Lecture 8

Outline

- Matrix norms
- Eckart–Young
- Rayleigh Quotients

Readings

- Strang I.9, I.10, I.11

Matrix norms

What does it mean for matrix A to be "close" to matrix B?

- <u>def</u> Let $\|\cdot\|$ be any vector norm. The corresponding operator norm is $\|A\| = \sup_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|}$

- <u>note</u> $\|Q\| = \sup_{\vec{x} \neq \vec{0}} \frac{\|Q\vec{x}\|}{\|\vec{x}\|} = 1$

distance $\|A - B\|$

Common matrix norms

for $A \in \mathbb{R}^{m \times n}$

- $\|A\|_2 = \Big[\max_{1 \le i \le n} \underbrace{\lambda_i(A^TA)}_{\text{eigenvalue } i \text{ of } A^TA}\Big]^{1/2}$ (square root of the largest eigenvalue of A^TA)

  "2-norm" or "spectral norm"

  also the largest singular value of A

- $\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$ (max 1-norm of rows)

- $\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|$ (max 1-norm of columns)

- $\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$ ("Frobenius norm")

Eckart-Young Theorem

If $A = U\Sigma V^T = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \dots + \sigma_r \vec{u}_r \vec{v}_r^T$,

then $A_k = \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_k \vec{u}_k \vec{v}_k^T$ is the "best" rank-k approximation to A.

More precisely, for some matrix norms, if B is an $m \times n$ matrix with rank k, then $\|A - A_k\| \leq \|A - B\|$.

This is true for $\|A\|_F$ (Frobenius) and $\|A\|_2$ (spectral norm / 2-norm)

ex What is the rank-2 matrix closest to $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$?

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{V^T}$$

$$A_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\|A - A_2\|_F = \left\| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\|_F = \sqrt{5}$$

$$\|A - A_2\|_2 = 2$$

Proof in 2-norm / spectral norm

Let $A \in \mathbb{R}^{m \times n}$. If $B \in \mathbb{R}^{m \times n}$ and $\operatorname{rank}(B) \leq k < r = \operatorname{rank}(A)$

we want to show that $\|A - B\|_2 \geq \|A - A_k\|_2 = \sigma_{k+1}$

① Choose $\vec{x} \neq \vec{0}$ such that: $B\vec{x} = \vec{0}$ and $\vec{x} = \sum_{i=1}^{k+1} c_i \vec{v}_i$

where $\vec{v}_1, \dots, \vec{v}_{k+1}$ are the first $k+1$ right singular vectors of A.

this is possible since

$$\dim(N(B)) \geq n-k$$

$$\dim(\operatorname{span}(\vec{v}_1, \dots, \vec{v}_{k+1})) = k+1$$

② Recall:

$$\|A-B\|_2 = \sup_{\vec{x} \neq \vec{0}} \frac{\|(A-B)\vec{x}\|_2}{\|\vec{x}\|_2}$$

Observe that

$$\|(A-B)\vec{x}\|_2 = \|A\vec{x} - B\vec{x}\|_2$$

$$= \|A\vec{x}\|_2 \qquad (\text{since } B\vec{x} = 0)$$

$$= \left\|A\left(\sum_{i=1}^{k+1} c_i \vec{v}_i\right)\right\|_2$$

$$= \left\|\sum_{i=1}^{k+1} c_i (A\vec{v}_i)\right\|_2$$

$$= \left\|\sum_{i=1}^{k+1} c_i (\sigma_i \vec{u}_i)\right\|_2 \qquad (A\vec{v}_i = \sigma_i \vec{u}_i \text{ from SVD})$$

$$= \sqrt{\sum_{i=1}^{k+1} c_i^2 \sigma_i^2 \|\vec{u}_i\|_2^2} \qquad (\text{since } \vec{u}_1, \dots, \vec{u}_{k+1} \text{ are orthogonal})$$

$$= \sqrt{\sum_{i=1}^{k+1} c_i^2 \sigma_i^2} \qquad (\text{since } \|u_i\|_2 = 1)$$

$$\geq \sqrt{\sum_{i=1}^{k+1} c_i^2 \sigma_{k+1}^2}$$

$$= \sigma_{k+1} \sqrt{\sum_{i=1}^{k+1} c_i^2}$$

$$= \sigma_{k+1} \|\vec{x}\|_2$$

So $\|(A-B)\vec{x}\|_2 \geq \sigma_{k+1} \|\vec{x}\|_2$

and $\|A-B\|_2 = \sup_{\vec{x} \neq \vec{0}} \dfrac{\|(A-B)\vec{x}\|_2}{\|\vec{x}\|_2} \geq \dfrac{\sigma_{k+1}\|\vec{x}\|_2}{\|\vec{x}\|_2} = \sigma_{k+1}$

Frobenius norm

① $\|A\|_F = \sqrt{|a_{11}|^2 + \dots + |a_{m1}|^2 + |a_{12}|^2 + \dots + |a_{mn}|^2}$

② $\|A\|_F = \sqrt{tr(A^T A)}$ (diagonal entries of $A^T A$ contain 2-norms of columns of A)

trace $(B) = tr(B)$ = sum of diagonal entries of B.

③ $\|A\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_r^2}$ (sqrt of sum of squared singular values)

So if $A = U\Sigma V^T$, $\|A\|_F = \|\Sigma\|_F$

In general if Q_1 and Q_2 are orthogonal matrices

$$\|A\|_F = \|Q_1 A Q_2\|_F$$

Proof of EY in Frobenius

we want to show that if B has rank $\le k$, that

$$\|A - A_k\|_F \le \|A - B\|_F$$

Trick: take the SVD of B.

Suppose B is the closest rank-k matrix to A.

$$B = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where D is k by k diagonal matrix.

$$A = U \underbrace{\begin{bmatrix} L+E+R & F \\ G & H \end{bmatrix}}_{U^T A V} V^T$$

where L is strictly lower tri
R is strictly upper tri
E is diagonal

Consider a rank $\le k$ matrix C

$$C = U \begin{bmatrix} L+D+R & F \\ 0 & 0 \end{bmatrix} V^T \qquad C' = \begin{bmatrix} L+D+R & 0 \\ G & 0 \end{bmatrix}$$

$$\|A-B\|_F^2 = \|L+E+R-D\|_F^2 + \|F\|_F^2 + \|G\|_F^2 + \|H\|_F^2$$

$$= \|L\|_F^2 + \|E-D\|_F^2 + \|R\|_F^2 + \|F\|_F^2 + \|G\|_F^2 + \|H\|_F^2$$

$$\|A-C\|_F^2 = \|E-D\|_F^2 + \|G\|_F^2 + \|H\|_F^2$$

Thus, $L, R, F = 0$. An analogous argument shows $G = 0$

$$A = U \begin{bmatrix} E & 0 \\ 0 & H \end{bmatrix} V^T \qquad B = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Since B is closest to A, $D = E$

Notes

- Singular values of H are the $r-k$ smallest singular values of A.

$$\|A-B\|_F = \|H\|_F = \sqrt{\sigma_{k+1}^2 + \dots + \sigma_r^2}$$

Rayleigh Quotients

Another way to understand SVD:

If we maximize $\dfrac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$, the maximum value is σ_1 at $\vec{x} = \vec{v}_1$

How do we derive this?

$$\frac{\|A\vec{x}\|_2^2}{\|\vec{x}\|_2^2} = \frac{\vec{x}^T A^T A \vec{x}}{\vec{x}^T \vec{x}} = \frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}$$

(Rayleigh quotient for S)

<u>def</u> The Rayleigh quotient for a symmetric matrix S is a function

$$R : \mathbb{R}^n \setminus \{\vec{0}\} \longrightarrow \mathbb{R} :$$

$$R(\vec{x}) = \frac{\vec{x}^T S \vec{x}}{\vec{x}^T \vec{x}}$$

<u>ex</u> $S = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ $\quad R(\vec{x}) = \dfrac{x_1^2 + 2x_2^2 + 6x_1x_2}{x_1^2 + x_2^2}$ $\qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$