

Last name \_\_\_\_\_

First name \_\_\_\_\_

**LARSON—MATH 550—CLASSROOM WORKSHEET 29**  
**The Sorting Example.**

**Concepts & Notation**

- Sec. 5.1: Binomial coefficients, the Binomial Theorem, key formulas, the polynomial argument, Vandermonde's convolution.
- Sec. 5.2: the Sorting example.
- Sec. 5.3: Duplication formula, inversion, derangements.
- Sec. 5.4: Convolutions, generating functions.

**Review**

1.  $\binom{n}{k}$  is the number of  $k$ -subsets of an  $n$ -set (for  $n, k \in \mathbb{Z}^{\geq 0}$ ).
2. We proved the *absorption identity*  $\binom{r}{k} = \frac{r}{k} \binom{r-1}{k-1}$  and the *absorption identity* variation  $k \binom{r}{k} = r \binom{r-1}{k-1}$  and  $(r-k) \binom{r}{k} = r \binom{r-1}{k}$ .
3. **(Negating the Upper index).** We proved:  $\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$ .
4. **(Summing on the Upper Index)** Prove (for  $n, n \in \mathbb{Z}$ ):

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}.$$

1. **(The Sorting Problem)** Simplify:

$$T = \sum_{k=0}^n k \binom{m-k-1}{m-n-1} / \binom{m}{n}.$$

Steps:

(a) Let

$$S = \sum_{k=0}^n k \binom{m-k-1}{m-n-1}.$$

(b) Rewrite  $k$  as  $m - (m - k)$ ,

(c) Use absorption to get:

$$S = mA - (m - n)B, \text{ where :}$$

$$A = \sum_{k=0}^n \binom{m-k-1}{m-n-1}, \text{ and } B = \sum_{k=0}^n \binom{m-k}{m-n}.$$

(d) Sum on the Upper Index to get:

$$B = \binom{m+1}{m-n+1}, \text{ and } A = \binom{m}{m-n}.$$

(e) Show:

$$S = \frac{n}{m-n+1} \cdot \binom{m}{m-n}, \text{ and } T = \frac{n}{m-n+1}.$$

### **The Duplication Formula**

(f) Show:

$$r^k \left(r - \frac{1}{2}\right)^k = (2r)^{2k} / 2^{2k} \text{ for } k \in \mathbb{Z}^{\geq}.$$