

I.6) Eigenvalues and Eigenvectors

$$Ax = \lambda x, \quad x \neq 0$$

(A is $n \times n$)

x is an eigenvector and λ is the corresponding eigenvalue.

Then $(A - \lambda I)x = 0$, so $A - \lambda I$ has linearly dependent columns, which means that $A - \lambda I$ is not invertible.

$$\therefore \det(A - \lambda I) = 0.$$

Note that $\det(A - \lambda I)$ is an n -th degree polynomial, so it has n roots, counting multiplicities, which are the eigenvalues of the matrix A .

Example: (#22)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A .

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - (-1)(-1) = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 1, 3$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\underline{\lambda=1}: (A - \lambda I)x = 0$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 - x_2 = 0$$

x_2 is free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{\lambda=3}: \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 + x_2 = 0$$

x_2 is free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3, \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Check:

$$Ax_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 x_1 \quad \checkmark$$

$$Ax_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \lambda_2 x_2 \quad \checkmark$$

Note that:

(1) The trace of A

$$\text{tr}(A) = a_{11} + \cdots + a_{nn}$$

(ie the sum of the diagonal entries)

is equal to

$$\lambda_1 + \cdots + \lambda_n$$

(ie the sum of the eigenvalues).

(2) The determinant of A is equal to

$$\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$$

(ie the product of the eigenvalues).

[Thus A is invertible iff A
does not have a zero eigenvalue.]

(3) Symmetric matrices ($A^T = A$)

have real eigenvalues.

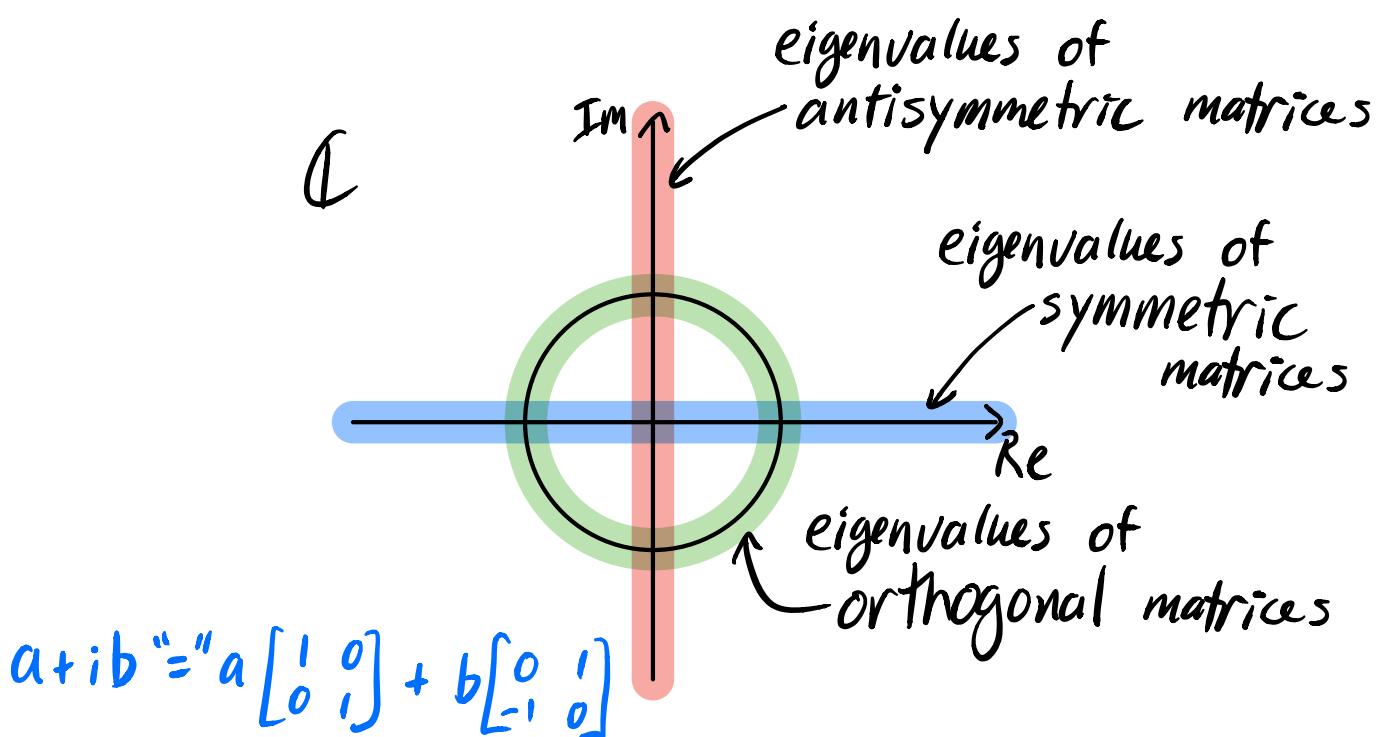
(4) Antisymmetric matrices ($A^T = -A$)

have imaginary eigenvalues.

(5) Orthogonal matrices ($Q^T Q = I$)

have $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$

(ie $|\lambda|=1$) eigenvalues.



(6) Eigenvectors of $A \in \mathbb{R}^{n \times n}$ are orthogonal iff $A^T A = A A^T$. *normal matrix*

\therefore Symmetric, antisymmetric, and orthogonal matrices have orthog. eigenvectors.

- (7) The eigenvalues of A^k are λ_i^k with eigenvectors x_i .
- (8) If A is invertible (ie $\lambda_i \neq 0, \forall i$) then the eigenvalues of A^{-1} are λ_i^{-1} with eigenvectors x_i .
- (9) The eigenvalues of $A+sI$ are λ_i+s with eigenvectors x_i .
- (10) The eigenvalues of $A+B$ are usually not $\lambda(A)$ plus $\lambda(B)$.
- (11) The eigenvalues of AB are usually not $\lambda(A)$ times $\lambda(B)$.
- (12) Eigenvectors for distinct eigenvalues are lin. indep.; a double eigenvalue $\lambda_1 = \lambda_2$ might not have two lin. indep. eigenvectors.

Diagonalizing a Matrix

If B is invertible, then BAB^{-1} has the same eigenvalues as A .

Pf:

$$\begin{aligned} & \det(BAB^{-1} - \lambda I) \\ &= \det(B(A - \lambda I)B^{-1}) \\ &= \det(B) \det(A - \lambda I) \det(B^{-1}) \\ &= \cancel{\det(B)} \det(A - \lambda I) \cancel{\det(B)^{-1}} \\ &= \det(A - \lambda I) \end{aligned}$$

We say that BAB^{-1} is similar to A .

If x is an eigenvector of A , then

Bx is an eigenvector of BAB^{-1} .

Pf:

$$\begin{aligned} Ax &= \lambda x \\ (BAB^{-1})(Bx) &= BA(B^{-1}B)x = BAIx \\ &= BAx = B(\lambda x) \\ &= \lambda(Bx) \end{aligned}$$

The eigenvalues of a triangular matrix are the diagonal entries.

So if we can find a triangular matrix that is similar to A , then we have found the eigenvalues of A .

If A has n linearly independent eigenvectors x_1, \dots, x_n having eigenvalues $\lambda_1, \dots, \lambda_n$ and

$$X = [x_1 \cdots x_n], \quad \Lambda = [\lambda_1 \cdots \lambda_n],$$

$Ax_i = \lambda_i x_i$

then $AX = X\Lambda$, so we have

$$A = X\Lambda X^{-1}.$$

$$X^{-1}AX = \Lambda$$

Then A is similar to the diagonal matrix Λ , so we have diagonalized the matrix A , and $A^k = X\Lambda^k X^{-1}$.

$\{x_1, \dots, x_n\}$ linearly independent implies that every n -dimensional vector v can be written as

$$v = c_1 x_1 + \dots + c_n x_n.$$

Then,

$$A^k v = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n.$$

Pf: $Ax_i = \lambda_i x_i$

$$A^2 x_i = \lambda_i A x_i = \lambda_i^2 x_i$$

⋮

$$[A^k x_i = \lambda_i^k x_i]$$

$$A^k v = A^k(c_1 x_1 + \dots + c_n x_n)$$

$$= c_1 A^k x_1 + \dots + c_n A^k x_n$$

$$= c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$$

$$A^k v = X \Lambda^k X^{-1} v \quad v = Xc \quad c = X^{-1} v$$

$$= X \Lambda^k c = X \begin{bmatrix} \lambda_1^{k c_1} \\ \vdots \\ \lambda_n^{k c_n} \end{bmatrix} = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$$

Example: (#22) (cont.)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Prove that

$$A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad A = XAX^{-1}$$

$$A^k = X A^k X^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$Q^{-1} = Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -3^k \\ 1 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

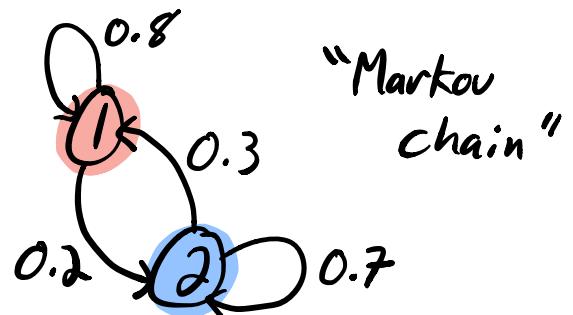
$$= \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix} \quad \checkmark$$



Example:

A Markov matrix is a square matrix with positive columns summing to one.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$



Find $A^\infty = \lim_{k \rightarrow \infty} A^k$.

$$A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2} \quad X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \|X_1\| = 1$$

$$\lambda_2 = 1 \quad X_2 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \|X_2\| = 1$$

$$A^k = X \Lambda^k X^{-1} \quad X = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} (\lambda_2)^k & 0 \\ 0 & 1^k \end{bmatrix} \cdot \frac{-1}{5} \begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix} \quad X^{-1} = \frac{-1}{5} \begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix}$$

$$= \frac{-1}{5} \begin{bmatrix} -(\lambda_2)^k & 3 \\ (\lambda_2)^k & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} -\frac{1}{2^{k-1}} - 3 & 3\left(\frac{1}{2}\right)^k - 3 \\ \frac{1}{2^{k-1}} - 2 & -3\left(\frac{1}{2}\right)^k - 2 \end{bmatrix}$$

$$\rightarrow -\frac{1}{5} \begin{bmatrix} 0-3 & 0-3 \\ 0-2 & 0-2 \end{bmatrix} = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix} = A^\infty.$$

$$\begin{aligned} A^k v &= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 \\ &= c_1 \left(\frac{1}{2}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 (1)^k \begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightarrow c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} Xc = v &\Rightarrow c = X^{-1}v = -\frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} (3v_2 - 2v_1)/5 \\ (v_1 + v_2)/5 \end{bmatrix} \end{aligned}$$

Assuming v_1 and v_2 are probabilities, $v_1 + v_2 = 1$.

$$\text{Then } c_2 = \frac{1}{5}, \text{ so } c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}.$$

$$\therefore A^\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}, A^\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix} \Rightarrow A^\infty = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix}$$