

Project 1

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Linear Algebra

1 Exercise 1

Exercise 1.1 Set up a system of linear equations based on this problem from the *Nine Chapters on the Mathematical Art*.

The text then shows three columns set up on a counting board (a tool for mathematical calculation) in the following manner:

The given augmented matrix is:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{array}$$

Solution:

Setting the system of linear equations:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned} \tag{1}$$

Exercise 1.2 Write the equations you found in Exercise 1.1 as an augmented matrix. How does your matrix compare with the numbers on the counting board?

Solution:

Setting the augmented matrix found in the **Exercise 1.1**:

$$\begin{vmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{vmatrix}$$

The major difference between the matrices is the order how the numbers are listed. According to the book *Nine Chapters on the Mathematical Art*, they vertically listed up the variables and created a matrix in this same fashion. The way how separating the coefficient variables is also following their writing order; from right to left and top to bottom. As this way of writing order used during Han Dynasty, creating matrix also followed the same method that they used for writing Chinese letters.

Exercise 1.3 Use row operations to get your augmented matrix from Exercise 1.2 into row-echelon form and solve the system of equations.

Work to reduced row-echelon form,
first with $j = 1$,

$$\begin{aligned} & \begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{bmatrix} \\ & \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 3 & 2 & 1 & 39 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & -4 & -8 & -39 \end{bmatrix} \end{aligned}$$

Now, with $j = 2$,

$$\xrightarrow{-4R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & 0 & 12 & 33 \end{bmatrix} \xrightarrow{2R_2 + R_1} \begin{bmatrix} 1 & 0 & -7 & -10 \\ 0 & -1 & -5 & -18 \\ 0 & 0 & 12 & 33 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 12 & 33 \end{bmatrix}$$

And finally, with $j = 3$,

$$\xrightarrow{\frac{1}{12}R_3} \begin{bmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 1 & \frac{11}{4} \end{bmatrix} \xrightarrow{-5R_3 + R_2} \begin{bmatrix} 1 & 0 & -7 & -10 \\ 0 & 1 & 0 & \frac{17}{4} \\ 0 & 0 & 1 & \frac{11}{4} \end{bmatrix} \xrightarrow{7R_3 + R_1} \begin{bmatrix} \boxed{1} & 0 & 0 & \frac{37}{4} \\ 0 & \boxed{1} & 0 & \frac{17}{4} \\ 0 & 0 & \boxed{1} & \frac{11}{4} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 & \frac{37}{4} \\ 0 & 1 & 0 & \frac{17}{4} \\ 0 & 0 & 1 & \frac{11}{4} \end{bmatrix}$$

Therefore, this system of equation is:

$$x_1 = \frac{37}{4} = 9.25$$

$$x_2 = \frac{17}{4} = 4.25$$

$$x_3 = \frac{11}{4} = 2.75$$

Exercise 1.4 Notice that the methods used in the *Nine Chapters* are very similar to our modern approach to solving a system of equations. Verify that the solution that you obtained in Exercise 1.3 is the same as the solution obtained when solving the equations given by the columns on the counting board shown below Exercise 1.3.

Solution:

The given first counting board found in the Exercise 1.1 is as follows,

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{array}$$

and we will call it as **A** which is written in an augmented matrix

$$\mathbf{A} = \left| \begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array} \right|$$

The given second counting board found in the Exercise 1.3 is as follows,

$$\begin{array}{ccc} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{array}$$

and we will call it as **B** which is written in an augmented matrix

$$\mathbf{B} = \left| \begin{array}{cccc} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{array} \right|$$

According to the definition of Row-Equivalent Matrices, we can transform the augmented matrices \mathbf{A} and \mathbf{B} into Reduced Row-Echelon Form

$$\mathbf{A} = \left| \begin{array}{cccc} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array} \right| \xrightarrow{\text{RREF}} \left| \begin{array}{cccc} \boxed{1} & 0 & 0 & \frac{37}{4} \\ 0 & \boxed{1} & 0 & \frac{17}{4} \\ 0 & 0 & \boxed{1} & \frac{11}{4} \end{array} \right|$$

$$\mathbf{B} = \left| \begin{array}{cccc} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{array} \right| \xrightarrow{\text{RREF}} \left| \begin{array}{cccc} \boxed{1} & 0 & 0 & \frac{37}{4} \\ 0 & \boxed{1} & 0 & \frac{17}{4} \\ 0 & 0 & \boxed{1} & \frac{11}{4} \end{array} \right|$$

Thus, two matrices \mathbf{A} and \mathbf{B} are row-equivalent, and these two augmented matrices which derived from the counting board are having the same solutions.

Exercise 1.5 Translate Borrel's language into a system of simultaneous linear equations using the variables A, B and C.

Borrel's language	System of simultaneous linear equations
3A 1B 1C [42 1ST	$3A + 1B + 1C = 42$
1A 4B 1C [32 2ST	$1A + 4B + 1C = 32$
1A 1B 5C [40 3ST	$1A + 1B + 5C = 40$
3A 12B 3C [96	$3A + 12B + 3C = 96$
3A 1B 1C [42	$3A + 1B + 1C = 42$
11B 2C [54	$11B + 2C = 54$
3A 3B 15C [120	$3A + 3B + 15C = 120$
3A 1B 1C [42	$3A + 1B + 1C = 42$
2B 14C [78	$2B + 14C = 78$
22B 154C [858	$22B + 154C = 858$
22B 4C [108	$22B + 4C = 108$
150C [750	$150C = 750$

Exercise 1.6 Verify that Borrel's solution is correct.

We can verify Borrel's solution using RREF.

To express matrix form from Borrel's language

$$\begin{bmatrix} 3 & 1 & 1 & 42 \\ 1 & 4 & 1 & 32 \\ 1 & 1 & 5 & 40 \end{bmatrix}$$

first with $j=1$,

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 5 & 40 \\ 1 & 4 & 1 & 32 \\ 3 & 1 & 1 & 42 \end{bmatrix} \xrightarrow{-R_1 + R_2} \begin{bmatrix} 1 & 1 & 5 & 40 \\ 0 & 3 & -4 & -8 \\ 3 & 1 & 1 & 42 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 1 & 5 & 40 \\ 0 & 3 & -4 & -8 \\ 0 & -2 & -14 & -78 \end{bmatrix}$$

Now, with $j = 2$,

$$\xrightarrow{R_2 \leftrightarrow -\frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 5 & 40 \\ 0 & 1 & 7 & 39 \\ 0 & 3 & -4 & -8 \end{bmatrix} \xrightarrow{-3R_2 + R_3} \begin{bmatrix} 1 & 1 & 5 & 40 \\ 0 & 1 & 7 & 39 \\ 0 & 0 & -25 & -125 \end{bmatrix} \xrightarrow{-R_2 + R_1} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 7 & 39 \\ 0 & 0 & -25 & -125 \end{bmatrix}$$

And finally, with $j = 3$,

$$\xrightarrow{-\frac{1}{25}R_3} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 7 & 39 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{7R_3 + R_2} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{2R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

This RREF tells the system of equations,

$$A = 11$$

$$B = 4$$

$$C = 5$$

Since this system of equations and Borrel's solution are same, Borrel's solution is correct.

Exercise 2.1 Use Cauchy's definition to find the determinants of the following:

Solution:

According to the paper, getting a determinant based on the matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{Det}(A)} ad - cb$$

$$\text{a. } \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \xrightarrow{\text{Det}(A)} (1)(-1) - (4)(2) = -1 - 8 = -9$$

$$\text{b. } \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} \xrightarrow{\text{Det}(A)} (3)(5) - (2)(-7) = 15 + 14 = 29$$

$$\text{c. } \begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{Det}(A)} (2)(2)(1) + (1)(3)(0) + (-2)(3)(-1) - (2)(3)(-1) - (-2)(2)(0) - (1)(3)(1) = 17$$

$$\text{d. } \begin{bmatrix} 5 & 0 & 4 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \xrightarrow{\text{Det}(A)} (5)(2)(1) + (1)(-2)(4) + (-1)(0)(-1) - (5)(-2)(-1) - (-1)(2)(4) - (1)(0)(1) = 0$$

$$\text{e. } \begin{bmatrix} 5 & 3 & 4 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{Det}(A)} (5)(2)(1) + (1)(3)(4) + (-2)(3)(-1) - (5)(3)(-1) - (-2)(2)(4) - (1)(3)(1) = 56$$

Exercise 2.2 For each of the matrices in Exercise 2.1 state whether or not it is invertible and explain why.

Solution:

All the matrices above are invertible. Each matrix is able to be written in RREF, and they are nonsingular matrices since they are square matrices and identity matrices.

$$\text{a. } \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 5 & 0 & 4 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\text{e. } \begin{bmatrix} 5 & 3 & 4 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

In addition, determinant plays the important role for determining an invertible matrix. Inverse of \mathbf{A} will be evaluated as

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{A}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

\mathbf{A}^{-1} won't be defined unless $\det(A)$ is nonzero. Thus, the matrices listed above are invertible since their determinants are nonzero.

Exercise 3.1 For the matrix $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{bmatrix}$ identify two different minors and their complements.

Solution:

1) The single element $a_{11} = -2$ and the Minor $\begin{Bmatrix} 2 & -1 \\ 3 & 1 \end{Bmatrix}$ are complementary to each other.

2) The single element $a_{22} = 2$ and the Minor $\begin{Bmatrix} -2 & 0 \\ -2 & 1 \end{Bmatrix}$ are complementary to each other.