Last name ______________________

First name ______________________

LARSON—MATH 610—CLASSROOM WORKSHEET 39
Annihilating Polynomials of a Linear Operator.

Concepts & Notation

- (Sec. 5.3) *permutation*, $\det A$.
- (Sec. 6.2) *characteristic value*, *characteristic vector*, *characteristic polynomial*, *diagonalizable* linear operator.
- (Sec. 6.3) *annihilating polynomial*, *minimal polynomial*.

Review

1. What does it mean for a polynomial to *annihilate* a linear operator T?
2. What is the *minimal polymonial* of a linear operator T over a finite-dimensional vector space T? (Does it exist? What does it tell us?)
3. (**Cayley-Hamilton Theorem**) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the characteristic polynomial p of A and check that $p(A) = 0$.

New

4. Use $p(A)$ to find A^{-1}.

5. (**Cayley-Hamilton Theorem**) Let $A \in \mathbb{F}^{n\times n}$. Let $p \in \mathbb{F}[x]$ be the characteristic polynomial of A. Then $p(A) = 0$.

6. (**Cayley-Hamilton Theorem**) Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and $T \in \mathcal{L}$. Let $p \in \mathbb{F}[x]$ be the characteristic polynomial of T. Then $p(T) = 0$.

The Special Case of Matrices over $\mathbb{R}$

7. We'll prove the following **claim:** The characteristic values of any symmetric matrix $A \in \mathbb{R}^{n\times n}$ are real.

   The following steps are a **proof**. Your job will be to explain the steps.

   Let $A \in \mathbb{R}^{n\times n}$.

   (a) Why is the characteristic polynomial of A, $\det(xI - A)$, guaranteed to have n complex roots?

   Let $c \in \mathbb{C}$, $\alpha \in \mathbb{C}^{n\times 1}$ ($\alpha \neq 0$) be such that $A\alpha = c\alpha$ (such a pair c, α must exist). And let $\bar{\alpha}$ be the $\mathbb{C}^{n\times 1}$ vector whose entries are the complex conjugates of the entries of α.

   (b) Argue that $\alpha^t\bar{\alpha}$ is real (or more precisely a 1×1 matrix with a real number entry).

   (c) Let $\bar{A}$ be the matrix who entries are the complex conjugates of the entries of A. Explain why $\bar{A} = A$.

   Let $\overline{A\alpha}$ be the matrix who entries are the complex conjugates of the entries of $A\alpha$.

   (d) Explain why $\overline{A\alpha} = \bar{A}\bar{\alpha}$. (And thus $\overline{A\alpha} = A\bar{\alpha}$).

   Let $\overline{c\alpha}$ be the matrix who entries are the complex conjugates of the entries of $c\alpha$.

   (e) Explain why $\overline{c\alpha} = \bar{c}\bar{\alpha}$.

   So, $\overline{A\alpha} = \overline{c\alpha}$ implies $A\bar{\alpha} = \bar{c}\bar{\alpha}$, and $\alpha^t A\bar{\alpha} = \alpha^t\bar{c}\bar{\alpha} = \bar{c}\alpha^t\bar{\alpha}$.

   Also, $A\alpha = c\alpha$ implies $(A\alpha)^t = (c\alpha)^t$, which implies $\alpha^t A = c\alpha^t$, and thus $\alpha^t A\bar{\alpha} = c\alpha^t\bar{\alpha}$.

   (f) Explain why $\bar{c}\alpha^t\bar{\alpha} = c\alpha^t\bar{\alpha}$.

   (g) Explain why $\bar{c} = c$.

   (h) Explain why c must be a real number (and thus every characteristic value of A is real).