Last name ______________________

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LARSON—MATH 610–CLASSROOM WORKSHEET 08
All Bases have the same Cardinality.

Concepts (Chp. 1): field, vector space, $\mathcal{P}$, $\mathbb{F}^n$, $\mathbb{M}_{m\times n}(\mathbb{F})$, subspace, null space, row(A), col(A), list of vectors, span of a list of vectors, linear independence, linear dependence, pivot column decomposition, direct sum $\mathcal{U} \oplus \mathcal{V}$, *orthogonal* matrix, *unitary* matrix.

Review:

(**Lemma 2.1.4. Replacement Lemma**). Let $\mathcal{V}$ be a non-zero $\mathbb{F}$-vector space and let r be a positive integer. Suppose that $\beta = \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_r$ spans $\mathcal{V}$. Let $v \in \mathcal{V}$ be non-zero and let

$$\hat{v} = \sum_{i=1}^{r} c_i \hat{u}_i$$

(a) $c_j \neq 0$ for some $j \in \{1, \ldots, r\}$,
(b) If $c_j \neq 0$ then the list $\hat{v}, \hat{u}_1, \ldots, \widehat{\hat{u}_j}, \ldots, \hat{u}_r$ spans $\mathcal{V}$,
(c) If β is a basis for $\mathcal{V}$ and $c_j \neq 0$ then the list in (b) is a basis for $\mathcal{V}$,
(d) If $r \geq 2$, β is a basis for $\mathcal{V}$, $\hat{v} \notin span\{\hat{u}_1, \ldots, \hat{u}_k\}$ for some $k \in \{1, 2, \ldots, r-1\}$, then there is an index $j \in \{k+1, k+2, \ldots, r\}$ such that

$$\hat{v}, \hat{u}_1, \ldots, \hat{u}_k, \hat{u}_{k+1}, \ldots, \widehat{\hat{u}_j}, \ldots, \hat{u}_r$$

is a basis for $\mathcal{V}$.

Chp. 2 of Garcia & Horn, Matrix Mathematics

(**Theorem 2.1.10**). Let $\mathcal{V}$ be an $\mathbb{F}$-vector space and let r and n be positive integers. Suppose that $\beta = \hat{u}_1, \ldots, \hat{u}_n$ is a basis for $\mathcal{V}$ and $\gamma = \hat{v}_1, \ldots, \hat{v}_r$ is linearly independent.

(a) $r \leq n$.

(b) If $r = n$ then γ is a basis for $\mathcal{V}$.

1. What is this theorem about?

2. Why is this true?

(**Corollary 2.1.11, All Bases have Same Cardinality**). If r and n are positive integers and $\hat{v}_1, \ldots, \hat{v}_r$ and $\hat{w}_1, \ldots, \hat{w}_n$ are bases of an $\mathbb{F}$-vector space $\mathcal{V}$ then $r = n$.

3. Why is this true?

(**Theorem 2.3.1**) Let $A \in \mathbb{M}_{m \times n}(\mathbb{F}$. Then

$$dim(col(A)) = dim(col(A^T)) = dim(row(A)) \leq \min\{m, n\}.$$

4. What is this theorem about?

5. Why is this true?

(**Theorem 2.3.7. Full Rank Factorization**). Let $A \in \mathbb{M}_{m \times n}(\mathbb{F})$ be non-zero, let $r = rank(A)$, and let the columns of $X \in \mathbb{M}_{m \times r}(\mathbb{F})$ be a basis for $col(A)$. Then there is a unique $Y \in \mathbb{M}_{r \times n}(\mathbb{F})$ such that $A = XY$. Moreover, $rank(Y) = r$, the rows of Y are a basis for $row(A)$ and $null(A) = null(Y)$.

6. What is this theorem about?

7. Why is this true?