

LARSON—MATH 610—CLASSROOM WORKSHEET 07

All Bases have the same Cardinality.

Concepts (Chp. 1): field, vector space, \mathcal{P} , \mathbb{F}^n , $\mathbb{M}_{m \times n}(\mathbb{F})$, subspace, null space, $\text{row}(A)$, $\text{col}(A)$, list of vectors, span of a list of vectors, linear independence, linear dependence, pivot column decomposition, direct sum $\mathcal{U} \oplus \mathcal{V}$, *orthogonal* matrix, *unitary* matrix.

Review:

(Theorem 1.7.5, Pivot Column Decomposition). Let $A = [\hat{a}_1 \dots \hat{a}_n] \in \mathbb{M}_{m \times n}(\mathbb{F})$ be non-zero, let $j_1 < j_2 < \dots < j_s$ be its pivot indices, and let $P = [\hat{a}_{j_1} \hat{a}_{j_2} \dots \hat{a}_{j_s}] \in \mathbb{M}_{m \times s}(\mathbb{F})$.

- (a) The s columns of P are linearly independent, $1 \leq s \leq n$ and $\text{col}(P) = \text{col}(A)$.
- (b) There is a unique $R = [\hat{r}_1 \dots \hat{r}_n] \in \mathbb{M}_{s \times n}(\mathbb{F})$ such that $A = PR$.
- (c) $\hat{r}_{j_k} = \hat{e}_k \in \mathbb{F}^s$, for each $k = 1, \dots, s$.
- (d) If $s > 1$, $2 \leq k \leq s$ and $j_{k-1} < k < j_k$ then $\hat{r}_j \in \text{span}\{\hat{e}_1 \dots \hat{e}_{j_{k-1}}\}$.
- (e) The rows of R are linearly independent and $\text{null}(A) = \text{null}(R)$.

Chp. 2 of Garcia & Horn, Matrix Mathematics

(Lemma 2.1.4. Replacement Lemma). Let \mathcal{V} be a non-zero \mathbb{F} -vector space and let r be a positive integer. Suppose that $\beta = \hat{u}_1, \hat{u}_2, \dots, \hat{u}_r$ spans \mathcal{V} . Let $v \in \mathcal{V}$ be non-zero and let

$$\hat{v} = \sum_{i=1}^r c_i \hat{u}_i$$

- (a) $c_j \neq 0$ for some $j \in \{1, \dots, r\}$,
- (b) If $c_j \neq 0$ then the list $\hat{v}, \hat{u}_1, \dots, \widehat{\hat{u}_j}, \dots, \hat{u}_r$ spans \mathcal{V} ,
- (c) If β is a basis for \mathcal{V} and $c_j \neq 0$ then the list in (b) is a basis for \mathcal{V} ,
- (d) If $r \geq 2$, β is a basis for \mathcal{V} , $\hat{v} \notin \text{span}\{\hat{u}_1, \dots, \hat{u}_k\}$ for some $k \in \{1, 2, \dots, r-1\}$, then there is an index $j \in \{k+1, k+2, \dots, r\}$ such that

$$\hat{v}, \hat{u}_1, \dots, \hat{u}_k, \hat{u}_{k+1}, \dots, \widehat{\hat{u}_j}, \dots, \hat{u}_r$$

is a basis for \mathcal{V} .

1. What is this theorem about? Consider the following example: $\beta = \hat{e}_1, \hat{e}_2, \hat{e}_3 \in \mathbb{R}^3$,

and $\hat{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$.

2. Suppose $\beta = \hat{v}_1, \hat{v}_2, \hat{v}_3$ is a basis for a vector space \mathcal{V} , and $\hat{v} \in \mathcal{V}$. What does the theorem say. Why is it true?
3. (**Theorem 2.1.10**). Let \mathcal{V} be an \mathbb{F} -vector space and let r and n be positive integers. Suppose that $\beta = \hat{u}_1, \dots, \hat{u}_n$ is a basis for \mathcal{V} and $\gamma = \hat{v}_1, \dots, \hat{v}_r$ is linearly independent.
- (a) $r \leq n$.
 - (b) If $r = n$ then γ is a basis for \mathcal{V} .
4. What is this theorem about?
5. (**Corollary 2.1.11, All Bases have Same Cardinality**). If r and n are positive integers and $\hat{v}_1, \dots, \hat{v}_r$ and $\hat{w}_1, \dots, \hat{w}_n$ are bases of an \mathbb{F} -vector space \mathcal{V} then $r = n$.
6. Why is this true?