Part I: Highlights of Linear Algebra

I.1:	Matrix-vector multiplication	Ax
I.2:	Matrix-matrix multiplication	AB
I.5:	Orthogonal matrices	$Q^TQ = I$
I.6:	Eigenvalues and eigenvectors	$Ax = \lambda x$
I.7:	Symmetric positive definite matrices	
I.8:	Singular value decomposition	$Av = \sigma u$
I.9:	Principal component analysis (PCA)	
I.11:	Norms of vectors and matrices	$\|x\|_2$

I.1) Multiplication Ax Using Columns of A

Let A be an $m \times n$ matrix and x is a vector of length n.

<u>Example:</u> $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \in \mathbb{R}^{3\times 2}$, $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

(3x2 matrix)

Two ways to multiply A times x:

① $$Ax = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

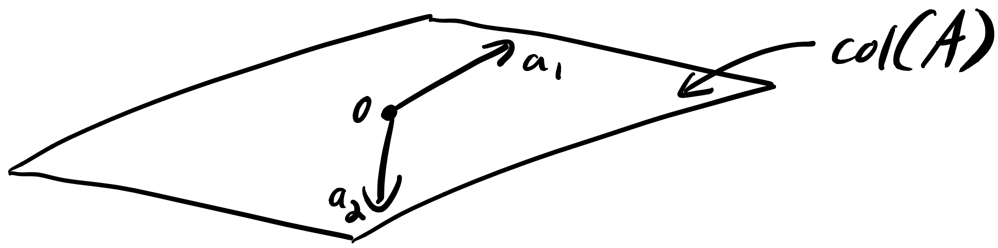
(<u>ie</u> <u>inner-product</u> of <u>rows</u> of A with x)

② $Ax = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

(ie Ax is a linear combination of the columns of A)

The column space of A is the set of all linear combinations of the columns of A.

eg



col(A) is a plane in $\mathbb{R}^3$

A vector $b \in \mathbb{R}^3$ is in col(A) if $b = Ax$ for some $x \in \mathbb{R}^2$.

Example: $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$

① Is $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \in \text{col}(A)$? Yes

$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow b = Ax$

② Is $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in col(A)$? Yes

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b = Ax$

③ Is $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \in col(A)$? Yes $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Ax = -1 \cdot \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -2+3 \\ -2+4 \\ -3+7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

④ Is $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in col(A)$? No

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \\ 3x_1 + 7x_2 &= 1 \end{aligned} \Rightarrow x_2 = 0 \Rightarrow \begin{aligned} x_1 &= 1/2 \\ x_1 &= 1/2 \\ x_1 &= 1/3 \end{aligned}$$

$\therefore$ not possible.

Example:

$A_2 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$

$col(A_2) = ?$ $col(A_3) = ?$

$col(A_2) = col(A)$ $col(A_3) \neq col(A)$

since $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \in col(A)$. $col(A_3) = \mathbb{R}^3$

Vectors a_1, a_2, a_3 are <u>linearly independent</u> if the only combination that gives the zero vector is $0 \cdot a_1 + 0 a_2 + 0 a_3$.

If, for example,

$$2a_1 + 3a_2 - a_3 = 0$$

then a_1, a_2, a_3 are <u>linearly dependent</u>.

If $A \in \mathbb{R}^{3\times 3}$ has lin. indep. columns

$$\left(\underline{\text{ie}} \quad Ax = 0 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

then A is an <u>invertible matrix</u>

$$\left(\underline{\text{ie}} \quad A^{-1}A = AA^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

and $Ax = b$ has exactly one solution $x = A^{-1}b$ for any $b \in \mathbb{R}^3$.

$$\boxed{A \in \mathbb{R}^{n\times n} \text{ is invertible} \iff col(A) = \mathbb{R}^n.}$$

Independent Columns and the Rank of A

By removing lin. dep. columns of $A \in \mathbb{R}^{m \times n}$, we can get a basis for col(A).

Putting the remaining r lin. indep. cols into a matrix C, we have col(A) = col(C).

$$\boxed{r = \text{rank}(A) = \text{dimension of col}(A).}$$

Example: Find the matrix C.

① $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$ rank(A) = 2

$C = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$ $r = 2$

② $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$ rank(A) = 3

$C = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$ $r = 3$

③ $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ $\quad \operatorname{rank}(A) = 1$

$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad r = 1$

Since each column of A is a lin. comb. of the columns of C, we have

$$\boxed{A = CR.}$$

<u>Example</u>: Find the matrix R.

① $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$A = CR$

② $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$\vdots$

$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = CR$

③ $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$A = CR$

$\vdots$

This gives us the following theorem:

$\boxed{\text{The number of independent columns equals the number of independent rows.}}$

The rows of R give us a basis for the row space of A: $\text{row}(A) = \text{row}(R)$.

$$\boxed{\text{Column rank} = \text{Row rank}}$$

$$\dim(\text{col}(A)) = \dim(\text{col}(A^T))$$