**TEXAS A&M UNIVERSITY**

Department of Statistics

# STAT626 - Methods in Time Series Analysis
# Homework #4

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Problem 1: **Do Problems 2.8, 2.11-2.15, 3.2 and 3.3 from the textbook.**

Solution: **Question (2.8)|: a)**

$$\begin{aligned}\gamma_y(0) = \gamma(y_t, y_t) &= cov(w_t - \theta w_{t-1} + u_t, w_t - \theta w_{t-1} + u_t)\\ &= cov(w_t, w_t) + \theta^2 cov(w_{t-1}, w_{t-1}) + cov(u_t, u_t)\\ &= var(w_t) + \theta^2 var(w_{t-1}) + var(u_t) = (1+\theta^2)\sigma_w^2 + \sigma_u^2\end{aligned}$$

**since $w_t, w_{t-1}, u_t$ are independent. Similarly, $\gamma_y(\pm 1) = -\theta\sigma_w^2$ leading;**

$$\gamma_y(h) = \begin{cases} (1+\theta^2)\sigma_w^2 + \sigma_u^2, & h = 0\\ -\theta\sigma_w^2, & |h| = 1\\ 0, & |h| > 1\end{cases}$$

**Therefore, the ACF of $y_t$ is given by;**

$$\rho_y(h) = \frac{\gamma(y_{t+h}, y_t)}{\sqrt{\gamma(y_{t+h}, y_{t+h})\gamma(y_t, y_t)}} = \begin{cases} 1, & h = 0\\ \dfrac{-\theta\sigma_w^2}{(1+\theta^2)\sigma_w^2 + \sigma_u^2}, & |h| = 1\\ 0, & |h| > 1\end{cases}$$

**b)**

$$\begin{aligned}\gamma_{xy}(h) = \gamma(x_{t+h}, y_t) &= cov(w_{t+h}, w_t - \theta w_{t-1} + u_t)\\ &= cov(w_{t+h}, w_t) - \theta cov(w_{t+h}, w_{t-1}) + cov(w_{t+h}, u_t)\\ &= \begin{cases} \sigma_w^2, & h = 0\\ -\theta\sigma_w^2, & h = -1\\ 0, & \textbf{otherwise}\end{cases}\end{aligned}$$

**Therefore, the CCF of $x$ and $y$ is given by;**

$$\rho_{xy}(h) = \frac{\gamma(x_{t+h}, y_t)}{\sqrt{\gamma(x_{t+h}, x_{t+h})\gamma(y_t, y_t)}} = \begin{cases} \dfrac{\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}, & h = 0\\ \dfrac{-\theta\sigma_w}{\sqrt{(1+\theta^2)\sigma_w^2 + \sigma_u^2}}, & h = -1\\ 0, & \textbf{otherwise}\end{cases}$$

**c)** $E[x_t] = E[w_t] = 0$ **(independent of time), and**

$$\gamma_x(h) = \begin{cases} \sigma_w^2, & h = 0\\ 0, & \textbf{otherwise}\end{cases}$$

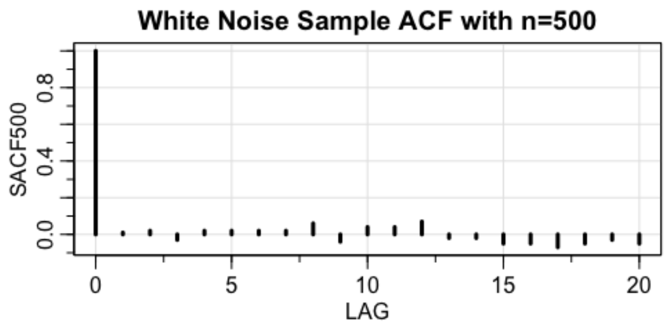
**implies $x_t$ is stationary. Similarly,** $E[y_t] = E[w_t] - \theta E[w_{t-1}] + E[u_t] = 0$**, and**

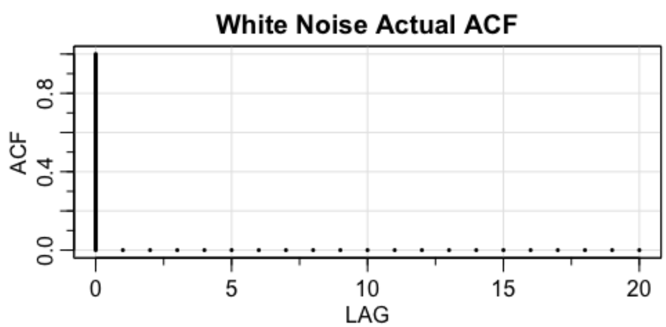
$$\gamma_y(h) = \begin{cases} (1+\theta^2)\sigma_w^2 + \sigma_u^2, & h = 0\\ -\theta\sigma_w^2, & |h| = 1\\ 0, & |h| > 1\end{cases}$$

**implies $y_t$ is stationary. Moreover, we have already shown that $\gamma_{xy}(h)$ is a function of lag $(h)$ and independent of time. Therefore, $x_t$ and $y_t$ are <u>jointly</u> stationary.**

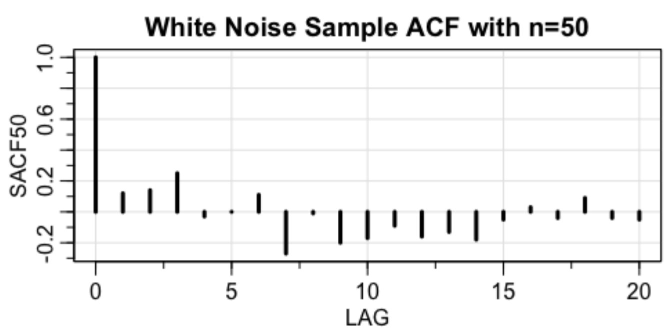
**Question(2.11)**|

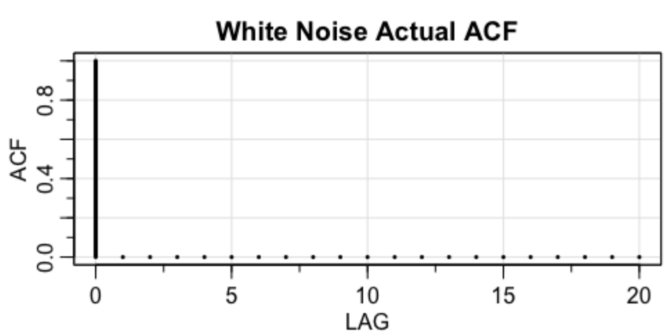
**a) For $n = 500$ Gaussian white noise observations, the plot of actual ACF vs SACF is given below:**



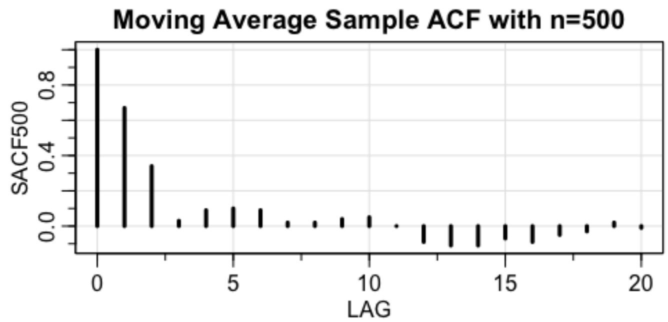

**b) For $n = 50$ Gaussian white noise observations, the plot of actual ACF vs SACF is given below:**

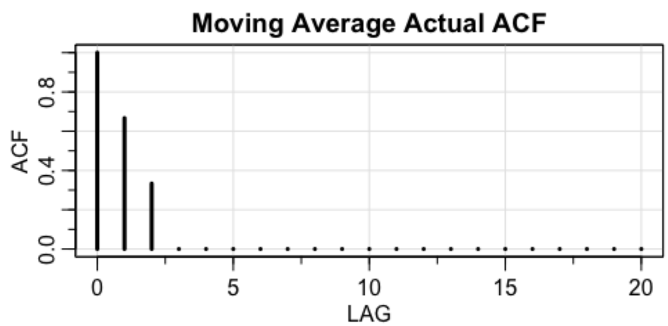


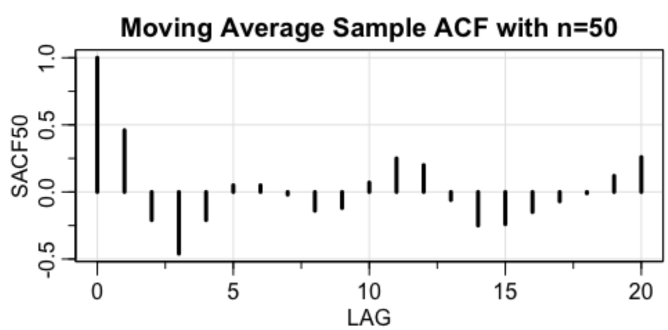

**Clearly, as $n$ increases Sample ACF gets closer to ACF.**

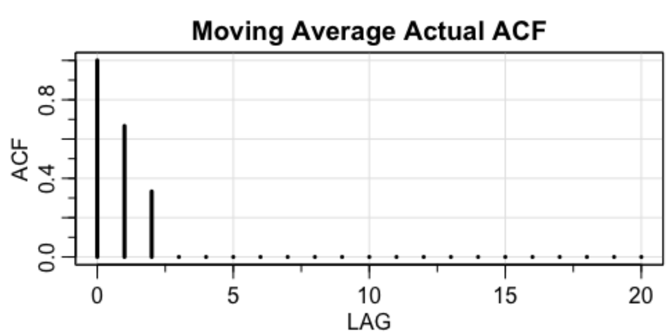
**Question(2.12)|**
**a) For $n = 500$ moving average observations, the plot of actual ACF vs SACF is given by the figure below:**



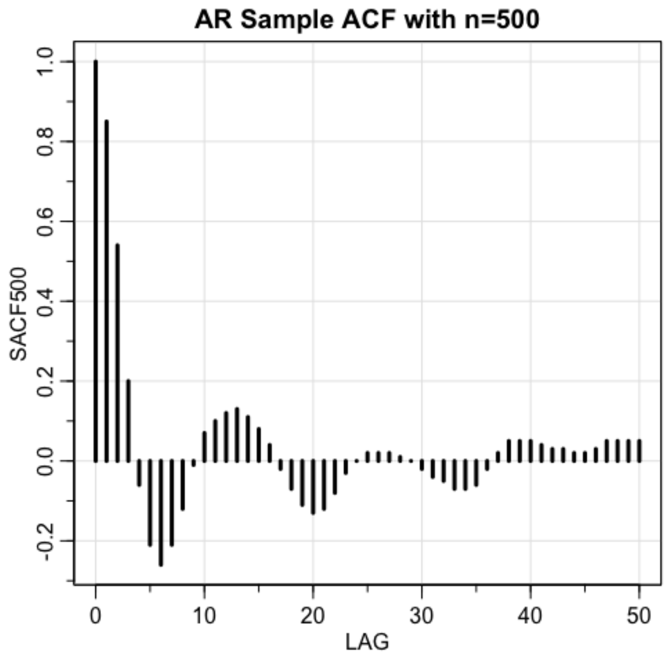

**b) For $n = 50$ moving average observations, the plot of actual ACF vs SACF is given by:**




**Again, as $n$ increases Sample ACF approaches to ACF, or deviations from the actual ACF decreases.**

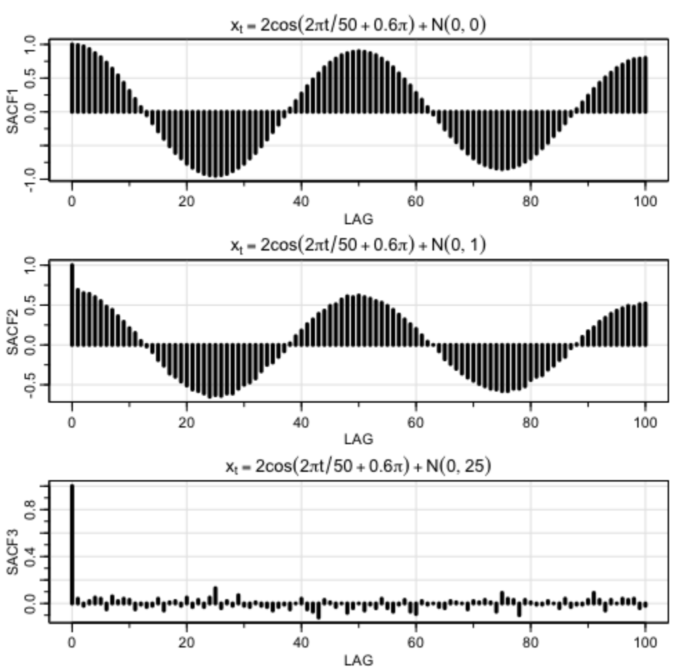
**Question(2.13)|**
**For $n = 500$ AR model observations, the plot of SACF is given by the figure below:**




**The sample ACF reflects the periodicity of the original data.**

**Question(2.14)|**

**For $n = 500$ observations from the signal-plus-noise model, the plots of SACF's for three series with (a)$\sigma_w = 0$, (b)$\sigma_w = 1$, and (c)$\sigma_w = 5$ is given by the figure below:**



**As the variance of white noise increases white noise dominates the Sample ACF, and as a result periodic behaviour of the Sample ACF disappears, or equivalently, Sample ACF data becomes smoother.**

**Question(2.15)|**

$$\begin{aligned}\gamma_y(h) = \gamma(y_{t+h}, y_t) &= cov(5 + x_{t+h} - 0.5x_{t+h-1}, 5 + x_t - 0.5x_{t-1}) \\ &= \cancelto{0}{cov(5,5)} + \cancelto{0}{cov(5,x_t)} - 0.5\cancelto{0}{cov(5,x_{t-1})} + \cancelto{0}{cov(x_{t+h},5)} - 0.5\cancelto{0}{cov(x_{t+h-1},5)} \\ &+ cov(x_{t+h}, x_t) - 0.5\big[cov(x_{t+h}, x_{t-1}) + cov(x_{t+h-1}, x_t)\big] + 0.25cov(x_{t+h-1}, x_{t-1})\end{aligned}$$

$$\gamma_y(h) = \begin{cases} 1.25\sigma_w^2, & h = 0 \\ -0.5\sigma_w^2, & |h| = 1 \\ 0, & \textbf{otherwise} \end{cases}$$

**Therefore the auto-correlation function of $y_t$ is given by:**

$$\rho_y(h) = \frac{\gamma_y(h)}{\gamma_y(0)} = \begin{cases} 1, & h = 0 \\ -0.4, & |h| = 1 \\ 0, & |h| > 1 \end{cases}$$

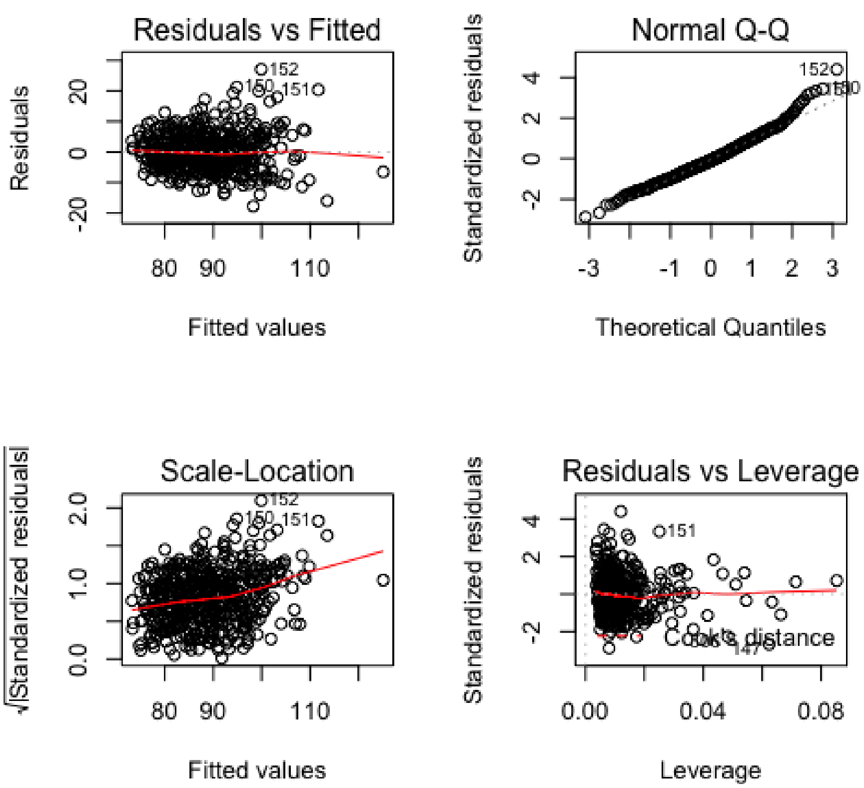
**Question(3.2)(a)|**

**The summary and anova tables of the final model are;**

```
Call:
lm(formula = cmort ~ trend + temp + temp2 + part + part4, na.action = NULL)

Residuals:
     Min       1Q   Median       3Q      Max 
-17.7459  -4.2014  -0.5341   3.7901  27.0358 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept) 2915.64607  193.57546  15.062  < 2e-16 ***
trend         -1.43715    0.09797 -14.670  < 2e-16 ***
temp          -0.48469    0.03066 -15.808  < 2e-16 ***
temp2          0.02432    0.00275   8.844  < 2e-16 ***
part           0.31560    0.02086  15.127  < 2e-16 ***
part4         -0.12122    0.02037  -5.952 4.96e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.177 on 502 degrees of freedom
Multiple R-squared:  0.6221,	Adjusted R-squared:  0.6183 
F-statistic: 165.3 on 5 and 502 DF,  p-value: < 2.2e-16
```

```
Analysis of Variance Table

Response: cmort
           Df  Sum Sq Mean Sq F value    Pr(>F)    
trend       1 10666.9 10666.9 279.530 < 2.2e-16 ***
temp        1  8606.6  8606.6 225.540 < 2.2e-16 ***
temp2       1  3428.7  3428.7  89.850 < 2.2e-16 ***
part        1  7476.1  7476.1 195.914 < 2.2e-16 ***
part4       1  1352.1  1352.1  35.432 4.963e-09 ***
Residuals 502 19156.3    38.2                      
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

**Moreover, diagnostic plots of the lagged final model is given by:**




**Comparison of key parameters of two model is given by the table below:**

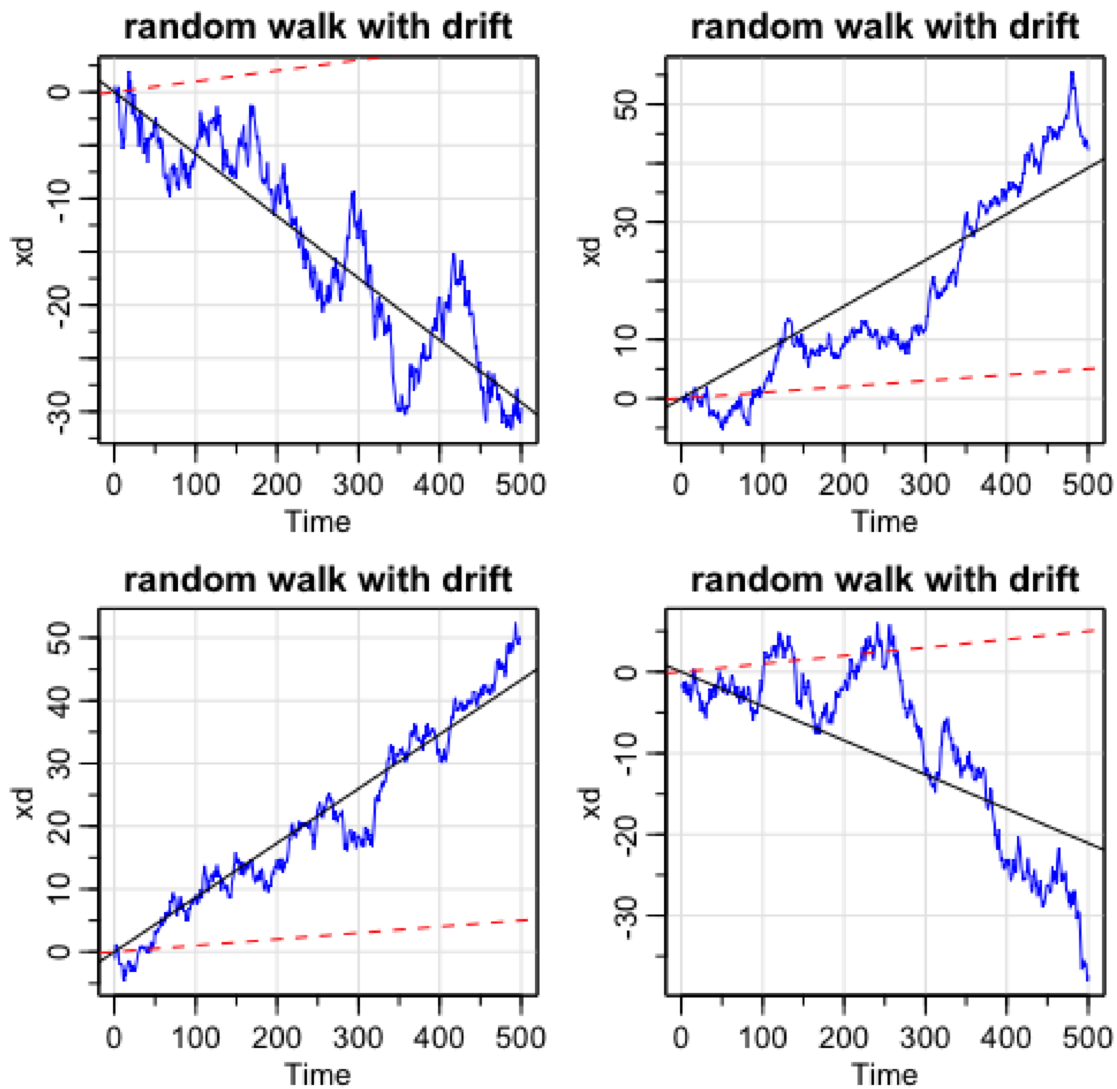
| Model | k | SSE | df | MSE | $R^2$ | AIC | BIC |
|---|---|---|---|---|---|---|---|
| (3.17) | 5 | 20,508 | 503 | 40.8 | 0.60 | 4.72 | 4.77 |
| Final | 6 | 19,156 | 502 | 38.2 | 0.62 | 4.66 | 4.72 |

**All the values in the table above clearly show that the model gained explanatory power (yields better results) by adding lagged terms.**

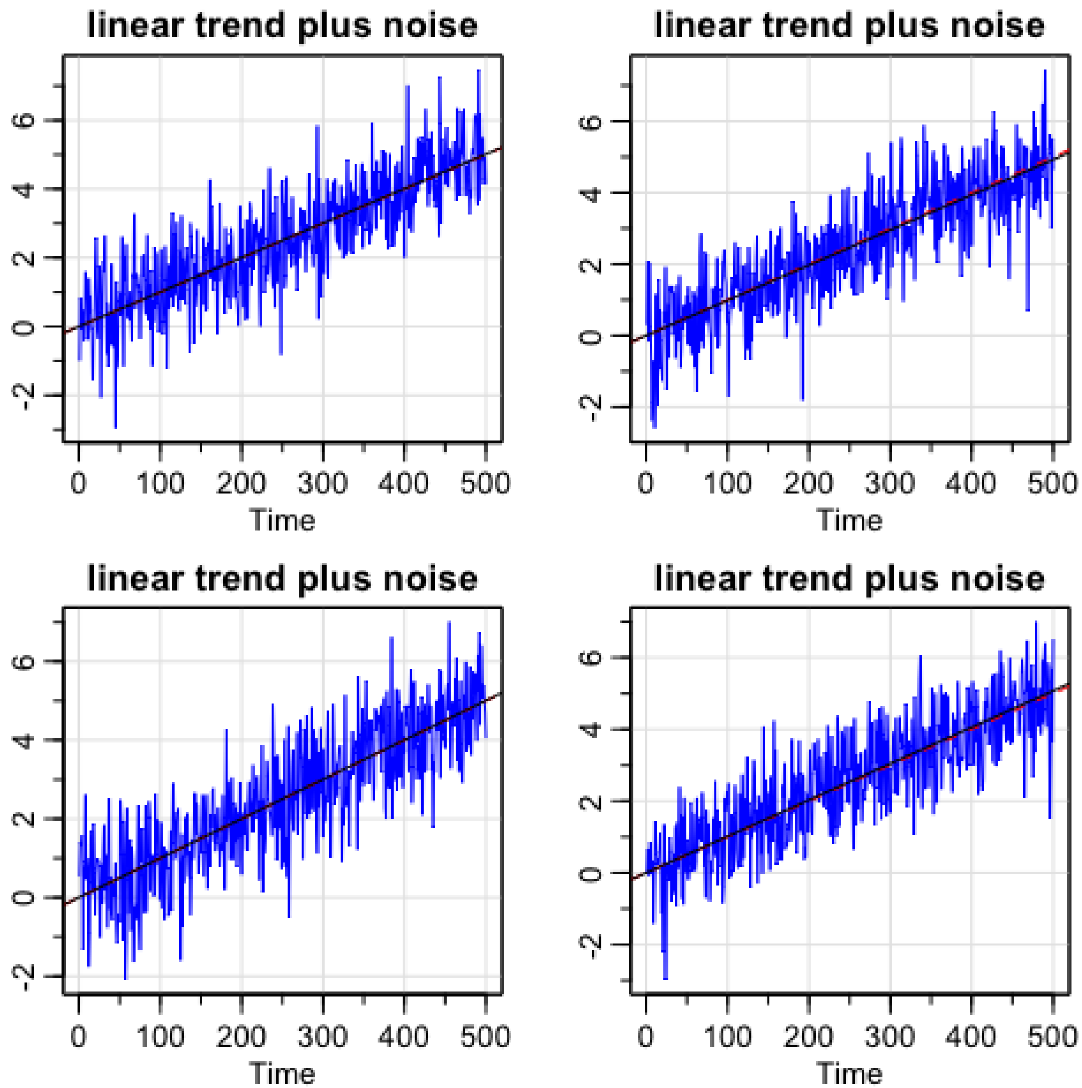
**(b) Since both AIC and BIC are reduced by the final model, it implies that the final model is an improvement over the final model (3.17) in Example 3.5.**

**Question(3.3)(a)**

**The plots for 4 series of random walk drift with the true mean and fitted lines:**




**(b) The plots for 4 series of linear trend plus noise with the true mean and fitted lines:**




**(c) The fitted regression line and the true mean function in the random walk with drift model are far away from each other, whereas they almost perfectly coincide for the linear trend plus noise model. Therefore, true mean function of the linear trend plus noise model is a good estimate for the regression line.**

Problem 2: (a) **Simulate two random walks $y_t$ , $x_t$, $t = 1, 2, ..., 100$, with initial values $x_0 = y_0 = 0$, using two independent $N(0, 1)$ white noises.**

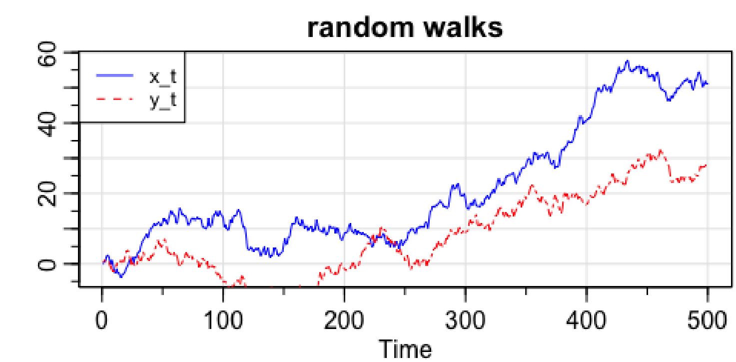
(i) Plot $y_t$ vs $x_t$ in the $(x, y)$-plane. Describe the pattern of dependence (if any) you notice in the scatter plot.

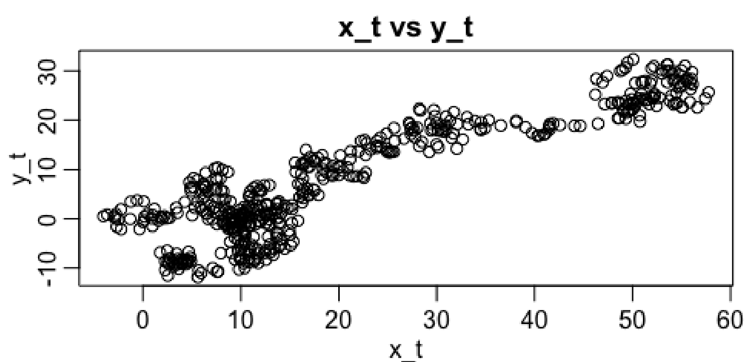
(ii) Consider the linear regression model $y_t = \beta_0 + \beta_1 x_t + w_t$, and the null hypothesis $H_0 : \beta_1 = 0$ vs $H_1 : \beta_1 \neq 0$ at $\alpha = 0.05$. Do you expect $H_0$ would be rejected? Why?

(iii) Perform the test and state your conclusion.

(b) **Repeat the above experiment $1000$ times and count the number of times $H_0$ is rejected in $(iii)$. Does it support your expectation in $(ii)$? If not, find an explanation for this phenomenon based on possible violations of assumptions of inference in regression models. (Hint: Recall the three assumptions of inference for linear regression models: (a) Independence, (b) Homogeneity of variances, (c) Normality. Check whether they hold for the data here.)**

Solution: (a) (i) The plot of $y_t$ vs $x_t$ is given by the plots below. Additionally, in this case there is a clear linear dependence between two models.

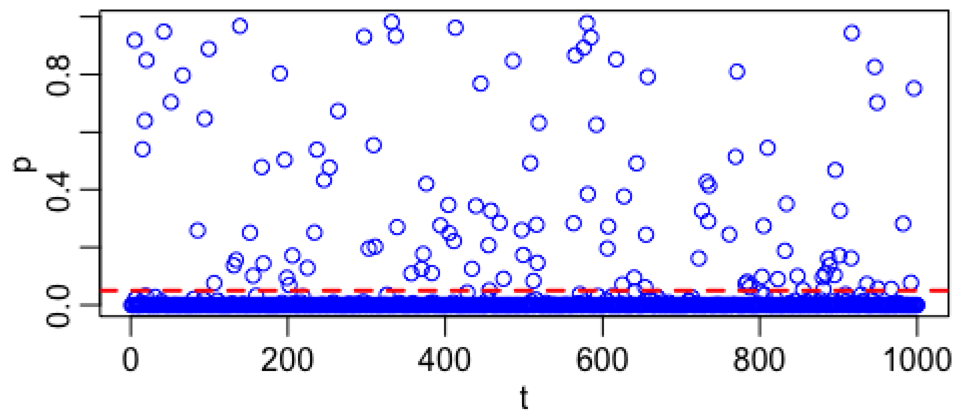


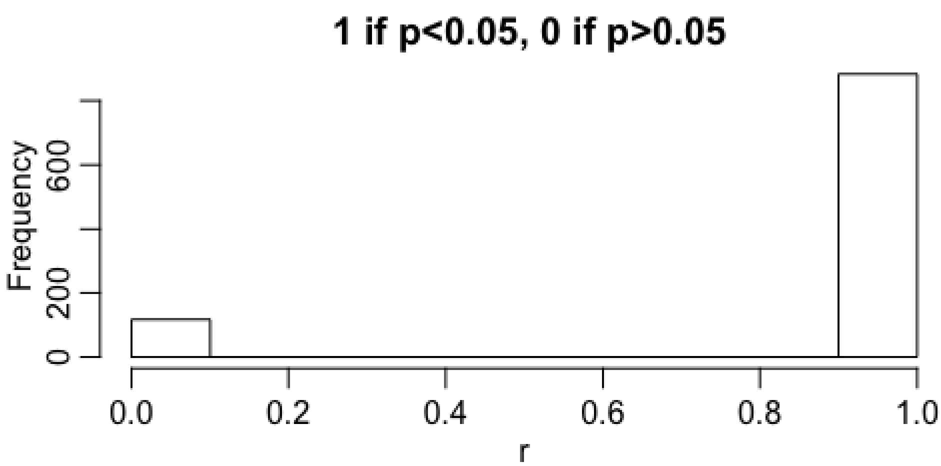




(ii) Yes, I would expect $H_0$ hypothesis to be rejected. Clear linear dependence imply it is expected that $\beta_1 \neq 0$.

(iii) $p - value$ for the slope estimate (from the summary of linear regression) is found $p = 2e - 16 < 0.05 \quad (***)$ and it shows the coefficient is significant, i.e., is not expected to be zero. Therefore, there is an evidence that $H_0$ hypothesis should be rejected.

(b) The number of times $H_0$ is rejected is 883 (i.e, the number of times $p < 0.05$) out of 1000. It supports my expectations in part (a). Also, here is a scatter plot and a histogram of $p - values$ of experiment with 1000 times:




Problem 3: **Multiple choice questions.**

Solution:

| Question | Answer |
|---|---|
| 1 | (d) |
| 2 | (b) |
| 3 | (d) |
| 4 | (d) |
| 5 | (b) |
| 6 | (c) |
| 7 | (a) |
| 8 | (a) |
| 9 | (e) |
| 10 | (d) |