MATH 250: Mathematical Data Visualization
Spring 2024

Homework 2: Linear algebra

due February 14, 2024

You may collaborate with your classmates and consult external resources, but you should write and submit your own answer. **Any classmates with whom you collaborate should be credited at the top of your submission. Similarly, if you consult any external references, you should cite them clearly and explicitly.**

1. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors, show that $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$. What are the lengths of those vectors?

2. Draw unit vectors $\mathbf{u}$ and $\mathbf{v}$ that are not orthogonal. Show that $\mathbf{w} = \mathbf{v} - \mathbf{u}(\mathbf{u}^\top \mathbf{v})$ is orthogonal to $\mathbf{u}$ (and add $\mathbf{w}$ to your picture).

3. Show that $(Q\mathbf{x})^\top (Q\mathbf{y}) = \mathbf{x}^\top \mathbf{y}$ for every vector $\mathbf{x}$ and $\mathbf{y}$ (so lengths and angles are not changed by Q).

4. Find an orthonormal basis for the column space of the following matrix A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 6 \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors for both of these matrices A and A^∞. Use your answers to explain why A^{100} is close to A^∞.

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \qquad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

6. Compute the eigenvalues and eigenvectors of A and A^{-1} for

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

   How do the eigenvalues and eigenvectors for A^{-1} compare to the eigenvalues and eigenvectors of A?

7. Show how the eigenvalues and eigenvectors of $B = A + 2I \in \mathbb{R}^{n\times n}$ compare with the eigenvalues and eigenvectors of $A \in \mathbb{R}^{n\times n}$.

8. Diagonalize the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

9. Show that the eigenvalues of a real symmetric matrix are real.

10. Write S in the form $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^\top + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^\top$, keeping $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

11. The energy $\mathbf{x}^\top S \mathbf{x} = 4x_1 x_2$ has a saddle point at $(0, 0)$. What symmetric matrix produces this energy and what are its eigenvalues?

12. Suppose S is positive definite with $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$.

    (a) What are the eigenvalues of $\lambda_1 I - S$? Is this matrix positive semidefinite?

    (b) Show that $\lambda_1 \mathbf{x}^\top \mathbf{x} \geq \mathbf{x}^\top \mathbf{x}$ for every $\mathbf{x}$ and compute the maximum value of $\mathbf{x}^\top S \mathbf{x} / \mathbf{x}^\top \mathbf{x}$.

13. Complete the **Matrix Algebra in R** lab and submit via Gradescope.