Lecture 9

- Rayleigh Quotients
- Applications of SVD
- Principal components analysis

Announcements

- Practice exam posted

Rayleigh Quotients

The Rayleigh quotient for symmetric matrix S is

$$R(\vec{x}) = \frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}$$

Given a matrix A, maximizing the Rayleigh quotient $\frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}$ where $S = A^T A$ gives a maximum of σ_1 (largest singular value of A) at $\vec{x} = \vec{v}_1$ (first right singular vector)

Suppose we want to maximize $\frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}$ ...

$$\frac{\partial}{\partial x_i}(\vec{x}^T \vec{x}) = \frac{\partial}{\partial x_i}\left(x_1^2 + x_2^2 + \dots + x_n^2\right) = 2x_i$$

$$\frac{\partial}{\partial x_i}(\vec{x}^T S \vec{x}) = \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij} x_i x_j\right) = 2\sum_{j=1}^{n} S_{ij} x_j = 2(S\vec{x})_i$$

Use the quotient rule

$$\frac{\partial}{\partial x_i}\left(\frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}\right) = \frac{2(\vec{x}^T\vec{x})(S\vec{x})_i - 2(\vec{x}^T S \vec{x})\, x_i}{\cdots}$$

Critical values when

$$(\vec{x}^T\vec{x})(S\vec{x})_i = (\vec{x}^T S \vec{x})\, x_i$$

$$(S\vec{x})_i = \left(\frac{\vec{x}^T S \vec{x}}{\vec{x}^T\vec{x}}\right) x_i$$

$$S\vec{x} = \left(\frac{\vec{x}^T S \vec{x}}{\vec{x}^T\vec{x}}\right) \vec{x}$$

eigenvectors of S are critical values...

We can show that $\vec{v}_1$ maximizes the Rayleigh quotient with value σ_1.

- also, the minimum value is the smallest eigenvalue.

Preview

Generalized Rayleigh Quotient

Given symmetric matrices S and M

a generalized RQ is $\dfrac{\vec{x}^T S \vec{x}}{\vec{x}^T M \vec{x}}$