M445 Final Exam Study Guide Solutions:

① The "typical" CI is better as it is <u>narrower</u>; thus, it provides a more precise estimate of the parameter.

② (a)

$$f(\underline{x}|\theta) = \prod_{i=1}^{n} \frac{e^{-i\theta}(i\theta)^{x_i}}{x_i!}$$

$$= \frac{e^{-\theta \Sigma i}\left(\prod_{i=1}^{n} i^{x_i}\right)\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$$

$$= \prod_{i=1}^{n}\left(\frac{i^{x_i}}{x_i!}\right) e^{-\theta \Sigma x_i}\,\theta^{\Sigma x_i}$$

(b) Let θ_0 denote the ML estimate of θ under H_0 and let θ_1 denote the usual ML estimate of θ.

$$T(\underline{x}) = \frac{f(\underline{x}|\theta_0)}{f(\underline{x}|\theta_1)} = \frac{e^{-\theta_0 \Sigma x_i}\,\theta_0^{\Sigma x_i}}{e^{-\theta_1 \Sigma x_i}\,\theta_1^{\Sigma x_i}} = e^{-\Sigma x_i(\theta_1-\theta_0)}\left(\frac{\theta_0}{\theta_1}\right)^{\Sigma x_i}$$

(c) The general LR test rejects H_0 if $T(\underline{x}) \le c$.

$$T(\underline{x}) = e^{-\Sigma x_i(\theta_1-\theta_0)}\left(\frac{\theta_0}{\theta_1}\right)^{\Sigma x_i} \le c$$

(taking the log)

$$-\Sigma x_i(\theta_1-\theta_0) + \Sigma x_i(\log\theta_0 - \log\theta_1) \le c$$

$$\Sigma x_i\left[(\log\theta_0 - \log\theta_1) - (\theta_1-\theta_0)\right] \le c$$

Reject if

$\Sigma x_i \le c$ if $\theta_0 > \theta_1$ (Based on the hypotheses, this is our situation)

$\Sigma x_i > c$ if $\theta_0 < \theta_1$

Now that we have our rejection rule we need to specify the constant, c.

$$P\left(\Sigma X_i \le c \mid H_0 \right) = e^{-3} = \alpha$$

Since X_i are indep. $Pois(\cdot;\theta)$ RVs, $\sum_{i=1}^{n} X_i \sim Pois\left(\sum_{i=1}^{n} ; \theta \right)$.

For $n=5$ and $\theta_0 = \frac{1}{5}$, $\Sigma X_i \sim Pois(3) \Rightarrow c = 0$.

(3) See class notes for the solution.

(4) (a) $f(\underline{x} \mid \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \Sigma x_i}$

Let $\hat{\lambda}$ denote the ML estimate of λ. Then

$$T(\underline{x}) = \frac{\lambda_0^n e^{-\lambda_0 \Sigma x_i}}{\hat{\lambda}^n e^{-\hat{\lambda} \Sigma x_i}} = \left(\frac{\lambda_0}{\hat{\lambda}} \right)^n e^{-(\lambda_0 - \hat{\lambda}) \Sigma x_i}$$

We will reject H_0 if, for some c, $T(\underline{x}) \le c$.

Solving for c:

$$\left(\frac{\lambda_0}{\hat{\lambda}} \right)^n e^{-(\lambda_0 - \hat{\lambda}) \Sigma x_i} \le c \implies e^{-(\lambda_0 - \hat{\lambda}) \Sigma x_i} \le \overbrace{\left(\frac{\hat{\lambda}}{\lambda_0} \right)^n}^{\text{constant!}} c$$

$$\implies -(\lambda_0 - \hat{\lambda}) \Sigma x_i \le c$$

Note that $\hat{\lambda} \ge \lambda_0 \qquad \implies \qquad \Sigma x_i \le c$

We know that $\sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$, so we find c

by find the α quantile of a $Gamma(n, \lambda)$ dsn.

(5)

(a) The support of the Lognormal dsn is strictly greater than zero.

(b) We can use the asymptotic result that

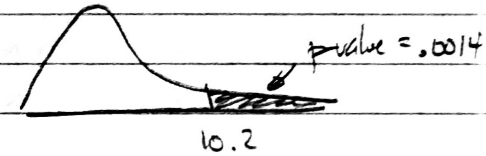
$\Lambda = -2 \log(T(x)) \,\dot{\sim}\, \chi^2_{df}$ for this problem

$$\Lambda = -2 \log(T(x)) = -2\left(\log(f(x|\beta_1=0,\hat{\beta}_0)) - \log(f(x|\hat{\beta}_1,\hat{\beta}_0))\right)$$

$$= -2(-374.43 - (-369.33))$$

$$= 10.20$$

Here, $\Lambda \,\dot{\sim}\, \chi^2_1$, so we find the pvalue as the below area under the χ^2_1 PDF:



There is very strong evidence that $\beta_1 \neq 0$; that is, that the amount someone spent on their last haircut depends on sex

(6)

(a) The SE of $\bar{X}_i$ will tell us how precisely we are estimating μ_i. Here, the experimental group has the smaller SE.

(b) If we do not assume that the variances are equal, then we carry out the two-sample t-test:

$$T = \frac{(10.7 - 10.3) - 0}{\sqrt{\frac{1.5^2}{10} + \frac{2.4^2}{20}}} \approx .558 \sim t_{12.66}$$

the pvalue $\approx .5866$

There is no evidence of a diff. in the average litter size between the control and treatment groups.

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(c)

Assumption	Method
Normality	Q-Q plots for each group
Independent groups	Think about the data collection process
Equal var (if you assumed it)	Boxplots, density plots etc by group

7 $\frac{\lambda}{2} \sum x_i$ is a pivotal quantity because it is a function of the data and the parameter of interest, but its distribution does not depend on any unknown parameter.

To find a $(1-\alpha)100\%$ CI for λ we consider

$$P\left(a < \frac{\lambda}{2} \sum_{i=1}^{n} X_i < b \right) = 1-\alpha$$

where a is the $\frac{\alpha}{2}$ quantile of Gamma$(n, \frac{1}{2})$
b is the $1-\frac{\alpha}{2}$ quantile of Gamma$(n, \frac{1}{2})$.

Thus, the interval is

$$\left(\frac{2a}{\sum_{i=1}^{n} X_i}, \frac{2b}{\sum_{i=1}^{n} X_i} \right)$$

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(a) $P(X \geq 3.2 \mid \lambda = 1) = \int_{3.2}^{\infty} e^{-x}\, dx = .04$

(b) $P(X \geq 3.2 \mid \lambda = \frac{1}{5}) = \int_{3.2}^{\infty} \frac{1}{5} e^{-\frac{1}{5}x}\, dx = .527$