Lecture 5

Announcements

- HW1 graded, solutions posted
- HW2 due next Wednesday

Readings

- Strang I.6 and I.7

Outline

- Similar matrices
- Diagonalizability
- Spectral Theorem
- Pos. def. and pos. semidef. matrices
- Application: optimization.

Similar matrices

recall $\vec{x}$ is an eigenvector of A if $A\vec{x} = \lambda\vec{x}$ for some number λ.

λ is called the eigenvalue of $\vec{x}$

A must be square $\in \mathbb{R}^{n\times n}$

def A matrix B is similar to matrix A if we can write $B = PAP^{-1}$. $(BP = PA)$

note Observe that B has the same eigenvalues as A. How do we show this?

Suppose λ is an eigenvalue of A. with eigenvector $\vec{x}$

$$A\vec{x} = \lambda\vec{x}$$

Want to show that there exists $\vec{y}$

$$B\vec{y} = \lambda\vec{y}$$

$$PAP^{-1}\vec{y} = \lambda\vec{y}$$

Let $\vec{y} = P\vec{x}$. Then $B\vec{y} = (PAP^{-1})P\vec{x} = PA\vec{x} = P(\lambda\vec{x}) = \lambda(P\vec{x}) = \lambda\vec{y}$

eigenvectors of A: $\vec{x}_1, \dots, \vec{x}_n$

eigenvectors of B: $P\vec{x}_1, \dots, P\vec{x}_n$

Diagonalizability

def A square matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if it is similar to a diagonal matrix. In other words, there exists an invertible matrix X and diagonal Λ such that

$$A = X\Lambda X^{-1} \qquad (\text{or equivalently } X^{-1}AX = \Lambda)$$

note if $X = \begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

then $AX = X\Lambda$

$$A\begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

therefore $A\vec{x}_1 = \lambda_1 \vec{x}_1$

$\vdots$

$A\vec{x}_n = \lambda_n \vec{x}_n$

The columns of X are eigenvectors of A

The diagonal of Λ are the corresponding eigenvalues.

Moreover the eigenvectors are linearly independent.

This is the eigendecomposition or spectral decomposition.

In general you can check that A has n linearly independent eigenvectors.

- solve $\det(A - \lambda I) = 0$ to get the n eigenvalues.
- Check that the Geometric Multiplicity (GM) of each eigenvalue equals the Algebraic Multiplicity (AM)
  - GM of an eigenvalue is the dimension of the space spanned by the corresponding eigenvectors
  - AM is number of times it appears as a solution $\det(A - \lambda I) = 0$

ex $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$ $\quad \lambda_1 = 10$, $\lambda_2 = 5$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1}$$

verify $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

note Why is this important?

- $A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 X^{-1}$ $\qquad A^{100} = X\Lambda^{100} X^{-1}$
- If A is invertible, A^{-1} is $X\Lambda^{-1}X^{-1}$

Symmetric Matrices

def S is a symmetric matrix if $S = S^T$ (S must be square)

note All symmetric matrices are diagonalizable.

In addition, they can be diagonalized by orthogonal matrices.

$$S = Q\Lambda Q^T$$

Theorem (Spectral)

If S is a real symmetric $n\times n$ matrix, S has n real eigenvalues and n orthonormal eigenvectors.

$$S = Q\Lambda Q^T \qquad SQ = Q\Lambda \qquad Q = \begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{bmatrix}$$

note We can rewrite S as a sum of rank-1 matrices

$$S = (Q\Lambda)Q^T = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \dots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$$

note We arrange the columns of Q and Λ so $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

proof

1. $\det(S-\lambda I)$ is an n^{th} degree polynomial with n roots (Fundamental Theorem of Algebra)
2. $\lambda_1, \dots, \lambda_n$ are all real if S is real + symmetric (Verify)
3. Eigenvectors to different eigenvalues are orthogonal (Verify)
4. Tricky: show that when an eigenvalue is repeated, it is still possible to produce orthogonal eigenvectors. when $S = S^T$

ex Find the spectral decomposition of $S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

1. Find λ_1 and λ_2

$$\det(S-\lambda I) = 0$$
$$(1-\lambda)^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0 \qquad \lambda_1 = 3 \quad \lambda_2 = -1 \qquad S = Q\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} Q^T$$

2. Find Q

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \qquad \begin{matrix} x + 2y = 3x \\ 2x + y = 3y \end{matrix} \Rightarrow x = y \qquad \vec{q}_1 = \cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. $$S = \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} \end{bmatrix}$$ (Verify)

Positive Definite + Positive Semidefinite Matrices

Two common definitions

① S is positive definite if the energy function $\vec{x}^T S \vec{x} > 0$ for all $\vec{x} \neq 0$

positive semidefinite if $\vec{x}^T S \vec{x} \geq 0$ for all $\vec{x} \neq 0$

ex Can we think of a positive definite matrix?

$S = I \quad \vec{x}^T S \vec{x} = \vec{x}^T \vec{x} > 0$ since $\vec{x} \neq 0$

ex $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ex $S = \begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix} \quad \vec{x}^T S \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + 8x_1x_2 + 9x_2^2$

$= 2(x_1 + 2x_2)^2 + x_2^2 > 0$

② S is positive (semi) definite if all eigenvalues > 0 (≥ 0)

Connecting ① and ②

For any eigenvector $\vec{q}$, $S\vec{q} = \lambda \vec{q}$

So $\vec{q}^T S \vec{q} = \lambda \vec{q}^T \vec{q}$. If $\lambda > 0$, $\vec{q}^T S \vec{q} > 0$

We want to show that for any vector $\vec{x}$, $\vec{x}^T S \vec{x} > 0$

Observe $\vec{x} = c_1 \vec{q}_1 + \dots + c_n \vec{q}_n$.

So, $\vec{x}^T S \vec{x} = (c_1\vec{q}_1 + \dots + c_n\vec{q}_n)^T S \, (c_1\vec{q}_1 + \dots + c_n\vec{q}_n)$

$= (c_1\vec{q}_1 + \dots + c_n\vec{q}_n)^T (c_1\lambda_1\vec{q}_1 + \dots + c_n\lambda_n\vec{q}_n)$

$= c_1^2 \lambda_1 \vec{q}_1^T \vec{q}_1 + \dots + c_n^2 \lambda_n \vec{q}_n^T \vec{q}_n \quad > 0$ if every $\lambda_i > 0$

Corollary of ①

If S_1 and S_2 are symmetric PD, so is $S_1 + S_2$ (Verify)

Other tests of PD

(3) $S = A^T A$ for A w/ independent columns

$S = Q \Lambda Q^T$ ... see what happens for $A = Q \Lambda^{1/2} Q^T$

(4) All pivots of S are positive (≥ 0)