Lecture 3

<u>Announcements</u>

- Geogebra
- HW1 due Friday
- HW2 edited

<u>Readings</u>

- Strang I.3, I.4, I.5
- Murphy Ch 7.

<u>Outline</u>

- $A = CR$
- matrix rank
- four fundamental subspaces
- Elimination and $A = LU$
- $Ax = b$.

$A = CR$

$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ What is $\dim(C(A))$? 2

- C is a matrix whose columns form a basis for $C(A)$

How to build C

- If column 1 of A is not all zero, put it in C
- If column 2 is not a multiple of col 1, put it into C
- If col 3 is not a lin. comb. of col 1 and col 2, put it into C
- continue...

- $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$

$\underset{3\times3}{A} = \underset{3\times2}{C}\,\underset{2\times3}{R}$

$m \times r$ where r = rank of A

<u>def</u> The rank(A) is the dimension of $C(A)$.

$A = C\,R$

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

R = row-reduced echelon form of A

If $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ what is $\underset{3\times1}{C}\,\underset{1\times3}{R}$?

Four fundamental subspaces

<u>def</u> A nonempty subset W is a linear subspace of $\mathbb{R}^n$ if for all $\vec{w}_1, \vec{w}_2 \in W$ and $c_1, c_2 \in \mathbb{R}$

$$c_1\vec{w}_1 + c_2\vec{w}_2 \in W$$

<u>ex</u>. $C(A)$

- $\{\vec{0}\}$
- $\mathbb{R}^n$

Any $m \times n$ matrix A is associated with four fundamental subspaces

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\quad m=2 \quad n=2$

$A = CR$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{R}$ $\quad \vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad A = \vec{u}\vec{v}^T$

1. The column space $C(A)$ contains all lin. comb. of columns of A
2. The row space $C(A^T)$ contains all lin. comb. of rows of A
3. The null space $N(A)$ contains all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$
4. The left nullspace $N(A^T)$ contains all solutions to $A^T\vec{y} = \vec{0}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

1. $C(A)$ is the line through $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
2. $C(A^T)$ is the line through $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3. $N(A)$ contains $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
4. $N(A^T)$ contains $\vec{y} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

If A is an $m \times n$ matrix, we can think of A as a map from $\mathbb{R}^n$ into $\mathbb{R}^m$.

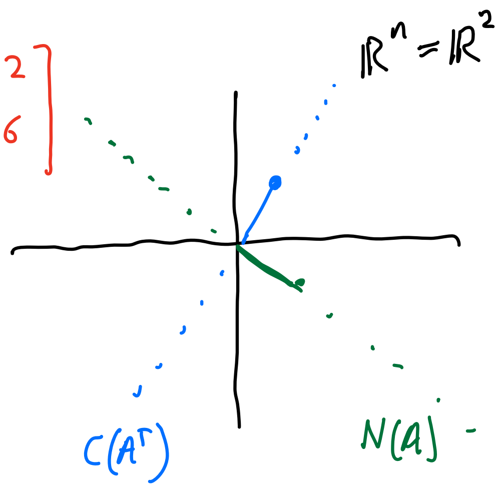
$$\underset{m\times n}{\begin{bmatrix} A \end{bmatrix}} \underset{\substack{n\times 1 \\ \mathbb{R}^n}}{\overset{\text{input}}{\begin{bmatrix} x \end{bmatrix}}} = \underset{\substack{m\times 1 \\ \mathbb{R}^m}}{\begin{bmatrix} b \end{bmatrix}}$$

$\mathbb{R}^n \longrightarrow \mathbb{R}^m$

contains $C(A^T)$ and $N(A)$ — contains $C(A)$ and $N(A^T)$

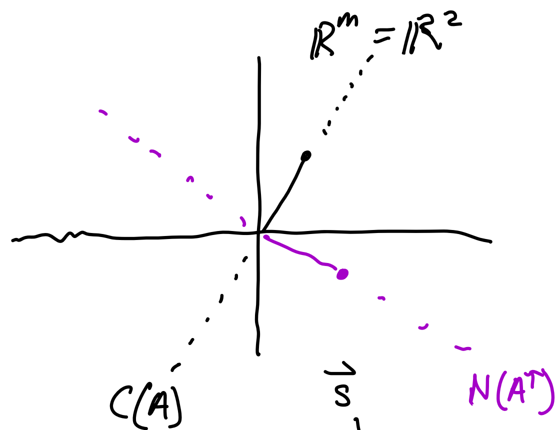
$$\underset{n\times m}{\begin{bmatrix} A^T \end{bmatrix}} \underset{m\times 1}{\begin{bmatrix} \vec{y} \end{bmatrix}} = \underset{n\times 1}{\begin{bmatrix} \vec{c} \end{bmatrix}}$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$



$C(A^T)$ and $N(A)$ are orthogonal

$C(A^T) \perp N(A)$



$C(A)$ and $N(A^T)$ are orthogonal subspaces

$\vec{r}$

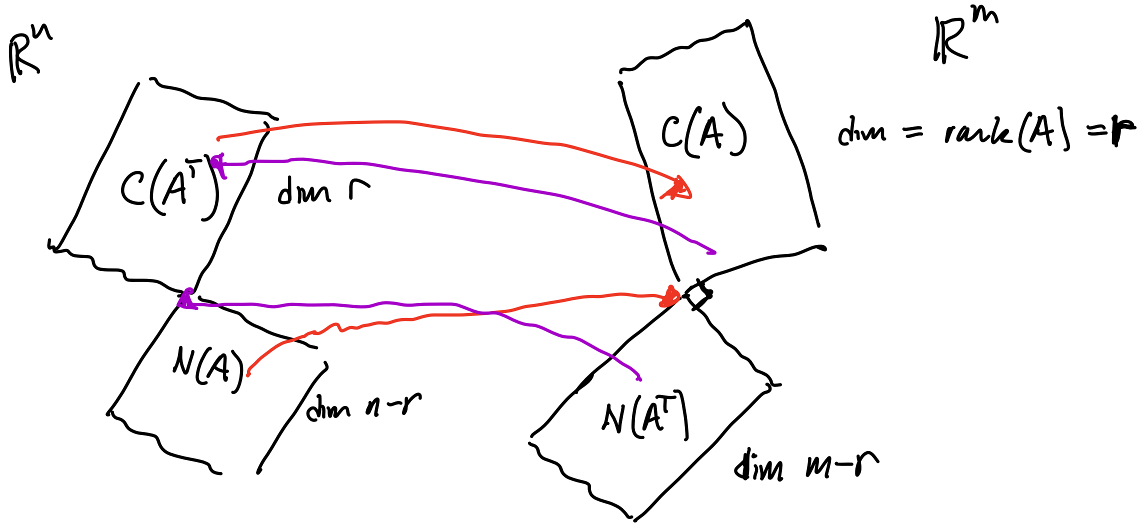
Perpendicular / orthogonal

$\vec{r}^T\vec{s} = \vec{r}\cdot\vec{s} = 0$

note this example is simple since $m=n=2$ and A is rank 1.

$B = \begin{bmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \end{bmatrix}$ Identify 4 fundamental subspaces.

A



Rank facts

① Rank of $AB \le$ Rank of A $\text{rank}(AB) \le \text{rank}(B)$

$C(AB) \subset C(A)$

② Rank of $A+B \le \text{rank}(A) + \text{rank}(B)$

③ Rank of A^TA = rank of A = rank of A^T = rank of AA^T

<u>note</u> A and A^TA have the same nullspace.

Elimination

Solving systems of linear equations

$$x + 2y + z = 4$$
$$y - z = -1$$
$$z = 2$$

$$X\vec{\beta} + \vec{\varepsilon} = \vec{y}$$

$$X\vec{\beta} = \vec{y}$$

$$A\vec{x} = \vec{b}$$

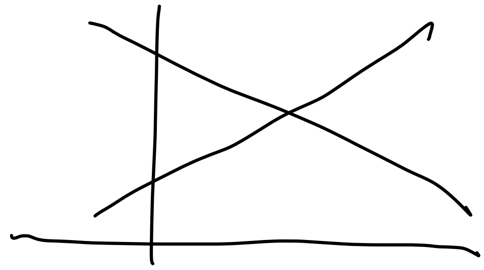
- If A is invertible, there is only one solution to $A\vec{x}=\vec{b}$.

  In this case, we can use the LU decomposition to solve $A\vec{x}=\vec{b}$ problems.

ex $\underbrace{\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 9 \end{bmatrix}}_{\vec{b}}$

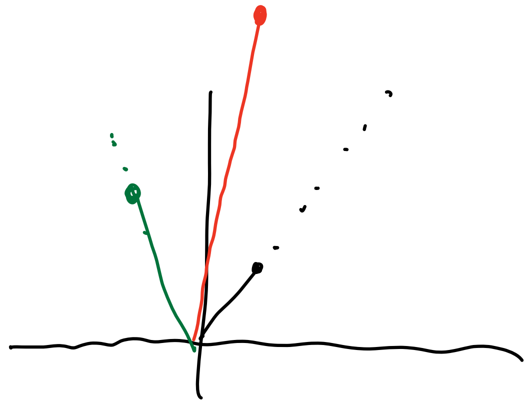
$x - 2y = 1$

$2x + 3y = 9$



$$\vec{x} = A^{-1}\vec{b}$$

$$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$



ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 7 & 8 \end{bmatrix}$

Goal: write $A = LU$

where L is a lower triangular matrix with 1s on the main diagonal

U is upper triangular

$$[A] = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} \diagdown \\ 0 \end{bmatrix}$$

Note $A = LU$ is not always possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \times \text{row } 1 \\ l_{21} \times \text{row } 1 \\ l_{31} \times \text{row } 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{A_2}} \\ 0 & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = LU$$

$A\vec{x} = \vec{b}$

$LU\vec{x} = \vec{b}$

$L\vec{y} = \vec{b}$ where $\vec{y} = U\vec{x}$

$U\vec{x} = \vec{y}$