

Homework 2: Linear algebra

due February 19, 2025

You may collaborate with your classmates and consult external resources, but you should write and submit your own answer. **Any classmates with whom you collaborate should be credited at the top of your submission. Similarly, if you consult any external references, you should cite them clearly and explicitly.**

1. If \mathbf{u} and \mathbf{v} are orthogonal unit vectors, show that $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$. What are the lengths of those vectors?
2. Draw unit vectors \mathbf{u} and \mathbf{v} that are not orthogonal. Show that $\mathbf{w} = \mathbf{v} - \mathbf{u}(\mathbf{u}^\top \mathbf{v})$ is orthogonal to \mathbf{u} (and add \mathbf{w} to your picture).
3. Show that if Q is an orthogonal matrix, $(Q\mathbf{x})^\top(Q\mathbf{y}) = \mathbf{x}^\top \mathbf{y}$ for every vector \mathbf{x} and \mathbf{y} (so lengths and angles are not changed by Q).
4. Find an orthonormal basis for the column space of the following matrix A .

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 6 \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors for both of these matrices A and A^∞ . Use your answers to explain why A^{100} is close to A^∞ .

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

6. Compute the eigenvalues and eigenvectors of A and A^{-1} for

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

How do the eigenvalues and eigenvectors for A^{-1} compare to the eigenvalues and eigenvectors of A ?

7. Show how the eigenvalues and eigenvectors of $B = A + 2I \in \mathbb{R}^{n \times n}$ compare with the eigenvalues and eigenvectors of $A \in \mathbb{R}^{n \times n}$.
8. Diagonalize the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

9. Show that the eigenvalues of a real symmetric matrix are real.
10. Write S in the form $\lambda_1 \mathbf{x}_1 \mathbf{x}_1^\top + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^\top$, keeping $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

11. The energy $\mathbf{x}^\top \mathbf{S} \mathbf{x} = 4x_1 x_2$ has a saddle point at $(0, 0)$. What symmetric matrix produces this energy and what are its eigenvalues?

12. Suppose S is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$.
- (a) What are the eigenvalues of $\lambda_1 I - S$? Is this matrix positive semidefinite?
 - (b) Show that $\lambda_1 \mathbf{x}^\top \mathbf{x} \geq \mathbf{x}^\top S \mathbf{x}$ for every \mathbf{x} and compute the maximum value of $\mathbf{x}^\top S \mathbf{x} / \mathbf{x}^\top \mathbf{x}$.
13. Complete the **Matrix Algebra in R** lab and submit via Gradescope.