Legendre Polynomials

$\mathbb{P}_n$: the set of all polynomials with degree less than or equal to n, $\forall n \in \mathbb{N}$

$(L_n)_{n\geq 0}$: the family of Legendre Polynomials $\boxed{L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$

Noting that the Legendre polynomials are orthogonal basis on $(-1,1)$

$\forall n, m \in \mathbb{N}$, $\int_{-1}^{1} L_n(x) L_m(x)\, dx = \frac{2}{2n+1} \delta_{nm}$ where $\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{Otherwise} \end{cases}$

The Legendre polynomials are solutions of the differential equation

$$\boxed{\left[(1-x^2) L_n'(x)\right]' + n(n+1) L_n(x) = 0 \, , \; n \geq 0}$$

and they satisfy the following three-term recurrence formula

$$L_0(x) = 1$$
$$L_1(x) = x$$
$$(n+1) L_{n+1}(x) = (2n+1) x L_n(x) - n L_{n-1}(x) \quad \text{for } n \geq 1.$$

Gauss – Legendre Quadrature

Numerical quadratures are efficient tools for computing an approximation of an integral.
For a smooth function φ

$$\int_{-1}^{1} \varphi(x)\, dx = \sum_{i=1}^{s} \varphi(x_i) w_i + R_s(\varphi) \quad \text{where}$$

1. The points x_i (the nodes) are the zeros of the Legendre polynomials L_s.
2. The real numbers w_i (the weights) are given by

$$w_i = \frac{2}{(1-x_i^2)(L_s'(x_i))^2}$$

3. The remainder is $R_s(\varphi) = \frac{2^{2s+1} (s!)^4}{(2s+1)((2s)!)^3} \cdot \varphi^{(2s)}(\xi)$ for $\xi \in (-1,1)$

The Gauss–Legendre quadrature of order s is the approximation.

$$\boxed{\int_{-1}^{1} \varphi(x)\, dx \approx \sum_{i=1}^{s} \varphi(x_i) w_i}$$

Legendre Series Expansion

For a function $f \in L^2((-1,1))$, its Legendre equation

$$\boxed{\mathcal{L}(f) = \sum_{j=0}^{\infty} \hat{f}_j L_j}$$

the Legendre coefficients $\hat{f}_j$ defined by $\boxed{\hat{f}_j = \left(\frac{2j+1}{2}\right) \int_{-1}^{1} f(x) L_j(x)\, dx}$

<u>the truncated expansion</u>

$$\boxed{\mathcal{L}_p(f) = \sum_{j=0}^{p} \hat{f}_j L_j(x)}$$