

This exam is closed book and closed notes, except for one standard size sheet of paper (8.5" by 11") that can have notes on both sides. No copying, cheating, collaborations, computers, or cell phones are allowed. Show your work and write complete and coherent answers. Be sure to clearly label and justify your solutions to these problems. Illegible or unjustified solutions will not receive full credit. After you have completed the exam, please reaffirm the Lawrence University Honor Code.

1. Let X have an Exponential distribution with parameter λ . Suppose we wish to test $H_0 : \lambda \geq 1$ vs. $H_a : \lambda < 1$ and will reject H_0 in favor of H_a if $X \geq 2$.

(a) Find the power of this test for a general value of λ (i.e., your answer should be left in terms of λ).

(b) Calculate the probability of making a type I error if $\lambda = 1$.

(c) What value of λ will result in the largest probability of making a type I error?

2. Let $Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} \text{Unif}[0, \theta]$.

(a) Write down the likelihood function.

(b) Suppose that you decide to use a Pareto prior distribution for θ that has PDF

$$\pi(\theta|\alpha, \beta) = \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}}, \quad \text{for } \theta \geq \beta > 0, \alpha > 0.$$

Find the posterior density of θ .

(c) Describe how you would create a 90% credible interval for θ from the posterior distribution.

3. The following problem is based on a study to compare the economic value of two types of information about Genetical Modified Organisms (GMO). Participants in the study read one page of information about GMO foods then participated in an experimental auction. Their bids during the auction provide information about the dollar value of the information. Participants are randomly assigned to read either a pro-GMO page of information produced by a major biotech company or an anti-GMO page of information produced by a major environmental organization. The response, measured on each participant, is the dollar value of the information they read. Forty (40) participants started the study. Three refused to bid and are omitted from the data summary.

Group	n_i	average	s.d.
pro-GMO	19	0.80	0.284
anti-GMO	18	1.23	0.310

(a) Calculate the pooled estimate of the standard deviation.

- (b) Calculate a 95% confidence interval for the difference in means (as pro-GMO - anti-GMO). (Plug in completely, but do not simplify. Be specific about what critical value(s) need to be found.)
- (c) Suppose that the interval you found in part (b) was $(-.628, -.232)$. Provide a one sentence interpretation of the confidence interval you calculated in part (b).
- (d) Is there a statistically significant difference in the economic value of pro- and anti-GMO information? If so, at what significance level?
- (e) What assumptions must hold for your confidence interval in part (b) to be appropriate?

4. Suppose that a single observation X_1, X_2, \dots, X_n is a random sample from a normal distribution with known mean μ and unknown variance σ^2 , and the following simple hypotheses are to be tested: $H_0 : \sigma^2 = 2$ vs. $H_a : \sigma^2 = 3$. Derive the most powerful test, using a general α .

5. Suppose that $Y \sim \text{Normal}(0, \sigma^2)$.

(a) Show that Y^2/σ^2 is a pivotal quantity.

- (b) Suppose that you observed $Y = 2$. Use the pivotal quantity Y^2/σ^2 and the R output below to find a 95% upper confidence bound for σ^2 .

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> qchisq(.025, df = 1)
[1] 0.0009820691
> qchisq(.975, df = 1)
[1] 5.023886
> qchisq(.05, df = 1)
[1] 0.00393214
> qchisq(.95, df = 1)
[1] 3.841459
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6. Explain the differences in the interpretations between a frequentist confidence interval and a Bayesian credible interval.

7. A certain type of electronic component has a lifetime X (in hours) with PDF given by

$$f(x|\theta) = \theta^{-2} x e^{-x/\theta}, \quad x > 0, \theta > 0$$

with mean $E(X) = 2\theta$ and variance $\text{Var}(X) = 2\theta^2$. Suppose that a random sample of n of these components is collected.

- (a) Derive the log-likelihood function for these data.

- (b) Find the maximum likelihood estimator for θ .

(c) Suppose that we wish to test $H_0 : \theta = \theta_0$ vs. $H_a : \theta \neq \theta_0$. Derive the formula for the asymptotic likelihood ratio test statistic.

(d) Specify the rejection region for the asymptotic likelihood ratio test described in part (c). If you would need R to do this, explain what you would need to find in R.

Name	Param.	PMF or PDF	Mean	Variance	MGF
Bernoulli	p	$P(X = 1) = p$ $P(X = 0) = q$	p	pq	$pe^t + q$
Binomial	n, p	$\binom{n}{k} p^k q^{n-k}$ $k \in \{0, 1, \dots, n\}$	np	npq	$(pe^t + 1 - p)^n$
Geometric	p	pq^k $k \in \{0, 1, 2, \dots\}$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qe^t}$
Negative Binomial	r, p	$\binom{r+n-1}{r-1} p^r q^n$ $k \in \{0, 1, 2, \dots\}$	$\frac{rq}{p}$	$\frac{rq}{p^2}$	$\left(\frac{p}{1 - qe^t}\right)^r$
Hypergeometric	w, b, n	$\frac{\binom{w}{k} \binom{b}{n-k}}{\binom{w+b}{n}}$ $k \in \{0, 1, \dots, n\}$	$\mu = \frac{nw}{w+b}$	$\left(\frac{w+b-n}{w+b-1}\right) n \frac{\mu}{n} \left(1 - \frac{\mu}{n}\right)$	
Poisson	λ	$\frac{e^{-\lambda} \lambda^k}{k!}$ $k \in \{0, 1, 2, \dots\}$	λ	λ	$e^{\lambda(e^t - 1)}$
Uniform	$a < b$	$\frac{1}{b-a}, \quad x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$
Normal	μ, σ^2	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ $x \in \mathbb{R}$	μ	σ^2	$e^{\mu t + (\sigma^2 t^2)/2}$
Exponential	λ	$\lambda e^{-\lambda x}, \quad x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$
Gamma	a, λ	$\frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}$ $x > 0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$	$\left(\frac{\lambda}{\lambda - t}\right)^a$
Beta	a, b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}$ $0 < x < 1$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$	$1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{a+r}{a+b+r}\right) \frac{t^k}{k!}$
Chi-Square	n	$\frac{1}{2^{n/2} \Gamma(n/2)} x^{n/2-1} e^{-x/2}$ $x > 0$	n	$2n$	$\left(\frac{1}{1-2t}\right)^{n/2}$