Newton's Method 11/21/19

Three root-finding methods have been discussed in lecture: intermediate value theorem, linear approximations, and Newton's method. There are many real problems for which these numeric (as opposed to analytic) methods are required. Recall that the 'recursive formula' or 'iteration step' of Newton's method to find solutions to $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

1. *Chi-Squared Distribution.* A statistician is interested in solutions x to $\int_0^x f(t)dt = 1-\alpha$, where $f(t) = \frac{1}{4}te^{-t/2}$. This $f(t)$ is a special case of a 'chi-squared probability density function.' These solutions x are useful in statistical analyses in most hard and social sciences.

   (a) Confirm that $-\frac{1}{2}te^{-t/2} - e^{-t/2}$ is an antiderivative for $f(t)$.

   (b) Get an equation relating x and α which has no integral.

   (c) Write the recursive formula for Newton's method to find x given α. A common choice for α is 0.05. Choose an initial value x_0.

2. *Finding Extrema.* Consider the function $f(x) = e^x + x^2$.

   (a) Show that $f(x)$ has a unique minimum value.

   (b) Determine a 'small' interval where the minimum of $f(x)$ lies.

   (c) Write down the recursive formula for Newton's method to find this minimum.

3. *Lambert W Function.* Problems in differential equations, physics, combinatorics, and biochemistry make use of the inverse of the function $g(x) = xe^x$, called the Lambert W function. That is, $W(c)$ is a solution to $xe^x = c$.

   (a) On what intervals does $g(x)$ have an inverse? We define W as the inverse of g on the rightmost interval.

   (b) Find a 'small' interval where $g(x) = 2$ has a solution. Use this to bound $W(2)$.

   (c) Write the recursive formula for Newton's method to find $W(2)$. Pick an initial value x_0.

4. *Finite Square Well.* In quantum mechanics, when solving for the wavefunction of a finite square well, one needs to solve for x in $x\tan(\alpha x) = \sqrt{\beta^2 - x^2}$ for constants $\alpha > 0$ and β depending on physical constants.

   (a) Write the recursive formula for Newton's method to find solutions to this equation.

   (b) Choose an initial guess for the smallest positive solution. Show that such a solution must always exist (either graphically or with the IVT).

   (c) For β large, approximately what is the smallest positive solution for x?