MATH 250: Mathematical Data Visualization
Spring 2025

Homework 3: Singular Value Decomposition and Principal Components Analysis

due March 5, 2025

For all questions below, you should show all work needed to reach your answer. You may collaborate with your classmates and consult external resources, but you should write and submit your own answer. **Any classmates with whom you collaborate should be credited at the top of your submission. Similarly, if you consult any external references, you should cite them clearly and explicitly.**

1. The Eckart-Young theorem is false in the matrix infinity norm $||A||_\infty =$ max row sum:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has } ||A||_\infty = \max \frac{||A\mathbf{x}||_\infty}{||\mathbf{x}||_\infty} = \max(|a| + |b|, |c| + |d|)$$

   Find a rank-1 matrix closer (in the infinity norm) to

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \text{ than } A_1 = \frac{3}{2}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

2. Assuming the SVD of A is $U\Sigma V^\top$, find the SVD of $A^\top$. If A is invertible, what is the SVD of A^{-1}?

3. **Complete the sentence**: If $\mathbf{v}$ is an eigenvector of $A^\top A$ with eigenvalue $\lambda \neq 0$, then _____ is an eigenvector of $AA^\top$. **Explain your answer**.

4. Find the SVD of the rank 1 matrix

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

   by hand. Factor $A^\top A$ into $Q\Lambda Q^\top$. **Show all work.**

5. For each of the following matrices, find the closest rank-1 approximation in the Frobenius norm:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

6. Find the closest rank-1 approximation in the L^2 norm to

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

7. In class, we discussed applications of SVD/PCA to genetics, politics, and image recognition. **Describe a dataset (not necessarily a real dataset) which you think would be interesting to explore with SVD. What do you think SVD/PCA could reveal?** For example, if you were interested in baseball, you could apply SVD to a dataset where each row represented a baseball player and each column represented different statistics (batting average, number of home runs). Be sure to address the following:

   (a) Describe the rows and columns of your data matrix.

   (b) Discuss the interpretation of the left singular vectors and the right singular vectors.

8. Complete the **SVD and PCA R Lab** and submit via Gradescope.