TEXAS A&M UNIVERSITY

DEPARTMENT OF STATISTICS

STAT626 - Methods in Time Series Analysis

Homework #2

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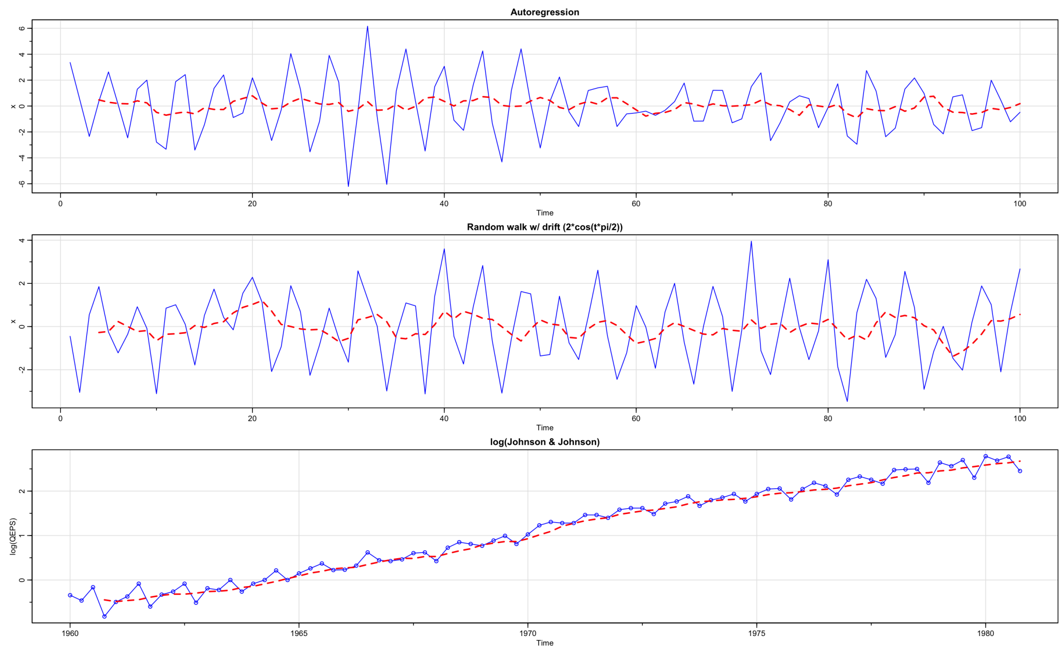
May 29, 2019

Problem 1: **Do problems** 1.1**,** 1.3**, and** 1.4 **from the textbook.**

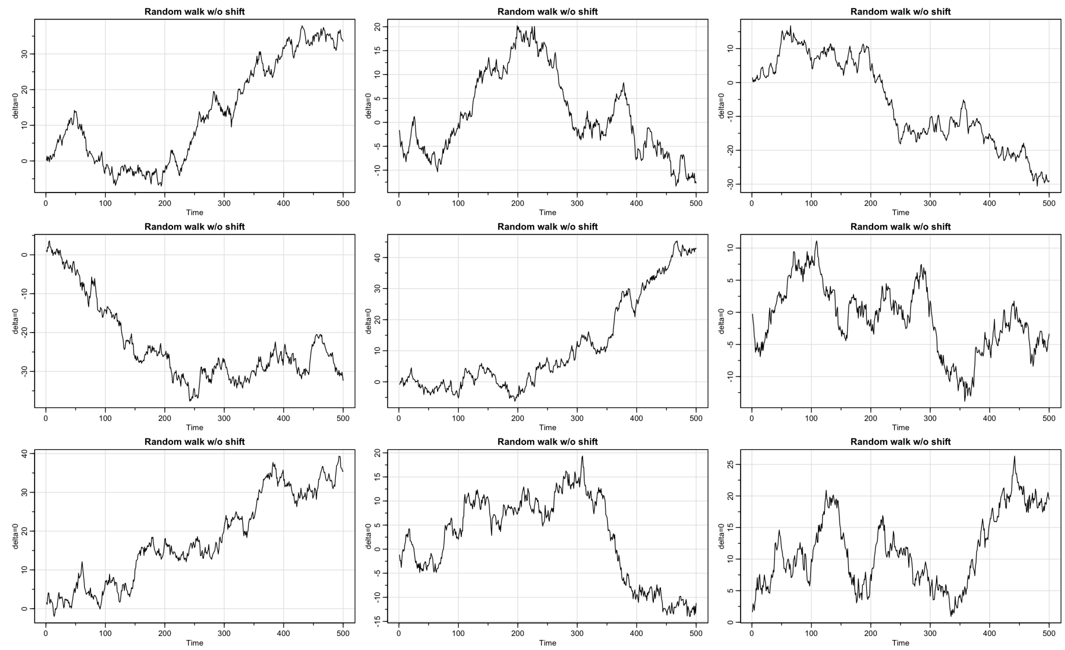
Solution: (1.1) The figures below are for (1.1) (a) (b) (c), respectively.



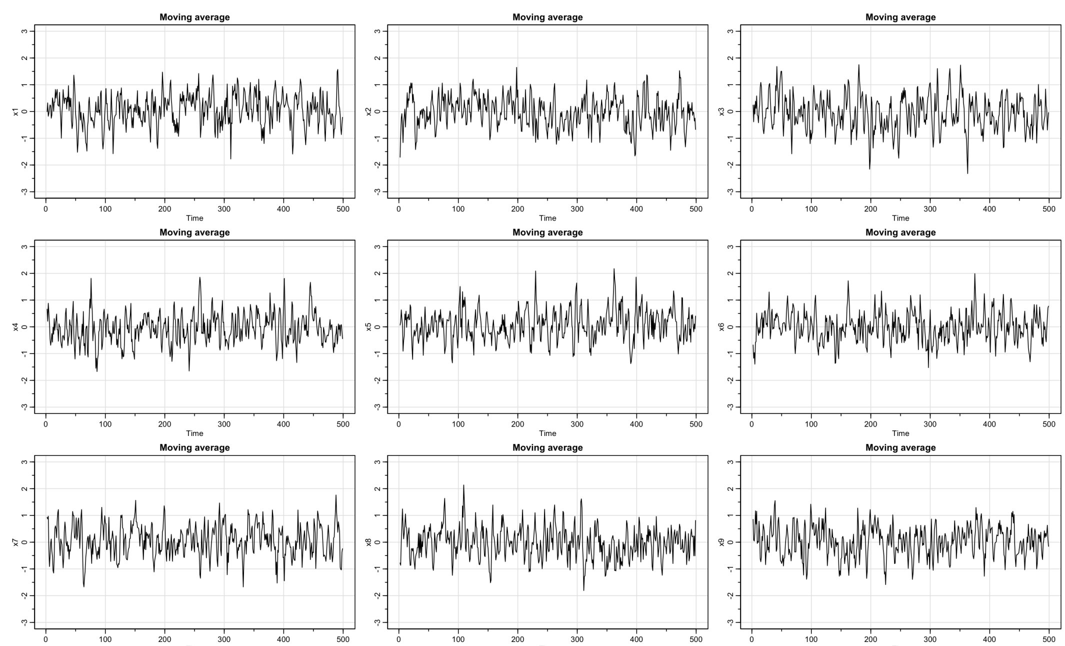
(d) Seasonal adjustment is a statistical method to measure and reduce/remove the influences of predictable seasonal patterns that changes from month to month. Over the course of a year, measures of the labor market activity reveal fluctuations due to seasonal events including changes in weather, harvests, major holidays, and school schedules. Because these seasonal events follow a more or less regular pattern each year, their influence on statistical trends can be eliminated by seasonally adjusting the statistics from month to month. These seasonal adjustments make it easier to observe the cyclical, underlying trend, and other nonseasonal movements in the series.

(e) First conclusion is that having negative coefficients in the auto-regression as well as signal plus noise will lead a seasonal (cyclic, periodic, sinusoidal) pattern in the data. Also, by (c) we see that these seasonal patterns can be reduced/removed using log(data) and simply non-linearity can be transform into a simple linear pattern (in this case it looks like a random walk with a drift).

(1.3(a)) The plots below are for (1.3) (a).

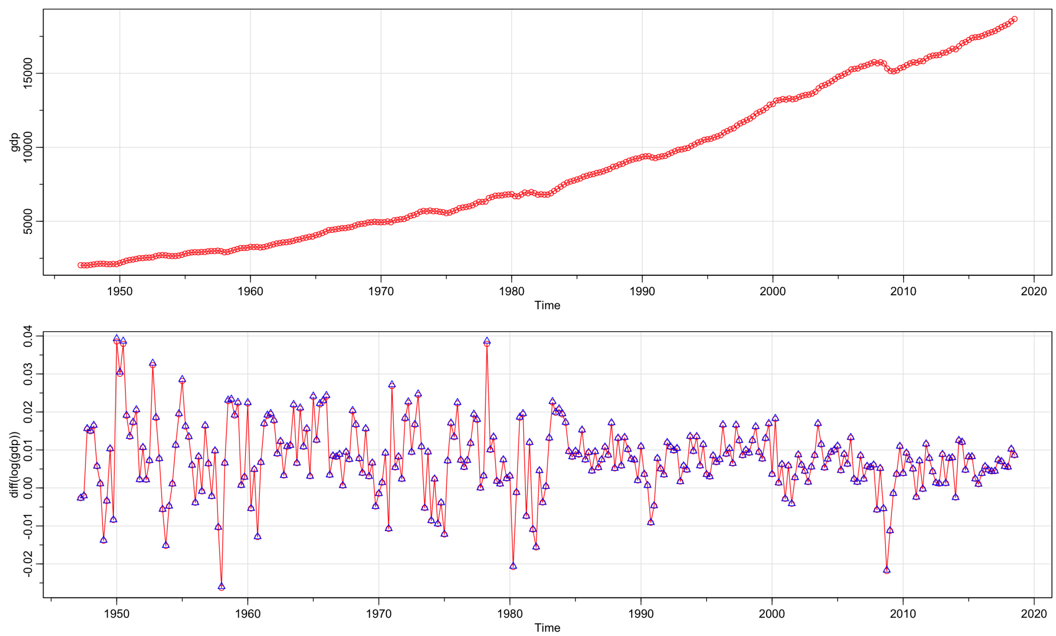


1.3(b) This plots below are for (1.3) (b).



(c) Taking averages of white noises makes the plots much smoother which is oscillating around zero. Also, increasing the number of elements in the averages decreases the range of plot (still oscillating around zero) and makes the plots even smoother.

(1.4) The figures below are for (1.4) (a) and (b), respectively.



(a) The first plot pretty much looks like a random walk with drift in which follows a line of higher slope that random walk without a drift.

(b) The first method finds the actual growth rate by finding the previous time series element to determine the growth, whereas the log-diff method produces an approximation to growth rate.

(c) A random walk with drift behaves exactly like the plot of data, whereas an autoregression with negative coefficients and a signal plus noise describes the plots of methods used. Therefore, random walk with drift describes the behaviour of data itself better.

Problem 2: **In the simple regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ one usually deals with pairs of observations $(x_i, y_i),\ i = 1, 2, ..., n$. Show that:**

(a) $\sum\limits_{i=1}^{n}(x_i - \bar{x}) = 0$, and $\sum\limits_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum\limits_{i=1}^{n}(x_i - \bar{x})y_i$

(b) $S_x^{-2}\sum\limits_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \sum\limits_{i=1}^{n} c_i y_i$, where $c_i = \dfrac{x_i - \bar{x}}{S_x^2},\ S_x^2 = \sum\limits_{i=1}^{n}(x_i - \bar{x})^2$.

Solution: (a) (i)

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \bar{x}) &= \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x} = \sum_{i=1}^{n} x_i - n\bar{x} \\ &= \sum_{i=1}^{n} x_i - \cancel{n}\frac{1}{\cancel{n}}\sum_{i=1}^{n} x_i = 0\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^{n}(x_i - \bar{x})y_i - \sum_{i=1}^{n}(x_i - \bar{x})\bar{y} \\ &= \sum_{i=1}^{n}(x_i - \bar{x})y_i - \bar{y}\cancelto{0\ by\ (i)}{\sum_{i=1}^{n}(x_i - \bar{x})} \\ &= \sum_{i=1}^{n}(x_i - \bar{x})y_i\end{aligned}$$

(b)

$$\begin{aligned}S_x^{-2}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) &= \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sum\limits_{i=1}^{n}(x_i - \bar{x})y_i}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} \\ &= \frac{(x_1 - \bar{x})y_1}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} + \frac{(x_2 - \bar{x})y_2}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} + ... + \frac{(x_n - \bar{x})y_n}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2} \\ &= \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{\sum\limits_{i=1}^{n}(x_i - \bar{x})^2}y_i = \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{S_x^2}y_i = \sum_{i=1}^{n} c_i y_i\end{aligned}$$

Problem 3: **(Bonus) For the simple linear regression model,** $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$,

(a) Derive the equations for minimizing the residual sum of squares (RSS):

$$Q(\beta_0, \beta_1) = \sum (y_i - \beta_0 - \beta_1 x_i)^2,$$

and give the formulae for the least squares estimates (LSE) $\hat{\beta}_0, \hat{\beta}_1$ of the regression coefficients.

(b) Show that $\hat{\beta}_1 = \sum_{i=1}^{n} c_i y_i$, with c_i's as above. If $\varepsilon_i \sim N(0, \sigma^2)$, what is the distribution of $\hat{\beta}_1$?

(c) Let $e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i, \ i = 1, 2, ..., n$ be the (estimated regression) residuals. Show that

$$\sum_{i=1}^{n} e_i = 0, \text{ and } \sum_{i=1}^{n} e_i x_i = 0.$$

What is the (geometrical) interpretation of the above identities?

Solution:

BONUS PROBLEM: $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

a) $$RSS = \sum_{i=1}^{n} \hat{e}_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\hat{\beta}_0 = \underset{\beta_0}{\arg\min}(RSS) \quad \text{where} \quad \left.\frac{\partial RSS}{\partial \beta_0}\right|_{\hat{\beta}_0} = 0 \quad \text{(minimizer)}$$

$$\hat{\beta}_1 = \underset{\beta_1}{\arg\min}(RSS) \quad \text{where} \quad \left.\frac{\partial RSS}{\partial \beta_1}\right|_{\hat{\beta}_1} = 0 \quad \text{(minimizer)}$$

$$\frac{\partial RSS}{\partial \beta_0} = -\cancel{2}\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0 \Rightarrow \boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}} \ (*)$$

$$\Rightarrow (i) \quad \underbrace{\sum_{i=1}^{n} \beta_0}_{\beta_0 \sum_{i=1}^{n} 1 = \beta_0 . n} + \beta_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \Rightarrow \boxed{\beta_0 = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i\right)} = \bar{y} - \beta_1 \bar{x}$$

$$\frac{\partial RSS}{\partial \beta_1} = -\cancel{2}\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\Rightarrow (ii) \quad \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

Multiplying (i) by $-\frac{1}{n}\sum_{i=1}^{n} x_i$ and summing up with (ii) yields;

$$\beta_1 \underbrace{\left(\sum_{i=1}^{n} x_i^2 - \underbrace{\frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}_{= n\bar{x}^2 = \sum_{i=1}^{n} \bar{x}^2}\right)}_{= \sum_{i=1}^{n}(x_i^2 - \bar{x}^2)} = \underbrace{\sum_{i=1}^{n} x_i y_i - \underbrace{\overbrace{\frac{1}{n}\sum_{i=1}^{n} x_i}^{\bar{x}} \sum_{i=1}^{n} y_i}_{-\sum_{i=1}^{n} \bar{x} y_i}}_{= \sum_{i=1}^{n} (x_i - \bar{x}) y_i}$$

$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{since} \quad \sum_{i=1}^{n} 2x_i\bar{x} = 2\bar{x}\underbrace{\sum_{i=1}^{n} x_i}_{n\bar{x}} = 2n\bar{x}^2 = \sum_{i=1}^{n} 2\bar{x}^2$$

Therefore;

$$\beta_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

which also proves the first part of (b).

$$\boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}) y_i}{\sum_{i=1}^{n} (x_i - \bar{x})^2}} \ (*)(*)$$

(b) $$\hat{B}_1 = \underbrace{\frac{\sum_{i=1}^{n} (x_i-\bar{x})y_i}{\sum_{i=1}^{n} (x_i-\bar{x})^2}}_{\text{by (a)}} = \sum_{i=1}^{n} c_i y_i \quad \text{where} \quad c_i = \frac{x_i-\bar{x}}{S_x^2}$$

$$\text{and} \quad S_x^2 = \sum_{i=1}^{n} (x_i-\bar{x})^2$$

$$E[\hat{B}_1 \mid X] = E\Big[\sum_{i=1}^{n} c_i y_i \mid X = x_i\Big] = \sum_{i=1}^{n} c_i \, E[y_i \mid X = x_i]$$

$$= \sum_{i=1}^{n} c_i(B_0 + B_1 x_i) = B_0 \sum_{i=1}^{n} \frac{(x_i - x)}{S_x^2} + B_1 \sum_{i=1}^{n} \frac{(x_i-\bar{x})}{S_x^2} . x_i$$

$$= B_0 \frac{1}{S_x^2} \underbrace{\cancel{\sum_{i=1}^{n} (x_i - x)}}_{0} + B_1 \underbrace{\cancel{\frac{1}{S_x^2}} \cancel{\sum_{i=1}^{n} (x_i - \bar{x}) x_i}}_{=1} = S_x^2$$

$$= B_1 \quad \text{(unbiased estimate)} \qquad \text{since} \sum_{i=1}^{n} (x_i-\bar{x})x_i = \sum_{i=1}^{n} x_i^2 - \bar{x}^2 = S_x^2$$

$$\operatorname{var}(\hat{B}_1 \mid X) = \operatorname{var}\Big[\sum_{i=1}^{n} c_i y_i \mid X = x_i\Big] = \sum_{i=1}^{n} c_i^2 \operatorname{var}(y_i \mid X = x_i)$$

$$= \sum_{i=1}^{n} c_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} \left(\frac{(x_i-\bar{x})}{S_x^2}\right)^2 = \sigma^2 \frac{\cancel{\sum_{i=1}^{n} (x_i-\bar{x})^2}}{(S_x^2)^{\cancel{2}}} = \frac{\sigma^2}{S_x^2}$$

Finally, since $e_i \mid X$ are normally distributed and $y_i = B_0 + B_1 x_i + e_i$; thus, $Y_i \mid X$ is normally distributed. Since $\hat{B}_1 \mid X$ is a linear combination of the y_i's, $\hat{B}_1 \mid X$ is normally distributed. $\Rightarrow$ $\boxed{\hat{B}_1 \sim N\left(B_1, \frac{\sigma^2}{S_x^2}\right)}$

(c) $$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{B}_0 - \hat{B}_1 x_i) = \sum_{i=1}^{n} (y_i - (\bar{y} - \hat{B}_1 \bar{x}) - \hat{B}_1 x_i$$

$$= \sum_{i=1}^{n} [(y_i - \bar{y}) - \hat{B}_1 (x_i - \bar{x})]$$

$$= \underbrace{\cancel{\sum_{i=1}^{n} (y_i-\bar{y})}}_{0} - \hat{B}_1 \underbrace{\cancel{\sum_{i=1}^{n} (x_i-\bar{x})}}_{0} = 0$$

$$\sum_{i=1}^{n} e_i x_i = \sum_{i=1}^{n} x_i y_i - x_i \bar{y} + \hat{B}_1 x_i \bar{x} - \hat{B}_1 x_i^2$$

$$= \sum_{i=1}^{n} x_i y_i - \bar{x} y_i - \hat{B}_1 (x_i - \bar{x}) x_i$$

$$\hat{B}_1 . S_x^2 = \sum_{i=1}^{n} (x_i - \bar{x}) y_i$$

$$= \sum_{i=1}^{n} (x_i - \bar{x}) y_i - \sum_{i=1}^{n} (x_i - \bar{x}) y_i = 0$$

$$\sum_{i=1}^{n} x_i \bar{y} = \sum_{i=1}^{n} x_i \Big(\frac{1}{n}\sum_{i=1}^{n} y_i\Big) = \Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big)\sum_{i=1}^{n} y_i = \bar{x} \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \bar{x} y_i$$

$0 = \sum_{i=1}^{n} e_i x_i = \vec{x}^T \vec{e} = \langle \vec{e}, \vec{x} \rangle = 0$ means $\vec{e} = \begin{bmatrix} e_1 \\ e_n \end{bmatrix} \perp \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\boxed{\text{orthogonal}}$

Also, $\vec{e}$ is the projection of $(y - \bar{y})$ on $(x - \bar{x})$ vector