Math 445
Final Exam
June 6, 2016

Name: KEY

73 pts

This exam is closed book and closed notes, except for one standard size sheet of paper (8.5" by 11") that can have notes on both sides. No copying, cheating, collaborations, computers, or cell phones are allowed. Show your work and write complete and coherent answers. Be sure to clearly label and justify your solutions to these problems. Illegible or unjustified solutions will not receive full credit. After you have completed the exam, please reaffirm the Lawrence University Honor Code.

1. Let X have an Exponential distribution with parameter λ. Suppose we wish to test $H_0 : \lambda \geq 1$ vs. $H_a : \lambda < 1$ and will reject H_0 in favor of H_a if $X \geq 2$.

4pts

(a) Find the power of this test for a general value of λ (i.e., your answer should be left in terms of λ).

+1 $P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(X \geq 2 \mid \lambda < 1)$

$$= \int_2^\infty \lambda e^{-\lambda} dx = 1 - \int_0^2 \lambda e^{-\lambda x} dx$$

$$= 1 - \left[-e^{-\lambda x} \Big|_0^2 \right]$$

$$= 1 - \left(-e^{-2\lambda} + e^{-\lambda 0} \right)$$

$$= e^{-2\lambda} \quad \text{for } \lambda < 1$$

3pts

(b) Calculate the probability of making a type I error if $\lambda = 1$.

$P(\text{Reject } H_0 \mid H_0 \text{ is true}) = P(X \geq 2 \mid \lambda = 1) = e^{-2}$

2pts

(c) What value of λ will result in the largest probability of making a type I error?

Notice that $e^{-2\lambda}$ is a decreasing function of λ; thus, $\lambda = 1$ results in the largest probability of making a type I error.

(+9)

2. Let $Y_1, \dots, Y_n \overset{\text{iid}}{\sim} \text{Unif}[0, \theta]$.

2pts (a) Write down the likelihood function.

$$f(y|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \left(\frac{1}{\theta}\right)^n$$

(b) Suppose that you decide to use a Pareto prior distribution for θ that has PDF

$$\pi(\theta|\alpha, \beta) = \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}}, \quad \text{for } \theta \geq \beta > 0,\ \alpha > 0.$$

3pts Find the posterior density of θ. +1

$$p(\theta|y) \propto \frac{1}{\theta^{\alpha+1}}\left(\frac{1}{\theta}\right)^n = \frac{1}{\theta^{n+\alpha+1}}$$

So the posterior of θ is Pareto$(n+\alpha, \beta)$.

2pts (c) Describe how you would create a 90% credible interval for θ from the posterior distribution.

To create a 90% credible interval for θ we can take the .05 and .95 quantiles from the posterior distribution.

3. The following problem is based on a study to compare the economic value of two types of information about Genetical Modified Organisms (GMO). Participants in the study read one page of information about GMO foods then participated in an experimental auction. Their bids during the auction provide information about the dollar value of the information. Participants are randomly assigned to read either a pro-GMO page of information produced by a major biotech company or an anti-GMO page of information produced by a major environmental organization. The response, measured on each participant, is the dollar value of the information they read. Fourty (40) participants started the study. Three refused to bid and are omitted from the data summary.

Group	n_i	average	s.d.
pro-GMO	19	0.80	0.284
anti-GMO	18	1.23	0.310

3pts (a) Calculate the pooled estimate of the standard deviation.

$$S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{18(.284)^2 + 17(.31)^2}{19+18-2} \approx .0882$$

$$\Longrightarrow S_p \approx .297$$

+10

~~Need to~~ give → (b) Calculate a 95% confidence interval for the difference in means (as pro-GMO - anti-GMO). (Plug in completely but don't simplify)

5 pts

$$(\bar{X}_1 - \bar{X}_2) \pm t_{1-\frac{\alpha}{2}, n_1+n_2-2} \cdot S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= (.8 - 1.23) \pm t_{.975, 35} \, (.297) \sqrt{\frac{1}{19} + \frac{1}{18}}$$

$$-.628 \text{ to } -.232$$

Give an interval here → (c) Provide a one sentence interpretation of the confidence interval you calculated in part (b).

4 pts

"We are 95% confident that..."

4 pts

(d) Is there a statistically significant difference in the economic value of pro- and anti-GMO information? If so, at what significance level?

Since zero is not contained in the 95% CI there is statistically signif. evidence of a diff. at the $\alpha = .05$ signif. level.

(e) What assumptions must hold for your confidence interval in part (b) to be appropriate?

4 pts

Assumptions

1. Indep. groups
2. Equal variances
3. iid normal obs. w/in each group

+17

8 pts

4. Suppose that a single observation $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with known mean μ and unknown variance σ^2, and the following simple hypotheses are to be tested: $H_0 : \sigma^2 = 2$ vs. $H_a : \sigma^2 = 3$. Derive the most powerful test, using a general α.

The Neyman-Pearson lemma says that the LR test is the UMP test of size α.

$$f(\underline{x} \mid \sigma^2, \mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2}$$

$$T(\underline{x}) = \frac{\left(\frac{1}{\sqrt{4\pi}}\right)^n e^{-\frac{1}{4}\Sigma(x_i-\mu)^2}}{\left(\frac{1}{\sqrt{6\pi}}\right)^n e^{-\frac{1}{6}\Sigma(x_i-\mu)^2}} \propto e^{\frac{1}{6}\Sigma(x_i-\mu)^2 - \frac{1}{4}\Sigma(x_i-\mu)^2}$$

Reject if $T(\underline{x}) \leq c \Rightarrow \log(T(\underline{x})) \leq c$

$\Sigma(x_i-\mu)^2 \left(\frac{1}{6}-\frac{1}{4}\right) \leq c$

$\Rightarrow$ Reject if $\Sigma(x_i-\mu)^2 \geq c$

and c is found by solving: $P(\Sigma(x_i-\mu)^2 \geq c \mid \sigma^2=2) = \alpha$.

So c is the $1-\alpha$ quantile of a χ^2_n dsn.

6 pts

5. Suppose that $Y \sim \text{Normal}(0, \sigma^2)$.

(a) Show that Y^2/σ^2 is a pivotal quantity.

$$P\left(\frac{Y^2}{\sigma^2} \leq u\right) = P(Y^2 \leq u\sigma^2) = P(-\sqrt{u\sigma^2} \leq Y \leq \sqrt{u\sigma^2})$$

$$= P(Y \leq \sqrt{u\sigma^2}) - P(Y \leq -\sqrt{u\sigma^2})$$

$$= 1 - 2P(Y \leq -\sqrt{u\sigma^2})$$

So $f(u) = -2\left(-\frac{1}{2}u^{-1/2}\sqrt{\sigma^2}\right)\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u\sigma^2}$

$\propto \frac{1}{\sqrt{2\pi}} u^{-1/2} e^{-u/2} = \frac{1}{\sqrt{2\pi}} u^{1/2-1} e^{-u/2}$

$\rightarrow \frac{Y^2}{\sigma^2} \sim \chi^2_1$, which does not depend on σ^2, so it is pivotal.

4 pts

(b) Suppose that you observed $Y = 2$. Use the pivotal quantity Y^2/σ^2 and the R output below to find a 95% upper confidence bound for σ^2.

```
> qchisq(.025, df = 1)
[1] 0.0009820691
> qchisq(.975, df = 1)
[1] 5.023886
> qchisq(.05, df = 1)
[1] 0.00393214
> qchisq(.95, df = 1)
[1] 3.841459
```

Use $P\left(\frac{Y^2}{\sigma^2} \geq q\right) = .95$

So $\frac{1}{\sigma^2} \geq \frac{q}{Y^2} \Rightarrow \sigma^2 \leq \frac{Y^2}{q} \approx \frac{4}{.0039} = \underline{1025.641}$

(+18)

6. A certain type of electronic component, which has a lifetime X (in hours) with PDF given by

$$f(\cancel{x}|\theta) = \theta^{-2} x e^{-x/\theta},\ x > 0,\ \theta > 0$$

with mean $E(X) = 2\theta$ and variance $\text{Var}(X) = 2\theta^2$. Suppose that a random sample of n of these components is collected.

3 pts (a) Derive the log-likelihood function for these data.

$$f(\cancel{x}|\theta) = \prod_{i=1}^{n} \theta^{-2} x_i e^{-x_i/\theta} = \theta^{-2n} \left(\prod_{i=1}^{n} x_i\right) e^{-\frac{1}{\theta}\sum x_i}$$

$$l(\theta) = -2n \log\theta + \sum \log x_i - \frac{1}{\theta}\sum x_i$$

4 pts (b) Find the maximum likelihood estimator for θ.

$$\frac{d}{d\theta} l(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum x_i \equiv 0$$

$$\frac{1}{\theta^2}\sum x_i = \frac{2n}{\theta}$$

$$\frac{1}{2}\bar{X}_n = \hat{\theta}$$

* Check 2nd derivative

3 pts (c) Suppose that we wish to test $H_0 : \theta = \theta_0$ vs. $H_a : \theta \neq \theta_0$. Derive the formula for the asymptotic likelihood ratio test statistic.

$$T(x) = \frac{f(x|\theta_0)}{f(x|\hat{\theta})}$$

$$-2\log T(x) = -2\left[\left(-2n\log\theta_0 + \sum \log x_i - \frac{1}{\theta_0}\sum x_i\right) - \left(-2n\log\hat{\theta} + \sum \log x_i - \frac{1}{\hat{\theta}}\sum x_i\right)\right]$$

$$= -2\left[-2n(\log\theta_0 - \log\hat{\theta}) - \sum x_i\left(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}\right)\right]$$

$$= 4n(\log\theta_0 - \log\hat{\theta}) + 2\sum x_i\left(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}\right)$$

3 pts (d) Specify the rejection region for an α-level test for the asymptotic likelihood ratio test described in part (c). If you would need R to do this, explain what you would need to find in R.

Reject Ho if $-2\log T(x) \geq \chi^2_{1,1-\alpha}$

+13

6pts

7. Explain the differences in the interpretations between a frequentist confidence interval and a Bayesian credible interval.

See notes. Answer revolves around the interpretation of probability and the postulates.

+6

Name	Param.	PMF or PDF	Mean	Variance	MGF
Bernoulli	p	$P(X=1)=p$ $P(X=0)=q$	p	pq	pe^t+q
Binomial	n,p	$\binom{n}{k}p^k q^{n-k}$ $k \in \{0,1,\ldots,n\}$	np	npq	$(pe^t+1-p)^n$
Geometric	p	pq^k $k \in \{0,1,2,\ldots\}$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1-qe^t}$
Negative Binomial	r,p	$\binom{r+n-1}{r-1}p^r q^n$ $k \in \{0,1,2,\ldots\}$	$\frac{rq}{p}$	$\frac{rq}{p^2}$	$\left(\frac{p}{1-qe^t}\right)^r$
Hypergeometric	w,b,n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$ $k \in \{0,1,\ldots,n\}$	$\mu = \frac{nw}{w+b}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}\left(1-\frac{\mu}{n}\right)$	
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$ $k \in \{0,1,2,\ldots\}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform	$a<b$	$\frac{1}{b-a},\ x \in (a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Normal	μ,σ^2	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ $x \in \mathbb{R}$	μ	σ^2	$e^{\mu t+(\sigma^2 t^2)/2}$
Exponential	λ	$\lambda e^{-\lambda x},\ x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$
Gamma	a,λ	$\frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}$ $x>0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^a$
Beta	a,b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ $0<x<1$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{a+r}{a+b+r}\right)\frac{t^k}{k!}$
Chi-Square	n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x>0$	n	$2n$	$\left(\frac{1}{1-2t}\right)^{n/2}$