Minimizing $\|A\vec{x} - \vec{b}\|_2^2$

$$\underset{n \times p}{\begin{bmatrix} \\ & A & \\ \\ \end{bmatrix}} \underset{p \times 1}{\begin{bmatrix} \vec{x} \end{bmatrix}} = \underset{n \times 1}{\begin{bmatrix} \vec{b} \end{bmatrix}}$$

$p > n$

$$\underset{n \times p}{\begin{bmatrix} & A & \end{bmatrix}} \underset{p \times 1}{\begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix}} = \underset{n \times 1}{\begin{bmatrix} \\ \vec{b} \\ \\ \end{bmatrix}}$$

Pseudo-inverse

$A^{-1}A = I$

Desired properties of A^+

- if A is invertible, $A^+ = A^{-1}$
- if $A \in \mathbb{R}^{m \times n}$ $A^+ \in \mathbb{R}^{n \times m}$
- $A^+A\vec{x} = \vec{x}$ when $\vec{x}$ is in row space of A
- $AA^+\vec{b} = \vec{b}$ when $\vec{b}$ is in column space of A

$A \in \mathbb{R}^{m \times n}$

column space $C(A) \subset \mathbb{R}^m$

row space $C(A^T) \subset \mathbb{R}^n$

null space $N(A) \subset \mathbb{R}^n$

row null space $N(A^T) \subset \mathbb{R}^m$

$\vec{x} \in N(A)$ $\quad A^+A\vec{x} \neq \vec{x}$ if $\vec{x} \neq \vec{0}$

$A \in \mathbb{R}^{m \times n}$ where $n > m$

$$\begin{bmatrix} & A & \end{bmatrix}$$

There must be $\vec{x} \neq \vec{0}$ s.t.

$A\vec{x} = \vec{0}$

$A^+A \in \mathbb{R}^{n \times n}$ $\quad$ $A^+A\vec{x} = \vec{0}$

If $\vec{x} \in C(A^+)$, $A^+A\vec{x} = \vec{x}$

<u>def</u> The Moore-Penrose pseudo-inverse of $A \in \mathbb{R}^{m \times n}$ satisfies

- $AA^+A = A$
- $A^+AA^+ = A^+$
- $(AA^+)^T = AA^+$
- $(A^+A)^T = A^+A$

<u>theorem</u> A^+ always exists and is unique.

<u>How do we compute A^+?</u>

If the SVD of A is $A = U\Sigma V^T$

if A is invertible

$A^{-1} = V\Sigma^{-1}U^T$

$A^+ = V\Sigma^+U^T$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \end{bmatrix}$$

Exercise: Check properties.

$$A^+ = V\Sigma^+U^T$$

Check $AA^+A = A$

$$(U\Sigma V^T)(V\Sigma^+U^T)(U\Sigma V^T)$$

$$= U\Sigma\Sigma^+\Sigma V^T$$

$$= U\Sigma V^T$$

Remark: The pseudo inverse allows us to compute the minimum norm least squares solution to $A\vec{x} = \vec{b}$.

Let $\hat{\vec{x}}^+ = A^+\vec{b}$. Then

- $\hat{\vec{x}}^+$ minimizes $\|\vec{b} - A\vec{x}\|_2^2$ (least sq.)
- if another $\hat{\vec{x}}$ also minimizes $\|\vec{b} - A\vec{x}\|_2^2$

  then $\|\hat{\vec{x}}^+\|_2 \leq \|\vec{x}\|_2$ (minimum norm)

So even if A^TA is not invertible, A^+ can be used to get a least squares solution.