Homework 5 Solutions, Math 445, Spring 2017

27 Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

(a) $\text{Bias}(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = \left(\frac{n-1}{n}\right)\sigma^2 - \sigma^2 = \sigma^2\left(\frac{-1}{n}\right)$

Note that
$$
\begin{aligned}
E(\hat{\sigma}^2) &= E\left[\frac{1}{n}\sum(X_i - \bar{X})^2\right] \\
&= E\left[\frac{1}{n}\sum\left(X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right)\right] \\
&= E\left[\frac{1}{n}\left(\sum X_i^2 - n\bar{X}^2\right)\right] \\
&= E\left(\frac{1}{n}\sum X_i^2\right) - E(\bar{X}^2) \\
&= \frac{1}{n}\sum E(X_i^2) - \left[\text{Var}(\bar{X}^2) + \{E(\bar{X})\}^2\right] \\
&= \frac{1}{n}\sum\left(\text{Var}(X_i) + \{E(X_i)\}^2\right) - \left[\frac{\sigma^2}{n} + \mu^2\right] \\
&= \frac{1}{n}\sum(\sigma^2 + \mu^2) - \frac{\sigma^2 + n\mu^2}{n} \\
&= \frac{n\sigma^2 + n\mu^2}{n} - \frac{\sigma^2 + n\mu^2}{n} \\
&= \left(\frac{n-1}{n}\right)\sigma^2
\end{aligned}
$$

(b) Theorem B.16 part 2 states that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, so if we can write this in terms of $\hat{\sigma}^2$ we can use the variance of χ^2_{n-1}.

Recall that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \Rightarrow (n-1)S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$

So $\frac{(n-1)S^2}{\sigma^2} = \frac{n\hat{\sigma}^2}{\sigma^2} \Longrightarrow \hat{\sigma}^2 = \left(\frac{\sigma^2}{n}\right)\cdot\frac{(n-1)S^2}{\sigma^2}$.

Finally, appeal to the variance:
$$
\begin{aligned}
\text{Var}(\hat{\sigma}^2) &= \text{Var}\left(\frac{\sigma^2}{n}\cdot\frac{(n-1)S^2}{\sigma^2}\right) \\
&= \left(\frac{\sigma^2}{n}\right)^2\cdot\text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) \\
&= \frac{\sigma^4}{n^2}\cdot 2(n-1)
\end{aligned}
$$

27 (continued)

(c) $$\mathrm{MSE}(\hat{\sigma}^2) = \mathrm{Var}(\hat{\sigma}^2) + [\mathrm{Bias}(\hat{\sigma}^2)]^2$$
$$= \frac{\sigma^4}{n^2} \cdot 2(n-1) + \left(\frac{-\sigma^2}{n}\right)^2$$
$$= \frac{\sigma^4}{n^2}(2n-1)$$

29 Let $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $x>0, \theta>0$ and $\hat{\theta}_1 = X_1$, $\hat{\theta}_2 = \frac{1}{2}(X_1+X_2)$, $\hat{\theta}_3 = \frac{1}{3}(X_1+2X_2)$

First, we will show that all three estimators are unbiased.

$$E(X) = \int_0^\infty x \cdot \frac{1}{\theta} e^{-x/\theta} dx = 1/{1/\theta} = \theta, \text{ as } X \sim Expo(\lambda = \tfrac{1}{\theta})$$

Alternatively, you can use integration by parts to show this

So $E(\hat{\theta}_1) = E(X_1) = \theta$

$$E(\hat{\theta}_2) = E[\tfrac{1}{2}(X_1+X_2)] = \tfrac{1}{2}[E(X_1)+E(X_2)] = \tfrac{1}{2}(2\theta) = \theta$$

$$E(\hat{\theta}_3) = E[\tfrac{1}{3}(X_1+2X_2)] = \tfrac{1}{3}[E(X_1)+2E(X_2)] = \tfrac{1}{3}(3\theta) = \theta,$$

establishing that all three estimators are unbiased.

Next, we calculate the variance of the estimators.

Note that $\mathrm{Var}(X) = \theta^2$

$$\mathrm{Var}(\hat{\theta}_1) = \theta^2$$

$$\mathrm{Var}(\hat{\theta}_2) = \left(\tfrac{1}{2}\right)^2[\mathrm{Var}(X_1)+\mathrm{Var}(X_2)] = \tfrac{1}{4}(2\theta^2) = \frac{\theta^2}{2}$$

$$\mathrm{Var}(\hat{\theta}_3) = \left(\tfrac{1}{3}\right)^2[\mathrm{Var}(X_1) + 2^2\mathrm{Var}(X_2)] = \left(\tfrac{1}{3}\right)^2[5\theta^2] = \tfrac{5}{9}\theta^2$$

Finally, we calculate the relative efficiencies:

$$RE(\hat{\theta}_1,\hat{\theta}_2) = \frac{\theta^2/2}{\theta^2} = \frac{1}{2}, \quad RE(\hat{\theta}_1,\hat{\theta}_3) = \frac{5\theta^2/9}{\theta^2} = \frac{5}{9}, \quad RE(\hat{\theta}_2,\hat{\theta}_3) = \frac{5\theta^2/9}{\theta^2/2} = \frac{10}{9}$$

$\Longrightarrow$ We see that $\hat{\theta}_2$ is the most efficient estimator of the three.

$\boxed{34}$ Let $f(x) = \frac{6}{\theta^6} x^5$, $0 \le x \le \theta$.

(a) From theorem 4.1 we know that

$$f_{max}(x) = n[F(x)]^{n-1} f(x) = n\left[\frac{1}{\theta^6}x^6\right]^{n-1}\left(\frac{6}{\theta^6}x^5\right) = n \cdot \frac{1}{\theta^{6n-6}} x^{6n-6}\left(\frac{6}{\theta^6}x^5\right)$$

$$= 6n \frac{1}{\theta^{6n}} x^{6n-1}, \quad 0 \le x \le \theta$$

Note: $F(x) = \int_0^x \frac{6}{\theta^6} t^5 dt = \frac{1}{\theta^6} t^6 \Big|_0^x = \frac{1}{\theta^6} x^6$

(b) $$E(X_{max}) = \int_0^\theta x \cdot 6n \frac{1}{\theta^{6n}} x^{6n-1} dx = \int_0^\theta 6n \frac{1}{\theta^{6n}} x^{6n} dx$$

$$= \frac{6n}{6n+1}\left(\frac{1}{\theta^{6n}}\right) x^{6n+1}\Big|_0^\theta = \frac{6n}{6n+1}\left(\frac{1}{\theta^{6n}}\right)\theta^{6n+1}$$

$$= \frac{6n}{6n+1} \cdot \theta$$

(c) $$\text{Bias}(X_{max}) = E(X_{max}) - \theta = \frac{6n}{6n+1}\theta - \theta = \theta\left[\frac{6n}{6n+1} - 1\right] = \theta\left[\frac{-1}{6n+1}\right]$$

(d) $$MSE(X_{max}) = Var(X_{max}) + [\text{Bias}(X_{max})]^2 = \frac{6n\theta^2}{6n+2} - \frac{(6n\theta)^2}{(6n+1)^2} + \frac{\theta^2}{(6n+1)^2}$$

$$Var(X_{max}) = E(X_{max}^2) - [E(X_{max})]^2 = \frac{6n}{6n+2}\theta^2 - \left(\frac{6n}{6n+1}\right)^2\theta^2$$

$$E(X_{max}^2) = \int_0^\theta x^2\, 6n \frac{1}{\theta^{6n}} x^{6n-1} dx = \int_0^\theta 6n \cdot \frac{1}{\theta^{6n}} x^{6n+1} dx = \frac{6n}{6n+2}\left(\frac{1}{\theta^{6n}}\right) x^{6n+2}\Big|_0^\theta$$

$$= \frac{6n}{6n+2}\left(\frac{1}{\theta^{6n}}\right)\theta^{6n+2} = \frac{6n}{6n+2}\theta^2$$

41 $X_1, \dots, X_n \overset{iid}{\sim} \text{Expo}(\lambda)$ and $\hat{\lambda} = \sum_{i=1}^{n} X_i$

$$E(\hat{\lambda}) = E(\sum X_i) = n\,E(X_1) = \frac{n}{\lambda}$$

$$Var(\hat{\lambda}) = Var(\sum X_i) = n\,Var(X_1) = \frac{n}{\lambda^2}$$

Since the $\lim_{n\to\infty} E(\hat{\lambda}) \neq \lambda$, we cannot use proposition 6.6 for this problem; rather we must appeal to the definition of consistency. Let $\epsilon > 0$, then

$$\lim_{n\to\infty} P(|\hat{\lambda}-\lambda| < \epsilon) = \lim_{n\to\infty} P(-\epsilon < \hat{\lambda}-\lambda < \epsilon)$$

$$= \lim_{n\to\infty} P(\lambda-\epsilon < \hat{\lambda} < \lambda+\epsilon)$$

If $\hat{\lambda}$ is a consistent estimator of λ, then this limit must be 1 for all $\epsilon > 0$.

Consider $\epsilon = \lambda$, then we have that

$$\hat{\lambda} = \sum X_i$$

$$P(\hat{\lambda} > 2\lambda) = P(\sum X_i > 2\lambda)$$

$$\geq P(X_1 > 2\lambda)$$

$$= \int_{2\lambda}^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_{2\lambda}^{\infty} = 0 - (-e^{-\lambda(2\lambda)})$$

$$= e^{-2\lambda^2}$$

So $\lim_{n\to\infty} P(0 < \hat{\lambda} < 2\lambda) = \lim_{n\to\infty} 1 - P(\hat{\lambda} \geq 2\lambda)$

$$\leq \lim_{n\to\infty} 1 - e^{-2\lambda^2}$$

$$< 1$$

$\Longrightarrow \hat{\lambda}$ is not a consistent estimator of λ

Problem 5 $\quad X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

(a) $L(p \mid x_1, \dots, x_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}$

$$l(p) = \Sigma x_i \log p + (n - \Sigma x_i)\log(1-p)$$

$$\frac{d}{dp} l(p) = \frac{\Sigma x_i}{p} - \frac{n - \Sigma x_i}{1-p}$$

Setting this equal to 0 we can solve to find a candidate for $\hat{p}$:

$$\frac{\Sigma x_i}{p} = \frac{n - \Sigma x_i}{1-p} \Longrightarrow \frac{1-p}{p} = \frac{n - \Sigma x_i}{\Sigma x_i} \Longrightarrow \frac{1}{p} - 1 = \frac{n}{\Sigma x_i} - 1 \Rightarrow \hat{p} = \overline{X}$$

Now, we must check the 2nd deriv and evaluate it at $\overline{X}$ and see if it is less than 0.

(b) $Var(\overline{X}) = \frac{Var(X_1)}{n} = \frac{p(1-p)}{n}$

To find the CRLB, start by find $I_1(p)$:

$$\log(f(x|p)) = \log\left(p^x(1-p)^{1-x}\right) = x\log p + (1-x)\log(1-p)$$

$$\frac{d}{dp} \log f(x|p) = \frac{x}{p} - \frac{1-x}{1-p}$$

$$\frac{d^2}{dp^2} \log(f(x|p)) = -\frac{x}{p^2} + \frac{1-x}{(1-p)^2}$$

$$I_1(p) = -E\left(\frac{d^2}{dp^2} \log f(x|p)\right) = -E\left(\frac{-x}{p^2} + \frac{1-x}{(1-p)^2}\right)$$

$$= -\left[-\frac{1}{p^2}E(X) + \frac{1}{(1-p)^2}(1 - E(X))\right] = -\left[-\frac{1}{p^2}(p) + \frac{1}{(1-p)^2}(1-p)\right]$$

$$= -\left[\frac{1}{(1-p)} - \frac{1}{p}\right] = -\left[\frac{p - (1-p)}{p(1-p)}\right] = \frac{1}{p(1-p)}$$

So the CRLB of $Var(\hat{p}) = \frac{1}{I_n(p)} = \frac{1}{n/p(1-p)} = \frac{p(1-p)}{n}$; thus, $\overline{X}$ attains the CRLB.