LARSON—MATH 511—CLASSROOM WORKSHEET 22
Low-Rank and Compressed Sensing

Outer-Product Expansion

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Check that:

$$AB = \hat{a}_1\hat{b}_1^T + \hat{a}_2\hat{b}_2^T,$$

where $\hat{a}_i$'s are the columns of A and $\hat{b}_i^T$ are the rows of B.

2. **Why** is it true that, for an $m \times n$ matrix A with columns $\hat{a}_1,\ldots,\hat{a}_n$, and $n \times t$ matrix B, with rows $\hat{b}_1^T,\ldots,\hat{b}_n^T$, that:

$$AB = \hat{a}_1\hat{b}_1^T + \hat{a}_2\hat{b}_2^T + \ldots + \hat{a}_n\hat{b}_n^T.$$

Changes in A^{-1} from Changes in A

3. (**Rank-1 changes**) Let $\hat{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\hat{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Find $M = I - \hat{u}\hat{v}^T$ (where it is implicit that $I = I_2$).

4. Find $I + \frac{\hat{u}\hat{v}^T}{1-\hat{u}^T\hat{v}}$.

5. Check that $M^{-1} = I + \frac{\hat{u}\hat{v}^T}{1-\hat{u}^T\hat{v}}$.

6. Show that if $M = I - \hat{u}\hat{v}^T$ is invertible then $M^{-1} = I + \frac{\hat{u}\hat{v}^T}{1-\hat{u}^T\hat{v}}$.

7. (**Rank-k changes**). Let $UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (rank-2). How does subtraction of UV^T change the inverse of I? (We know the inverse of I is itself. What is the inverse of $I - UV^T$?)

8. (**Sherman-Morrison formula**) Show that if $M = I - UV^T$ (with rank-k U) is invertible then $M^{-1} = I_n + U(I_k - V^TU)^{-1}V^T$.

9. (**Sherman-Morrison-Woodbury formula**) Show that if $M = A - UV^T$ (with rank-k U) is invertible then $M^{-1} = A^{-1} + A^{-1}U(I - V^TA^{-1}U)^{-1}V^TA^{-1}$.

Sage/CoCalc

10. (a) Start the Chrome browser.
    (b) Go to `http://cocalc.com`
    (c) Login (likely using **your VCU email address**).
    (d) You should see an existing Project for our class. Click on that.
    (e) Click "New", then "Sage Worksheet", then call it **c22**.

11. (**Rank-1 changes**) Let $\hat{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\hat{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Find $M = I - \hat{u}\hat{v}^T$ (where it is implicit that $I = I_2$).

12. Check that $M^{-1} = I + \frac{\hat{u}\hat{v}^T}{1-\hat{u}^T\hat{v}}$.

13. (**Rank-k changes**). Let $UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (rank-2). How does subtraction of UV^T change the inverse of I? (We know the inverse of I is itself. What is the inverse of $I - UV^T$?)

14. (**Sherman-Morrison formula**) Check the formula $M^{-1} = I_n + U(I_k - V^TU)^{-1}V^T$ with $M = I - UV^T$ and U, V^T from the previous example.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c22 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!