

VII.3 | Linear Programming, ~~Game Theory~~, and Duality

We saw that a constrained optimization problem (The primal problem)

$$(P) \quad \min f(x) \text{ subject to } g(x) = 0$$

can be written as

$$(P) \quad \min_x \max_{\lambda} L(x, \lambda)$$

where $\lambda \in \mathbb{R}^m$ are the Lagrange multipliers for the constraints $g(x) = 0$ (here $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$), and

$$L(x, \lambda) = f(x) + \lambda^T g(x)$$

is the Lagrangian.

By changing the order of the min and the max, we obtain the dual problem

(D)

$$\max_{\lambda} \min_x L(x, \lambda)$$

which satisfies

$$\max_{\lambda} \min_x L(x, \lambda) \leq \min_x \max_{\lambda} L(x, \lambda).$$

This is weak duality: The dual gives a lower bound on the optimal value of the primal.

If we can find λ^* and x^* such that $\min_x L(x, \lambda^*) = \max_{\lambda} L(x^*, \lambda)$,

then λ^* is opt. for (D), x^* is opt. for (P), and their opt. values match.

Thus, λ^* is a certificate that x^* is opt. for (P) and x^* is a certificate that λ^* is opt. for (D). This is strong duality: There is an opt. certificate λ^* and

$$\max_x L(x, \lambda^*) = \min_x \max_{\lambda} L(x, \lambda).$$

It is possible that strong duality does not hold for a primal/dual pair of problems. In this case, there is a duality gap:

$$\max_{\lambda} \min_x L(x, \lambda) < \min_x \max_{\lambda} L(x, \lambda).$$

This can happen in semidefinite programming, for example, but not in linear programming.

Linear Programming

The primal problem is

$$(P) \quad \min c^T x \text{ subject to } Ax=b, x \geq 0.$$

(Any opt. problem with a linear objective and linear equality/inequality constraints can be written in this form.)

We let $y \in \mathbb{R}^m$ be the dual multipliers for the constraints $Ax=b$ (here $A \in \mathbb{R}^{m \times n}$). The Lagrangian is

$$L(x, y) = c^T x + y^T(b - Ax).$$

Then (P) is

$$\min_{x \geq 0} \max_y L(x, y).$$

The dual problem is

$$\max_y \min_{x \geq 0} L(x, y).$$

There is a hidden constraint on y .

Player Y wants to prevent

Player X from driving $L(x, y)$ to $-\infty$. To find the hidden constraint, we rewrite the Lagrangian by combining the x terms:

$$L(x, y) = c^T x + y^T(b - Ax)$$

$$L(x, y) = b^T y + x^T(c - A^T y).$$

If Player Y chooses y such that the i^{th} component of the vector $c - A^T y$ is negative, then Player X can let $x_i \rightarrow +\infty$, causing $x_i(c - A^T y)_i$ to go to $-\infty$.

Player Y needs to avoid this situation, so must always choose y so that $c - A^T y \geq 0$. This is the hidden constraint. Then

$$0 = \min_{x \geq 0} x^T(c - A^T y),$$

from which we conclude that

$$\min_{x \geq 0} L(x, y) = \begin{cases} b^T y, & \text{if } A^T y \leq c, \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore, the dual problem is

$$(D) \quad \boxed{\max b^T y \text{ subject to } A^T y \leq c}.$$

Let's check weak duality. Let x be feasible for (P) and y feasible for (D); ie $Ax=b$, $x \geq 0$, $A^T y \leq c$.

Then

$$b^T y = (Ax)^T y = (A^T y)^T x \leq c^T x.$$

Weak duality can be used to conclude that if one problem is unbounded, then the other problem is infeasible.

Strong duality always holds: if one problem has an optimal sol'n, then so does the other, and their optimal values agree:

$$c^T x^* = b^T y^*.$$

Equal objective values is equivalent to $x^T(c - A^T y) = 0$, which holds iff $x_i(c - A^T y)_i = 0$, for $i = 1, \dots, n$. This is complementary slackness.

Note: $x_i > 0 \Rightarrow (c - A^T y)_i = 0,$
 $(c - A^T y)_i > 0 \Rightarrow x_i = 0.$

The optimality conditions are:

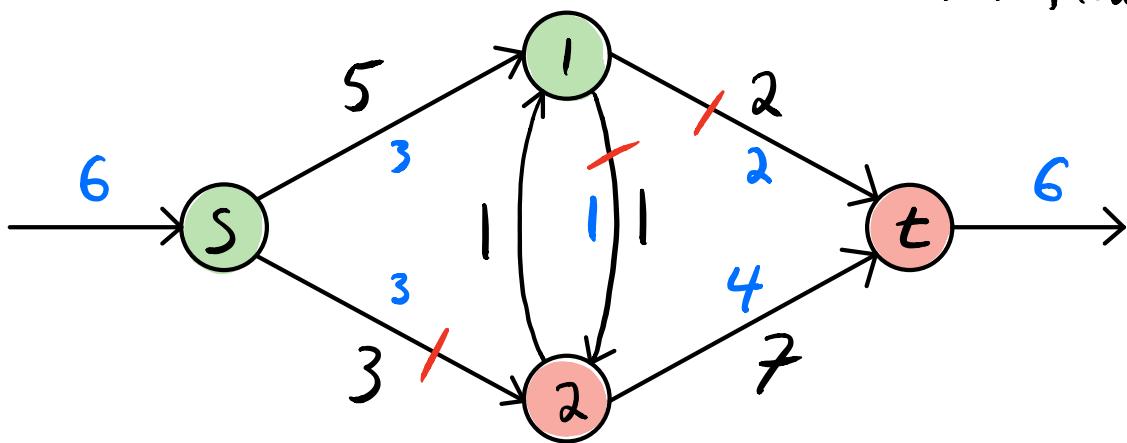
$$\left[\begin{array}{l} (1) \quad Ax = b, \quad x \geq 0 \\ (2) \quad A^T y + s = c, \quad s \geq 0 \\ (3) \quad x_i s_i = 0, \quad i=1, \dots, n. \end{array} \right]$$

The Simplex method moves from corner to corner on the polyhedral primal feas. set, keeping (1) and (3) satisfied, until (2) is satisfied.

Interior point methods keep (1) and (2) satisfied with $x > 0$ and $s > 0$ (i.e interior), making all the $x_i s_i$ terms smaller and smaller.

Max flow - min cut

max-flow = 6



$$x, u \in \mathbb{R}^6$$

$$\begin{aligned} X_{ij} &= \text{flow along edge } ij \\ U_{ij} &= \text{capacity of edge } ij \end{aligned} \quad \Rightarrow \boxed{x \leq u}$$

Flow-in = flow-out at nodes ① and ②:

$$\textcircled{1}: \quad X_{s1} + X_{21} = X_{12} + X_{1t}$$

$$\textcircled{2}: \quad X_{s2} + X_{12} = X_{21} + X_{2t}$$

Max-flow is a linear programming problem:

$\text{Max} \quad X_{1t} + X_{2t}$ $\text{s.t.} \quad X_{s1} + X_{21} - X_{12} - X_{1t} = 0$ $X_{s2} + X_{12} - X_{21} - X_{2t} = 0$ $X \leq u$
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This problem has the form

$$\begin{aligned} \text{Max } & C^T X \\ \text{s.t. } & A X = 0 \\ & X \leq u \end{aligned}$$

where

$$A = \begin{bmatrix} s_1 & s_2 & 1_2 & 1_t & 2_1 & 2_t \\ 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}, \quad \begin{array}{l} \textcircled{1} \\ \textcircled{2} \end{array}, \quad X = \begin{bmatrix} X_{s_1} \\ X_{s_2} \\ X_{1_2} \\ X_{1_t} \\ X_{2_1} \\ X_{2_t} \end{bmatrix}, \quad u = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \\ 1 \\ 7 \end{bmatrix}.$$

$$C^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^T$$

Let $y \in \mathbb{R}^2$ and $z \in \mathbb{R}_+^6$ (i.e $z \geq 0$) be the dual variables for $Ax=0$ and $X \leq u$, respectively. Then

$$L(x, y, z) = C^T x + y^T A x + z^T (u - x)$$

satisfies

$$\min_{\substack{y \\ z \geq 0}} L(x, y, z) = \begin{cases} C^T x & \text{if } Ax=0, \quad x \leq u \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is

$$\min_{\substack{y \\ z \geq 0}} \max_x L(x, y, z).$$

Note that

$$\max_x L(x, y, z) = \begin{cases} u^T z & \text{if } A^T y + c = z \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, the dual problem is

$$\boxed{\begin{aligned} & \min u^T z \\ \text{s.t. } & A^T y + c = z \\ & z \geq 0. \end{aligned}}$$

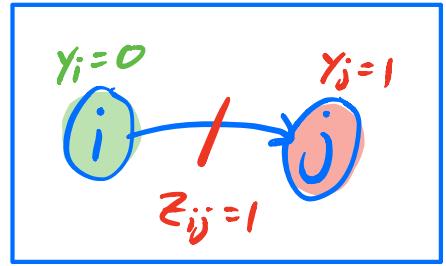
The dual variables

$$y = [y_1 \ y_2]^T, \quad z = [z_{S1} \ z_{S2} \ z_{12} \ z_{1t} \ z_{21} \ z_{2t}]^T$$

that represent cutting the graph into two disjoint pieces, S and T , with $s \in S$ and $t \in T$.

We have

$$y_i = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \in T, \end{cases}$$



$$z_{ij} = \begin{cases} 1 & \text{if } i \in S \text{ and } j \in T \\ 0 & \text{otherwise.} \end{cases}$$

$$U^T Z$$

$$\min 5z_{s1} + 3z_{s2} + z_{12} + 2z_{1t} + z_{21} + 7z_{2t}$$

s.t.

$$y_1 = z_{s1}$$

$$y_2 = z_{s2}$$

$$-y_1 + y_2 = z_{12}$$

$$-y_1 + 1 = z_{1t}$$

$$y_1 - y_2 = z_{21}$$

$$A^T y + c = z$$

$$-y_2 + 1 = z_{2t}$$

$$z \geq 0.$$

