Lecture 4

Announcements

- HW2 + HW2 Lab posted

Readings

- Strang I.5 and I.6

Today

- vector norms
- orthogonality
- eigenvalues + eigenvectors

Vector norm

How do we quantify the "size" or "length" of a vector?

A vector norm maps vectors in $\mathbb{R}^n$ to a single number in $\mathbb{R}$.

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 10 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

ex. $\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$ (1-norm)

- $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ (2-norm)
- $\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ (p-norm) $p>0$
- $\|\vec{x}\|_\infty = \max_{1 \le i \le n} |x_i|$ (max-norm)

$\|\vec{x}\|_1 = 12$

$\|\vec{y}\|_1 = 15$

$\|\vec{x}\|_2 = \sqrt{102}$

$\|\vec{y}\|_2 = \sqrt{75}$

$\|\vec{x}\|_\infty = 10$

$\|\vec{y}\| = 5$

note $\|\vec{x}\|_2^2 = \vec{x}^T\vec{x} = \vec{x} \cdot \vec{x} = \langle \vec{x}, \vec{x} \rangle$

Exercise Create more examples of vectors where norms disagree. Is it always possible?

Orthogonality

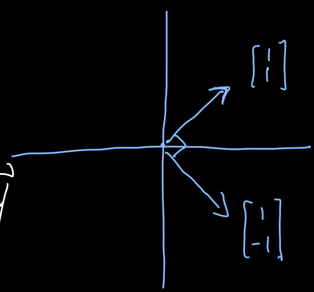
def Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are orthogonal if

$$\vec{x}^T\vec{y} = \vec{y}^T\vec{x} = 0$$

In $\mathbb{R}^n$ this is equivalent to them being perpendicular.

note $\vec{x}^T\vec{y} = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta$

where θ is the angle between $\vec{x}$ and $\vec{y}$



exercise What are three orthogonal vectors in $\mathbb{R}^3$?

$$\left\{ \underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$$

note These vectors form a basis for $\mathbb{R}^3$

theorem Let $\vec{q}_1, \dots, \vec{q}_n$ be non-zero orthogonal vectors.

Then they are linearly independent.

proof outline proof by contradiction

① Assume $\vec{q}_1, \dots, \vec{q}_n$ are non-zero and orthogonal

AND that they are linearly dependent

② Linear dependence:

$$c_1\vec{q}_1 + \dots + c_n\vec{q}_n = \vec{0}$$

where not all $c_i = 0$

③ ... What now?

Choose i such that $c_i \neq 0$

$$(c_1 \vec{q}_1 + \dots + c_n \vec{q}_n) \cdot \vec{q}_i$$

$$= c_1 \vec{q}_1^T \vec{q}_i + c_2 \vec{q}_2^T \vec{q}_i + \dots + c_n \vec{q}_n^T \vec{q}_i$$

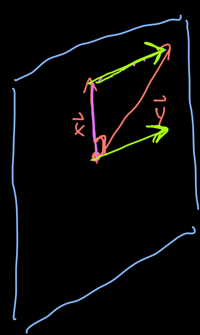
For all $j \neq i$, $\vec{q}_j^T \vec{q}_i = 0$

$$= c_i \vec{q}_i^T \vec{q}_i \neq 0$$

<u>Theorem</u> Every subspace of $\mathbb{R}^n$ has an orthogonal basis.

<u>proof</u> $\rightarrow$ Gram-Schmidt

<u>ex</u> A plane in $\mathbb{R}^3$ spanned by independent vectors $\vec{x}$ and $\vec{y}$.



To get an orthogonal basis take $\vec{x}$ and the residual after projecting $\vec{y}$ on $\vec{x}$.

(This idea leads to Gram-Schmidt orthogonalization)

Orthonormal vectors

Let $\vec{q}_1, \ldots, \vec{q}_n$ be orthogonal vectors with all having length 1. (so $\|\vec{q}_j\|_2 = 1$ for all j)

Then they are called <u>orthonormal</u>.

In particular, if $\vec{q}_1, \ldots, \vec{q}_n$ are n orthonormal vectors in $\mathbb{R}^n$, then they form an <u>orthonormal basis</u> for $\mathbb{R}^n$.

Orthogonal matrices

<u>def</u> A square matrix $Q \in \mathbb{R}^{n \times n}$ with orthonormal columns is called an orthogonal matrix.

<u>properties</u>

- $Q^T Q = I$
- $Q Q^T = I$
- $Q^{-1} = Q^T$
- $\|Q\vec{x}\|_2 = \|\vec{x}\|_2$

$$\underbrace{\begin{bmatrix} — & \vec{q}_1^{\,T} & — \\ & \vdots & \\ — & \vec{q}_n^{\,T} & — \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} | & & | \\ \vec{q}_1 & \cdots & \vec{q}_n \\ | & & | \end{bmatrix}}_{Q} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ & & & & & \end{bmatrix}}_{I}$$

$$\begin{aligned} \|Q\vec{x}\|_2^2 &= (Q\vec{x})^T (Q\vec{x}) \\ &= \vec{x}^T Q^T Q \vec{x} \\ &= \vec{x}^T \vec{x} \\ &= \|\vec{x}\|_2^2 \end{aligned}$$

exercise

Show that $Q_3 = \frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is orthogonal.

ex $Q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

Is this orthogonal? not really.

$Q_1^T Q_1 = I_1$

$Q_1 Q_1^T = \,?$

↳ it turns out that $Q_1 Q_1^T$ is the orthogonal projection matrix onto the column space of Q_1.

Note In general, given an orthonormal basis $\vec{q}_1, \ldots, \vec{q}_n$ for $\mathbb{R}^n$ any vector $\vec{x} \in \mathbb{R}^n$ can be written

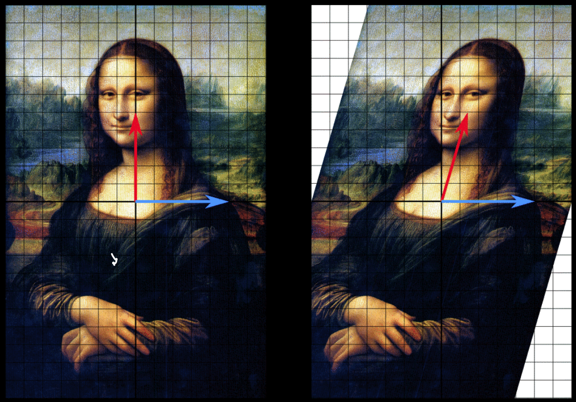
$$\vec{x} = c_1 \vec{q}_1 + \ldots + c_n \vec{q}_n \quad \text{where} \quad c_1 = \vec{q}_1^{\,T} \vec{x}, \ \ldots, \ c_n = \vec{q}_n^{\,T} \vec{x}$$

proof as exercise!

Eigenvalues and Eigenvectors

def A vector $\vec{x}$ is an eigenvector of A if $A\vec{x} = \lambda \vec{x}$ for some number λ. λ is called an eigenvalue of $\vec{x}$ and of A

note: if $\vec{x}$ is an eigenvector of A, it is also an eigenvector of A^2



$$A^2\vec{x} = AA\vec{x} = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$$

ex $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

exercise what are the eigenvalues?

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_1 = 3$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \lambda_2 = 1$$

ex $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has imaginary eigenvalues i and $-i$, $i = \sqrt{-1}$

$$\begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}$$