Last name ________________________

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LARSON—MATH 550—CLASSROOM WORKSHEET 34
Inversion, Derangements & Generating Functions.

Concepts & Notation

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

Inversion

Let $f(n)$ be a function defined on $\mathbb{Z}^{\geq 0}$, and define:

$$g(n) = \sum_k \binom{n}{k} (-1)^k f(k).$$

We **proved**:

$$f(n) = \sum_k \binom{n}{k} (-1)^k g(k).$$

Derangements

A *derangement* of n objects is an ordering (listing, or permutation) of those objects so that none is in its original place; or, more formally, it is a bijection $\phi : [n] \to [n]$ such that $\phi(i) \neq i$, for every $i \in [n]$.

Let ¡n be the number of derangements of n objects.

We showed that:

$$n! = \sum_k \binom{n}{k} (-1)^k [(-1)^k \text{ ¡}k],$$

and then we used the Inversion Theorem to get:

$$(-1)^n \text{ ¡}n = \sum_k \binom{n}{k} (-1)^k k!,$$

and simplified to get:

$$\text{¡}n = n! \sum_k \frac{(-1)^k}{k!}.$$

1. What is $\frac{1}{e}$?

2. KGP claim that ¡n converges to $\frac{n!}{e}$. What is their argument?

3. If ¡n is the number of ways to arrange n hats so that the hat of the i^{th} person to check their hat doesn't match the i^{th} hat, then what interpretation can $\frac{\text{¡}n}{n!}$ be given?

Generating Functions

A sequence $\langle a_0, a_1, a_2, \ldots \rangle$ can be represented as the coefficinets of the formal power series,

$$A(z) = a_0 + a_1 z^1 + a_2 z^2 + \ldots = \sum_{k \geq 0} a_k z^k,$$

called the *generating function* of the sequence, with notation $a_n = [z^n]A(z)$.

4. Find the generating function $F(z)$ of the sequence $\langle 0, 1, 1, 2, 3, 5 \ldots \rangle$, and $[z^n]F(z)$.

5. Let $B(z)$ be the generating function of the sequence $\langle b_0, b_1, b_2, \ldots \rangle$. Find the coefficient of z^n in $A(z)B(z)$, that is, find c_n where $c_n = [z^n]A(z)B(z)$.