LARSON—MATH 511—CLASSROOM WORKSHEET 12
SVD Image Compression!

Sage/CoCalc

1. (a) Start the Chrome browser.
   (b) Go to `http://cocalc.com`
   (c) Login (likely using **your VCU email address**).
   (d) You should see an existing Project for our class. Click on that.
   (e) Click "New", then "Sage Worksheet", then call it **c12**.

2. Open your CoCalc project Handouts folder, click on "SVD_image_compression.sage".

3. We will run the code here step-by-step in your c12 worksheet.

SVD Algorithm (general case, A has rank r)

Suppose A is **any** $m \times n$ matrix.

- Find A^TA (its $n \times n$). This matrix has rank r, is symmetric and is positive semi-definite.
- Find the positive eigenvalues and corresponding (unit) eigenvectors λ, $\hat{v}$ (So $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$). The remaining eigenvalues $\lambda_{r+1}, \ldots, \lambda_n$ are 0. The remaining (unit) eigenvectors $\hat{v}_{r+1}, \ldots, \hat{v}_n$ are a basis for the nullspace of A^TA. Vectors corresponding to different eigenvalues are orthogonal. For each eigenspace with dimension greater than 1, Gram-Schmidt can be used to find an orthogonal basis.
- Let $V = [\hat{v}_1 \ldots \hat{v}_r \ldots \hat{v}_n]$. V is orthogonal.
- Let $\sigma_i = \sqrt{\lambda_i}$ and let Σ be the $m \times n$ matrix with diagonal entries $\sigma_1, \ldots, \sigma_r$.
- Find AA^T (its $m \times m$). This matrix has rank r, is symmetric and is positive semi-definite..
- Check that $A\hat{v}_i$ is an eigenvector for AA^T with corresponding eigenvalue λ_i. This means that these are all the non-zero eigenvalues for AA^T. Let $u_i = \frac{1}{\sigma_i}A\hat{v}_i$ for $i = 1 \ldots r$. Check that these $\hat{u}_i$ are unit.
- The remaining eigenvalues of AA^T, $\lambda_{r+1}, \ldots, \lambda_m$ are 0. The remaining (unit) eigenvectors $\hat{u}_{r+1}, \ldots, \hat{u}_m$ are a basis for the nullspace of AA^T.
- Let $U = [\hat{u}_1 \ldots \hat{u}_r \ldots \hat{u}_m]$.
  U is orthogonal.
- $AV = U\Sigma$ and $A = U\Sigma V^T$.

Remaining Claims

4. If A is a rank r matrix then $A^T A$ is rank r.

5. If A is a rank r matrix then AA^T is rank r.

6. Show: the $\hat{u}$'s in U are orthogonal.

7. Show: $AV = U\Sigma$ and $A = U\Sigma V^T$.

Low Rank Approximation

8. Why does $A = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_r \hat{u}_r \hat{v}_r^T$?

9. For $k \leq r$ let $A_k = \sigma_1 \hat{u}_1 \hat{v}_1^T + \ldots + \sigma_k \hat{u}_k \hat{v}_k^T$.

10. **Coming Soon!** A_k is the "best" low rank approximation to A.

Getting your classwork recorded

When you are done, before you leave class...

1. Click the "Make pdf" (Adobe symbol) icon and make a pdf of this worksheet. (If CoCalc hangs, click the printer icon, then "Open", then print or make a pdf using your browser).

2. Send me an email with an informative header like "Math 511—c12 worksheet attached" (so that it will be properly recorded).

3. Remember to attach today's classroom worksheet!