Lecture 5

Readings

Strang I.6 and I.7

Today

- Eigenvalues + eigenvectors
- Similar matrices
- Diagonalizability
- Spectral Theorem
- Positive definite + positive semidefinite matrices

Eigenvalues and eigenvectors

def A non-zero vector $\vec{x}$ is an eigenvector of a matrix A if $A\vec{x} = \lambda\vec{x}$ for some number λ. λ is called an eigenvalue of A.

notes
- $A\vec{x} = \lambda\vec{x} \implies A$ is square
- eigenvectors are not unique: if $\vec{x}$ is an eigenvector, $2\vec{x}$ is also an eigenvector

$$A(2\vec{x}) = 2A\vec{x} = 2\lambda\vec{x} = \lambda(2\vec{x})$$

Computing eigenvalues

If $A\vec{x} = \lambda\vec{x}$,

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\vec{x} \in N(A - \lambda I)$$

Since $\vec{x}$ is non-zero, $(A-\lambda I)$ is not invertible and must be singular.

Thus $\det(A-\lambda I)=0$

This is an n^{th} degree polynomial in the variable λ.

ex $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad-bc$

$A-\lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ $\quad$ $\det(A-\lambda I) = (a-\lambda)(d-\lambda)-bc$

$$= \lambda^2-(a+d)\lambda+(ad-bc)$$

If we solve $\lambda^2-(a+d)\lambda+(ad-bc)=0$

we will get two (possibly complex) roots λ_1 and λ_2.

ex $A=\begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$ $\quad A-\lambda I = \begin{bmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (8-\lambda)(7-\lambda)-6$$
$$= \lambda^2-15\lambda+50$$
$$=(\lambda-10)(\lambda-5)$$

The two roots are $\lambda_1=10$, $\lambda_2=5$

$A\vec{x} = \lambda_1 \vec{x} = 10\vec{x}$

$$\begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

exercise: solve for x_1 and x_2

Similar matrices

def A matrix B is similar to matrix A if we can write $B = PAP^{-1}$ $\quad (BP = PA)$

note Observe that B has the same eigenvalues as A if B and A are similar.

How do we show this?

Suppose λ is an eigenvalue of A with eigenvector $\vec{x}$.

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

Want to show (WTS) that there exists $\vec{y}$ such that

$$B\vec{y} = \lambda\vec{y}$$

WTS $\boxed{PAP^{-1}\vec{y} = \lambda\vec{y}}$

Try $\vec{y} = P\vec{x}$

$$B\vec{y} = B(P\vec{x}) = PAP^{-1}P\vec{x} = PA\vec{x} = P(\lambda\vec{x}) = \lambda(P\vec{x}) = \lambda\vec{y}$$

eigenvectors of A are $\vec{x}_1, \dots, \vec{x}_n$

eigenvectors of B are $P\vec{x}_1, \dots, P\vec{x}_n$

Diagonalizability

def A square matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable if it is similar to a diagonal matrix.

In other words, there exists an invertible matrix X and diagonal matrix Λ such that

$$A = X \Lambda X^{-1} \qquad (X^{-1} A X = \Lambda)$$

note if $X = \begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

$$A = X \Lambda X^{-1}$$

$$\Rightarrow AX = X\Lambda$$

$$A \begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix} = \underset{n \times n}{\begin{bmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}$$

$$\begin{bmatrix} | & & | \\ A\vec{x}_1 & \cdots & A\vec{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 \vec{x}_1 & \lambda_2 \vec{x}_2 & \cdots & \lambda_n \vec{x}_n \\ | & | & & | \end{bmatrix}$$

- The columns of X are eigenvectors of A where the eigenvalues are the corresponding entries on the diagonal of Λ

Note The eigenvectors are linearly independent if A is diagonalizable.

Note This is called the eigendecomposition or spectral decomposition.

Note If $A = X\Lambda X^{-1}$, $A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 X^{-1}$

$$A^{100} = X\Lambda^{100}X^{-1}$$

$$\Lambda^{100} = \begin{bmatrix} \lambda_1^{100} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{100} \end{bmatrix}$$

Note If A is invertible, $A^{-1} = (X\Lambda X^{-1})^{-1} = X\Lambda^{-1}X^{-1}$.

How do we know if a matrix is diagonalizable?

1. Solve $\det(A - \lambda I) = 0$ to get n eigenvalues

2. Then we need to show that there are n linearly independent eigenvectors ... not so easy!

We will come back to this.

Symmetric matrices

def S is a symmetric matrix if $S = S^T$ (S must be square)

Theorem (Spectral)

If S is a real symmetric $n \times n$ matrix:

- S has n real eigenvalues
- S has n orthonormal eigenvectors
- $S = Q \Lambda Q^T$ $\quad\quad (SQ = Q\Lambda)$

note All real symmetric matrices are diagonalizable (by orthogonal matrices)

note If $S = Q\Lambda Q^T$

$$= \begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & \vec{q}_1^T & - \\ & \vdots & \\ - & \vec{q}_n^T & - \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \lambda_1 \vec{q}_1 & \cdots & \lambda_n \vec{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \vec{q}_1^T & - \\ & \vdots & \\ - & \vec{q}_n^T & - \end{bmatrix}$$

$$= \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \ldots + \lambda_n \vec{q}_n \vec{q}_n^T$$

we can write S as a sum of rank-1 matrices

<u>note</u> We arrange the columns of Q and Λ so

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

<u>proof | outline</u>

1. $\det(S - \lambda I)$ is an n^{th}-degree polynomial so it has n roots (Fundamental Theorem of Algebra)

2. If S is real + symmetric the roots $\lambda_1, \dots, \lambda_n$ are all real (need to verify)

3. If S is real + symmetric, eigenvectors corresponding to distinct eigenvalues are orthogonal (verify)

4. If an eigenvalue is repeated, it is still possible to produce orthogonal eigenvectors when $S = S^T$ (tricky – need to verify)

<u>ex</u> let $S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ what are the eigenvalues?

1. $\det(S - \lambda I) = 0$
   $(1-\lambda)^2 - 4 = 0$
   $\lambda^2 - 2\lambda - 3 = 0$

   $\lambda_1 = 3$
   $\lambda_2 = -1$

   $S = Q \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} Q^T$

2. Find eigenvector for λ_1:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$x + 2y = 3x$
$2x + y = 3y$

$\Rightarrow x = y$ $\quad \cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ $\quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

③ $$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$ (verify)

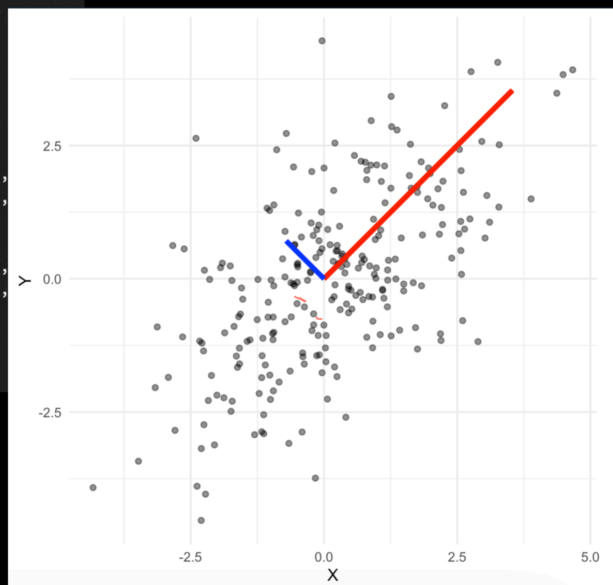
```r
library(mvtnorm)
library(ggplot2)

# generate bivariate normal data
set.seed(123)
Sigma <- matrix(c(3, 2, 2, 3), nrow = 2)
dat <- rmvnorm(250, sigma = Sigma) |>
  as.data.frame()

# calculate eigenvector and eigenvalues of covariance
eig <- eigen(Sigma)

# plot data and eigenvectors
ggplot(dat, aes(x = V1, y = V2)) +
  geom_point(alpha = 0.5) +
  geom_segment(x = 0, y = 0,
               xend = eig$values[1]*eig$vectors[1, 1],
               yend = eig$values[1]*eig$vectors[2, 1],
               color = "red", linewidth = 1.4) +
  geom_segment(x = 0, y = 0,
               xend = eig$values[2]*eig$vectors[1, 2],
               yend = eig$values[2]*eig$vectors[2, 2],
               color = "blue", linewidth = 1.4) +
  theme_minimal() +
  coord_fixed() +
  xlab("X") + ylab("Y")
```

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right)$$



Positive definite and positive semidefinite matrices

Def 1 A symmetric matrix S is positive definite if its energy function $f(\vec{x}) = \vec{x}^T S \vec{x} > 0$ for all $\vec{x} \neq 0$

S is positive semidefinite if $\vec{x}^T S \vec{x} \geq 0$ for all $\vec{x} \neq 0$

ex Can we think of an example of a positive definite matrix?

$S = I$ $\quad \vec{x}^T S \vec{x} = \vec{x}^T \vec{x} > 0$ if $\vec{x} \neq 0$