Last name ______________________

First name ______________________

LARSON—MATH 610—CLASSROOM WORKSHEET 13
Eigenvalues and Eigenvectors.

Concepts & Notation

- (Chp. 1) *field* $\mathbb{F}$, *list*, *vector space*, $\mathbb{F}^n$, $\mathbb{F}^S$, $\mathbb{F}^\infty$, *subspace*, sums of subspaces, *direct sum*.
- (Chp. 2) *linear combination*, *span*, *finite-dimensional* vector space, *linear independence*, *basis*.
- (Chp. 3) *linear map*, *null space*, *range*, *injective*, *surjective*, *invertible*, *isomorphism*, , *isomorphism*.
- (Chp. 4) *polynomial*, *root*.
- (Chp. 5) *eigenvalue*, *eigenvector*, *invariant subspace*, *minimal polynomial*.

Eigenvalues and Eigenvectors

1. **Linearly independent eigenvectors**: Suppose $T \in \mathcal{L}(V)$. Every list of eigenvectors of T corresponding to distinct eigenvalues is linearly independent.
2. (**Claim**:) A finite-dimensional vector space V has at most *dim* V eigenvalues.
3. If $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$, what is $p(T)$?
4. (**Claim**:) Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.
5. (**Existence, uniqueness, and degree of minimal polynomial**). If V is a finite-dimensional vector space, and $T \in \mathcal{L}(V)$, then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree with $p(T) = 0$. Alsp *deg* $p \leq$ *dim* V.
6. What is the *minimal polynomial* of $T \in \mathcal{L}(V)$ (for finite-dimensional V)?

7. What is an *invariant subspace* of $T \in \mathcal{L}(V)$?

8. For $T \in \mathcal{L}(V)$ the *eigenspace* $U = \{v : T(v) = \lambda v\}$ corresponding to an eigenvalue λ of T is an invariant subspace of T.

9. (**Claim**:) Suppose $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_n)$ is a basis of V. Then the following are equivalent:

   (a) the matrix of T with respect to $(v_1, \ldots, v_n)$ is upper-triangular;

   (b) $T(v_k) \in span(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$;

   (c) $span(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

10. (**Claim**:) Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.