I.6 <u>Eigenvalues and Eigenvectors</u>

$$\boxed{Ax = \lambda x, \quad x \neq 0} \qquad (A \text{ is } n \times n)$$

x is an <u>eigenvector</u> and λ is the corresponding <u>eigenvalue</u>.

Then $(A - \lambda I)x = 0$, so $A - \lambda I$ has <u>linearly dependent columns</u>, which means that $A - \lambda I$ is not invertible.

$$\therefore \boxed{\det(A - \lambda I) = 0.}$$

Note that $\det(A - \lambda I)$ is an <u>n-th degree polynomial</u>, so it has <u>n roots</u>, counting multiplicities, which are the <u>eigenvalues</u> of the matrix A.

Example: (#22)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - (-1)(-1) = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 1, 3$$

$\underline{\lambda = 1}$: $(A - \lambda I)x = 0$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0$

x_2 is free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 3}$: $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 + x_2 = 0$

x_2 is free

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 1, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3, \quad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

<u>Check:</u>

$$Ax_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 x_1 \quad \checkmark$$

$$Ax_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \lambda_2 x_2 \quad \checkmark$$

__Note that:__

(1) The __trace of A__

$$tr(A) = a_{11} + \cdots + a_{nn}$$

(__ie__ the sum of the diagonal entries)

is equal to

$$\lambda_1 + \cdots + \lambda_n$$

(__ie__ the sum of the eigenvalues).

(2) The __determinant of A__ is equal to

$$\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$$

(__ie__ the product of the eigenvalues).

[Thus A is invertible __iff__ A does not have a zero eigenvalue.]

(3) __Symmetric matrices__ ($A^T = A$) have __real eigenvalues__.

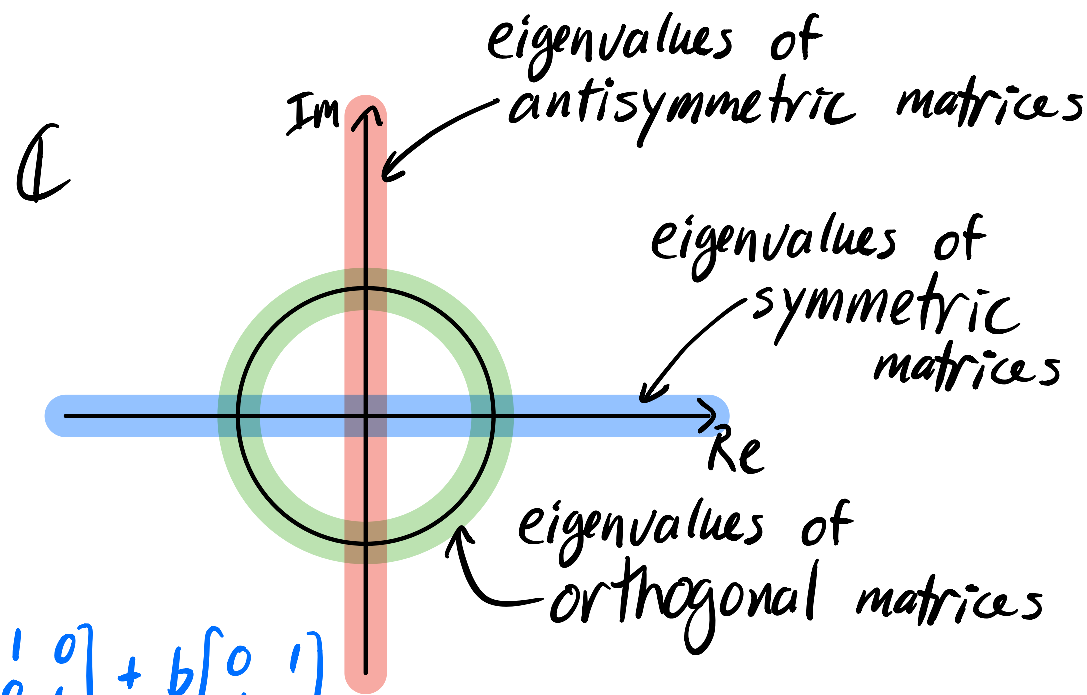
(4) <u>**Antisymmetric matrices**</u> ($A^T = -A$)

have <u>imaginary eigenvalues</u>.

(5) <u>**Orthogonal matrices**</u> ($Q^TQ = I$)

have $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$

(<u>ie</u> $|\lambda| = 1$) eigenvalues.



$a + ib$ "=" $a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

(6) Eigenvectors of $A \in \mathbb{R}^{n\times n}$ are orthogonal <u>iff</u> $A^TA = AA^T$. normal matrix

$\therefore$ symmetric, antisymmetric, and orthogonal matrices have orthog. eigenvectors.

(7) The eigenvalues of A^k are λ_i^k with eigenvectors x_i.

(8) If A is invertible (ie $\lambda_i \neq 0, \forall i$) then the eigenvalues of A^{-1} are λ_i^{-1} with eigenvectors x_i.

(9) The eigenvalues of $A+sI$ are $\lambda_i + s$ with eigenvectors x_i.

(10) The eigenvalues of $A+B$ are usually not $\lambda(A)$ plus $\lambda(B)$.

(11) The eigenvalues of AB are usually not $\lambda(A)$ times $\lambda(B)$.

(12) Eigenvectors for distinct eigenvalues are lin. indep.; a double eigenvalue $\lambda_1 = \lambda_2$ might not have two lin. indep. eigenvectors.

Diagonalizing a Matrix

If B is invertible, then BAB^{-1} has the same eigenvalues as A.

Pf: $\det(BAB^{-1} - \lambda I)$

$= \det(B(A-\lambda I)B^{-1})$

$= \det(B)\det(A-\lambda I)\det(B^{-1})$

$= \cancel{\det(B)}\det(A-\lambda I)\cancel{\det(B)^{-1}}$

$= \det(A-\lambda I)$

We say that BAB^{-1} is <u>similar</u> to A.

If x is an eigenvector of A, then <u>Bx</u> is an eigenvector of BAB^{-1}.

Pf: $Ax = \lambda x$

$(BAB^{-1})(Bx) = BA(B^{-1}B)x = BAIx$

$= BAx = B(\lambda x)$

$= \lambda(Bx)$

The eigenvalues of a <u>triangular matrix</u> are the diagonal entries.
So if we can find a triangular matrix that is similar to A, then we have found the eigenvalues of A.

If A has <u>n linearly independent eigenvectors</u> $x_1, \ldots, x_n$ having eigenvalues $\lambda_1, \ldots, \lambda_n$ and

$$X = [x_1 \cdots x_n], \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

$Ax_i = \lambda_i x_i$

then $AX = X\Lambda$, so we have

$$\boxed{A = X\Lambda X^{-1}.}$$

$X^{-1}AX = \Lambda$

Then A is similar to the diagonal matrix Λ, so we have <u>diagonalized</u> the matrix A, and $A^k = X\Lambda^k X^{-1}$.

$\{x_1, \dots, x_n\}$ linearly independent implies that every n-dimensional vector v can be written as

$$v = c_1 x_1 + \cdots + c_n x_n.$$

Then,

$$A^k v = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n.$$

Pf: $Ax_i = \lambda_i x_i$

$$A^2 x_i = \lambda_i A x_i = \lambda_i^2 x_i$$

$$\vdots$$

$$\left[A^k x_i = \lambda_i^k x_i\right]$$

$$\begin{aligned} A^k v &= A^k (c_1 x_1 + \cdots + c_n x_n) \\ &= c_1 A^k x_1 + \cdots + c_n A^k x_n \\ &= c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n \end{aligned}$$

$$A^k v = X \Lambda^k X^{-1} v \qquad v = Xc \qquad c = X^{-1} v$$

$$= X \Lambda^k c = X \begin{bmatrix} \lambda_1^k c_1 \\ \vdots \\ \lambda_n^k c_n \end{bmatrix} = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$$

Example: (#22) (cont.)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Prove that

$$A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

$X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A = X\Lambda X^{-1}$

$A^k = X\Lambda^k X^{-1}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$Q^{-1} = Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^k = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 3^k \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

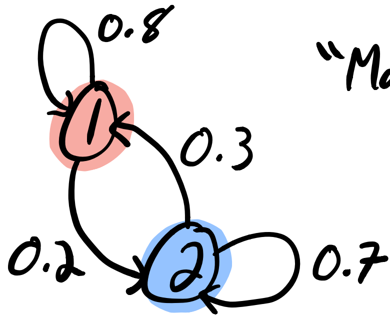
$$= \frac{1}{2}\begin{bmatrix} 1 & -3^k \\ 1 & 3^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix} \checkmark$$

Example:

A Markov matrix is a square matrix with positive columns summing to one.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$



"Markov chain"

Find $A^{\infty} = \lim_{k \to \infty} A^k$.

$$A^{\infty} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

$\lambda_1 = \frac{1}{2}$ $\quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$ $\quad x_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$