Lecture 4

Announcements

- HW2 + HW2 Lab posted

Readings

- Strang I.5 and I.6

Today

- Vector norms
- Orthogonality (vectors, bases, subspaces, matrices, examples + properties)
- Eigenvalues and eigenvectors

ex

$$X = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_p \\ | & | & & | \end{bmatrix}$$

$n \times p$

n observation $\times$ p variables

What if we have rows that are linearly dependent?

ex one row is equal to another row

" " " " columns

Vector Norms

How do we quantify the "size" or "length" of a vector?

- a vector norm maps vectors in $\mathbb{R}^n$ to a single number in $\mathbb{R}$

ex. $\cdot\ \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$ (1-norm)

$\cdot\ \|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ (2-norm)

$\cdot\ \|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ (p-norm, $p>0$)

$\cdot\ \|\vec{x}\|_\infty = \max_{1\le i \le n} |x_i|$ (max-norm)

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 10 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$

$\cdot\ \|\vec{x}\|_1 = 12$

$\|\vec{y}\|_1 = 15$

$\|\vec{x}\|_2 = \sqrt{102}$

$\|\vec{y}\|_2 = \sqrt{75}$

$\|\vec{x}\|_\infty = 10$

$\|\vec{y}\|_\infty = 5$

note $\|\vec{x}\|_2^2 = \vec{x}^T\vec{x} = \vec{x}\cdot\vec{x} = \langle \vec{x}, \vec{x} \rangle$

Orthogonality

def Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are orthogonal if $\vec{x}^T\vec{y} = \vec{y}^T\vec{x} = 0$

$\vec{x}^T\vec{y} = x_1y_1 + \dots + x_ny_n = 0$

In $\mathbb{R}^n$ this is equivalent to them being perpendicular.

note $\vec{x}^T\vec{y} = \|\vec{x}\|\|\vec{y}\|\cos\theta$ where θ is the angle between $\vec{x}$ and $\vec{y}$.

What are three orthogonal vectors in $\mathbb{R}^3$?

$\left\{ \underset{\vec{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \right\}$

note These vectors form a basis for $\mathbb{R}^3$

theorem Let $\vec{q}_1, \dots, \vec{q}_n$ be orthogonal vectors (so $\vec{q}_i^T\vec{q}_j = 0$ for $i \ne j$)

Then they are linearly independent.

proof outline Assume linear dependence and prove by contradiction.

theorem Every subspace of $\mathbb{R}^n$ has an orthogonal basis.

ex A plane in $\mathbb{R}^3$ spanned by independent vectors $\vec{x}$ and $\vec{y}$.

- Set the first vector of our basis $\vec{q}_1 = \vec{x}$.
- define $\vec{q}_2 = \vec{y} - \frac{\vec{y}^T\vec{x}}{\vec{x}^T\vec{x}}\vec{x}$ (residual from projecting $\vec{y}$ on $\vec{x}$)

In general, this idea leads to Gram-Schmidt orthogonalization

Orthogonal Subspaces

ex The row space of A is orthogonal to the null space of A.
The column space of A is orthogonal to the null space of A^T

$$\underbrace{\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_m & - \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \vec{0} \\ \vdots \\ \vec{0} \end{bmatrix} = \vec{0}$$

Orthonormal vectors

Let $\vec{q}_1, \dots, \vec{q}_n$ be orthogonal with all having length 1 so $\|\vec{q}_j\|_2 = 1$ for all j.

Then they are called orthonormal.

In particular if $\vec{q}_1, \dots, \vec{q}_n$ are n orthonormal vectors in $\mathbb{R}^n$, they form an orthonormal basis for $\mathbb{R}^n$.

Orthogonal Matrices

A square matrix $Q \in \mathbb{R}^{n\times n}$ with orthonormal columns is called an orthogonal matrix.

properties

- $Q^TQ = I$
- $QQ^T = I$
- $Q^{-1} = Q^T$

$$\underbrace{\begin{bmatrix} - & \vec{q}_1^T & - \\ & \vdots & \\ - & \vec{q}_n^T & - \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & 0 \\ 0 & & 0 & 1 \end{bmatrix}}_{I}$$

property

$$\|Q\vec{x}\|_2 = \|\vec{x}\|_2$$

$$\|Q\vec{x}\|_2^2 = (Q\vec{x})\cdot(Q\vec{x}) = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^T\overbrace{Q^TQ}^{I}\vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|_2^2$$

ex

$$Q_3 = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Confirm this is orthogonal

$$Q_3^TQ_3 = I_3$$
$$Q_3Q_3^t = I_3$$

$$Q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Is this orthogonal? Not really

$$Q_1^TQ_1 = I_1$$

$$Q_1Q_1^T = ?$$

It turns out that $Q_1Q_1^T$ is the orthogonal projection matrix onto the column space of Q_1.

$$\underset{Q_1}{\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}} \underset{Q_1^T}{\begin{bmatrix} 2/3 & 2/3 & -1/3 \end{bmatrix}} \underset{\vec{b}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

compare with $\dfrac{\vec{q}_1^T\vec{b}}{\vec{q}_1^T\vec{q}_1}\vec{q}_1$

In general, given an orthonormal basis $\vec{q}_1, \dots, \vec{q}_n$ for $\mathbb{R}_n$ any vector $\vec{x} \in \mathbb{R}^n$ can be written

$$\vec{x} = c_1\vec{q}_1 + \dots + c_n\vec{q}_n \qquad c_1 = \vec{q}_1^T\vec{x} \quad c_2 = \vec{q}_2^T\vec{x} \quad , \dots , \quad c_n = \vec{q}_n^T\vec{x}$$

Proof exercise

Eigenvalues and Eigenvectors

def A vector $\vec{x}$ is an eigenvector of A if $\boxed{A\vec{x} = \lambda\vec{x}}$ for some number λ. λ is called an eigenvalue of $\vec{x}$ and of A.

note if $\vec{x}$ is an eigenvector of A, it is also an eigenvector of A^2

$$A^2\vec{x} = AA\vec{x} = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x}$$

ex $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

What are the eigenvalues?

$\lambda_1 = 3$

$S\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$S\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ex $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has imaginary eigenvalues i and $-i$ $\quad i = \sqrt{-1}$

$Q\begin{bmatrix} 1 \\ -i \end{bmatrix} = i\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad Q\begin{bmatrix} 1 \\ i \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix}$

Computing eigenvalues

Note that $A\vec{x} = \lambda\vec{x}$ so $A\vec{x} - \lambda\vec{x} = \vec{0} \Rightarrow (A - \lambda I)\vec{x} = 0$

Assuming $\vec{x}$ is non-zero $(A - \lambda I)$ is not invertible and must be singular.

$$\det(A - \lambda I) = 0$$

This is an n^{th}-degree polynomial in λ.

$\rightarrow \det(A - \lambda I) = 0$ has n solutions.

ex $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ $\quad \det(A - \lambda I) = (a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

Solve this eqn to get two roots λ_1 and λ_2.

ex $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$

$\det(A - \lambda I) = (8-\lambda)(7-\lambda) - 6$

$= \lambda^2 - 15\lambda + 50$

$A - \lambda I = \begin{bmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{bmatrix}$

$\boxed{\begin{matrix} \lambda_1 = 10 \\ \lambda_2 = 5 \end{matrix}}$

$A\vec{x} = 10\vec{x}$ $\qquad \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$