Last name ______________________

First name ______________________

LARSON—MATH 556—CLASSROOM WORKSHEET 27
Ford-Fulkerson imples Kőnig!

Review

- What is a *network*?
- What is a *flow* in a network?
- What is the *value* of a flow in a network?
- What is the *capacity* of a cut $\nabla^+(A)$ (or a separator A) in a network?
- (**Claim:**) A flow f is maximum if and only if there are no f-augmenting paths.
- (**Max-Flow Min-Cut Theorem:**) The value of a maximum flow in a network equals the capacity of a minimum cut.
- (**Flow Integrality Theorem**:). If the capacities of a network are integers, then there exists a maximum flow which is integral on every line.

1. Let G be the milkbone graph. It bipartite, so $G = (A, B)$. Build a network G' by directing all lines of G from A to B, adding a new point s (the "source") joined to all points of A and a new point t (the "sink") to which all points of B are joined, and then assigning capacity ∞ to all lines of G and capacity 1 to all new lines of G'.

2. Find a maximum flow. What do you notice?

Max-Flow Min-Cut implies Kőnig's Minimax Theorem

Let $G = (A, B)$ be a bipartite graph. Build a network G' by directing all lines of G from A to B, adding a new point s (the "source") joined to all points of A and a new point t (the "sink") to which all points of B are joined, and then assigning capacity ∞ to all lines of G and capacity 1 to all new lines of G'.

We want to show $\tau = \nu$. We know $\tau \geq \nu$, so we need to show $\tau \leq \nu$.

Let f be a maximum flow in G'. Let A_f be s together with the points that can be reached from s by a flow-augmenting path. Let $X = A \setminus A_f$ and let $Y = B \cap A_f$.

We proved A_f is a minimum cut. So $val(f) = cap(\nabla^+(A_f))$.

3. Let M be the lines in G with positive flow. Argue: that M is a matching.

4. Argue: M is a *maximum* matching (so $\nu = |M|$).

5. Argue: $val(f) = \nu$.

6. Argue: There is no flow from X to Y.

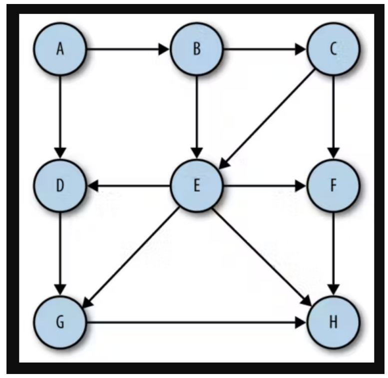
7. Argue: Y covers $A_f \cap A$.

8. Argue: X is matched to $B \setminus Y$.

9. Argue: There are no lines in G from $A \cap A_f$ to $B \setminus Y$.

10. Argue: $X \cup Y$ covers G.

11. Argue: $cap(\nabla^+(A_f)) = |X \cup Y|$. So, $\nu = cap(\nabla^+(A_f)) = |X \cup Y| \geq \tau$.



12. Find the maximum number of directed paths you can find from point B to point H, which are *line disjoint* (that is, no two directed paths share a common line).

13. Find the minimum number of directed lines you need to remove in order to destroy all $B - H$ directed paths.

14. (**Menger's Theorem**:) Let x, y be points in a directed graph D. The maximum number of line-disjoint directed paths from point x to point y equals the minimum number of directed lines you need to remove in order to destroy all $x - y$ directed paths.

    How can you apply the Max Flow-Min Cut Theorem in order to prove Menger's Theorem.