VI.3 Linear Programming, ~~Game Theory~~, and Duality

We saw that a constrained optimization problem (the <u>primal</u> problem)

$$(P) \quad \boxed{\min f(x) \text{ subject to } g(x) = 0}$$

can be written as

$$(P) \quad \boxed{\min_x \max_\lambda L(x, \lambda)}$$

where $\lambda \in \mathbb{R}^m$ are the Lagrange multipliers for the constraints $g(x) = 0$ (here $g : \mathbb{R}^n \to \mathbb{R}^m$), and

$$\boxed{L(x, \lambda) = f(x) + \lambda^T g(x)}$$

is the Lagrangian.

By changing the order of the min and the max, we obtain the <u>dual</u> problem

$$(D) \quad \boxed{\max_{\lambda} \min_{x} L(x,\lambda)}$$

which satisfies

$$\max_{\lambda} \min_{x} L(x,\lambda) \leqslant \min_{x} \max_{\lambda} L(x,\lambda).$$

This is <u>weak duality</u>: the dual gives a lower bound on the optimal value of the primal.

If we can find λ^* and x^* such that $\min_{x} L(x,\lambda^*) = \max_{\lambda} L(x^*,\lambda)$, then λ^* is opt. for (D), x^* is opt. for (P), and their opt. values match.

Thus, λ^* is a certificate that x^* is opt. for (P) and x^* is a certificate that λ^* is opt. for (D). This is <u>strong duality</u>: there is an opt. certificate λ^* and

$$\max_x L(x, \lambda^*) = \min_x \max_\lambda L(x, \lambda).$$

It is possible that strong duality does not hold for a primal/dual pair of problems. In this case, there is a <u>duality gap</u>:

$$\max_\lambda \min_x L(x, \lambda) < \min_x \max_\lambda L(x, \lambda).$$

This can happen in <u>semidefinite programming</u>, for example, but not in <u>linear programming</u>.

Linear Programming

The primal problem is

$$\text{(P)} \quad \boxed{\min\ c^T x \text{ subject to } Ax = b,\ x \geq 0}.$$

(Any opt. problem with a linear objective and linear equality/inequality constraints can be written in this form.)

We let $y \in \mathbb{R}^m$ be the dual multipliers for the constraints $Ax = b$ (here $A \in \mathbb{R}^{m \times n}$). The Lagrangian is

$$\boxed{L(x, y) = c^T x + y^T (b - Ax)}.$$

Then (P) is

$$\min_{x \geq 0} \max_{y} L(x, y).$$

The <u>dual problem</u> is

$$\max_{y} \min_{x \geq 0} L(x,y).$$

There is a <u>hidden constraint</u> on y. Player Y wants to prevent Player X from driving $L(x,y)$ to $-\infty$. To find the hidden constraint, we rewrite the Lagrangian by combining the x terms:

$$L(x,y) = b^T y + x^T(c - A^T y).$$

If Player Y chooses y such that the i^{th} component of the vector $c - A^T y$ is negative, then Player X can let $x_i \to +\infty$, causing $x_i(c-A^T y)_i$ to go to $-\infty$.

Player Y needs to avoid this situation, so must always choose y so that $c - A^T y \geq 0$. This is the <u>hidden constraint</u>. Then

$$0 = \min_{x \geq 0} x^T (c - A^T y),$$

from which we conclude that

$$\min_{x \geq 0} L(x,y) = \begin{cases} b^T y, & \text{if } A^T y \leq c, \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore, the <u>dual problem</u> is

$$(D) \quad \boxed{\max\ b^T y \text{ subject to } A^T y \leq c}.$$

Let's check <u>weak duality</u>. Let x be feasible for (P) and y feasible for (D); <u>ie</u> $Ax = b$, $x \geq 0$, $A^T y \leq c$.

Then

$$b^T y = (Ax)^T y = (A^T y)^T x \leq c^T x.$$

Weak duality can be used to conclude that if one problem is <u>unbounded</u>, then the other problem is <u>infeasible</u>.

<u>Strong duality</u> always holds: if one problem has an optimal sol'n, then so does the other, and their optimal values agree:

$$c^T x^* = b^T y^*.$$

Equal objective values is equivalent to $x^T(c - A^T y) = 0$, which holds <u>iff</u> $x_i(c - A^T y)_i = 0$, for $i = 1, \ldots, n$. This is <u>complementary slackness</u>.

<u>Note:</u> $x_i > 0 \Rightarrow (c - A^T y)_i = 0,$
$(c - A^T y)_i > 0 \Rightarrow x_i = 0.$

The optimality conditions are:

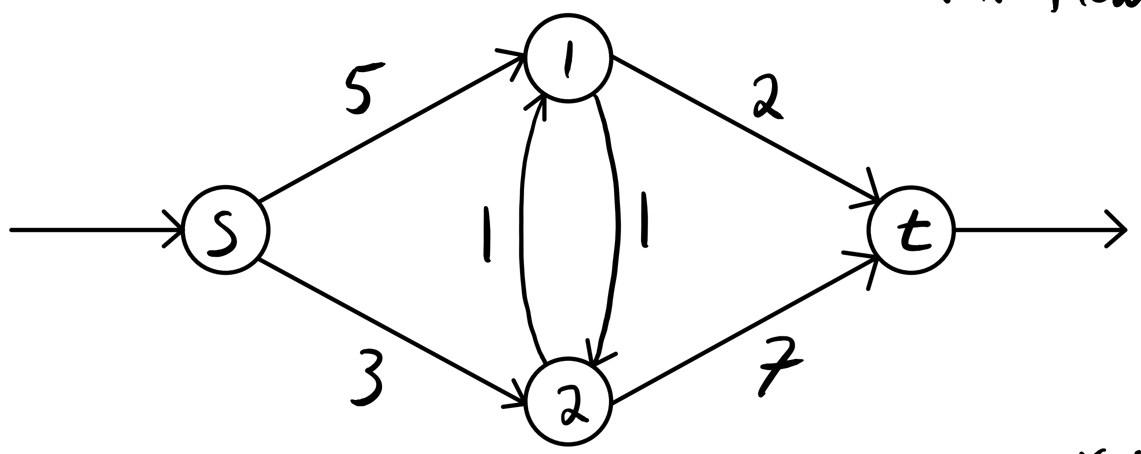
$$\left[\begin{array}{ll} (1) & Ax = b, \quad x \geq 0 \\ (2) & A^T y + s = c, \quad s \geq 0 \\ (3) & x_i s_i = 0, \quad i = 1, \ldots, n. \end{array}\right]$$

The <u>simplex method</u> moves from corner to corner on the polyhedral primal feas. set, keeping (1) and (3) satisfied, until (2) is satisfied.

<u>Interior point methods</u> keep (1) and (2) satisfied with $x > 0$ and $s > 0$ (<u>ie</u> interior), making all the $x_i s_i$ terms smaller and smaller.

Max flow - min cut

max-flow =



$x, u \in \mathbb{R}^6$

$$\left.\begin{array}{l} x_{ij} = \text{flow along edge } ij \\ u_{ij} = \text{capacity of edge } ij \end{array}\right\} \Rightarrow \boxed{x \le u}$$

Flow-in = flow-out at nodes 1 and 2:

1 : $\quad x_{s1} + x_{21} = x_{12} + x_{1t}$

2 : $\quad x_{s2} + x_{12} = x_{21} + x_{2t}$

Max-flow is a linear programming problem:

$$\boxed{\begin{array}{ll} \max & x_{1t} + x_{2t} \\ \text{s.t.} & x_{s1} + x_{21} - x_{12} - x_{1t} = 0 \\ & x_{s2} + x_{12} - x_{21} - x_{2t} = 0 \\ & x \le u \end{array}}$$

This problem has the form

$$\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = 0 \\ & x \le u \end{array}}$$

where

$$A = \begin{array}{c} \begin{array}{cccccc} s1 & s2 & 12 & 1t & 21 & 2t \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array} \begin{array}{c} ① \\ ② \end{array}, \quad x = \begin{bmatrix} x_{s1} \\ x_{s2} \\ x_{12} \\ x_{1t} \\ x_{21} \\ x_{2t} \end{bmatrix}, \quad u = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \\ 1 \\ 7 \end{bmatrix}.$$

$$c^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

Let $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^6_+$ (ie $z \geq 0$) be the dual variables for $Ax = 0$ and $x \le u$, respectively. Then

$$L(x, y, z) = c^T x + y^T A x + z^T (u - x)$$

satisfies

$$\min_{\substack{y \\ z \geq 0}} L(x, y, z) = \begin{cases} c^T x & \text{if } Ax = 0,\ x \le u \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is

$$\min_{\substack{y \\ z \geq 0}} \max_{x} L(x, y, z).$$

Note that

$$\max_{x} L(x, y, z) = \begin{cases} u^T z & \text{if } A^T y + c = z \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, the dual problem is

$$\boxed{\begin{aligned} \min \quad & u^T z \\ \text{s.t.} \quad & A^T y + c = z \\ & z \geq 0. \end{aligned}}$$

The dual variables

$y = [y_1 \ \ y_2]^T$, $z = [z_{s1} \ z_{s2} \ z_{12} \ z_{1t} \ z_{21} \ z_{2t}]^T$

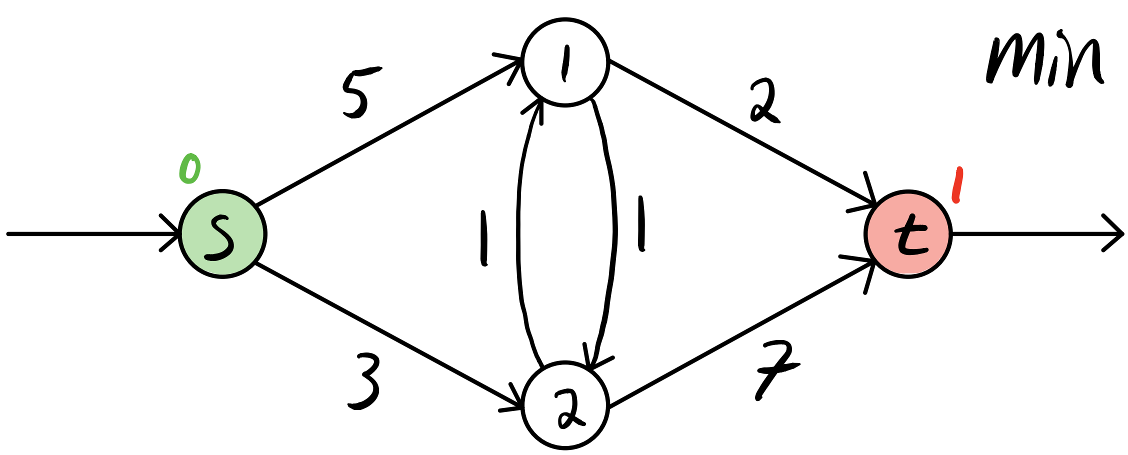
that represent cutting the graph into two disjoint pieces, S and T, with $s \in S$ and $t \in T$.

We have

$$y_i = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \in T, \end{cases}$$

$$z_{ij} = \begin{cases} 1 & \text{if } i \in S \text{ and } j \in T \\ & \text{or } j \in S \text{ and } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\begin{array}{ll} \min & 5z_{s1} + 3z_{s2} + z_{12} + 2z_{1t} + z_{21} + 7z_{2t} \\ \text{s.t.} & y_1 = z_{s1} \\ & y_2 = z_{s2} \\ & -y_1 + y_2 = z_{12} \\ & -y_1 + 1 = z_{1t} \\ & y_1 - y_2 = z_{21} \\ & -y_2 + 1 = z_{2t} \\ & z \geq 0. \end{array}}$$



min cut =