MATH 250: Mathematical Data Visualization
Spring 2025

Homework 1: Linear algebra

due February 5, 2025

This assignment covers the prerequisite linear algebra material used in this course. You may collaborate with your classmates and consult external resources, but you should write and submit your own answer. **Any classmates with whom you collaborate should be credited at the top of your submission. Similarly, if you consult any external references, you should cite them clearly and explicitly.**

1. For the below matrices A and B and vector $\mathbf{x}$, compute the following quantities. If the following quantities cannot be computed (ex. due to mismatching dimensions), write **not computable**.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

   (a) $A\mathbf{x}$
   (b) $B\mathbf{x}$
   (c) $A^{-1}\mathbf{x}$
   (d) $A^\top A$
   (e) $AA^\top$
   (f) AB
   (g) BA
   (h) $B^{-1}A$
   (i) $\det(A)$
   (j) $\det(B)$

2. Rewrite

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j$$

   as a matrix-vector expression in terms of the n by n matrix $A = (a_{ij})$ and length n vector $\mathbf{x} = (x_1, \dots, x_n)^T$.

3. Let A be an $m \times n$ matrix with entries $a_{ij} = 2^i 3^j$. Express A as the outer product of two appropriately defined vectors.

4. Factor the following matrix into $A = CR$. The matrix C will contain independent columns of A and R will contain the numbers multiplying the columns of C to recover A.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ 4 & 12 & -8 \end{bmatrix}$$

5. Consider the linear subspace spanned by combinations of $\mathbf{x} = (1, 1, 0)^\top$ and $\mathbf{y} = (1, 0, 1)^\top$.

   (a) What is the dimension of the space spanned by $\mathbf{x}$ and $\mathbf{y}$? Draw a picture to illustrate this space.

(b) Find a vector $\mathbf{z}$ that is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. Reminder: two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if $\mathbf{a}^\top \mathbf{b} = 0$. Add it to your illustration.

(c) Show that any vector $\mathbf{w}$ contained in the span of $\mathbf{x}$ and $\mathbf{y}$ is perpendicular to $\mathbf{z}$.

6. Give an example of three non-zero vectors in $\mathbb{R}^4$ that are linearly dependent. Write your example in the form of a matrix equation $A\mathbf{x} = \mathbf{0}$. What are the shapes of A, $\mathbf{x}$, and $\mathbf{0}$?

7. Show that the nullspace of $A^\top A$ is the same as the nullspace of A for any matrix A.

8. A matrix A for which $A^2 = A$ is called **idempotent**. Show for any $m \times n$ matrix X that $I_n - X(X^\top X)^{-1} X^\top$ is idempotent.

9. For the following matrix A, find:

   (a) $\text{rank}(A)$

   (b) a basis for each of the four fundamental subspaces of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 6 \end{bmatrix}$$

10. Compute the LU decomposition for this matrix A:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

11. Compute the LU decomposition for this symmetric matrix A:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four nonzero pivots.

12. The sequence of Fibonacci numbers is defined such that the nth number is the sum of the previous two numbers (starting with 0 and 1, so the sequence is 0, 1, 1, 2, 3, 5,...). If x_i represents the ith Fibonacci number, we can represent the Fibonacci sequence recursively:

$$\begin{bmatrix} x_i \\ x_{i-1} \end{bmatrix} = A \begin{bmatrix} x_{i-1} \\ x_{i-2} \end{bmatrix}$$

where A is a 2 by 2 matrix. What is A?