Numerical ML Estimates

Math 445

Spring 2016

Overview

- It's not always possible (or sometimes its just overly tedious) to find closed for solutions for ML estimators
- In this case, we use numerical methods to find the parameter estimates that maximize the likelihood

Gamma Example

In class we derived the likelihood function for a random sample drawn from a Gamma(α, λ) population distribution.

$$L(\alpha, \lambda | X_1, \dots, X_n) = \left(\frac{\lambda^\alpha}{\Gamma(\alpha)} \right)^n \left(\prod_{i=1}^{n} X_i^{\alpha-1} \right) \exp -\lambda \sum_{i=1}^{n} X_i$$

Then we found the log-likelihood

$$\ell(\alpha, \lambda) = -n \log(\Gamma(\alpha)) + n\alpha \log(\lambda) + (\alpha - 1) \textstyle\sum_{i=1}^{n} \log(X_i) - \lambda \sum_{i=1}^{n} X_i$$

Gamma Example

Followed by the λ-partial

$$\widehat{\lambda} = \frac{\alpha}{\bar{X}_n}$$

and (plugging this in) the α-partial we found:

$$\log(\hat{\alpha}) - \log \bar{X}_n - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \frac{1}{n}\sum_{i=1}^{n} \log X_i = 0$$

which is a non-linear equation with no closed-form solution...

Numerical Methods in R

- We can use the `optim` function to maximize a function that has multiple parameters at once
- Check `?optim` for more info
- To do this, we need to write a log-likelihood function in R

Gamma Log-Likelihood in R

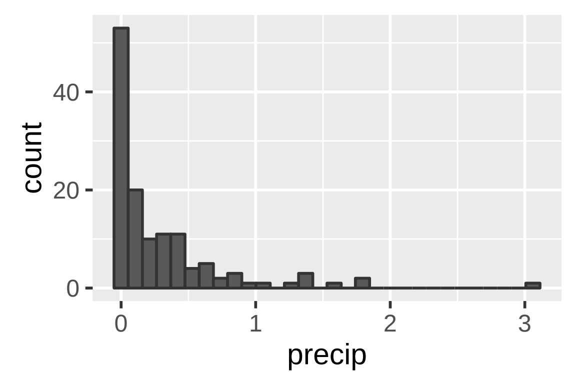
Key arguments:

- A vector of parameter values
- The observed data

```
gamma_loglik <- function(theta, x) {
  n <- length(x)
  alpha <- theta[1]
  lambda <- theta[2]
  RES <- n * (alpha * log(lambda) - log(gamma(alpha)) +
              (alpha - 1) * mean(log(x)) - lambda * mean(x))
  return(RES)
}
```

Data: Boston Storms

- `BostonStorms2013.csv` contains daily rainfall for each day that it rained in Boston in 2013
- Gamma distribution seems reasonable to precipitation



Finding the ML Estimates

- We need to provide starting values to `optim`.
- MoM estimates are reasonable

```
n <- length(storm$precip)
xbar <- mean(storm$precip)
s2 <- var(storm$precip)
sigma2_hat <- s2 * (n - 1)/n

lambda_mom <- xbar/sigma2_hat
lambda_mom
```

```
## [1] 1.459674
```

```
alpha_mom <- xbar^2/sigma2_hat
alpha_mom
```

```
## [1] 0.4325839
```

Finding the ML Estimates

```
theta <- optim(par = c(alpha_mom, lambda_mom), fn = gamma_loglik,
               control = list(fnscale = -1), x = storm$precip)
theta
```

```
## $par
## [1] 0.5919581 1.9973441
##
## $value
## [1] 42.54922
##
## $counts
## function gradient
##      101       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

An Alternative Gamma Log-Likelihood

- Instead of writing our own log-likelihood functions, we can often take advantage of the distributions that are already available in R
- To see a list, type `?Distributions`

An Alternative Gamma Log-Likelihood

```
gamma_loglik_alt <- function(theta, x) {
  sum(dgamma(x, shape = theta[1], rate = theta[2], log = TRUE))
}

theta2 <- optim(par = c(alpha_mom, lambda_mom), fn = gamma_loglik_alt,
                control = list(fnscale = -1), x = storm$precip)
theta2
```

```
## $par
## [1] 0.5919581 1.9973441
##
## $value
## [1] 42.54922
##
## $counts
## function gradient
##      101       NA
##
## $convergence
```