Solving an Advection–Diffusion Equation by a Finite Element Method 09/04/2024

(The Short Explanation of the Method. Detailed version is following this page)

Define $u: [0,1] \to \mathbb{R}$

$(*)\ -\epsilon u'' + \lambda u' = f(x)$, $x \in [0,1]$, $\epsilon > 0$, $u(0)=0, u(1)=0$

ϵ: the diffusivity constant, λ: the speed of a fluid

A_FEM_app.py for python code!

Variational Formulation

$\forall v \in V$ where V denotes the space of functions $v: [0,1] \to \mathbb{R}$ s.t. the integrals $\int_0^1 |v|^2 dx$, and $\int_0^1 |v'|^2 dx$ are bounded.

so we have $-\epsilon u'' v + \lambda u' v = f(x) \cdot v$ integrating both sides from 0 to 1

$$-\epsilon \int_0^1 u'' v\, dx + \lambda \int_0^1 u' v\, dx = \int_0^1 f(x) v\, dx$$

Applying Integration by parts, we have $(\ddagger)$ $\boxed{\epsilon \int_0^1 u' v' dx + \lambda \int_0^1 u' v\, dx = \int_0^1 f(x) v\, dx}$ which is called the weak form of the differential equation $(*)$. We have also one more important remark coming from Integration by parts is $v(0)=0$, $v(1)=0$.

A P1 Finite Element Method

Discretization of $(**)$

$$\epsilon \underbrace{\int_0^1 u' v' dx}_{B} + \lambda \underbrace{\int_0^1 u' v\, dx}_{C} = \underbrace{\int_0^1 f(x) v\, dx}_{b}$$

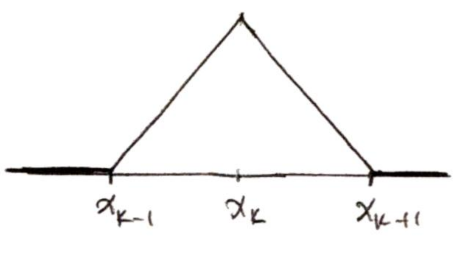
where $B = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 \cdots 0 \\ 0 & \cdots & \cdots & 0 & -1 \ 2 \end{bmatrix}$ $\quad C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}$ $\quad b = h\begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}$, $k \in \mathbb{R}$.

Choosing $f(x) = k$, $k \in \mathbb{R}$ is a constant function.

$\boxed{A = \epsilon B + \lambda C}$ so the solution of $Au = b$ gives A P1 FEM approximation

B, and C matrices are coming from integration of the basis function φ.

The shape of basis function φ is given by



and defined by

$$\varphi = \begin{cases} \frac{x - x_{k-1}}{h}, & x \leq x_k \\ \frac{x_{k+1} - x}{h}, & x \geq x_k \end{cases} \quad \text{and}$$

its derivative

$$\varphi' = \begin{cases} 1/h, & x \leq x_k \\ -1/h, & x \geq x_k. \end{cases}$$

Solving an Advection – Diffusion Equation by a Finite Element Method

$u : [0,1] \to \mathbb{R}$, the problem

$$(*) \quad \boxed{\begin{array}{l} -\varepsilon u'' + \lambda u' = f(x), \quad x \in [0,1], \quad \varepsilon > 0 \\ u(0) = 0, \quad u(1) = 0. \end{array}}$$

Aim: To approximate the solution using a continuous piecewise polynomial function.

The differential equation $(*)$ is an advection-diffusion equation. where λ is the speed of a fluid and ε is the diffusivity constant. The ratio $\theta = \lambda/\varepsilon$ measures the importance of the advection compared to the diffusion.

Variational Formulation

A solution u of the boundary value problem $(*)$ is also solution of the following problem

$$\boxed{\begin{cases} \text{Find } u \in V \text{ s.t.} \\ \text{for all } v \in V \end{cases} \quad a(u,v) = \int_0^1 f(x) v(x)\,dx}$$

Here, V denotes $H_0^1(]0,1[)$, the space of functions $v : [0,1] \to \mathbb{R}$ s.t. the integrals $\int_0^1 |v|^2 dx$ and $\int_0^1 |v'|^2 dx$ are bounded and $v(0) = v(1) = 0$

The bilinear form a is defined on $V \times V$ by

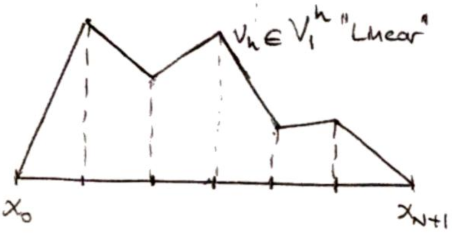
$$(**) \quad \boxed{a(u,v) = \varepsilon \int_0^1 u'(x) v'(x)\,dx + \lambda \int_0^1 u'(x) v(x)\,dx} \quad \text{"The Variational formulation of } (*)\text{"}$$

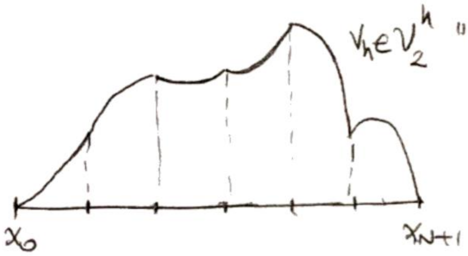
The finite element method is based on the computation of the solution of the variational problem while finite difference methods are based on a direct discretization of equation $(*)$.

Given $n > 0$, we divide the interval $[0,1]$ into $n+1$ subintervals I_i.

For $\ell > 0$, we denote by $\mathbb{P}_\ell(I_i)$ the set of algebraic polynomials of degree less than or equal to ℓ on I_i.

V_ℓ^h : the set of continuous functions defined on $[0,1]$ whose restriction to each interval I_i belongs to $\mathbb{P}_\ell(I_i)$.





The finite element method consists in searching for an approximation $u_h \in V_\ell^h$ of the function u, defined as the solution of the following problem:

$$\boxed{\begin{cases} \text{Find } u_h \in V_\ell^h \text{ s.t.} \\ \text{for all } v_h \in V_\ell^h \end{cases} \quad a(u_h, v_h) = \int_0^1 f(x) v_h(x)\,dx}$$

For the hard part of integrations, we will use some quadrature rules

1. Trapezoidal rule $\displaystyle\int_a^b g(x)\,dx \approx (b-a)\frac{(g(b)+g(a))}{2}$.

This method is of order 1, that is, it is exact for all $g \in \mathbb{P}_1([a,b])$.

2. The Simpson quadrature rule $\int_a^b g(x)\,dx \approx \frac{b-a}{6}\left(g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right)$

This method is of order 3, so it is exact for all $g \in \mathbb{P}_3([a,b])$

Exercises. (5.1)

1. Derive explicit formulas for the solution u of the problem (*)

$$-\varepsilon u'' + \lambda u' = f(x) \; ; \; x \in [0,1], \; \varepsilon > 0$$

$$u(0) = 0, \; u(1) = 0$$

$u'' - \frac{\lambda}{\varepsilon} u' = -\frac{1}{\varepsilon} f(x)$ and say $K = -\frac{\lambda}{\varepsilon}$ and $F = -\frac{1}{\varepsilon} f(x)$ then the equation becomes the form $u'' + Ku' = F$

Multiplying e^{Kx} from both sides and integrate

$$u'' \cdot e^{Kx} + Ke^{Kx} u' = F \cdot e^{Kx} \rightarrow (u' \cdot e^{Kx})' = F \cdot e^{Kx} \rightarrow u' \cdot e^{Kx} = F\left(\frac{1}{K} e^{Kx} + c_1\right)$$

so $u'(x) = \frac{F}{K} + Fc_1 e^{-Kx}$ and integrate $u(x) = \frac{Fx}{K} + Fc_1\left(-\frac{1}{K} e^{-Kx}\right) + c_2$ *

Inserting the Boundary conditions into (*), we get

$$u(0) = Fc_1\left(-\frac{1}{K}\right) + c_2 = 0 \quad \text{so} \quad \boxed{c_2 = \frac{Fc_1}{K}}$$

$$u(1) = \frac{F}{K} + Fc_1\left(-\frac{1}{K} e^{-K}\right) + c_2 = 0 \rightarrow \frac{F}{K} - \frac{Fc_1}{K} e^{-K} + \frac{Fc_1}{K} = 0$$

so $\boxed{c_1 = \frac{1}{e^{-K} - 1}}$ and $\boxed{c_2 = \frac{F}{K} \frac{1}{e^{-K} - 1}}$

so our solution takes the form

$$u(x) = \frac{F}{K} x - \frac{F}{K} \cdot \frac{1}{e^{-K} - 1} e^{-Kx} + \frac{F}{K} \frac{1}{e^{-K} - 1}$$

$$= \frac{F}{K}\left(x - \frac{e^{-Kx} - 1}{e^{-K} - 1}\right) \quad \text{so} \quad \boxed{u(x) = \frac{f}{\lambda}\left(x - \frac{e^{\lambda x/\varepsilon} - 1}{e^{\lambda/\varepsilon} - 1}\right)}$$

2. Prove the existence of $x_\theta \in]0,1[$ depending only on the ratio $\theta = \lambda/\varepsilon$ such that the function u is strictly increasing (respectively decreasing) over $]0, x_\theta[$ (respectively $]x_\theta, 1[$). Determine $\lim_{|\theta|\to\infty} x_\theta$.

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$$u(x) = \frac{f}{\lambda}\left(x - \frac{e^{\theta x}-1}{e^\theta - 1}\right) \rightarrow u'(x) = \frac{f}{\lambda}\left(1 - \frac{\theta \cdot e^{\theta x}}{e^\theta - 1}\right) = 0 \Leftrightarrow \boxed{x_\theta = \frac{1}{\theta}\ln\left(\frac{e^\theta - 1}{\theta}\right)}$$

$$x_\theta = \frac{1}{\theta}\left(\theta + \ln\left(\frac{1-e^{-\theta}}{\theta}\right)\right) \Leftrightarrow \boxed{x_\theta = 1 + \frac{1}{\theta}\ln\left(\frac{1-e^{-\theta}}{\theta}\right)}$$

$$\lim_{\theta\to+\infty} x_\theta = \lim_{\theta\to\infty}\left(1 + \frac{1}{\theta}\ln\left(\frac{1-e^{-\theta}}{\theta}\right)\right) = 1.$$

$$\lim_{\theta\to-\infty} x_\theta = \lim_{\theta\to-\infty}\left(\frac{1}{\theta}\ln\left(\frac{e^\theta - 1}{\theta}\right)\right) = 0.$$

A P1 Finite Element Method

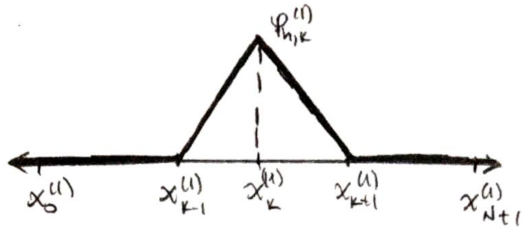
For $n \in \mathbb{N}$, we define the points

$$x_k^{(1)} = kh, \quad k = 0, \dots, n+1$$

and intervals

$$I_k =]x_k^{(1)}, x_{k+1}^{(1)}[, \quad k = 0, \dots, n$$

A hat function $\varphi_{h,k}^{(1)}$ of V_1^k



with grid size $h = 1/(n+1)$. We also define k "hat functions" $\varphi_{h,k}^{(1)}$ $(k=1,\dots,n)$ such that

$$\varphi_{h,k}^{(1)} \in V_1^k \quad \text{and} \quad \varphi_{h,k}^{(1)}(x_j^{(1)}) = \delta_{j,k}, \quad \forall j = 1, 2, \dots, n$$

with $\delta_{j,k}$ the Kronecker symbol. Note that the support of the function $\varphi_{h,k}^{(1)}$ is the union of two intervals I_{k-1} and I_k

In the finite element methods, the points $x_k^{(1)}$ are called <u>nodes</u> and the intervals I_k are called <u>cells</u>. We seek an approximation $u_h^{(1)} \in V_1^h$ of the function u, a solution of the problem

$$\begin{cases} \text{Find } u_h^{(1)} \in V_1^h \text{ s.t.} \\ \text{for all } v_h \in V_1^h \end{cases} \quad a(u_h^{(1)}, v_h) = \int_0^1 f(x)\, v_h\, dx.$$

Exercises (5.2)

1. Prove that the functions $(\varphi_{h,k}^{(1)})_{k=1}^{n}$ form a basis of V_1^h.

The linear space V_1^h has dimension n since there is an obvious isomorphism from this space onto $\mathbb{R}^n$: to each $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$ corresponds a function $v \in V_1^h$ defined by $v(x_i) = u_i$. From the identities $\varphi_{h,k}^{(1)}(x_j) = \delta_{j,k}$, we deduce that functions $(\varphi_{h,k}^{(1)})_{k=1}^{n}$ are linearly independent.

$$\left(\sum_{k=1}^{n} c_k \varphi_{h,k}^{(1)} = 0, \forall x\right) \Rightarrow \left(\sum_{k=1}^{n} c_k \varphi_{h,k}^{(1)}(x_j) = 0, \forall j\right) \Rightarrow (c_j = 0, \forall j)$$

Since V_1^h has dimension n, $(\varphi_{h,k}^{(1)})_{k=1}^{n}$ form a basis of this space.

Here are analytical expressions for $\varphi_{h,k}^{(1)}$ and its derivative

for $x \notin I_{k-1} \cup I_k$, $\varphi_{h,k}^{(1)}(x) = \varphi_{h,k}^{(1)\prime}(x) = 0$

for $x \in I_{k-1}$, $\varphi_{h,k}^{(1)}(x) = \dfrac{x - x_{k-1}}{h}$ and $\varphi_{h,k}^{(1)\prime}(x) = \dfrac{1}{h}$

for $x \in I_k$, $\varphi_{h,k}^{(1)}(x) = \dfrac{x_{k+1} - x}{h}$ and $\varphi_{h,k}^{(1)\prime}(x) = -\dfrac{1}{h}$

3. By expanding $u_h^{(1)}$ in the basis $(\varphi_{h,k}^{(1)})_{k=1}^{n}$,

$$u_h^{(1)} = \sum_{m=1}^{n} \alpha_m \varphi_{h,m}^{(1)}$$

show that $\alpha_k = u_h(x_k^{(1)})$. Show that $\tilde{u}_h^{(1)} = (u_h^{(1)}(x_1^{(1)}), \dots, u_h^{(1)}(x_n^{(1)}))^T$ is a solution of the linear system $A_h^{(1)} \tilde{u}_h^{(1)} = b_h^{(1)}$ where $A_h^{(1)}$ is the real matrix of size $n \times n$ defined by $\boxed{(A_h^{(1)})_{k,m} = a(\varphi_{h,m}^{(1)}, \varphi_{h,k}^{(1)}),\ 1 \le m, k \le n}$ and $b_h^{(1)}$ is the vector of $\mathbb{R}^n$ with elements $\boxed{(b_h^{(1)})_k = \int_0^1 f(x)\, \varphi_{h,k}^{(1)}(x)\, dx,\ 1 \le k \le n}$

Show that $\boxed{A_h^{(1)} = \varepsilon B_h^{(1)} + \lambda C_h^{(1)}}$ where $B_h^{(1)}$ is a tridiagonal symmetric matrix and $C_h^{(1)}$ a tridiagonal skew-symmetric matrix.

Exercises (5.3) 08/04/2024

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1. Derive the following explicit formulas for $B_h^{(1)}$ and $C_h^{(1)}$.

Computation of A_h. Recall that the supports of two sufficiently distant basis functions $\varphi_{h,j}^{(1)}$ and $\varphi_{h,k}^{(1)}$ are disjoint. More precisely, defining

$b_{kj} = \int_0^1 \varphi_{h,k}^{(1)'} \varphi_{h,j}' \, dx$ and $c_{jk} = \int_0^1 \varphi_{h,k}^{(1)'} \varphi_{h,j}$ we get

$\rightarrow$ for $|k-j|>1$, $b_{kj} = c_{kj} = 0$

$\rightarrow$ for $k=j$, $b_{kk} = \int_0^1 (\varphi_{hk}^{(1)'})^2 dx = \int_{I_{k-1}} (\varphi_{hk}^{(1)'})^2 dx + \int_{I_k} (\varphi_{hk}^{(1)'})^2 dx = \frac{2}{h}$

$$c_{kk} = \int_0^1 \varphi_{hk}^{(1)'} \varphi_{h,k} \, dx = \int_{I_{k-1}} \varphi_{hk}^{(1)'} \varphi_{hk} \, dx + \int_{I_k} \varphi_{hk}^{(1)'} \varphi_{hk} \, dx = 0$$

$\rightarrow$ for $k = j+1$

$$b_{j+1,j} = b_{j,j+1} = \int_0^1 \varphi_{h,j+1}^{(1)'} \varphi_{h,j}^{(1)'} \, dx = -\frac{1}{h} \quad \text{'Symmetric''}$$

$$c_{j+1,j} = -c_{j,j+1} = \int_0^1 \varphi_{h,j+1}^{(1)'} \varphi_{h,j} \, dx = -\frac{1}{2} \quad \text{''Skew-Symmetric''}.$$

So, we have

$$B_h^{(1)} = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & & & -1 & 2 & -1 \\ \vdots & & & & & \\ 0 & \cdots & & & -1 & 2 \end{bmatrix}, \quad C_h^{(1)} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ -1 & 0 & 1 & \cdots & & 0 \\ 0 & & & & & \vdots \\ \vdots & & 0 & -1 & 0 & 1 \\ 0 & & & 0 & -1 & 0 \end{bmatrix}$$

3. Use the trapezoidal rule to compute the components of the vector $b_h^{(1)}$.

$$(b_h^{(1)})_k = \int_0^1 f \varphi_{h,k}^{(1)} \, dx = \int_{x_k^{(1)}}^{x_{k+1}^{(1)}} f \varphi_{h,k}^{(1)} \, dx + \int_{x_{k-1}^{(1)}}^{x_k^{(1)}} f \varphi_{h,k}^{(1)} \, dx$$

$$\approx \frac{h}{2} \left[f(x_{k-1}^{(1)}) \varphi_{hk}^{(1)}(x_{k-1}^{(1)}) + 2 f(x_k^{(1)}) \varphi_{h,k}^{(1)}(x_k^{(1)}) + f(x_{k+1}^{(1)}) \varphi_{h,k}^{(1)}(x_{k+1}^{(1)}) \right]$$

$$= h f(x_k^{(1)})$$