Part I: Highlights of Linear Algebra

I.1:	Matrix-vector multiplication	Ax
I.2:	Matrix-matrix multiplication	AB
I.5:	Orthogonal matrices	$Q^TQ = I$
I.6:	Eigenvalues and eigenvectors	$Ax = \lambda x$
I.7:	Symmetric positive definite matrices	
I.8:	Singular value decomposition	$Av = \sigma u$
I.9:	Principal component analysis (PCA)	
I.11:	Norms of vectors and matrices	$\|x\|_2$

I.1) Multiplication Ax Using Columns of A

Let A be an $m \times n$ matrix and x is a vector of length n.

Example: $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \in \mathbb{R}^{3\times 2}$, $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

(3x2 matrix)

Two ways to multiply A times x:

① $$Ax = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

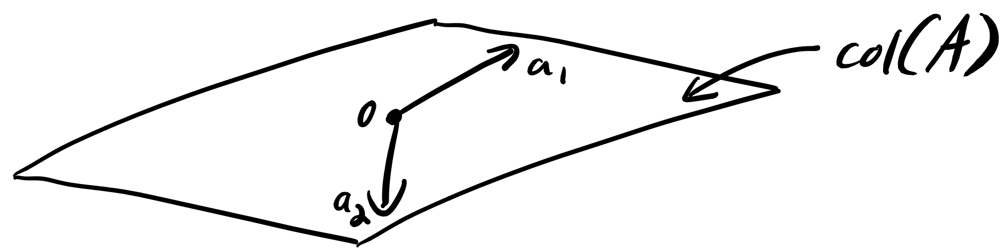
(ie inner-product of rows of A with x)

② $Ax = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

(ie Ax is a linear combination of the columns of A)

The column space of A is the set of all linear combinations of the columns of A.

eg



col(A) is a plane in $\mathbb{R}^3$

A vector $b \in \mathbb{R}^3$ is in col(A) if $b = Ax$ for some $x \in \mathbb{R}^2$.

Example: $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$

① Is $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} \in \mathrm{col}(A)$? Yes

$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow b = Ax$

(2) Is $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in col(A)$? Yes

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow b = Ax$

(3) Is $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \in col(A)$? Yes $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow Ax = -1 \cdot \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -2+3 \\ -2+4 \\ -3+7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

(4) Is $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in col(A)$? No

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \\ 3x_1 + 7x_2 &= 1 \end{aligned} \Rightarrow x_2 = 0 \Rightarrow \begin{aligned} x_1 &= 1/2 \\ x_1 &= 1/2 \\ x_1 &= 1/3 \end{aligned}$$

$\therefore$ not possible.

Example:

$A_2 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$

$col(A_2) = ?$

$col(A_3) = ?$

$col(A_2) = col(A)$

since $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \in col(A)$.

$col(A_3) \neq col(A)$

$col(A_3) = \mathbb{R}^3$

Vectors a_1, a_2, a_3 are <u>linearly independent</u> if the only combination that gives the zero vector is $0 \cdot a_1 + 0 a_2 + 0 a_3$.

If, for example,

$$2a_1 + 3a_2 - a_3 = 0$$

then a_1, a_2, a_3 are <u>linearly dependent</u>.

If $A \in \mathbb{R}^{3\times 3}$ has lin. indep. columns

$$\left(\underline{\text{ie}} \quad Ax = 0 \implies x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

then A is an <u>invertible matrix</u>

$$\left(\underline{\text{ie}} \quad A^{-1}A = AA^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

and $Ax = b$ has exactly one solution $x = A^{-1}b$ for any $b \in \mathbb{R}^3$.

$$\boxed{A \in \mathbb{R}^{n\times n} \text{ is invertible} \iff \operatorname{col}(A) = \mathbb{R}^n.}$$

Independent Columns and the Rank of A

By removing lin. dep. columns of $A \in \mathbb{R}^{m \times n}$, we can get a basis for $col(A)$.

Putting the remaining r lin. indep. cols into a matrix C, we have $col(A) = col(C)$.

$$\boxed{r = \text{rank}(A) = \text{dimension of } col(A).}$$

Example: Find the matrix C.

① $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$ $\quad$ $\text{rank}(A) = 2$

$C = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$ $\quad$ $r = 2$

② $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$ $\quad$ $\text{rank}(A) = 3$

$C = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$ $\quad$ $r = 3$

③ $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ $\quad \mathrm{rank}(A) = 1$

$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad r = 1$

Since each column of A is a lin. comb. of the columns of C, we have

$$\boxed{A = CR.}$$

Example: Find the matrix R.

① $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$A = CR$

② $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

$\vdots$

$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = CR$

③ $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$A = CR$

$\vdots$

This gives us the following theorem:

> The number of independent columns equals the number of independent rows.

The rows of R give us a basis for the row space of A: $\text{row}(A) = \text{row}(R)$.

$$\boxed{\text{Column rank} = \text{Row rank}}$$

$$\dim(\text{col}(A)) = \dim(\text{col}(A^T))$$

I.2 Matrix-Matrix Multiplication AB

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{23} \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$A \qquad B \quad = \quad C$$

$$c_{23} = 1 \cdot 1 + 4 \cdot 4 + 3 \cdot 1 = 20$$

The <u>inner-product</u> formula for $C = AB$ is

$$c_{ij} = a_{i1} b_{1j} + \cdots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

(<u>ie</u> <u>row</u> of A times <u>column</u> of B)

Another way to mult. A and B is using <u>outer-products</u> (<u>ie</u> <u>column</u> of A times <u>row</u> of B).

<u>Example:</u> $u = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

The <u>inner-product</u> of u and v is:

$$u^T v = \begin{bmatrix} 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 2\cdot 1 + 1\cdot 3 + 4\cdot 3 = 17$$

The <u>outer-product</u> of u and v is:

$$u v^T = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 6 & 3 & 12 \\ 6 & 3 & 12 \end{bmatrix} \leftarrow \text{rank} = 1$$

The <u>outer-product</u> formula for AB is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix} = a_1 b_1^T + a_2 b_2^T + a_3 b_3^T$$

(a_1, a_2, a_3 are the columns of A,
b_1^T, b_2^T, b_3^T are the rows of B)

$$\boxed{AB = \text{sum of rank one matrices}}$$

Example: Use both inner- and outer-product formulas to multiply the matrices.

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4\cdot 4+3\cdot 1 & 4\cdot 0+3\cdot 3 \\ 2\cdot 4+0\cdot 1 & 2\cdot 0+0\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 9 \\ 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 8 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 8 & 0 \end{bmatrix}$$

Matrix Factorizations

In the previous section we saw how to factor A into C times R.

$$A = CR = c_1 r_1^T + \cdots + c_k r_k^T$$

Factoring a matrix is the reverse of multiplying matrices and usually takes more effort and time.

Using the outer-product formula for matrix multiplication, the factorization splits the matrix up in some number of rank-one pieces.

We can use this to identify the most important pieces in our data, revealing structure and information in our data.

Six important factorizations are:

① $A = LU$ (from Gaussian elimination)

L is lower-triangular with ones on the diagonal.

U is upper-triangular.

② $A = LDL^T$ and $A = R^T R$ (Cholesky)

If A is symmetric, $A = LU$ becomes $A = LDL^T$, where D is a diagonal matrix. If the eigenvalues of A are all positive then $A = R^T R$ is possible.

③ $A = QR$ (from orthogonalizing the columns of A, as in "Gram-Schmidt")

Q is orthogonal ($Q^T Q = I$).

R is upper-triangular.

④ $A = Q \Lambda Q^T$ (eigenvalue decomposition)

A is symmetric, so eigenvalues are real.

Q is orthogonal, so $AQ = Q\Lambda$.

Λ is diagonal. Thus, $Aq_i = \lambda_i q_i$,

so the columns of Q are the eigenvectors and the diagonal entries of Λ are the eigenvalues.

⑤ $A = X \Lambda X^{-1}$ (diagonalization)

If A is nonsymmetric but has lin. indep. eigenvectors, then A can be diagonalized.

⑥ $A = U\Sigma V^T$ (singular value decomposition)
$(m \times n)$

U and V are othogonal.

Σ is diagonal with singular values

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ on its diagonal.

$(r = \text{rank}(A))$

Thus, $Av_i = \sigma_i u_i$.

u_i are the left-singular vectors.

v_i are the right-singular vectors.

Big Idea: The outer-product multiplication formula tells us that

$$A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T$$

Since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, we see that $\sigma_1 u_1 v_1^T$ is the most important rank-one piece of A.

(Numerical demonstration)