Lecture 2/3

Announcements

- HW1 due Wednesday

Readings

Strang I.3, I.4, I.5

Murphy Ch 7

Outline

- matrix-matrix multiplication
- span + basis
- matrix rank
- four fundamental subspaces
- $A = CR$
- $A = LU$

Matrix-Matrix multiplication

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} \qquad C = AB = \begin{bmatrix} 2 & 4 \\ 6 & 17 \end{bmatrix}$$

$m \times n \qquad n \times p$

"column-row multiplication"

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 17 \end{bmatrix}$$

<u>note</u> the usual way: $m \times p$ dot products, n multiplications each

the column-row way n outer products, mp multiplications each

<u>ex</u> Give an example of two matrices A and B such that $AB = BA$.

→ also try an example where neither is diagonal.

<u>$A = CR$</u>

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$
$\quad\ \ \vec{c}_1 \ \ \vec{c}_2 \ \ \vec{c}_3$

what is the $\dim(C(A))$? <u>2</u>

$\vec{c}_3 = 2\vec{c}_1 + 2\vec{c}_2$ is a linear combination of the other two columns

<u>def</u> The <u>span</u> of a set of vectors $\vec{v}_1, \ldots, \vec{v}_N$ is the set of all linear combinations of the vectors.

$$\vec{v} = \sum_{i=1}^{N} c_i \vec{v}_i$$

→ the column space of A is the span of its columns.

<u>def</u> A set of vectors $\vec{b}_1, \vec{b}_2, \ldots$ in a vector space V is a <u>basis</u> for a subspace $S \subseteq V$ if

- $\vec{b}_1, \vec{b}_2, \ldots$ are linearly independent
- $\text{span}(\{\vec{b}_1, \vec{b}_2, \ldots\}) = S$

<u>def</u> A nonempty subset S is a <u>subspace</u> of $\mathbb{R}^n$ if for all $\vec{s}_1, \vec{s}_2 \in S$ and $c_1, c_2 \in \mathbb{R}$

$c_1\vec{s}_1 + c_2\vec{s}_2 \in S$

<u>ex</u> $C(A)$, $\{\vec{0}\}$, $\mathbb{R}^n$

<u>A=CR</u> motivation

- C will be a matrix whose columns form a basis for $C(A)$

<u>How to build C</u>

1. If col 1 of A is not all zeros, put it in C
2. If col 2 is not a multiple of col 1, put it into C
3. If col 3 is not a lin. comb. of col 1 and col 2, put it into C
4. Continue...

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} - & \vec{r}_1^{\,*} & - \\ - & \vec{r}_2^{\,*} & - \end{bmatrix}$$

$$3 \times 3 \qquad\qquad 3 \times 2 \qquad\qquad 2 \times 3$$

$$= \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}}_{R}$$

R = row-reduced echelon form

ex $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ What is CR?

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{3 \times 1} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$$

note the number of columns in C is $\dim(C(A))$

def the rank of A, $\operatorname{rank}(A)$ is the dimension of $C(A)$

Four fundamental subspaces of A

- We've seen $C(A)$ is a subspace.

Any $m \times n$ matrix A is associated with four fundamental subspaces:

ex $A = \begin{bmatrix}1&2\\3&6\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}$

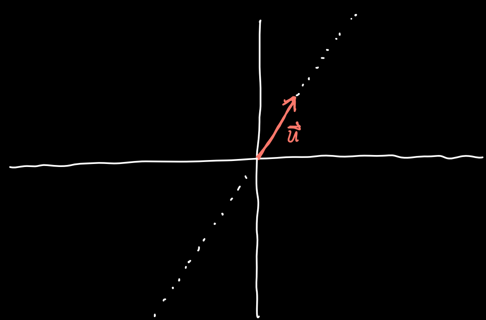
① The column space $C(A)$: all lin. comb. of columns of A

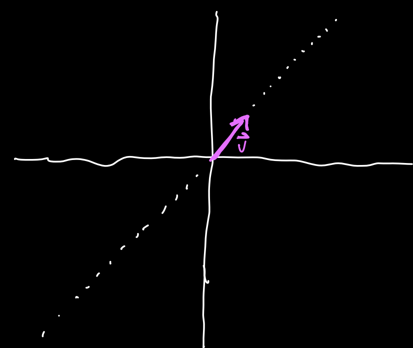
② The row space $C(A^T)$: all lin. comb. of rows of A

③ The null space $N(A)$: all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$

④ The left null space $N(A^T)$: all solutions $\vec{y}$ to $A^T\vec{y} = \vec{0}$

ex $A = \begin{bmatrix}1&2\\3&6\end{bmatrix}$





① $C(A)$ is the line through $\vec{u} = \begin{bmatrix}1\\3\end{bmatrix}$

② $C(A^T)$ is the line through $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$

③ $N(A)$ contains $\vec{x} = \begin{bmatrix}2\\-1\end{bmatrix}$

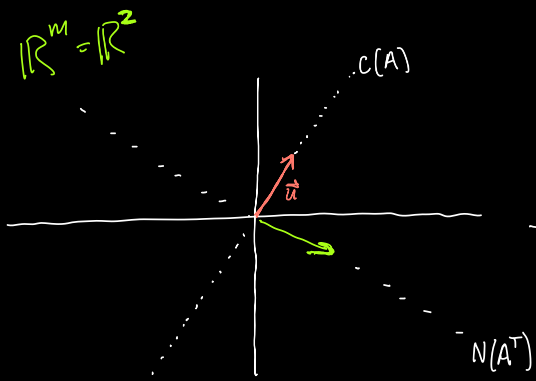
$$\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

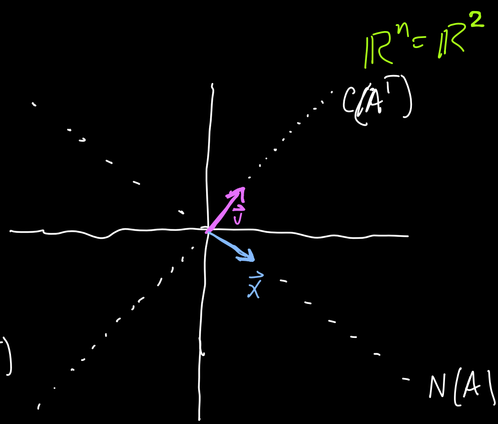
④ $N(A^T)$ contains $\vec{y} = \begin{bmatrix}3\\-1\end{bmatrix}$

note If A is an $m \times n$ matrix, we can think of A as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\underset{m\times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}\underset{n\times 1}{\begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} \\ \vec{b} \\ \\ \end{bmatrix}} \qquad \underset{n\times m}{\begin{bmatrix} & & \\ & A^T & \\ & & \end{bmatrix}}\underset{m\times 1}{\begin{bmatrix} \\ \vec{y} \\ \\ \end{bmatrix}} = \underset{n\times 1}{\begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix}}$$

A maps from $\mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $A^T: \mathbb{R}^m \to \mathbb{R}^n$





Elimination

Solve systems of linear equations

$$\begin{aligned} x + 2y + z &= 4 \\ y - z &= -1 \\ z &= 2 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \underset{=}{=} \underset{\vec{b}}{\begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} & & \\ & X & \\ & & \end{bmatrix} \begin{bmatrix} \\ \vec{\beta} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \vec{y} \\ \\ \end{bmatrix}$$

$$X\vec{\beta} = \vec{y}$$

- If A is invertible, there is only one solution to $A\vec{x}=\vec{b}$

In this case, we can use the LU decomposition to solve $A\vec{x}=\vec{b}$ problems.

ex. $A=\begin{bmatrix}1&2&3\\2&5&7\\2&7&8\end{bmatrix}$

<u>Goal</u> solve problems of the form

$$A\vec{x}=\vec{b}$$

<u>Method</u> Decompose A as the product LU where L is lower triangular and U is upper triangular

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix}1&&0\\&\ddots&\\&&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}\ddots&&\\&\ddots&\\0&&\ddots\end{bmatrix}}_{U}$$

<u>Note</u> This is not always possible

<u>ex</u>

$$A=\begin{bmatrix}1&2&3\\2&5&7\\2&7&8\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}}_{U}$$

$A\vec{x}=\vec{b}$

$LU\vec{x}=\vec{b}$

$\hookrightarrow L\vec{y}=\vec{b}$ where $\vec{y}=U\vec{x}$

$\hookrightarrow U\vec{x}=\vec{y}$

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$