I.6 Eigenvalues and Eigenvectors

$$\boxed{Ax = \lambda x, \quad x \neq 0} \qquad (A \text{ is } n\times n)$$

x is an eigenvector and λ is the corresponding eigenvalue.

Then $(A-\lambda I)x = 0$, so $A-\lambda I$ has linearly dependent columns, which means that $A-\lambda I$ is not invertible.

$$\therefore \boxed{\det(A-\lambda I) = 0.}$$

Note that $\det(A-\lambda I)$ is an n-th degree polynomial, so it has n roots, counting multiplicities, which are the eigenvalues of the matrix A.

<u>Example:</u> (#22)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Note that:

(1) The trace of A

$$tr(A) = a_{11} + \cdots + a_{nn}$$

(ie the sum of the diagonal entries)

is equal to

$$\lambda_1 + \cdots + \lambda_n$$

(ie the sum of the eigenvalues).

(2) The determinant of A is equal to

$$\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$$

(ie the product of the eigenvalues).

Thus A is invertible iff A does not have a zero eigenvalue.

(3) Symmetric matrices $(A^T = A)$ have real eigenvalues.

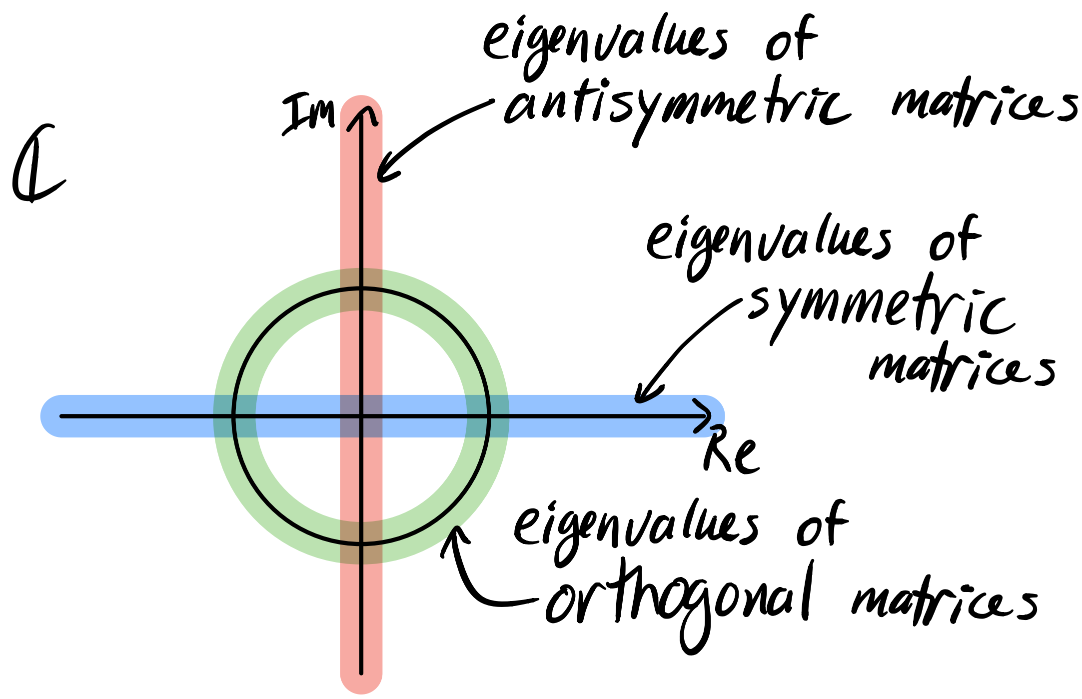
(4) **Antisymmetric matrices** ($A^T = -A$)

have imaginary eigenvalues.

(5) **Orthogonal matrices** ($Q^T Q = I$)

have $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$

(ie $|\lambda| = 1$) eigenvalues.



(6) Eigenvectors of $A \in \mathbb{R}^{n \times n}$ are orthogonal iff $A^T A = A A^T$.

$\therefore$ symmetric, antisymmetric, and orthogonal matrices have orthog. eigenvectors.

(7) The eigenvalues of A^k are λ_i^k with eigenvectors x_i.

(8) If A is invertible (ie $\lambda_i \neq 0, \forall i$) then the eigenvalues of A^{-1} are λ_i^{-1} with eigenvectors x_i.

(9) The eigenvalues of $A+sI$ are λ_i+s with eigenvectors x_i.

(10) The eigenvalues of $A+B$ are usually not $\lambda(A)$ plus $\lambda(B)$.

(11) The eigenvalues of AB are usually not $\lambda(A)$ times $\lambda(B)$.

(12) Eigenvectors for distinct eigenvalues are lin. indep.; a double eigenvalue $\lambda_1=\lambda_2$ might not have two lin. indep. eigenvectors.

Diagonalizing a Matrix

If B is invertible, then BAB^{-1} has the same eigenvalues as A.

Pf:

We say that BAB^{-1} is <u>similar</u> to A.

If x is an eigenvector of A, then ______ is an eigenvector of BAB^{-1}.

Pf:

The eigenvalues of a <u>triangular matrix</u> are the diagonal entries.

So if we can find a triangular matrix that is similar to A, then we have found the eigenvalues of A.

If A has <u>n linearly independent eigenvectors</u> $x_1, \ldots, x_n$ having eigenvalues $\lambda_1, \ldots, \lambda_n$ and

$$X = [x_1 \cdots x_n], \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

then $AX = X\Lambda$, so we have

$$\boxed{A = X\Lambda X^{-1}.}$$

Then A is similar to the diagonal matrix Λ, so we have <u>diagonalized</u> the matrix A, and $A^k = X\Lambda^k X^{-1}$.

$\{x_1, \dots, x_n\}$ linearly independent implies that every n-dimensional vector v can be written as

$$v = c_1 x_1 + \cdots + c_n x_n.$$

Then,

$$A^k v = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n.$$

Pf:

Example: (#22) (cont.)

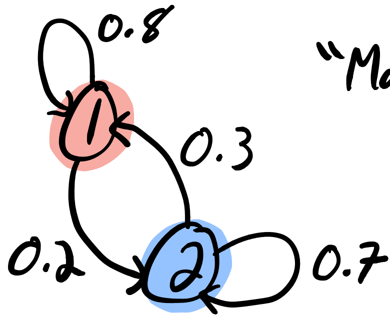
Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Prove that

$$A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

Example:

A **Markov matrix** is a square matrix with positive columns summing to one.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$



"Markov chain"

Find $A^{\infty} = \lim_{k \to \infty} A^k$.