Lecture 6

Readings

Strang I.8

Outline

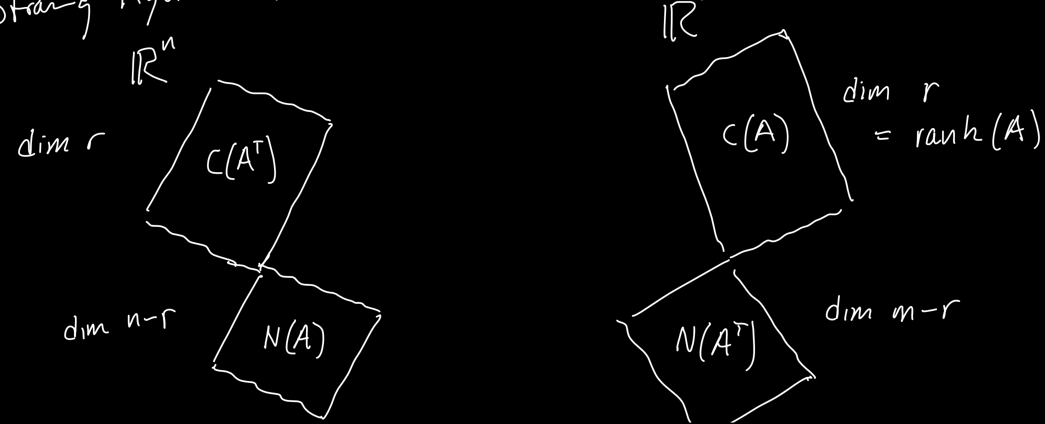
- Review
- Positive definite matrices
- Singular value decomposition

Four fundamental subspaces

- We can think of an real $m \times n$ matrix A as mapping vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$\underset{m \times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} \vec{b} \\ \\ \\ \end{bmatrix}}$$

Strang Figure 1.3



$N(A)$ is the subspace of all vectors $\vec{x}$ such that $A\vec{x} = 0$

- Show that the column space $C(A)$ is orthogonal to $N(A^T)$.

Review : Diagonalizability

$A \in \mathbb{R}^{n\times n}$ if $\underbrace{A = X \Lambda X^{-1}}$ where Λ is diagonal

eigendecomposition
spectral decomposition

Symmetric matrices are all diagonalizable
(by orthogonal matrices)

$$S = Q \Lambda Q^T$$

* HW2 #9 — see Strang p. 44-45
A real symmetric matrix has real eigenvalues

(Symmetric)
Positive definite / semidefinite matrices

① S is pos. def. if the energy function
$f(\vec{x}) = \vec{x}^T S \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$

S is pos. semidef. if $\vec{x}^T S \vec{x} \geq 0$ for all $\vec{x} \neq \vec{0}$

ex $S = I$ is PD ... $\vec{x}^T I \vec{x} = \vec{x}^T \vec{x} > 0$ if $\vec{x} \neq \vec{0}$

ex $S = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ is PD ex $S = \begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix}$

$$\vec{x}^T S \vec{x} = \underset{1\times 2}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$= 2x_1^2 + 8x_1x_2 + 9x_2^2$$

$$= 2(x_1 + 2x_2)^2 + x_2^2 > 0$$

so S is PD.

(2) S is positive definite if all eigenvalues > 0

positive semidefinite if all eigenvalues ≥ 0

Connecting (1) and (2) assuming S is symmetric

For any eigenvector $\vec{q}$, $S\vec{q} = \lambda \vec{q}$

So, $\vec{q}^T S \vec{q} = \lambda \vec{q}^T \vec{q}$

If all eigenvalues > 0, $\lambda > 0$ and thus $\vec{q}^T S \vec{q} > 0$

We want to show that for any vector $\vec{x}$, $\vec{x}^T S \vec{x} > 0$

Since S is symmetric, its eigenvectors form a basis for $\mathbb{R}^n$

Thus, $\vec{x} = c_1\vec{q}_1 + \dots + c_n\vec{q}_n$

Then $\vec{x}^T S \vec{x} = (c_1\vec{q}_1 + \dots + c_n\vec{q}_n)^T S (c_1\vec{q}_1 + \dots + c_n\vec{q}_n)$

$$= c_1^2 \vec{q}_1^T S \vec{q}_1 + \boxed{\cancel{c_1c_2\vec{q}_1^T S\vec{q}_2}} + \dots$$

$$+ \cancel{c_2c_1\vec{q}_2^T S\vec{q}_1} + c_2^2 \vec{q}_2^T S \vec{q}_2 + \dots$$

$$+ \dots$$

$$+ \cancel{c_nc_1\vec{q}_n^T S\vec{q}_1} + \dots$$

$c_1c_2\vec{q}_1^T S \vec{q}_2$

$= c_1c_2\lambda_2 \vec{q}_1^T\vec{q}_2 = 0$

$$= c_1^2 \vec{q}_1^T S \vec{q}_1 + \dots + c_n^2 \vec{q}_n^T S \vec{q}_n > 0 \text{ if every } \lambda_i > 0$$

How do we get from ① to ②?

Corollary ①

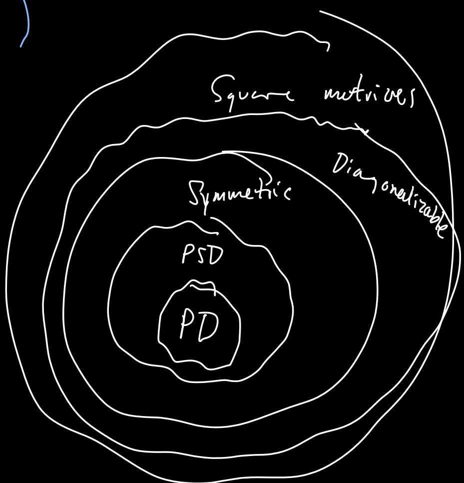
If S_1 and S_2 are PD, so is $S_1 + S_2$ (exercise: verify)

Other definitions of PD

③ $S = A^T A$ for A with independent columns.

$S = Q \Lambda Q^T$ ... see what happens for $A = Q \Lambda^{1/2} Q^T$.

$$
\begin{aligned}
A^T A &= (Q \Lambda^{1/2} Q^T)^T (Q \Lambda^{1/2} Q^T) \\
&= Q \Lambda^{1/2} Q^T Q \Lambda^{1/2} Q^T \\
&= Q \Lambda^{1/2} \Lambda^{1/2} Q^T \\
&= Q \Lambda Q^T
\end{aligned}
$$



Singular value decomposition

- Real symmetric matrix S is diagonalizable so $S = Q \Lambda Q^T$

What about for a non-square matrix?

Consider $A \in \mathbb{R}^{m \times n}$.

$\cancel{A\vec{x} = \lambda \vec{x}}$

$\mathbb{R}^n$ $\mathbb{R}^m$

For a real $m \times n$ matrix, instead of constructing a set of orthogonal eigenvectors,

we will construct two sets of orthogonal singular vectors

n right singular vectors $\vec{v}_1, \ldots, \vec{v}_n$

m left singular vectors $\vec{u}_1, \ldots, \vec{u}_m$

These will form bases for the row and column spaces of A.

Key equations

For singular vectors

For eigenvalue/eigenvectors

$A\vec{x} = \lambda\vec{x}$

$$A\vec{v}_i = \sigma_i \vec{u}_i$$

row space

column space

In particular

$$A\vec{v}_1 = \sigma_1 \vec{u}_1, \ldots, A\vec{v}_r = \sigma_r \vec{u}_r \quad \text{where } r = \text{rank}(A)$$

$$A\vec{v}_{r+1} = \ldots = A\vec{v}_n = \vec{0}$$

so the last $n-r$ $\vec{v}$ vectors are in the null space of A

the last $m-r$ $\vec{u}$ vectors are in the null space of A^T

$$\underset{m \times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \underset{\substack{V \\ n \times n}}{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}} = \underset{\substack{U \\ m \times m}}{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}} \underset{\substack{\Sigma \\ m \times n}}{\left[\begin{array}{ccc:c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hdashline & 0 & & 0 \end{array}\right]}$$

$AV = U\Sigma$

(compare with $SQ = Q\Lambda$)

$AV = U\Sigma$ gives us $A = U\Sigma V^{-1} = U\Sigma V^T$

Singular value decomposition

$$A = U\Sigma V^T$$

(sometimes $A = UDV^T$)

- V is an $n \times n$ matrix of orthogonal right singular vectors
- U is an $m \times m$ matrix of orthogonal left singular vectors
- Σ is the matrix of singular values $(m \times n)$

ex $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

$$\underset{A}{\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}} \underset{V}{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}$$

- A is not symmetric (if it were $V = U$)
- $\det(\Sigma) = \det(A) = 15$
- V and U are orthogonal matrices