Announcements

HW 1 due Friday 11:59 on Gradescope

HW2 posted

Outline

- vectors, matrices
- matrix-vector multiplication
- linear independence
- column space of a matrix
- rank of a matrix
- $A = CR$ decomposition
- matrix-matrix multiplication

Readings

Strang I.1 I.2

Murphy Ch 7

Note

Assume all vectors and matrices have real number components

Vector

In this class, think of a vector as a point in $\mathbb{R}^n$ (set of n-tuples)

We will use boldface lowercase letters for vectors

ex $\vec{x} = (x_1, x_2, x_3)^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ The i^{th} element of $\vec{x}$ is denote x_i.

Special vectors

- $\vec{0}_n = (0, \dots, 0)^T \in \mathbb{R}^n$ (zero vector)
- $\vec{e}_i = (\underset{\text{position } i}{0, \dots, 0, 1}, 0, \dots, 0)^T$ (canonical basis vector in $\mathbb{R}^n$) $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

Matrices

Think of a matrix as a rectangular grid of real numbers.

In particular a real-valued matrix with m rows and n columns is said to be an element of $\mathbb{R}^{m \times n}$.

We use uppercase letters for matrices:

ex. $A = (a_{ij}) \in \mathbb{R}^{m \times n}$

The entry of A in the i^{th} row and j^{th} column is denoted a_{ij} (sometimes $A(i,j)$ or A_{ij})

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

What is m? 2

What is n? 3

What is a_{22}? 5

Special matrix

- Identity matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$ $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Beyond vectors and matrices: Tensors

We can generalize to multidimensional arrays: Tensor

Matrix-Vector Multiplication

ex $A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ What is $A\vec{x}$? $\vec{x} \in \mathbb{R}^2$ $A\vec{x} \in \mathbb{R}^3$ $A\vec{x} = \begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix}$

Two approaches to matrix-vector multiplication:

By rows

rows of the matrix must be same length as vector

$\begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix}$

$\mathbb{R}^{3\times 2}$ $\mathbb{R}^2$

equivalent to computing three dot products

row · column $\begin{bmatrix} 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1x_1 + 5x_2$

By columns

$$\begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix}$$

$A\vec{x}$ is a linear combination of the columns of A.

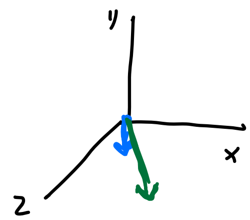
Column space

def the column space of A is the vector space of all linear combinations of the columns of A, denoted $C(A)$ (also span of columns of A or image of A)

$A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix}$ What is the column space of A? a plane

$\vec{a}_1$ $\vec{a}_2$ What is its dimension? 2

$$c_1\vec{a}_1 + c_2\vec{a}_2 \in C(A)$$



$\vec{a}_1 \in \mathbb{R}^3$ $c_1\vec{a}_1 + c_2\vec{a}_2 \in \mathbb{R}^3$

$\vec{a}_2 \in \mathbb{R}^3$

The column space is a plane in $\mathbb{R}^3$

- includes $\vec{0}$
- $\vec{b}$ is $C(A)$ if and only if $\boxed{A\vec{x} = \vec{b} \text{ has a solution } \vec{x}}$

Is $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the column space of A? $A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

What is the column space of

$$A_2 = \begin{bmatrix} 1 & 5 & 6 \\ 1 & 6 & 7 \\ 2 & 8 & 10 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 5 & 1 \\ 1 & 6 & 1 \\ 2 & 8 & 1 \end{bmatrix} ?$$

$$C(A_2) = C(A) \qquad C(A_3) = \mathbb{R}^3$$

Possible column spaces in $\mathbb{R}^3$

- a point (zero vector)
- a 1D line
- a 2D plane
- all of $\mathbb{R}^3$

def The row space of A is the column space of A^T

$$C(A^T)$$

Linear independence

def A set of vectors $\vec{a}_1, \ldots, \vec{a}_k \in \mathbb{R}^n$ are linearly independent if

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n = \vec{0} \text{ is only satisfied for } c_1 = \ldots = c_n = 0$$

note Three independent vectors in $\mathbb{R}^3$ produce a matrix whose column space is all of $\mathbb{R}^3$

$A = CR$

We can easily (?) find a basis for the column space of A and produce a matrix decomposition $A = CR$

def A basis for a subspace is a full set of independent vectors such that all vectors in the subspace are (non-trivial) linear combinations of the basis vectors.

Matrix-matrix multiplication AB

Visual strategy

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} & & \\ & & c_{23} \\ & & \end{bmatrix} \qquad c_{23} = a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33}$$

$A \in \mathbb{R}^{3\times3}$ $\quad B \in \mathbb{R}^{3\times3}$ $\quad C \in \mathbb{R}^{3\times3}$

(inner/dot products)

in general $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

The other way

ex. $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$ $\qquad \vec{u}\vec{v}^T = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix}$

$\vec{u}$ $\quad \vec{v}$ $\qquad 3\times1 \quad 1\times3$

note all columns of $\vec{u}\vec{v}^T$ are multiples of $\vec{u}$

all rows of $\vec{u}\vec{v}^T$ are multiples of $\vec{v}^T$

column-row multiplication

$$\begin{array}{c} \vec{a}_1 \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \cdot \begin{array}{cc} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} & \begin{array}{l} \vec{b}_1^* \\ \\ \\ \end{array} \end{array}$$

$A \in \mathbb{R}^{3\times3}$ $\quad B \in \mathbb{R}^{3\times3}$

$AB = \vec{a}_1\vec{b}_1^* + \vec{a}_2\vec{b}_2^* + \dots + \vec{a}_n\vec{b}_n^*$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} \quad AB = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 17 \end{bmatrix}$$

try multiplying the other way.

If A is $m \times n$ and B is $n \times p$, C is $m \times p$.

note

the usual way: mp dot products, n multiplications each

the outer product way: n outer products, mp multiplications each.