Axiom of Choice

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Up until the early 20th century naive set theory has been in prominent use without a need for a formal construction. This changed starting from 1903 when paradoxes in set theory were discovered by Bertrand Russell and Jules Richard. These paradoxes were known as Russel's paradox and Richard's paradox respectively. To illustrate Russel's paradox, consider a set $R = \{x|x \notin x\}$ i.e, any element x where x is not an element of itself. This poses a problem when you consider whether or not $R \in R$. If $R \in R$ then by its own definition implies $R \notin R$. Similarly if $R \notin R$ then $R \in R$ by definition. Richard's paradox comes about similarly but with a focus on defining sets with the English language. Suppose $R = \{x|x \text{ is the least positive integer not definable in one sentence}\}$. Observe that the way x is defined in the set is in one sentence so we have a contradiction. Both of these paradoxes gave question to what should be considered a set and how we should restrict sets so these kinds of sets can't be allowed.

In 1904 a mathematician named Ernst Zermelo created the axiom of choice for his proof of the Well-Ordering Theorem. More axioms were created and in 1908 Zermelo axiomatized set theory. Later in 1922 Adolf Fraenkel and Thoralf Skolem improved the axioms to what is known today as the Zermelo-Fraenkel Axioms. This axiomatic system is usually denoted as ZFC with the C to indicate the inclusion of the axiom of choice and other times ZF for its exclusion. In the same year of 1922, Kazimierz Kuratowski proved an equivalent statement of the axiom of choice known as Zorn's Lemma but Max Zorn himself didn't prove it independently later until 1935.

The result of the ZFC axioms is to restrict objects such as the set of all sets to be considered "too big" to be a set and such objects are instead called a proper class. The sets defined for the two paradoxes mentioned in the beginning are now unable to be considered sets under the ZFC axioms which resolves the concern of constructing contradicting sets. A consequence of the ZFC axioms is that all numbers are sets in its construction.

Zermelo originally created the axiom of choice to prove the Well-Ordering theorem, but now we will demonstrate that the converse also holds true after introducing the following.

Definition 1. Linearly Ordered Set: A linearly ordered set, say X, is an ordering with the binary relation, $\leq$ and is denoted $(X, \leq)$. The binary relation is anti-symmetric, transitive, reflexive, and total. Another way to look at it is a partial ordering with the additional property of totality.

Definition 2. Well-Ordered Set: A set X is well-ordered if X is linearly ordered and for every non-empty subset $A \subset X$ there is a least element $s \in A$.

Note that the $\leq$ binary relation with our usual intuition of the well-ordering of numbers makes the concept of a least element clear. However it is possible for other unintuitive well-orderings to be established.

Now we will state the Axiom of Choice and Well-Ordering Theorem below.

Definition 3. Axim of Choice (3): For any non-empty set A, there exists a choice function F such that the domain of F is the collection of non-empty subsets of A and such that $F(B) \in B$ for every non-empty subset $B \subset A$.

Essentially, the choice function will take every non-empty subset of a set and output an element from that subset.

Theorem 1. Well-Ordering Theorem: For every set X, there exists a well-ordering with domain X.

Previously mentioned was that the Well-Ordering Theorem was an implication of the Axiom of Choice, but now we will now show the converse holds true as well.

Proof. Let Y be a non-empty collection of sets. Define a well ordering W over the union of the non-empty sets of Y, i.e By W, every non-empty set $x \in Y$ will have a least element. Define a choice function G that maps every non-empty set $x \in Y$ to its least element. So $\forall x \in Y, G(x) = a, a \in x$ where a is the respective least element of x. □

So far we have just gone over one statement of the Axiom of Choice but many equivalent forms exist. One of the most famous is Zorn's Lemma which has been used to prove many results. One example is that every vector space has a basis. Another statement of the axiom of choice says that the cartesian product of any collection of non-empty sets is non-empty. More recently, in 1975 Diaconescu's theorem showed that the axiom of choice implies the law of the excluded middle. This essentially tells us any propositional statement has to be either true or false.

As promising as these results may seem, there are also controversial and unintuitive results that have come about from the axiom of choice. One famous example is the Banach-Tarski Paradox which tells us that a solid ball in 3-dimensional space can be decompositioned into finite disjoint subsets and then reformed to create two of the same balls. The paradox refers to how the theorem breaks normal intuition in that creating another ball entirely seems unbelievable. Lastly there is the obvious question of applying the Well-Ordering Theorem to the real numbers. The set of real numbers is applicable to a well ordering but there is no explicitly known ordering. If we try to work in our usual ordering, well ordering the reals is impossible, but there does exist one, perhaps not even definable in the language of set theory.

The axiom of choice has inevitably given us a choice. The axiom of choice guarantees us results and properties that we would like with the addition of results and properties that we may not like. However, without the axiom of choice we would not be able to get many of these results at all. Today the axiom of choice has become more accepting in modern mathematics than it has been in the mid 1950's after its inception. Whether any groundbreaking contradictions arise as a result of this has yet to be seen.