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Randomness in Shuffling Cards

In mathematics, the Gilbert–Shannon–Reeds model is a probability distribution on riffle shuffle permutations of playing cards that has been reported to be a good match for experimentally observed outcomes of human shuffling. It is named after the work of Edgar Gilbert, Claude Shannon, and J. Reeds. Their work was reported in a technical report by Gilbert in 1955 called *Theory of shuffling* and an unpublished manuscript by Reeds in 1981. This model will serve as the foundation for this topic. Let's first cover some important concepts.

First, a "riffle shuffle" is a shuffle in which a deck of cards is split in two, and the 2 smaller decks are interweaved with each other. The Gilbert-Shannon-Reeds model states that if there are *n* cards in a deck, then the probability of selecting *k* cards in the first deck, Deck A, and *n*-*k* in the second deck, Deck B is $\frac{\binom{n}{k}}{2^n}$. With our 2 decks, one card at a time is moved from the bottom of one deck to the top of our new shuffled deck. The probability of Deck A and Deck B having the next card moved from it is $\frac{x}{x+y}$ and $\frac{y}{x+y}$ respectively, where *x* and *y* are the cards remaining in the respective decks.

Next, **Random Walks**, or **Random Processes**, describe a path that consists of a succession of random steps on some mathematical space such as the integers. A **Markov Chain**, or **Markov Process**, is a random walk that satisfies that Markov property, which is that predictions can be made for the future of a process based solely on its present state.

Next, **Total Variation Distance**, D_{VT}, describes how similar or dissimilar two probability distributions are. When D_{VT} = 0, the distributions are identical to each other, and when D_{VT} = 1, the distributions never generate the same values as each other. This is important because the Gilbert-Shannon-Reeds model is compared to the uniform distribution. The uniform distribution is chosen for comparison because ”random” is when the probability of each possible outcome is equal. Interestingly, the Gilbert-Shannon-Reeds model is not uniform. Each permutation has a probability of $\frac{1}{2^n}$ except for the identity permutation, which has a probability of $\frac{n+1}{2^n}$.

Lastly, the **Cutoff Phenomenon** describes a sharp transition in the convergence of finite Markov chains to stationary. Used with total variation distance, we can find when something seemly random will become uniform, if at all. The cutoff point is represented by

$$\tau = inf(m \geq 1 : d(m) < \frac{1}{\epsilon})$$

where $d(m)$ is the total variation distance after m shuffles. Here, $d(m) \to 1$ for all $m < \tau$, drops sharply at τ, then $d(m) \to 0$ for all $m > \tau$.

Now, the Gilbert-Shannon-Reeds model is the foundation of this idea and has been expanded on by David Aldous, Dave Bayer, and Persi Diaconis. Aldous was the first to expand on the aforementioned model. His idea was as follows:

Let X_m be the state of a deck after m shuffles.

If $m = (1 - \epsilon)(\frac{3}{2}log_2 n)$ then $d(m) \to 1$

and

If $m = (1 + \epsilon)(\frac{3}{2}log_2 n)$ then $d(m) \to 0$

This tell us that our cutoff is $\tau = \frac{3}{2}log_2 n$

Bayer and Diaconis expanded the model and Aldous’ work even further. Their idea was:

After m shuffles,

$$P(X_m = \sigma) = \frac{1}{2^{mn}} \binom{2^m + n - R(\sigma)}{n}$$

where $P(X_m = \sigma)$ is the probability that the state of the deck after m shuffles will be a permutation, $\sigma, and R(\sigma)$ is the number of rising sequences in σ.

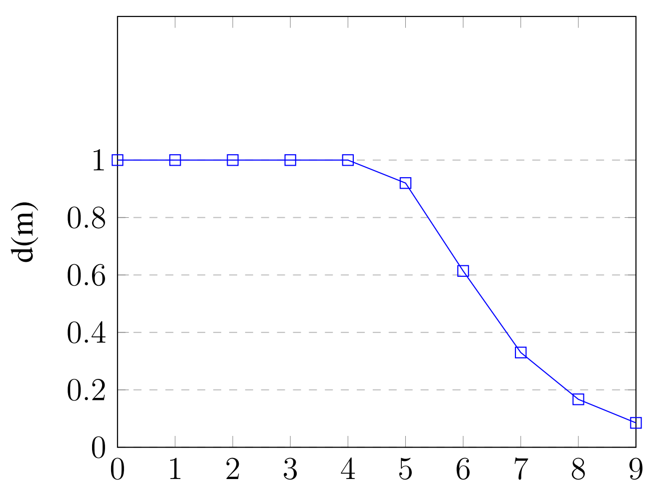
A **Rising Sequence** is a set of cards that are consecutive and increasing in value. Foe example, the set of cards that reads 1 2 3 4 5 6 7 8 is a rising sequence. This set can be split to create two rising sequences, e.g. 1 2 3 and 4 5 6 7 8.

Using these formulas, Aldous determined that 8 to 9 shuffles should suffice in randomizing a deck. However, with Bayer's and Diaconis' work, it was determined that only 7 shuffles are needed.

m	5	6	7	8	9
d(m)	0.92	0.614	0.33	0.167	0.085

They found that only after 5 shuffles will $d(m) < 1$ and after 6 shuffles, there is a sharp decline as $d(m) \to 0$. After 7 shuffles the decline begins to lessen and thus, 7 is considered the cutoff.

cutoff of n=52 xlabel



In conclusion, decks can be manipulated to seem like they're being randomized however, unless the shuffler shuffles at least 7 times, the deck cannot be considered random. There have been exceptions to this by Persi Diaconis (source?) where a deck that is split perfectly in half and riffle shuffled perfectly 8 times will be back to its original permutation.

Works Cited

Aldous, David and Diaconis, Persi. ”Strong Uniform Times and Finite Random Walks.” *Advances in Applied Mathematics*. 1987

Bayer, Dave and Diaconis, Persi. ”Trailing the Dovetail Shuffle to its lair”, *The Annals of Applied Probability*. 1992