Announcements

- Library Course Reserves on Canvas
- HW1 .tex file on Canvas

Readings

- Strang I.1 and I.2
- Murphy Ch. 7

Outline

- basic definitions
- matrix vector multiplication
- linear independence
- column space of a matrix
- rank of a matrix
- matrix-matrix multiplication
- $A = CR$ decomposition

Note

- Assume that all vectors and matrices have real number components

Vectors

Think of a vector as a point in $\mathbb{R}^n$ ← set of n-tuples of real numbers

We will use boldface lowercase letters for vectors

ex. $\vec{x} = (x_1, x_2, x_3)^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

The i^{th} element of $\vec{x}$ is denoted x_i

Special vectors

- $\vec{0}_n = (0, \ldots, 0)^T \in \mathbb{R}^n$
- $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (canonical basis vector in $\mathbb{R}^n$)

  ith position

Matrices

- Think of a matrix as a rectangular grid of real numbers.
- In particular, a real-valued matrix with m rows and n columns is said to be an element of

  $$\mathbb{R}^{m \times n}$$

- We will use uppercase letters for matrices

ex. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$

The entry of A in the ith row and jth column is denoted a_{ij} (sometimes $A(i,j)$, A_{ij})

ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

What is m? 2

What is n? 3

What is a_{22}? 5

Special matrices

- Identity matrix

$$I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Beyond vectors/matrices: Tensors

A tensor is a multidimensional array.

Matrix-vector multiplication

$$A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What is $A\vec{x}$?

Two approaches for matrix-vector multiplication

"By rows"

$$\underset{\mathbb{R}^{3\times 2}}{\begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix}} \underset{\mathbb{R}^{2}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\mathbb{R}^{3}}{\begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix}}$$

equivalent to computing three dot products

"By columns"

$$\begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix} = \underset{\mathbb{R}^{3}}{\begin{bmatrix} x_1 + 5x_2 \\ x_1 + 6x_2 \\ 2x_1 + 8x_2 \end{bmatrix}}$$

$A\vec{x}$ is a linear combination of the columns of A.

Column space

def The column space of a matrix A is the vector space of all linear combinations of the columns of A, denoted $C(A)$.

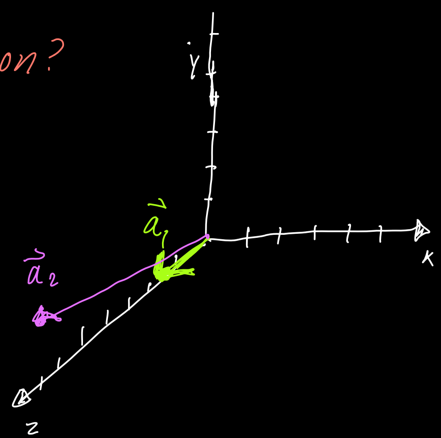
- span of columns of A
- image of A

ex. $A = \begin{array}{cc} & \\ \left[\begin{array}{cc} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{array}\right] \\ \vec{a}_1 \quad \vec{a}_2 \end{array}$ What is the column space of A?

What is its dimension?

$$c_1\vec{a}_1 + c_2\vec{a}_2 \in C(A)$$

The column space is the plane containing all vectors that are linear combinations of $\vec{a}_1$ and $\vec{a}_2$



- The column space includes $\vec{0}$.
- $\vec{b} \in C(A)$ if and only if $A\vec{x} = \vec{b}$ has a solution $\vec{x}$.

ex $A = \begin{bmatrix} 1 & 5 \\ 1 & 6 \\ 2 & 8 \end{bmatrix}$ Is $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the column space of A?

No.

$c_1 \vec{a}_1 + c_2 \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} c_1 + 5c_2 \\ c_1 + 6c_2 \\ 2c_1 + 8c_2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Impossible to satisfy all 3 equations $\rightarrow$ no solution

What is the column space of

$A_2 = \begin{bmatrix} 1 & 5 & 6 \\ 1 & 6 & 7 \\ 2 & 8 & 10 \end{bmatrix}$? $\quad C(A_2) = C(A)$

$A_3 = \begin{bmatrix} 1 & 5 & 1 \\ 1 & 6 & 1 \\ 2 & 8 & 1 \end{bmatrix}$? $\quad C(A_3) = \mathbb{R}^3$

Possible column spaces in $\mathbb{R}^3$

- A single point (zero vector)
- A 1D line
- A 2D plane
- all of $\mathbb{R}^3$

example

Create 3×3 matrices for each of these types of column space

Row space

def The row space of A is the column space of A^T.

Linear independence

def A set of vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$ are linearly independent

if $c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots + c_n\vec{a}_n = \vec{0}$

is only satisfied when $c_1 = c_2 = \ldots = c_n = 0$

note Three linearly independent vectors in $\mathbb{R}^3$ produce a matrix whose column space is all of $\mathbb{R}^3$.

Matrix–matrix multiplication

Usual strategy

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$A \in \mathbb{R}^{3\times3}$ $\quad B \in \mathbb{R}^{3\times2}$ $\quad C = AB \in \mathbb{R}^{3\times2}$

$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$ (inner product of row 1 of A and col 1 of B)

Outer products

$$\underset{\vec{u}}{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}} \quad \underset{\vec{v}}{\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}}$$

The outer product will be

$$\vec{u}\vec{v}^T = \underset{3\times1}{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 3 & 4 & 6 \end{bmatrix}} = \begin{bmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{bmatrix}$$

<u>note</u> all columns of $\vec{u}\vec{v}^T$ are multiples of $\vec{u}$

all rows of $\vec{u}\vec{v}^T$ are multiples of $\vec{v}^T$

<u>"The other way" of matrix multiplication</u>

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \vec{a}_1 \end{array} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{33} \end{bmatrix} \vec{b}_1^*$$

$$AB = \vec{a}_1\vec{b}_1^* + \vec{a}_2\vec{b}_2^* + \vec{a}_3\vec{b}_3^*$$

<u>ex</u> Show that $C(AB)$ is always a subset of $C(A)$.