MATH 250: Mathematical Data Visualization
Spring 2025

Homework 4: Linear systems and least squares

due March 26, 2025

For all questions below, you should show all work needed to reach your answer. You may collaborate with your classmates and consult external resources, but you should write and submit your own answer. **Any classmates with whom you collaborate should be credited at the top of your submission. Similarly, if you consult any external references, you should cite them clearly and explicitly.**

1. Compute the condition number of the following matrix A in the ℓ^2 norm:

$$A = \begin{bmatrix} 1 & 2.01 \\ 2 & 4 \end{bmatrix}$$

2. If A is square, what can we say about the eigenvalues and eigenvectors of A and A^+?

3. Suppose that A has independent columns.

   (a) Describe the shape and entries of the matrix Σ in the singular value decomposition $A = U\Sigma V^\top$. How many non-zero entries are in Σ?

   (b) Show that $\Sigma^\top\Sigma$ is invertible by finding its inverse.

   (c) Write down the n by m matrix $(\Sigma^\top\Sigma)^{-1}\Sigma^\top$ and show that it must be the pseudo-inverse Σ^+.

   (d) Substitute $A = U\Sigma V^\top$ into $(A^\top A)^{-1}A^\top$ and identify that matrix as A^+. In other words, show that $A^+ = (A^\top A)^{-1}A^\top$ if A has independent columns.

4. Suppose $A \in \mathbb{R}^{m\times n}$ with its orthogonal decomposition given by

$$A = Q_1 \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} Q_2^\top$$

   where Q_1 and Q_2 are orthogonal matrices and L is a nonsingular lower triangular matrix. Recall that $A^+ \in \mathbb{R}^{n\times m}$ is the Moore-Penrose pseudo-inverse of A if the following conditions hold:

   (i) $AA^+A = A$

   (ii) $A^+AA^+ = A^+$

   (iii) $(AA^+)^\top = AA^+$

   (iv) $(A^+A)^\top = A^+A$

   Prove that

$$A^+ = Q_2 \begin{bmatrix} L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_1^\top$$

   Hint: Write

$$A^+ = Q_2 \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} Q_1^\top$$

   and show that $G_{11} = L^{-1}$ using condition (i), that $G_{22} = 0$ using (ii), and that $G_{12} = G_{21} = 0$ using (iii) and (iv).

5. Let $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. A Householder matrix $H_{\mathbf{v}} \in \mathbb{R}^{n \times n}$ is

$$H_{\mathbf{v}} = I - 2\frac{\mathbf{v}\mathbf{v}^\top}{||\mathbf{v}||^2} = I - 2\mathbf{u}\mathbf{u}^T$$

   where $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$.

   (a) Show that $H_{\mathbf{v}}$ is symmetric and orthogonal.

   (b) Show that if $\mathbf{v} = \mathbf{a} - \mathbf{r}$ with $||\mathbf{a}|| = ||\mathbf{r}||$, $H_{\mathbf{v}}\mathbf{a} = \mathbf{r}$.

   (c) Show that $\mathbf{v}$ is an eigenvector of $H_{\mathbf{v}}$. What is the eigenvalue for $\mathbf{v}$?

   (d) Show that any $\mathbf{w}$ that is orthogonal to $\mathbf{v}$ is an eigenvector of $H_{\mathbf{v}}$. What are the corresponding eigenvalues?

6. Complete the **Linear Systems and Least Squares R Lab** and submit via Gradescope.