LARSON—MATH 511—CLASSROOM WORKSHEET 23
Low-Rank and Compressed Sensing

Changes in A^{-1} from Changes in A

1. (**Sherman-Morrison-Woodbury formula**, rank-k changes) Show that if $M = A - UV^T$ (with rank-k U) is invertible then

$$M^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1}U)^{-1}V^T A^{-1}.$$

Vandermonde Matrices

2. You know you can find a unique line that passes through any 2 points. Did you know you can find a unique parabola through any 3 points? Find a parabola through the points $(1, 5), (2, 3), (4, 7)$.

3. Claim: Given $n+1$ points you can find a unique degree-n polynomial that fits them; that is, given points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, you can find a unique function $f(x) = c_0 + c_1 x^1 + \ldots + c_n x^n$ such that $f(x_0) = y_0, f(x_1) = y_1, \ldots, f(x_n) = y_n$ (assuming all x_i's are different of course).

4. (**Philosophical Implications**). What is the next term in the sequence $1, 2, 3, 4 \ldots$?

5. Show that Vandermonde Matrices are invertible.

The Derivative of A^{-1}

6. Let $A(t) = \begin{bmatrix} t & \frac{1}{t} \\ t^2 & t^2 + 1 \end{bmatrix}$. Find $A(1)$, $A(2)$.

7. Find $\frac{dA}{dt}$.

8. Let $A = A(1)$ and $B = A(2)$. Are they invertible?

9. Let $\Delta A = B - A$. Find ΔA.

10. (**A Very Useful Formula**). Check: $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$.

11. Use this to find $\frac{\Delta A^{-1}}{\Delta t}$ and $\frac{dA^{-1}}{dt}$.

Sage/CoCalc

12. (a) Start the Chrome browser.
    (b) Go to `http://cocalc.com`
    (c) Login (likely using **your VCU email address**).
    (d) You should see an existing Project for our class. Click on that.
    (e) Click "New", then "Sage Worksheet", then call it **c23**.

13. (**Rank-k changes**). Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Let $UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (rank-2). How does subtraction of UV^T change the inverse of A? (We know the inverse of A. What is the inverse of $A - UV^T$?)

14. (**Sherman-Morrison formula**) Check the formula

$$M^{-1} = A^{-1} + A^{-1}U(I - V^T A^{-1} U)^{-1} V^T A^{-1}$$

with $M = A - UV^T$ and U, V^T from the previous example.

15. (**Vandermonde Matrices**) What is the next term in the sequence $1, 2, 3, 4 \ldots$? Choose any number for the next term. Find the Vandermonde matrix V for your sequence, find $V^{-1}\hat{y}$ where $\hat{y}$ contains the sequence entries, form a degree-4 polynomial that fits these sequence terms.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).
2. Send me an email with an informative header like "Math 511—c23 worksheet attached" (so that it will be properly recorded).
3. Remember to attach today's classroom worksheet!