LARSON—MATH 511—CLASSROOM WORKSHEET 27
Low-Rank and Other Matrix Updates

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c27**.

Interlacing Eigenvalues for Symmetric Matrices

2. First we'll make a random matrix S: evaluate `S=random_matrix(RDF, 5, 5)`. Evaluate S to see what we have.

3. Evaluate S to see what we have.

4. Let's make it symmetric by adding the corresponding upper and lower triangular entries together:

```
for i in range(5):
    for j in range(i,5):
        value=S[i,j]+S[j,i]
        S[i,j]=value
        S[j,i]=value
```

   Evaluate S to see what we have. Is it symmetric?

5. Let's find the eigenvalues of S: evaluate `S.eigenvalues()`. Notice that they are (probably) not ordered.

6. We'll want to view them to compare them to the eigenvalues of other matrices. So lets sort them from largest to smallest.

```
Seigs = S.eigenvalues()
Seigs.sort(reverse=True)
```

   Now evaluate `Seigs` to see what we have.

7. Now let T be the matrix consisting of the first 4 rows and columns of S (so its a *principal submatrix*): evaluate `T = S.matrix_from_rows_and_columns([0..3],[0..3])`. Evaluate T to see that its what we wanted.

8. Let's get T's eigenvalues (sorted from largest to smallest):

```
Teigs = T.eigenvalues()
Teigs.sort(reverse=True)
```

   Now evaluate `Teigs` to see what we have. Do they "interlace" with the eigenvalues of S?

9. Now let's add a rank-1 update/signal to S and see how that effects the eigenvalues.

```
U=matrix(RDF,5,1,[1,1,1,1,1])
Ut=U.transpose()
theta = 5
signal = theta*U*Ut
T=S+signal
```

What are the eigenvalues of the signal matrix? Find and sort the eigenvalues of T. Check that they interlace with S.

Interlacing Eigenvalues for Symmetric Matrices

10. (**Review**) If S is a symmetric symmetric $n \times n$ matrix with (real) eigenvalues $\lambda_1 \geq \ldots \geq \ldots \geq \lambda_n$, then for *any* vector $\hat{x} \in \mathbb{R}^n$, $\lambda_n \leq \frac{\hat{x}^T S \hat{x}}{\hat{x}^T \hat{x}} \leq \lambda_1$.

11. (A Rayleigh-Ritz-type formula for symmetric matrices). Let S be a symmetric $n \times n$ matrix with (real) eigenvalues $\lambda_1 \geq \ldots \geq \lambda_q \geq \ldots \lambda_p \geq \ldots \geq \lambda_n$, and corresponding eigenvectors $\hat{u}_1, \ldots, \hat{u}_n$. If $\hat{x}$ is a unit eigenvector in $Span(\{\hat{u}_p, \ldots, \hat{u}_q\})$ then $\lambda_p \leq \hat{u}^T S \hat{u} \leq \lambda_q$.

12. (**Cauchy's Interlacing Theorem**) If A is a $(n-1) \times (n-1)$ principle submatrix of a symmetric matrix S with eigenvalues $\mu_1 \geq \ldots \geq \mu_{n-1}$ then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \lambda_n.$$

13. (**Weyl's Inequalities**) Strang claims that if S is a symmetric matrix and we add a rank-1 matrix $\theta \hat{u}\hat{u}^T$, the resulting eigenvalues will all be larger than the eigenvalues of T. Here's a relevant theorem. Let T be a symmetric matrix with eigenvalues $\mu_1 \geq \ldots \mu_n$ and let $\lambda_i(S+T)$ be the i^{th} eigenvalue of $S+T$. Then:

$$\lambda_i + \mu_n \leq \lambda_i(S+T) \leq \lambda_i + \mu_1.$$

What does this say when T is a rank-1 matrix (like our `signal` matrix)?

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c26 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!