal graph is induced P_6-free. Bedrossian [5] showed that every P_6-free, claw-free, 2-connected graph is Hamiltonian; as nontrivial graphs with radius equal to diameter are 2-connected [1], this completes the proof. □

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[1] While *outerplanar* seems to be the standard modern terminology, such graphs are also frequently called *circular planar*.

4. Strongly Regular Graphs

In this section, we'll prove that all (connected) bipartite strongly-regular graphs are Hamiltonian. Our original proof of Theorem 5 required a much longer argument; this greatly simplified argument is due to a referee.

Recall that a graph is (n, k, λ, μ) *strongly-regular* if it has n vertices, every vertex has degree k[2], every pair of adjacent vertices have exactly λ common neighbors, and every pair of non-adjacent vertices have exactly μ common neighbors.

Theorem 5. *If G is a bipartite, connected, (n, k, λ, μ)-strongly-regular graph, then G is a complete balanced bipartite graph.*

Proof. Assume that G is as stated, with bipartition (X, Y). It is well-known that every strongly-regular graph has diameter 2. Every bipartite graph of diameter 2 is a complete bipartite graph. Since G is regular and complete bipartite, G is also balanced. □

Of course, such complete bipartite graphs are Hamiltonian, and so we reach the corollary which was our goal.

Corollary 6. *Every (connected) bipartite (n, k, λ, μ)-strongly-regular graph is Hamiltonian.*

We also conjecture a considerable generalization of this result. We call a graph *distance-regular* if for every pair of vertices u, v and integers i, j, the number of vertices at distance i from u and distance j from v depends only on i, j, and $d(u, v)$ – not on the particular choice of vertices u, v (for many basic properties and equivalent definitions, see [11]). Note that an (n, k, λ, μ)-strongly-regular graphs are trivially distance-regular, whenever $\mu \neq 0$. The following, then, is a considerable generalization of the preceding corollary. This is perhaps the most intriguing of our conjectures which was left open.

Conjecture 7. *Every (connected) bipartite distance-regular graph is Hamiltonian.*

It is worth reporting that the evidence for this conjecture extends beyond our database of stored graphs—which contains very few distance-regular graphs (let alone bipartite ones). In order to find more, we began constructing instances of random regular graphs (via the configuration model of Bollobás [7]), and determining whether they were distance-regular or not. Among these still quite few were bipartite; by constructing the bipartite double of such a graph (see, e.g., [11]) we generated a list of several thousand bipartite distance-regular graphs – and each was Hamiltonian. This is largely indicative of our process for attempting to find a counterexample to a new conjecture – check it against easily accessed graphs (i.e., small graphs, or graphs in a particular class), and then broaden our search space to find many more graphs meeting the conditions (often, using a random graph model as our starting point).

While Conjecture 7 is in many ways the most attractive, we also list here a number of other related conjectures which were made during our project; for each, we were able to find neither proof nor counterexample. In order to avoid making

[2]hence the *regular* part of the name

these statements unbearably long, we use standard terminology but do not give definitions[3].

Conjecture 8. *Every graph which is both planar and vertex transitive is Hamiltonian.*

Conjecture 9. *Every graph which is both Eulerian and vertex transitive is Hamiltonian.*

Conjecture 10. *Every graph which is vertex transitive and contains an induced C_4 is Hamiltonian.*

Conjecture 11. *Every graph which is both distance regular and perfect is Hamiltonian.*

Conjecture 12. *Every graph which is semi-symmetric is Hamiltonian.*

You may notice that the first four conjectures above all involve a variation on the property of being *vertex transitive*. This is related to an old conjecture of Lovász[23] that every vertex transitive graph contains a Hamilton path. As to Hamilton cycles, there is some disagreement – Thomassen [26] has conjectured that there all but finitely many vertex transitive graphs are Hamiltonian, while Babai [2, 3] has conjectured that there are infinitely many non-Hamiltonian vertex transitive graphs. Only five such graphs are currently known: K_2, the Petersen graph, the Coxeter graph, and the triangle-replaced Petersen and Coxeter graphs. With regard to either truth, Conjecturing is working hard to find properties which weed out the five known non-Hamiltonian vertex-transitive graphs in some way. In Section 6, we'll show that every planar vertex-transitive graph is Hamiltonian.

5. Cartesian Products

Our main theorem is now Corollary 15. A referee points out that this follows from Theorem 14 of Jha and Slutzki (below). Both our original argument and that of Jha and Slutzki both materially use the following lemma, which can easily be deduced from Kuratowski's Theorem [20]. Our argument is sufficiently similar to Jha and Slutzki's to not be worth repeating.

Lemma 13. *A graph is outerplanar if and only if it is $K_{2,3}$-minor free and K_4-minor-free.*

Theorem 14. [19] *The Cartesian product of two non-trivial connected outerplanar graphs is outerplanar if and only if one graph is a path and the other is K_2.*

Corollary 15. *If G is a nontrivial Cartesian product which is outerplanar, then it is Hamiltonian.*

6. Planar-Transitive Graphs

A graph automorphism is an isomorphism between the graph and itself. That is, it is a map $\phi : V(G) \to V(G)$ such that $xy \in E(G)$ iff $\phi(x)\phi(y) \in E(G)$. A graph G is **vertex-transitive** if for every pair of vertices $u, v \in V(G)$, there is a graph automorphism ϕ with $\phi(u) = \phi(v)$.

[3]For careful definitions, the interested reader can view the Graph Brain Project's github at `http://math1um.github.io/objects-invariants-properties/`

Here, we'll prove that every planar vertex-transitive graph is Hamiltonian[4]. This builds on a number of results, several of which are highly non-trivial in their own right. Nevertheless, their combined power allows us to prove this quite strong theorem in a streamlined way.

Observation 16. *Every vertex-transitive graph is d-regular for some $d \in \mathbb{N}$.*

Theorem 17. [24] *If G is a d-regular vertex-transitive graph with connectivity k, then $\frac{2(d+1)}{3} \leq k$.*

Theorem 18. [27] *Every 4-connected planar graph is Hamiltonian.*

Theorem 19. [29] *The only 3-regular vertex-transitive simple planar graphs are the tetrahedron, the dodecahedron, the n-sided prisms (for $n \geq 3$), the tricone graph, the truncated cube, truncated octahedron, trunctated dodecahedron, truncated icosahedron (bucky ball), and the Great Rhombicosidodecahedral Graph.*

With our tools in place, we prove the following theorem.

Theorem 20. *Every vertex-transitive planar graph is Hamiltonian.*

Proof. By Theorem 18, it is enough for us to consider graphs with connectivity at most three. By Theorem 17, such graphs are regular of degree at most $7/2$; that is, they either have all vertices of degree two or of degree three. Since a two-regular connected graph is a cycle (and thus Hamiltonian), we focus only on the three regular case. But, these graphs are characterized by Theorem 19. The only infinite family of graphs here are the prisms (which are trivially Hamiltonian − just walk around all but a single edge on one of the end polygons, across the adjacent face, around the opposing end, and back along the same face). The other graphs are easily tested to be Hamiltonian in `Sage`[5]. □

7. Necessary Conditions

Finding interesting necessary conditions for hamiltonicity (beyond, say, two-connectedness and a few spectral results) is a much more difficult endeavor than finding sufficient conditions; there are very few in the literature. Nevertheless, we present one such a condition here for the class of cubic graphs. We call a graph G *class one* if it has chromatic index equal to maximum degree.

Theorem 21. *Every cubic Hamiltonian graph is class one; that is, the chromatic index is equal to the maximum degree.*

Proof. Assume G is Hamiltonian and cubic. Then G has an even number of vertices, which are spanned by a cycle. The edges of this cycle can be colored by alternating colors 1 and 2. Since G is cubic, the remaining edges of G form a perfect matching, and can thus be given color 3. □

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[4]Recall that we are only considering graphs with at least three vertices in this paper, so K_2 is not a counterexample to this theorem

[5]...and we have done so.

Appendix: Automated Conjecturing

It was mentioned in the Introduction that an automated conjecture-making program was used in this research. We consider this to be extremely useful—especially in the context of working with students new-to-research. In particular, as the program produces conjectures, students are confronted with the task of either producing a proof of a statement or producing a counterexample. While the creativity of producing conjectures is off-loaded to the computer, there is much left for the researcher to do.

The following description of the CONJECTURING program is abbreviated. A complete description of the properties-relations version of this program may be found in [22]. Our goal here is only to say something useful and relevant to this specific project, with the idea that it might suggest how analogous investigations could be organized.

The CONJECTURING program may be used to make sufficient (or necessary) conditions conjectures for any graph property. All graph properties are represented as functions that input a graph and return a boolean: `True` if the input graph has the specified property and `False` otherwise. The user starts by specifying a graph property to be investigated. Other inputs to CONJECTURING include other graph properties—boolean functions of these these will appear as conjectured sufficient conditions, as well as a list of pre-coded graphs. The program has two heuristics for outputting conjectured sufficient conditions; the first of these is a requirement that any produced conjecture be true for every graph in the list of input graphs.

The program maintains an internal list of current "best" conjectures. These will be updated as the program considers further possible expressions representing sufficient conditions, and once-stored conjectures may be removed if they are superseded by better conjectures. The second heuristic the program uses is to require that any added conjecture say something "new" about at least one stored object: in the case of a sufficient condition for a specified graph property, this means that the considered property P must be true for at least one input graph and not true for any other currently stored conjecture. In this case property P is considered to be "significant" (with respect to the currently stored conjectures and input graphs). When a conjecture is added the list of stored conjectures is re-evaluated. It may be the case that a stored conjecture is no longer significant; in this case it is removed.

This second heuristic—the "significance heuristic" (which is Fajtlowicz's Dalmatian heuristic)—also suggests a way of making use of existing theorems. It is also possible to add theoretical knowledge as an input to the program; in the case of sufficient condition conjectures this would be existing sufficient conditions for the investigated graph property. Adding *theory* to the program's input forces the program to only produce conjectures that are significant with respect to the input theory: produced conjectures must each be true for at least one input graph but is false for each input theoretical sufficient condition. For instance, if a graph is a clique (with at least three vertices) then it is Hamiltonian. The property of "being a clique" (represented by the function `is_clique` in the program) can be added in the `theory` input; then any conjectured sufficient condition must be true for at least one input graph that is *not* a clique.

An interesting feature of working with the CONJECTURING program is that the program will often make conjectures between seemingly unrelated properties. Of course many of these conjectures are false. But some are true. The program simply generates every possible expression of complexity 1, and then of complexity 2, and so on, and then tests the generated conjecture with its two heuristics. (The *complexity* of an expression here is simply the number of nodes in the tree representing the expression.) Humans might miss considering certain expressions—but since the program generates them *all* there is the possibility for surprising relations to be produced—and sometimes verified.

It is also worth noting that we made essential use of the Graph Brain Project code base [6]; this includes code for more than one hundred graph properties and more than 500 graphs (that have been compiled from graph theory research papers or that appeared as counterexamples in our previous research efforts). The significance of this code base is two-fold: first, the properties can easily be used as inputs to the CONJECTURING program, to be elements of the boolean expressions representing conjectured necessary or sufficient conditions and, second, the coded graph examples can be used as a first important test for our conjectures—it is impossible to ever test a conjecture against all graphs and even infeasible to check them against all graphs up to any order more than 10, so these curated graphs, previously important, provide a first hurdle for a conjecture to overcome.

As examples, we discuss the early conjectures made by the program in our investigation. In every investigation, the program will produce simple true facts. While these conjectures seem uninspired, in fact they both provide evidence that the program is in fact working as intended and, importantly, these basic facts will be added to the `theory` of the investigation, and be built on by the program—and new conjecture must be true for at least one graph that is not covered by the input theory.

Most of the reported early conjectures are straightforward – once the proper definitions are given, the results follow easily; other early conjectures involve graph properties which were introduced in the graph hamiltonicity literature exactly because they are sufficient conditions for a graph to be Hamiltonian. Note that our results are ordered by the complexity of the statement as output by CONJECTURING – theorems with just a single condition come before those with several conditions[7].

We report these theorems in two forms – as presented by the conjecturing software (in `teletype font`) using undefined properties, as well as in an expanded form that looks more like a traditional theorem and explains the properties used.

Observation 22. `(is_clique)->(is_hamiltonian)`
If G is a complete graph, then G is Hamiltonian.

Recall that we have specifically excluded K_1 and K_2 as a condition for every theorem and observation.

Theorem 23 (Chvatal-Erdős [12]). `(is_chvatal_erdos)->(is_hamiltonian)`
Let G be a connected graph on at least 3 vertices. If the connectivity of G is larger than the independence number of G, then G is Hamiltonian.

Here we encounter a graph property which was clearly added because of its connection to hamiltonicity – and as one would hope, CONJECTURING discovered

[6]Open source, and available on GitHub: http://math1um.github.io/objects-invariants-properties/

[7]To be slightly more formal, each expression is stored as a boolean tree; the complexity of a statement is the number of nodes in its associated tree.

its usefulness quickly. The following theorems are similar; somewhat complicated properties, which on their own serve as sufficient conditions for hamiltonicity.

Theorem 24 (Ore [25]). `(is_ore)->(is_hamiltonian)`
If G is a graph which satisfies $d(u) + d(v) \geq |G|$ for every pair of non-adjacent vertices u, v, then G is Hamiltonian.

Theorem 25 (Haggkvist-Nicoghossian [18]). `(is_haggkvist_nicoghossian)->(is_hamiltonian)`
If G is 2-connected and has minimum degree at least $\frac{|G|+\kappa(G)}{3}$ (where $|G|$ is the order of the graph and $\kappa(G)$ its (vertex-)connectivity), then G is Hamiltonian.

Theorem 26 (Faudree, Gould, Jacobson, Schelp [14]).
`(is_generalized_dirac)->(is_hamiltonian)`
If G is a 2-connected n-vertex graph in which every pair of nonadjacent vertices u, v satisfy $|N(u) \cup N(v)| \geq \frac{2n-1}{3}$, then G is Hamiltonian.

Theorem 27 (Fan [13]). `(is_genghua_fan)->(is_hamiltonian)`
If G is 2-connected and every pair of vertices u, v with $d(u, v) = 2$ satisfies $\max\{d(u), d(v)\} \geq \frac{|G|}{2}$, then G is Hamiltonian.

Observation 28. `((is_cycle)|(is_clique))->(is_hamiltonian)`
If G is either a complete graph or a cycle, then G is Hamiltonian.

Here | is the *logical or*; this statement then has greater complexity (the expression tree for this property has three nodes) than the preceding statements. Although nearly as trivial as Observation 22, it is interesting that both were discovered by CONJECTURING – this result is strictly stronger than Observation 22, and so the only way that both could have been discovered is by changing the object list or property list.

The sufficient condition theorems for hamiltonicity in the body of the paper are, by the design of this research, improvements on these observations. Each new theorem is true for at least one graph that was not covered by any of these sufficient conditions.

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Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA 23284, USA.

(1) Department of Mathematics and Computer Science, Hampden-Sydney College, Hampden Sydney, VA 23943, USA.

(2) Department of Applied Mathematics, Computer Science and Statistics, Ghent University, 9000 Ghent, Belgium.