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STAT 626 HW3

Question 1: Stationary requires regularity in the mean and autocorrelation function so that (at least) they could be estimated by averaging. The assumption of stationary is critical, and it allows the use of averaging to estimate the population mean and covariance functions.

Question 2: $x_t = \beta_0 + \beta_1 t + w_t$

(a) i) $\mu_{x_t} = E[\beta_0 + \beta_1 t + w_t] = \beta_0 + \beta_1 t$ (depends on time), therefore it is not stationary.

(b) i) $y_t = x_t - x_{t-1} = \cancel{\beta_0} + \cancel{\beta_1 t} + w_t - \cancel{\beta_0} - \beta_1(t-1) - w_{t-1} = \beta_1 + w_t - w_{t-1}$

$\mu_{y_t} = E[\beta_1 + w_t - w_{t-1}] = \beta_1$ (constant) ✓

ii) $\gamma(s,t) = cov(\beta_1 + w_s - w_{s-1}, \beta_1 + w_t - w_{t-1})$

$= E[((\beta_1 + w_s - w_{s-1}) - \beta_1)((\beta_1 + w_t - w_{t-1}) - \beta_1)]$

$= E[(w_s - w_{s-1}), (w_t - w_{t-1})] = \begin{cases} 2\sigma_w^2 & , s-t=0 \\ -\sigma_w^2 & , |s-t|=1 \\ 0 & , |s-t|>1 \end{cases}$

$(= E[(w_{t+h} - w_{t+h-1}, w_t - w_{t-1})]$

Therefore, y_t is stationary.

(c) $\mu_{v_t} = E\left[\frac{1}{3}(x_{t-1} + x_t + x_{t+1})\right] = \frac{1}{3}E\left[3\beta_0 + \beta_1((t-1)+t+(t+1)) + w_{t-1} + w_t + w_{t+1}\right]$

$= \frac{1}{3}(3\beta_0 + 3\beta_1 t) + \underbrace{E[w_{t-1}]}_{0} + \underbrace{E[w_t]}_{0} + \underbrace{E[w_{t+1}]}_{0} = \beta_0 + \beta_1 t$

Question 3: $\gamma_x(t,t) = cov\left(\frac{1}{4}(w_{t-1} + 2w_t + w_{t+1}), \frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})\right)$

$= \frac{1}{16}\left[cov(w_{t-1}^2) + 4cov(w_t^2) + cov(w_{t+1}^2)\right] = \frac{6}{16}\sigma_w^2$

$\gamma_x(s,t) = \gamma(t+h,t) = \begin{cases} \frac{6}{16}\sigma_w^2 & , h=0 \\ \frac{4}{16}\sigma_w^2 & , |h|=1 \\ \frac{1}{16}\sigma_w^2 & , |h|=2 \\ 0 & , |h|>2 \end{cases}$

Let $\boxed{s = t+h}$

NOTE:

$\sqrt{\gamma(s,s)\gamma(t,t)}$

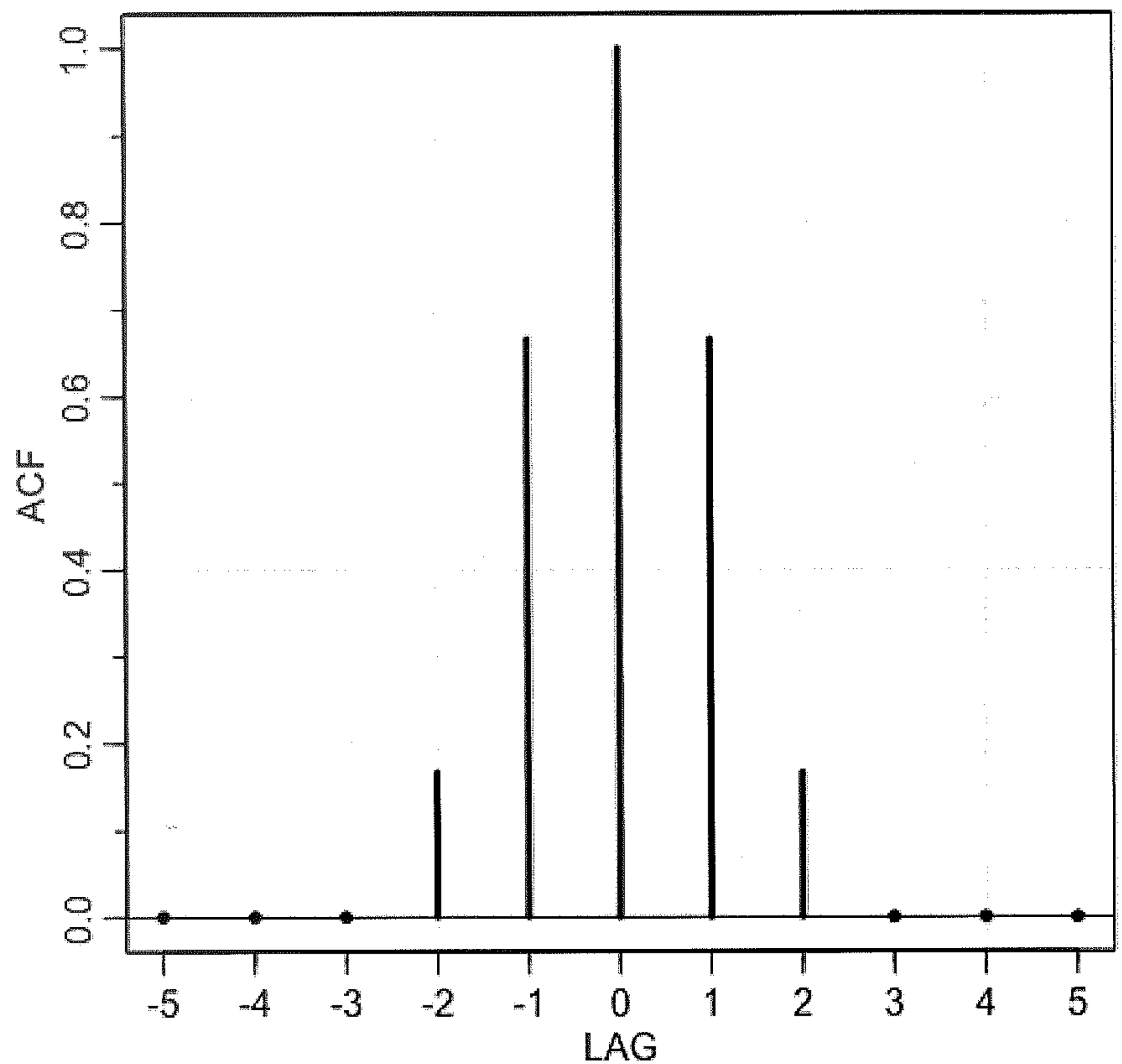
$= \sqrt{\gamma(0)\gamma(0)}$

$= \frac{6}{16}\sigma_w^2$

$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$

$\rho_x(s,t) = \rho(t+h,t) = \begin{cases} 1 & , h=0 \\ \frac{2}{3} & , |h|=1 \\ \frac{1}{6} & , |h|=2 \\ 0 & , |h|>2 \end{cases}$

Use code in $\boxed{\text{Figure 2.1}}$ I attached the figure of ACF.


ACF
0.0
0.2
0.4
0.6
0.8
1.0
-5
-4
-3
-2
-1
0
1
2
3
4
5
LAG

Question 4: $X_t = \phi X_{t-1} + W_t$ $\quad W_t \sim wn(0,1)$ $\quad$ | X_t stationary, $cov(X_{t-1}, W_t) = 0$

(a) $M_x = E[\phi X_{t-1} + W_t] = \phi E[X_{t-1}] = \phi M_x$

$\Rightarrow M_x(1-\phi) = 0 \Rightarrow M_x = 0$ since $\boxed{\phi \neq 1}$ or $|\phi| < 1$

(b) $\gamma_x(0) = \gamma(t,t) = cov(\phi X_{t-1} + W_t, \phi X_{t-1} + W_t) = \phi^2 \overbrace{cov(X_{t-1}, X_{t-1})}^{\gamma_x(0)} + \overbrace{cov(W_t, W_t)}^{var}$

$\gamma_x(0) = \phi^2 \gamma_x(0) + 1$

$\Rightarrow \boxed{\gamma_x(0) = var(X_t) = 1/(1-\phi^2)}$ $\quad$ $|\phi| < 1$ since ($0 \leq var(X_t) < \infty$ by stationary condition)

(c) Since X_t is stationary,

$0 < var(X_t) < \infty \Rightarrow 0 < \frac{1}{(1-\phi^2)} < \infty \Rightarrow \boxed{|\phi| < 1}$ makes sense only

(d) $$\rho_x(1) = \frac{\gamma_x(t+1,t)}{\sqrt{\gamma_x(t+1,t+1)\gamma_x(t,t)}} = \frac{cov(\phi X_t + W_{t+1}, \phi X_{t-1} + W_t)}{\gamma_x(0)}$$

$$= \frac{\phi^2 \gamma_x(1)}{1/(1-\phi)^2} = \phi^2(1-\phi^2)\gamma_x(1)$$

Question 5: $X_t = \delta + X_{t-1} + W_t$, $X_0 = 0$, $W_t \sim N(0, \sigma_w^2)$

(a) $X_1 = \delta + \cancel{X_0}^{0} + W_1 = \delta + W_1$

$X_2 = \delta + X_1 + W_2 = \delta + (\delta + W_1) + W_2 = 2\delta + (W_1 + W_2)$

$X_3 = \delta + X_2 + W_3 = \delta + (2\delta + (W_1 + W_2)) + W_3 = 3\delta + (W_1 + W_2 + W_3)$

$\vdots$

$X_t = \delta + X_{t-1} + W_t = \delta + ((t-1)\delta + (W_1 + \dots + W_{t-1})) + W_t = \delta t + \sum_{k=1}^{t} W_k$

(b) $M_{X_t} = E[\delta t + \sum_{k=1}^{t} W_k] = \delta t + \sum_{k=1}^{t} E[\cancel{W_k}]^{0} = \delta t$ (clearly not stationary)

$\gamma_x(s,t) = cov(X_{t+h}, X_t) = cov(\delta(t+h) + \sum_{k=1}^{t+h} W_k, \delta t + \sum_{k=1}^{t} W_k)$

Let $\boxed{s = t+h}$ $= E[(\cancel{\delta(t+h)} + \sum_{k=1}^{t+h} W_k - \underbrace{\cancel{\delta(t+h)}}_{M_{X_{t+h}}}), (\cancel{\delta t} + \sum_{k=1}^{t} W_k - \underbrace{\cancel{\delta t}}_{M_{X_t}})]$

$= E[(W_1 + \dots + W_t) + W_{t+1} + \dots + W_{t+h})(W_1 + \dots + W_t)]$

$= E[W_1^2] + \dots + E[W_t^2] = \min\{s,t\}\sigma_w^2$

(c) Not stationary clearly.

(d) $\lim_{t\to\infty} \rho_x(-1) = \lim_{t\to\infty} \sqrt{\frac{t-1}{t}} = \lim_{t\to\infty} \sqrt{1 - \frac{1}{t}} = \sqrt{1 - \lim_{t\to\infty} \frac{1}{t}}^{\to 0} = \sqrt{1} = 1$

meaning as t increases the relation between successive terms $(t-1, t)$ becomes linear.

(e) Let $y_t = X_t - X_{t-1} = \delta t + \sum_{k=1}^{t} W_k - \delta(t-1) \sum_{k=1}^{t-1} W_k = \delta + W_t$

i) $E[y_t] = E[\delta + w_t] = \delta + E[\cancel{w_t}]_{0} = \delta$ (independent of time) ✓

ii) $\gamma_y(h) = \gamma(y_{t+h}, y_t) = cov(\delta + w_{t+h}, \delta + w_t) = \underbrace{cov(\delta, \delta)}_{var(\delta)\,0} + \cancel{cov(\delta + w_t)}_{0} + \cancel{cov(w_{t+h}, \delta)}_{0} + cov(w_{t+h}, w_t)$

$$= \begin{cases} \delta_w^2, & h = 0 \\ 0, & \text{otherwise} \end{cases} \quad \left(\begin{array}{c}\text{independent} \\ \text{of time, function} \\ \text{of lag } (h)\end{array}\right) \Rightarrow y_t \text{ is stationary.}$$

2.6

Question 6: Stationary time series have constant mean and time independent autocovariances with a finite variance. In figure 1.2. it is clear that data has an upward (increasing) trend with sharp anomalies (increase & decreases) (locally in time; therefore, the data clearly doesn't have a constant mean value function and it depends on the time. Therefore, the global temperature data is not-stationary. Also, it can be seen that the variance of the data $var(x_t) = cov(x_t, x_t)$ is changing over time in the data which; clearly, implies the non-stationary.

2.7

Question 7: $X_1 = U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)$

i) $E[x_t] = E[U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)]$

$= \cancel{E[U_1]}_{0} \sin(2\pi w_0 t) + \cancel{E[U_2]}_{0} \cos(2\pi w_0 t) = 0$ (time independent) ✓

ii) $\gamma(h) = \gamma(x_{t+h}, x_t) = cov\left(\begin{array}{l} U_1 \sin(2\pi w_0(t+h)) \\ +U_2 \cos(2\pi w_0(t+h)) \end{array}, \begin{array}{l} U_1 \sin(2\pi w_0(t)) \\ +U_2 \cos(2\pi w_0(t)) \end{array}\right)$

$= [\sin(2\pi w_0(t+h)) \cos(2\pi w_0 t) + \sin(2\pi w_0 t) \cos(2\pi w_0(t+h))]\, \cancel{cov(U_1, U_2)}_{0}$

$+ \underbrace{\sin(2\pi w_0(t+h))}_{a} . \underbrace{\sin(2\pi w_0 t)}_{b} \underbrace{cov(U_1, U_1)}_{var(U_1) = \sigma^2} + \underbrace{\cos(2\pi w_0(t+h))}_{a} \underbrace{\cos(2\pi w_0 t)}_{b} \underbrace{cov(U_2, U_2)}_{var(U_2) = \sigma^2}$

$= \sigma^2 (\sin a . \sin b + \cos a . \cos b) = \sigma^2 (\cos(a-b))$ (time independent function of lag h) ✓

$= \sigma^2 (\cos(2\pi w_0(t+h) - 2\pi w_0 t)) = \sigma^2 \cos(2\pi w_0 h)$

Therefore, X_t is "weakly" stationary.

$E[XY] = E[X]E[Y] + cov(XY)$

2.9

Question 8: $X_t = W_t W_{t-1}$ $\boxed{\text{NOTE:}}$ $cov(XY) = E[XY] - E[X]E[Y]$

i) $E[X_t] = E[W_t W_{t-1}] = E[\cancel{W_t}]_{0} E[\cancel{W_{t-1}}]_{0} + cov(\cancel{W_t, W_{t-1}})_{0} = 0$

ii) $\gamma(h) = \gamma(X_{t+h}, X_t) = cov(W_{t+h} W_{t+h-1}, W_t W_{t-1})$

$= E[W_{t+h} W_{t+h-1} W_t W_{t-1}] - E[\cancel{W_{t+h} W_{t+h-1}}]_{0} E[\cancel{W_t W_{t-1}}]_{0}$ by (i)

$$= \begin{cases} \sigma_w^4, & h = 0 \quad \text{since } E[W_t^2 W_{t-1}^2] = E[W_t^2] E[W_{t-1}^2] = \sigma_w^4 \\ 0, & |h| > 0 \quad \left(\text{because } E[W_t^2 W_{t-1}^2] = E[W_t^2] E[W_{t-1}^2] + cov(\cancel{W_t^2, W_{t-1}^2})_{0}\right) \end{cases}$$

2.10

Question 9: $x_t = \mu + w_t + \theta w_{t-1}$ where $w_t \sim wn(0, \sigma_w^2)$.

(a) $E[x_t] = E[\mu + w_t + \theta w_{t-1}] = \mu + E[w_t]^{\,0} + \theta E[w_{t-1}]^{\,0} = \mu$

(b) $\gamma_x(h) = \gamma(x_{t+h}, x_t) = cov(\mu + w_{t+h} + \theta w_{t+h-1}, \mu + w_t + \theta w_{t-1})$

$$= \underbrace{\cancel{cov(\mu,\mu)}}_{var(\mu)=0} + \cancel{cov(\mu, w_t + \theta w_{t-1})} + \cancel{cov(w_{t+h} + \theta w_{t+h-1}, \mu)} + cov(w_{t+h} + \theta w_{t+h-1}, w_t + \theta w_{t-1})$$

$$= cov(w_{t+h}, w_t) + \theta cov(w_{t+h}, w_{t-1}) + \theta cov(w_{t+h-1}, w_t) + \theta^2 cov(w_{t+h-1}, w_{t-1})$$

$$\left(\begin{array}{lll} h=0 & \Rightarrow & var(w_t^2) + \theta^2 var(w_{t-1}^2) = (1+\theta^2)\sigma_w^2 \\ |h|=1 & \Rightarrow & \theta cov(w_{t\pm1}, w_{t\pm1}) = \theta\sigma_w^2 \\ |h|>1 & \Rightarrow & 0 \quad \text{(no common terms)} \end{array}\right) = \begin{cases} (1+\theta^2)\sigma_w^2 & , h=0 \\ \theta\sigma_w^2 & , |h|=1 \\ 0 & , |h|>1 \end{cases}$$

(c) Since $E[x_t]$ is time independent and $\gamma_x(h)$ is a time independent function of lag (h), changing (parameter) $\theta \in \mathbb{R}$ doesn't change the fact that $\gamma_x(h)$ is time independent, therefore x_t is stationary for $\forall \theta \in \mathbb{R}$.

(d) $\theta = 0 \Rightarrow \gamma_x(h) = \begin{cases} \sigma_w^2 & , h=0 \\ 0 & , \text{otherwise} \end{cases}$

$\theta = +1 \Rightarrow \gamma_x(h) = \begin{cases} 2\sigma_w^2 & , h=0 \\ \sigma_w^2 & , |h|=1 \\ 0 & , |h|>1 \end{cases}$

$\theta = -1 \Rightarrow \gamma_x(h) = \begin{cases} 2\sigma_w^2 & , h=0 \\ -\sigma_w^2 & , |h|=1 \\ 0 & , |h|>1 \end{cases}$

Then,

(i) $\theta = 0 \Rightarrow var(\bar{x}) = \frac{1}{n}\sum_{h=-n}^{n}\left(1 - \frac{|h|}{n}\right)\gamma_x(h) = \frac{1}{n}\underbrace{(1-0)\sigma_w^2}_{h=0} = \frac{\sigma_w^2}{n}$

(ii) $\theta = +1 \Rightarrow var(\bar{x}) = \frac{1}{n}\left(\underbrace{2\left(1-\frac{1}{n}\right)\sigma_w^2}_{h=\pm1} + \underbrace{(1-0)2\sigma_w^2}_{h=0}\right) = \frac{2\sigma_w^2}{n}\left[\left(1-\frac{1}{n}\right)+1\right]$

(iii) $\theta = -1 \Rightarrow var(\bar{x}) = \frac{1}{n}\left(\underbrace{2\left(1-\frac{1}{n}\right)(-\sigma_w^2)}_{h=\pm1} + \underbrace{(1-0)2\sigma_w^2}_{h=0}\right) = \frac{2\sigma_w^2}{n}\left[1 - \left(1-\frac{1}{n}\right)\right]$

$$var(\bar{x}) = \begin{cases} \frac{\sigma_w^2}{n} & \text{if } \theta = 0 \\ \frac{2\sigma_w^2}{n}\left[\left(\frac{n-1}{n}\right)+1\right] & \text{if } \theta = +1 \\ \frac{2\sigma_w^2}{n}\left[1 - \left(\frac{n-1}{n}\right)\right] & \text{if } \theta = -1 \end{cases}$$

(e) As $n \to \infty$ $\quad \frac{(n-1)}{n} \approx 1$, then

$$
\operatorname{var}(\bar{x}) = \begin{cases} \dfrac{\sigma_w^2}{n}, & \text{when } \theta = 0 \\ \dfrac{2\sigma_w^2}{n} \cdot 2 = \dfrac{4\sigma_w^2}{n}, & \text{when } \theta = +1 \\ \dfrac{2\sigma_w^2}{n} \cdot 0 = 0, & \text{when } \theta = -1 \end{cases}
$$

Similarly, for $\theta = 2$, $\quad \operatorname{var}(\bar{x}) = \frac{1}{n}\Big(\overbrace{2\underbrace{(1-\tfrac{1}{n})}_{\approx 1} 2\sigma_w^2}^{h=\pm 1} + \overbrace{(1-0)\, 5\sigma_w^2}^{h=0}\Big) = \frac{9\sigma_w^2}{n}$

Therefore, $\quad \operatorname{var}(\bar{x}) = (1+\theta)^2 \frac{\sigma_w^2}{n} \quad$ since $\quad (1+\theta^2) + 2\theta = 1 + 2\theta + \theta^2 = (1+\theta)^2$ for $\forall \theta \in \mathbb{R}$.

BONUS PROBLEM: $\quad w_t \sim \text{i.i.d } N(0, \sigma_w^2)$, $\quad X_t = \underbrace{w_{t-1} w_{t-2}}_{Y_t} \underbrace{(w_{t-1} + w_t + t)}_{Z_t}$

i) $E[X_t] = E[Y_t Z_t] = E[w_{t-1} w_{t-2} \cdot w_{t-1}] + E[w_{t-1} w_{t-2} w_t] + tE[w_{t-1}$

$= E[w_{t-1} w_{t-2}]\left(E[w_{t-1} + w_t + t]\right) = E[Y_t]E[Z_t]$

$= \underbrace{E[w_{t-1}]}_{0} \underbrace{E[w_{t-2}]}_{0} \left(\underbrace{E[w_{t-1}]}_{0} + \underbrace{E[w_t]}_{0} + t\right) = 0$

Notice, since $E[Y_t Z_t] = E[Y_t]E[Z_t] \Rightarrow \operatorname{cov}(Y_t, Z_t) = 0$
(with $E[Y_t] = 0$, $E[Z_t] = t$)

ii) $\gamma_x(h) = \gamma(x_{t+h}, x_t) = \operatorname{cov}(x_{t+h}, x_t)$

$= \operatorname{cov}(y_{t+h} z_{t+h}, y_t z_t)$

$\boxed{\text{At least}}$ for $h=0$

$\gamma_x(0) = \operatorname{cov}(y_t z_t, y_t z_t) = \operatorname{var}(y_t z_t) = E[y_t^2 z_t^2] - E[\underbrace{\cancel{y_t z_t}}_{0}]^2$

$= E[w_{t-1}^2 w_{t-2}^2 (w_{t-1}^2 + w_t^2 + t^2 + 2w_{t-1} t + 2w_t t + 2w_{t-1} w_t)]$

$= E[w_{t-1}^4 w_{t-2}^2] + E[w_{t-1}^2 w_{t-2}^2 w_t^2] + E[w_{t-1}^2 w_{t-2}^2 t^2]$

$+ E[2w_{t-1}^2 w_{t-2}^2 (2w_{t-1} + 2w_t) t] + E[w_{t-1}^2 w_{t-2}^2 w_t]$

Notice, $\quad E[w_{t-1}^2 w_{t-2}^2 t^2] = E[(w_{t-1} w_{t-2})^2] E[t^2] = \operatorname{Var}(w_{t-1} w_{t-2}) t^2$

and it is time dependent. Therefore, $\quad X_t$ is not stationary. //