Basic Introduction Lie Groups

Joshua-Michael Tomiyama

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1 Preface

In introductory differential equations classes, one is often taught how to solve linear systems of ordinary differential equations. These techniques are very useful for these relatively simple types of equations. However, in life one may be faced with a nonlinear system of differential equations in say the orbit of planets space or the interaction of sub-atomic particles. In this situation, the techniques taught in the introductory courses are useless.

I wanted to know if techniques existed to handle these nonlinear systems, and in my search I found Lie Groups and Algebras to be very powerful for these specific problems. However, while reading the material it was quickly apparent that I would need at least a semester and more mathematical background to truly comprehend the ideas presented.

Thus my objective with this paper is to provide an introduction into Lie Groups that students similar to myself at this point in time will find accessible. At the same time, I hope this provides inspiration to further pursue study of this subject and help others appreciate how vast mathematics is.

Recommended for further reading: Peter J. Olver Application of Lie Groups to Differential Equations

2 Background

Before going into further detail, it is necessary for one to have at least background knowledge in linear algebra. Knowledge in abstract algebra and topology would also be immensely helpful before studying Lie Groups and Algebras. Without further ado, behold some definitions

Definition 2.1. Group A group is a set G of objects that are closed under a single operation denoted $\bullet$. A group satisfies the following axioms:

1. Closure under multiplication: For every a, b in $G, a \bullet b$ is in G

2. Associativity: For ever a,b,c in G $a \bullet (b \bullet c) = (a \bullet b) \bullet c$

3. Identity Element: There exists an inverse element in G

4. Inverse Element: Every element in G has an inverse

Remark. Normally the $\bullet$ is dropped. Hence $a \bullet b = ab$. This convention will be used for the rest of this paper.

Definition 2.2. Manifold (informal) Manifolds are fundamental objects that are locally Euclidean but maybe not globally.

Definition 2.3. Lie Group (informal) A Lie group has both the properties of a group and a manifold. This allows the Lie groups to embody continuous symmetries rather than the normal discrete symmetries other groups.

3 Basic Definitions

Definition 3.1. General Linear Group The general linear group, denoted GL(n;$\mathbb{R}$), is the group of invertible n x n matrices

Definition 3.2. Matrix Lie Group A **closed** subgroup of GL(n;$\mathbb{R}$). In other words this subgroup contains all of its limit points.

Remark. A matrix Lie group may not seem to be a manifold at first, however, it can be shown that every matrix Lie group is an embedded manifold.

3.1 The Classical Groups

The following matrix Lie groups are known as the classical groups. Though **not every Lie group is a matrix Lie Group**, many interesting problems are often related to these groups.

Definition 3.3. Special Linear Group Group of invertible matrices with determinant 1 denoted SL(n;$\mathbb{F}$)

Definition 3.4. Unitary Group/Special Unitary Group The unitary group is the group of unitary matrices denoted U(n). The **special unitary group** are unitary matrices with determinant one denoted SU(n)

Definition 3.5. Orthogonal Group/Special Orthogonal Group The orthogonal group is the group of orthogonal matrices denoted O(n). The **special orthogonal group** are orthogonal matrices with determinant one, denoted SO(n)

4 Why Study Lie Groups

The following exercise illustrates the inspiration for creating Lie Groups and the ideas used in solving nonlinear differential equations.

Exercise. Suppose you have n numbers $a_1, ..., a_n$ arranged on a circle. You have a transformation A such that $a_1 = \frac{a_n + a_2}{2}$, $a_2 = \frac{a_1 + a_3}{2}$, and so on. If you do this sufficiently many times, will the numbers be roughly equal?

It is clear that A is a linear transformation, therefore we can represent A with the matrix where for $0 \leq i, j \leq n-1$,

$$A_{i,j} = \begin{cases} \frac{1}{2} & if\ j = (i \pm 1) mod\ n + 1 \\ 0 & O.W. \end{cases} \tag{1}$$

Finding the eigenvalues and eigenvectors of A will allow us to predict the long term behavior of A. However, normal computation for the eigenvalues is very tedious for $n > 3$. Luckily, there is rotational symmetry that can be used to simplify the problem. Behold a linear transformation B that rotates the points once:

$$B(a_1, a_2, ..., a_n) = (a_2, a_3, ..., a_n, a_1) \tag{2}$$

Observe that if we rotate the points n-times then we are exactly back at the original order of points. In other words, $B^n = I$. Therefore the eigenvalues of B are ε^k where $\varepsilon = e^{\frac{2\pi i}{n}}$ and $k = 0, ..., n-1$ with corresponding eigenvectors $v_k = (1, \varepsilon^k, \varepsilon^{2k}, ..., \varepsilon^{(n-1)k})$.

One can verify that AB = BA, so the eigenvectors of B are also eigenvectors of A. Thus we have both the eigenvectors and eigenvalues of A and predict if all the numbers converge to a multiple of the vector (1,1,1,...,1) after many iterations.

Thus, we can find the eigenvalues of A by using the symmetries of the system. However, if the symmetry was the entire circle then this exact method fails because we would have uncountably infinitely large vectors and linear transformations, but this is not possible. For example, no matter how small the rotation transformation is there exists a smaller rotation that could accomplish the same thing (recall that between any real numbers there is another real number).

The ideas presented here are the basic techniques for nonlinear differential equations. Rather than trying to solve the problem with regular techniques, one can notice the symmetry in a differential equation to simplify it and solve. However, this often involves continuous symmetries which cannot be expressed with regular groups. Please read the recommended text in section 1 for more information on how the generalize the techniques in this example to differential equations.

References

[1] Hall, B. (2015) *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Springer.

[2] Kirillov, A. (2008). *An introduction to Lie groups and Lie algebras* (Vol. 113). Cambridge University Press.