Math 445 Spring 2014 — Midterm Solutions

1 (a) False — an unbiased estimator for θ will not necessarily have a smaller MSE than a biased estimator for the same θ.

Example: $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.. and consider two estimators of σ^2:

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \quad \text{and}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

You should be able to verify this.

It can be shown that $\hat{\sigma}^2$ has a smaller MSE than S^2 $\forall$ μ and σ^2. Recall that $\hat{\sigma}^2$ is a biased estimator of σ^2 while S^2 is unbiased!

(b) True — the "U" in MVUE stands for unbiased

(c) False — Counterexample:
Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ and define $\hat{\theta} = \frac{X_1 + X_2}{2}$

$E(\hat{\theta}) = \mu \implies \hat{\theta}$ is unbiased, but

$Var(\hat{\theta}) = \frac{1}{2}\sigma^2$

$\implies \lim_{n\to\infty} Var(\hat{\theta}) \neq 0 \implies \hat{\theta}$ is not consistent

(d) True — In class we showed that for $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ MLE for σ^2 is $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)$, which is biased!

(e) True

$\boxed{2}$ (a) $L(\theta|\underline{x}) = f(x_1,\dots,x_n|\theta) = \prod_{i=1}^{n} \theta 2^{-\theta} X_i^{\theta-1}$

$$= \theta^n 2^{-n\theta} \left(\prod_{i=1}^{n} X_i^{\theta-1}\right)$$

$$l(\theta|\underline{x}) = \log L(\theta|\underline{x}) = n\log\theta - n\theta\log 2 + (\theta-1)\sum_{i=1}^{n}\log X_i$$

$$\frac{d}{d\theta} l(\theta|\underline{x}) = \frac{n}{\theta} - n\log 2 + \sum_{i=1}^{n}\log X_i \overset{\text{set}}{=} 0$$

$$\hat{\theta} = \frac{n}{n\log 2 - \sum_{i=1}^{n}\log X_i} = \frac{1}{\log 2 - \frac{1}{n}\sum_{i=1}^{n}\log X_i}$$

$\frac{d^2}{d\theta^2} l(\theta|\underline{x}) = -\frac{n}{\theta} < 0$ so $\hat{\theta}$ is indeed a maximom.

(b) $E(X) = \int_0^2 x \cdot \theta 2^{-\theta} x^{\theta-1}\,dx$

$$= \int_0^2 \theta 2^{-\theta} x^{\theta}\,dx$$

$$= \frac{\theta 2^{-\theta}}{\theta+1} x^{\theta+1}\Big|_0^2$$

$$= \frac{\theta}{\theta+1} 2^{-\theta}\left(2^{\theta+1}\right)$$

$$= \frac{2\theta}{\theta+1}$$

The MOM is given by

$$E(X_1) = \overline{X}_n \iff \frac{2\tilde{\theta}}{\tilde{\theta}+1} = \overline{X}_n$$

$$\iff \frac{2}{\overline{X}_n} = 1 + \frac{1}{\tilde{\theta}}$$

$$\iff \tilde{\theta} = \frac{\overline{X}_n}{2-\overline{X}_n}$$

2 (c) From the WLLN we know that $\bar{X}_n \xrightarrow{P} E(X_1) = \dfrac{2\theta}{\theta+1}$ as $n \to \infty$.

Let $g(y) = y(2-y)^{-1}$, which is a continuous function for $y \neq 2$.

Then $g(\bar{X}_n) = \bar{X}_n (2-\bar{X}_n)^{-1} = \tilde{\theta} \xrightarrow{P} g(E(X_1)) = \theta$,

so $\tilde{\theta}$ is a consistent estimator of θ.

3 $\frac{\lambda}{2} \sum_{i=1}^{n} X_i$ is a pivotal quantity because it is a function of the data and the parameter of interest, and its distribution does not depend on any unknown parameters.

To find a $(1-\alpha)100\%$ CI for λ we consider the statement:

$$P\left(a < \frac{\lambda}{2} \sum_{i=1}^{n} X_i < b \right) = 1-\alpha$$

Here a is the $\frac{\alpha}{2}$ quantile of $Gamma(n, \frac{1}{2})$, and
b is the $1-\frac{\alpha}{2}$ quantile of $Gamma(n, \frac{1}{2})$.

Thus, a $(1-\alpha)100\%$ is given by

$$2a\left(\sum_{i=1}^{n} X_i\right)^{-1} < \lambda < 2b\left(\sum_{i=1}^{n} X_i\right)^{-1}$$

or

$$\left(\frac{2a}{\sum_{i=1}^{n} X_i}, \frac{2b}{\sum_{i=1}^{n} X_i} \right)$$

4 For an estimator to be a MVUE, it must be unbiased and have smaller variance than other UEs.

$E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \lambda \implies \bar{X}_n$ is an UE of λ

Now we must show that $\bar{X}_n$ attains the CRLB, that is

$$Var(\bar{X}_n) = \frac{1}{nI_1(\lambda)}$$

$$Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{\lambda}{n}$$

$$\begin{aligned}
I_1(\lambda) &= E\left(\left[\frac{\partial}{\partial\lambda}\log(f(X_1|\lambda))\right]^2\right) \\
&= E\left(\left[\frac{\partial}{\partial\lambda} -\lambda + X_1\log\lambda - \log(X_1!)\right]^2\right) \\
&= E\left(\left[-1 + \frac{X_1}{\lambda}\right]^2\right) \\
&= \frac{1}{\lambda^2} E\left(\left[X_1 - \lambda\right]^2\right) \\
&= \frac{1}{\lambda^2} Var(X_1) \\
&= \frac{1}{\lambda}
\end{aligned}$$

So the CRLB $= \frac{1}{nI_1(\lambda)} = \frac{1}{n(1/\lambda)} = \frac{\lambda}{n}$. Thus,

$Var(\bar{X}_n) = CRLB$.

Since $\bar{X}_n$ is unbiased and $V(\bar{X}_n)$ attains the CRLB, $\bar{X}_n$ is a MVUE.