Solution to Chapter 4 practice problems

9) $f(x) = \frac{3}{16}(x-4)^2 \quad , \quad 2 \le x \le 6$

The CLT states that $\bar{X} \mathrel{\dot\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$, so we must first find

$E(X) = \mu$ and $Var(X) = \sigma^2$.

$$\mu = E(X) = \int_2^6 x \cdot \frac{3}{16}(x-4)^2\,dx = 4$$

Recall that $Var(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_2^6 x^2 \cdot \frac{3}{16}(x-4)^2\,dx = 18.4$$

So $Var(X) = 18.4 - 4^2 = 2.4$

We now have that $\bar{X} \mathrel{\dot\sim} N(4, 2.4/244)$

$$P(\bar{X} \ge 4.2) = P\left(Z \ge \frac{4.2-4}{\sqrt{2.4/244}}\right)$$

$$= 1 - \Phi(2.02)$$

$$\approx .022$$

10) $X \sim Bin(n=800,\ p=.286)$

$X \,\dot{\sim}\, N(800(.286),\ 800(.286)(1-.286))$ or $X \,\dot{\sim}\, Bin(228.8,\ 163.4)$

$P(220 \le X \le 230) \approx P(219.5 \le X \le 230.5)$ (continuity correction)

$$= \Phi\left(\frac{230.5-228.8}{\sqrt{163.4}}\right) - \Phi\left(\frac{219.5-228.8}{\sqrt{163.4}}\right)$$

$$= \Phi(.13) - \Phi(-.73)$$

$$= .319$$

Using pbinom, we find $P(220 \le X \le 230) = \sum_{k=220}^{230} \binom{800}{k} .286^{k} (1-.286)^{800-k}$

$$= .3208$$

or pbinom, we find $P(220 \le X \le 230) = P(X \le 230) - P(X < 220)$

$$= .3208$$

13a) $X_i \overset{iid}{\sim} N(20, 8^2)$ and $Y_j \overset{iid}{\sim} N(16, 7^2)$

We know that $\bar{X} \sim N(20,\ 8^2/10)$ and $\bar{Y} \sim N(16,\ 7^2/15)$ by Corollary A.2 (p. 370).

Further, the sum of normal RVs is normal (p 370, Thm A.11),

and $E(\bar{X}+\bar{Y}) = E(\bar{X}) + E(\bar{Y}) = 20 + 16 = 36$, and

$Var(\bar{X}+\bar{Y}) = Var(\bar{X}) + Var(\bar{Y})$ (if indep, which should have been stated in the problem)

$$= 8^2/10 + 7^2/15$$

20) Thm 4.1 states that

$$f_{min}(x) = n[1-F(x)]^{n-1} f(x)$$

$$f_{max}(x) = n[F(x)]^{n-1} f(x)$$

Prf: First, let's consider the max.

$$P(X_{max} \le x) = P(X_1 \le x, X_2 \le x, \dots, X_n \le x)$$

$$= P(X_1 \le x)P(X_2 \le x) \cdots P(X_n \le x) \quad \text{by indep.}$$

$$= [P(X_1 \le x)]^n \quad \text{b/c } X_1, \dots, X_n \text{ are identically distributed}$$

$$= [F(x)]^n \quad \text{by def'n of CDF}$$

Taking the deriv. w.r.t. x we find the PDF:

$$f_{max}(x) = \frac{d}{dx}[F(x)]^n = n[F(x)]^{n-1} f(x)$$

Now, consider the minimum.

$$1-F_{min}(x) = P(X_{min} > x) = P(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$= P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) \quad \text{by indep}$$

$$= [P(X_1 > x)]^n \quad \text{b/c ident. dist. RVs}$$

$$= [1-F(x)]^n$$

$$\Rightarrow F_{min}(x) = 1-[1-F(x)]^n$$

Taking the derivative wrt x completes the proof

$$f_{min}(x) = \frac{d}{dx} 1-[1-F(x)]^n = n[1-F(x)]^{n-1} f(x)$$

23) $X_1, \ldots, X_{10} \overset{iid}{\sim} \text{Expo}(\lambda = 12)$

By thm 4.1 we know that

$$f_{max}(x) = n\left[F(x)\right]^{n-1} f(x)$$

$$= 10\left[1 - e^{-12x}\right]^{9} \cdot 12e^{-12x}, \quad x > 0.$$

$$= 120e^{-12x}\left[1 - e^{-12x}\right]^{9}, \quad x > 0$$

25) $X_1, \ldots, X_{10} \overset{iid}{\sim} \text{Pois}(3)$ and $X = \sum_{i=1}^{10} X_i$

From Math 240 we know that $X \sim \text{Pois}\left(\sum_{i=1}^{10} \lambda_i\right)$, so

$X \sim \text{Pois}(30)$.