Fundamentals of Electrical and Computer Engineering

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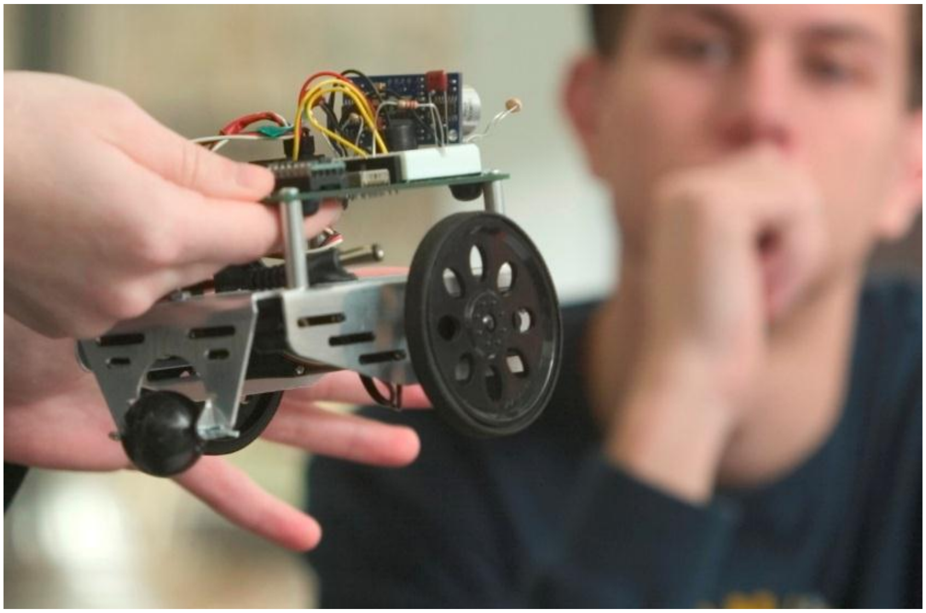

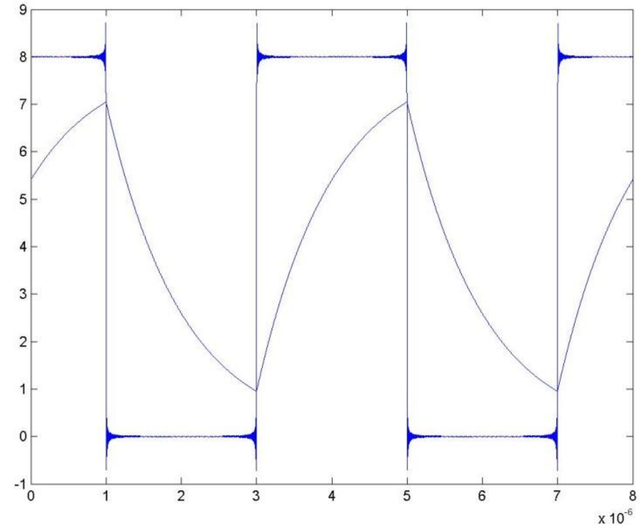

Version 2.1

Fundamentals of Electrical and Computer Engineering

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Chapter 1

Introduction

1.1 Engineering

Engineering is the acquisition and processing of information to design products and processes that improve human life quality. Science and mathematics are used as tools to solve problems requiring the engineer to consider standards and constraints in the problem solution process. Good engineering design practice takes into consideration many factors such as cost, environmental impact, manufacturability, health and safety, ethics, as well as the social and political impacts of the product or process being created. **Science** is a systematic approach to the acquisition of knowledge based on testable explanations for phenomena in the universe enabling predictions of event outcomes. Science differs from engineering. Both are important to the process of learning about how the world works, and the exercise of one discipline often involves the other. Engineering goes beyond seeking to understand how the world works. Engineers design, build and test devices, systems and processes that involve the creation of something new. There is an inherent process of invention in generating new products and processes that make the lives of humans better and the state of the world better. Some of the inventions of engineers have led to problems in the environment and general state of the planet, but a responsible engineer is committed to considering the environmental impact of his/her products and is bound by ethical principles to produce devices and systems that promote a greener world and a safe environment in which our children are raised. Engineering is an exciting and satisfying career and provides an excellent foundation for any career endeavor.

1.2 Electrical and Computer Engineering

Electrical and Computer Engineering (ECE) is a profession that integrates several sub-disciplines including analog and digital circuits and devices, signal processing and communication systems, computer architecture and networking, micro and nanodevices, quantum computing, photonics, sensing, waves and metamaterials, required to solve engineering problems to improve human life quality. The list of ECE sub-disciplines presented is not exhaustive, but reflects many of the strengths possessed by the faculty at Duke University. There are other areas of ECE, such as power system design that are strengths of other university's ECE programs. It is very unusual to find an ECE program that excels in all areas of ECE, but depending on the expertise of the faculty, the various sub-disciplines of ECE are represented to some degree.

1.3 Fundamentals of ECE

Fundamentals of Electrical and Computer Engineering is the first core course in the ECE curriculum at Duke University. It is intended to provide a rigorous introduction to the field of ECE, enabling informed selection of areas of concentration while investigating the theory and design of sensor systems. The course is centered on an **Integrated Design Challenge (IDC)** that requires teams of students to build a mobile sensing platform (robot) to navigate an obstacle course (arena) while acquiring and processing information to achieve a designated goal based upon a theme that is familiar to the undergraduate experience at Duke University. Students utilize a suite of sensors (e.g. infrared, ultrasonic, tactile, magnetic, pressure, photo-sensitive – to name a few) enabling their robot to follow lines, avoid collisions, measure the shape and color of objects and/or identify the location of a magnetic field source in the context of achieving the goal of the IDC. In the process of designing their robots, students undergo a rigorous process of analysis and design of analog and digital circuits, and are introduced to the fundamentals of ECE. For greater detail, the reader should examine [1].

1.4 Engineering Notation

It is common in the practice of ECE to encounter both large and small quantities. A set of engineering prefixes have been defined to efficiently deal with large and small quantities. The use of these engineering prefixes is called **engineering notation**. Engineering notation is similar to scientific notation, but the powers of ten are multiples of three. All engineering students should be familiar with the notation of engineering prefixes from 10^{-15} (femto) to 10^{15} (Peta). The following table should be common knowledge to all engineering students.

Engineering Notation	Engineering Prefix	Abbreviated Engineering prefix	Example usage
10^{-15}	femto	f	fs
10^{-12}	pico	p	pF
10^{-9}	nano	n	nm
10^{-6}	micro	μ	μH
10^{-3}	milli	m	mW
10^{3}	kilo	k	km
10^{6}	mega	M	MΩ
10^{9}	giga	G	GB
10^{12}	tera	T	TB
10^{15}	peta	P	PHz

Figure 1.1 Table of engineering prefixes and their definitions.

The **notation** used in this textbook for voltage and current expresses all time domain quantities with a **lower case** v for voltage and i for current. **Upper case** V and I are reserved for sinusoidal (AC) voltage maxima and current maxima respectively. A time domain voltage or current may be explicitly written as a function of time as in $v(t)$ or simply v with the understanding that the voltage may be a constant (DC) or time varying.

1.5 Plagiarism

Every instructor has a different set of expectations regarding what is considered acceptable collaboration. It is the obligation of the instructor to make collaboration expectations known to the students. And it is the students' responsibility to follow these expectations. The first document on the ECE 110L website provides a detailed description of the collaboration policy for this course.

Weekly homework assignments, tests, and a final comprehensive examination will sometimes require students to express their understanding of certain topics in writing. For example, students may be required to learn about batteries and the electrochemical reactions that take place within different batteries. At least two sources are required to be examined and cited properly. It is very important that students paraphrase what they learn by reading the sources and setting the sources aside as they write. If text is taken directly from a source, the **text must be in quotations**. **It is plagiarism to have a source of information open and simultaneously transcribe and paraphrase sequences of sentences taken from that source**. The source may be re-examined to ensure correctness of content, but the user of that information is **required to cite the source**. On tests and the final examination, students must be prepared to answer essay questions without having access to any sources. Throughout the textbook, definitions are provided. **Students are required to know these definitions and be able to provide the definitions in their own words accurately and completely**.

1.6 Order of Topics Presented

It may appear at first glance that the order of topics presented in this textbook is peculiar. The order of topical coverage is quite intentional and is driven by the need for information about a particular topic to support concurrent activities in the laboratory. The order is not haphazard. The order of topics is intended to be streamlined and as seamless as possible while supporting the concurrent activities in the lab. It is the belief of the author that students learn best when the lecture content is presented in synchronization with the laboratory exercises.

1.7 Recognition

The electric circuit concepts presented in this book were in large part learned from Dr. Donald R. Rhodes (1923-), University Professor Emeritus, North Carolina State University. Don Rhodes in turn learned a large portion of his conceptual understanding of electric circuits from Charles Steinmetz (1856-1923). In particular, the author is indebted to Don Rhodes for his understanding of the branch current method (BCM) of solving circuits, which uses physical currents instead of fictitious loop currents. To the author's knowledge, there is no textbook on electric circuits that utilizes the branch current method. Hence, the continued "life" of the BCM may be hinged upon the understanding of this technique by the readers of this textbook.

The author's understanding of semiconductor physics continues to be developed through discussions with Dr. Hisham Massoud, Professor of Electrical and Computer Engineering, Duke University. Numerous graduate and undergraduate students have provided suggestions for the improvement of the textbook and for identifying errata. Rodger Dalton, who earned his B.S. in electrical engineering at NCSU in Raleigh, NC in 1992 and his M.S. degree in ECE from Duke University in Durham, NC in 2006, has provided significant input to the current version of the textbook.

ECE 110L, *Fundamentals of Electrical and Computer Engineering*, was created by Dr. Lisa Huettel, Associate Chair, Director of Undergraduate Studies and Associate Professor of the Practice of ECE at Duke University. Several Duke ECE faculty members contributed to the creation of the laboratory manual and associated laboratory activities. The Integrated Design Challenge (IDC) was created by Dr. Lisa Huettel with assistance from Mr. Kip Coonley, ECE Undergraduate Lab Manager at Duke University. Several faculty contributed to the ECE undergraduate curriculum redesign that commenced in 2006 with the pilot offering of ECE 110L by Dr. Lisa Huettel. A detailed description of the ECE curriculum redesign at Duke University is presented in [2].

[1] Huettel L.G., Brown A.S., Coonley A.D., Gustafson M.R., Kim J., Ybarra G.A., and Collins L.M., "Fundamentals of ECE: A Rigorous Introduction to Electrical and Computer Engineering," *IEEE Transactions on Education*, vol 50 no. 3 (2007), pp. 174-181.

[2] G. Ybarra, L. Collins, L. Huettel, K. Coonley, H. Massoud, J. Board, S. Cummer, R. Roy Choudhury, M. Gustafson, N. Jokerst, M. Brooke, R. Willet, J. Kim, and M. Absher, "Integrating Sensing and Information Processing Theme-Based Redesign of the Undergraduate Electrical and Computer Engineering Undergraduate Curriculum at Duke University, Advances in Engineering Education, Summer 2011, Vol. 2, Number 4.

Chapter 2

Digital Logic Circuits

2.1 Introduction

Digital logic circuits utilize only two states, high and low. These two states can represent voltage levels (e.g. 5 V is a logical high and 0 V is a logical low) or any two distinct states of any signal (e.g 100 °F is a logical high and 80 °F is a logical low). Arithmetic using binary numbers was formalized by George Boole in 1847 and is termed Boolean algebra. There are six basic gates: NOT, NAND, NOR, AND, OR, XOR. NAND and NOR gates are both universal gates. This means that any logic function can be implemented by either a sequence of NAND gates or NOR gates. It is impossible to realize or implement an arbitrary logic function with just AND or OR gates. However, if one can combine XOR with AND gates, this combination is universal. This set of six fundamental gates contains no memory and the function produced by their interconnection is called **combinational logic**. When the inputs to a combinational logic circuit change state, a chain reaction occurs as the digital logic signal traverses the logic circuit. Each gate has a non-zero transition time called the propagation delay, which range from 10's of nanoseconds to picoseconds. In complex logic circuits, the propagation delay can become significant. However, in this course, gate propagation delay is considered to be zero. Although this is an idealization, the fundamental concepts associated with digital logic circuits are still conveyed.

2.2 The Binary System and Conversion from Decimal to Binary

Decimal numbers (base 10) are used almost exclusively in our lives. In digital logic, we are interested in manipulating binary numbers (base 2). In the decimal system, beginning from the least significant digit to the left of the decimal point are 1's, 10's, 100's, 1000's, 10,000's etc. In a similar manner, the numbering structure of bits (binary digits) beginning with the least significant bit to the left of the binary point are 1's, 2's, 4's, 8's, 16's, 32's etc. As an example of converting a decimal number to binary, consider the decimal number $a = 23{,}571$.

Example 2.1

Convert the decimal number $a = 23{,}571$ to its equivalent binary number

Solution

Make a list of powers of 2. Entries will include 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192 and 16,384 etc. Find the largest power of 2 that will divide into a, which is 16,384 $= 2^{14}$, which places a 1 in the 15th position, and subtract the value from a to get 7,187 and repeat the process. The largest power of 2 that divides in 7,187 is 4,096 $= 2^{12}$, which places a 1 in the 13th position. Subtracting 4,096 from 7,187 is 3,091. The largest power of 2 dividing into 3,091 is 2,048 $= 2^{11}$. Subtracting 2,048 from 3,091 is 1,043. The power of 2 dividing into 1,043 is

$1{,}024 = \ 2^{10}$. Subtracting gives 19, which, when converted to binary gives 10011. Putting this all together gives

$$a = 101110000010011$$

This result can be verified by converting the binary number to decimal: $2^{14} + \ 2^{12} + \ 2^{11} + \ + 2^{10} + 2^4 + \ 2^1 + \ 2^0 = 16{,}384 + 4{,}096 + 2{,}048 + 1{,}024 + 16 + 3 = 23{,}571$

2.2 Boolean Algebra and Complexity

Axioms
1. 0 · 0 = 0
2. 1 · 1 = 1
3. 1 + 1 = 1
4. 0 + 0 = 0
5. 0 · 1 = 1 · 0 = 0
6. 1 + 0 = 0 + 1 = 1
7. If x = 0, then $\bar{x}$ = 1
8. If x = 1, then $\bar{x}$ = 0

Single-Variable Theorems (derived from axioms)
1. $x \cdot 0 = 0$
2. $x + 1 = 1$
3. $x \cdot 1 = x$
4. $x + 0 = x$
5. $x \cdot x = x$
6. $x + x = x$
7. $x \cdot \bar{x} = 0$
8. $x + \bar{x} = 1$
9. $\bar{\bar{x}} = x$

Properties
1. $xy = yx$ Commutative
2. $x + y = y + x$ Commutative
3. $x(yz) = (xy)z$ Associative
4. $x + (y + z) = (x + y) + z$ Associative
5. $x(y + z) = xy + xz$ Distributive
6. $(x + y)(x + z) = x + (yz)$ Distributive

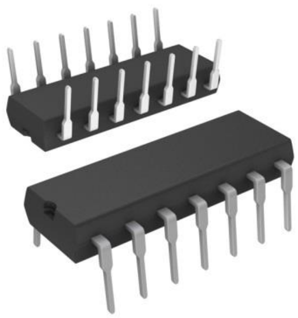
DeMorgan's Theorems
1. $\overline{x \cdot y} = \bar{x} + \bar{y}$
2. $\overline{x + y} = \bar{x} \cdot \ \bar{y}$

Figure 2.1 Boolean algebra axioms, properties and DeMorgan’s theorems

Boolean algebra can be used to manipulate logical expressions into a desired form or to simplify logical expressions. You might ask, what is the simplest form for a logical expression? To answer this question requires a metric or measure for **complexity** as well as the set of gates available. There are multiple definitions of complexity. We will use the one that is most common and is the **sum of the number of gates and the number of inputs**.

2.3 Fundamental Digital Logic Gates and Their Truth Tables

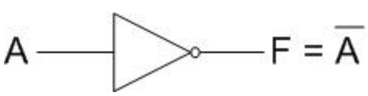
Basic logic gates may be obtained in a dual inline package (DIP) as shown in figure 2.2. The DIP is fabricated with an integrated circuit (IC) in a plastic enclosure with metal pins extending from the sides of the package that connect the integrated circuit inside the package with external circuitry. The metal pins extending from the sides of the DIP are bent downward so that the DIP can fit into a breadboard for rapid prototyping of a circuit.



All digital logic gates get their power from an external DC power supply, which can be a battery or a bench top DC power supply. Although most ICs are digital, there is a growing development of analog ICs and hybrid ICs that process mixed signals (i.e. both analog and digital signals). The pins surrounding the DIP provide electrical connections to the device inputs and outputs along with the DC power supply.

Figure 2.2 Generic dual in line package (DIP) often referred to as a "chip."

NOT Gate

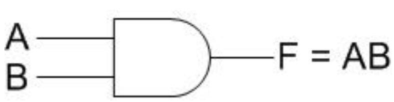


Truth Table

A	F
0	1
1	0

The output F of the NOT gate (or inverter) is the complement (inverse) of the input, $\bar{A}$. A truth table contains a list of all possible combinations of inputs and the value of the output as a function of the inputs. For a gate with n inputs, there will be 2^n rows in the **truth table**. It is general good practice to count in binary to obtain all of the possible combinations of inputs.

AND Gate

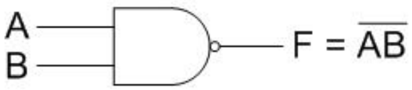


Truth Table

A	B	F
0	0	0
0	1	0
1	0	0
1	1	1

The output is high if and only if both inputs are high.

NAND Gate

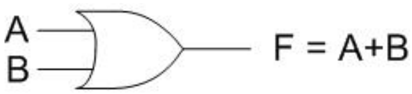


Truth Table

A	B	F
0	0	1
0	1	1
1	0	1
1	1	0

The output is low if and only if both inputs are high. The NAND function is the complement of the AND function.

OR Gate

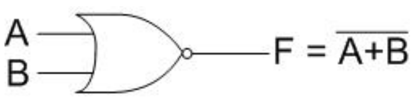


Truth Table

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

The output is high if and only if either of the inputs is high including the case when both inputs are high.

NOR Gate

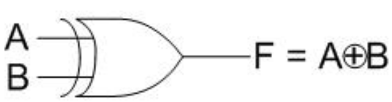


Truth Table

A	B	F
0	0	1
0	1	0
1	0	0
1	1	0

The NOR gate produces a high output if and only if both inputs are low. It produces the complement of the OR gate function.

XOR Gate



Truth Table

A	B	F
0	0	0
0	1	1
1	0	1
1	1	0

The output of an XOR gate is high if and only if a single input is high.

These gates have been presented with two inputs. Gates exist with several inputs, but the chips are more costly to fabricate. It is useful to consider the case when a logic function must be implemented using gates with only two inputs. The NOT gate never has more than one input.

2.4 Logic Function Analysis

Example 2.2

Given the logic circuit in figure 2.3, determine its output. Calculate its complexity before any manipulation of *F*. Using Boolean algebra, simplify *F* and calculate its complexity (without using XOR gates). The other five fundamental gates may be used, and multiple inputs are allowed.

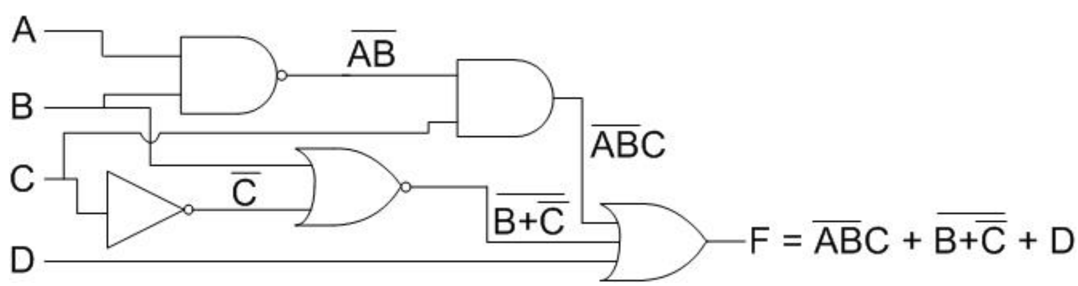


Figure 2.3 Logic circuit for analysis in Example 2.2

Solution: Examining the logic circuit, it is clear that there are four input variables, A, B, C, D. Starting from the inputs, move through the logic circuit labeling the signal at each node.
The final function is

$$F = \overline{AB}C + \overline{B + \bar{C}} + D \tag{2.1}$$

Its complexity without any simplification using multi-input gates (except avoiding XOR gates) is obtained from the logic circuit schematic, which has 5 gates and 10 inputs for a total complexity of 15. If Boolean algebra is used to simplify *F*,

$$F = \bar{A}C + \bar{B}C + D \tag{2.2}$$

which has a complexity of 14. Are there other expressions that result in a complexity fewer than 14?

2.5 Logic Function Synthesis (Implementation)

The calculation of complexity in Example 2.2 involves logic circuit synthesis, because the question of how many gates and inputs are needed to design the logic circuit is a question of synthesis. The question remains how to synthesize an arbitrary logic function. Ultimately what is needed is a truth table indicating the high values (1's) produced by the logic function.

Example 2.3

Consider a logic function of four variables $F(A,B,C,D)$ that is equal to 1 whenever three or more inputs are a 1. Write the unminimized sum of products expression for *F*. Calculate the complexity of *F*.

Solution: In order to create the truth table, we must list all possible combinations of the input states and indicate whether the output is a 1 or a 0 for each case. To ensure that all possible combinations of 1's and 0's are present as rows in the truth table, we will count from 0 to 15 in binary. There is value in counting in binary in groups of four. This process helps you determine whether or not you have included all 16 of the truth table rows. Then once the rows have been completed for every possible combination of inputs, the remaining column, the output function, *F* can be completed and the logic function written in **sum of products (SOP) form**. The first thing we must do is to create the input side of the table, which will include all possible combinations of the input states.

A	B	C	D	F
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Figure 2.4 Truth table for example 2.3.

The rows contain all combinations of 1's and 0's for the inputs. To ensure that all combinations are present, the rows are generated by counting in binary from 0 to 15. Bold horizontal lines have been inserted every four rows to illustrate how the four sections are delineated. This helps to ensure that all combinations of 1's and 0's are present. Next, the output function column is completed by placing 1's for F in all rows in which there are at least three inputs with 1's, in any order. 0's are entered in all other output cells. The output function may now be extracted from the truth table in the form of a sum of products (SOP):

$$F = \bar{A}BCD + A\bar{B}CD + AB\bar{C}D + ABC\bar{D} + ABCD \tag{2.3}$$

It may be apparent that this function can by simplified using Boolean algebra, which it can, but as shown in (2.3), the expression is referred to as the **unminimized** sum of products. The complexity is the sum of the number of gates and inputs. There are 4 inverters, 5 AND gates, and 1 OR gate, for a total of 10 gates. There are a total of 4 inputs to the inverters, 20 inputs to the AND gates and 5 inputs to the OR gate, for a total of 29 inputs. The complexity is 10 + 29 = 39.

Example 2.4

Implement the unminized SOP form for F in (2.3).

Solution:

All of the input logic variables are utilized in their uncomplemented and complemented forms. These signals are generated on the left and the terms in the unminimized logic function are formed from these signals. Finally, the output function F is obtained by a final OR function. Note that there is no restriction on the number of gate inputs.

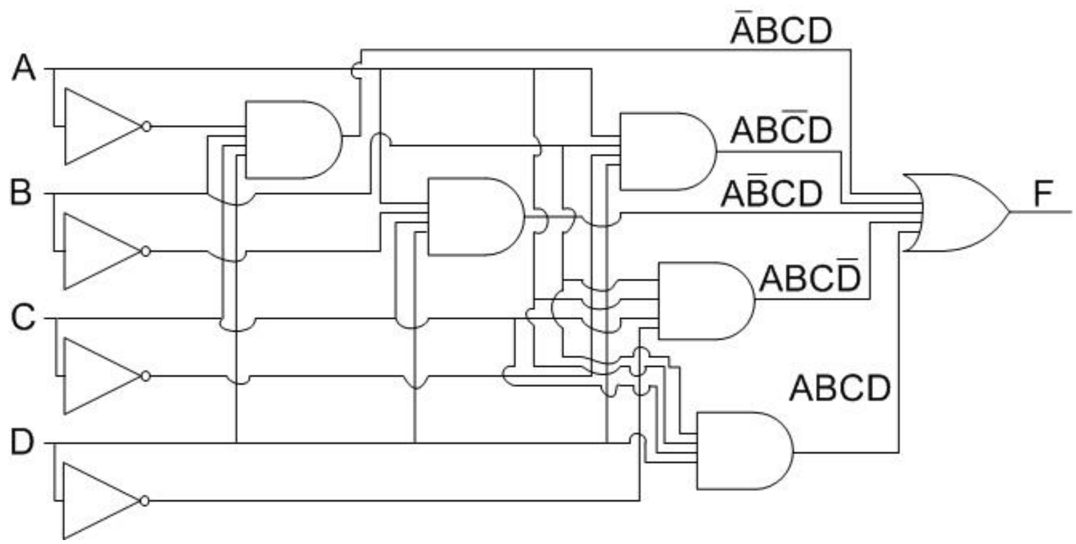


Figure 2.5 The unminimized sum of products implementation of the function in (2.3).

2.6 Logic Function Minimization and the use of the Karnaugh Map

Boolean algebra can be used to reduce the complexity of any logic function to some absolute minimum. Computers use a brute force approach. A computer examines exhaustively all realizations of logical expression and chooses the result that has the least complexity. A technique for finding the minimum **sum of products (MSOP)** is to use a Karnaugh or K-map. A K-map utilizes a geometric grid in which each row of a truth table is mapped to a unique cell in the K-map. By proper structuring of the meaning of each cell, it is possible to identify groups of 2^n (n integer) adjacent cells and directly use the map, which is equivalent to the use of the Boolean operation of absorption.

Two examples of K-maps are shown below for four input logic variables A, B, C, and D.

CD \ AB	$\bar{A}\bar{B}$	$\bar{A}B$	AB	$A\bar{B}$
$\bar{C}\bar{D}$	m_0	m_4	m_{12}	m_8
$\bar{C}D$	m_1	m_5	m_{13}	m_9
CD	m_3	m_7	m_{15}	m_{11}
$C\bar{D}$	m_2	m_6	m_{14}	m_{10}

CD \ AB	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

Figure 2.7 K-maps showing the explicit input variables and their logical values along the outside of the maps with the mapping indices shown in each cell.

The mapping indices shown in figure 2.7 are the decimal value associated with each row of the truth table. As an example, consider the unminimized SOP expression in (2.3).

As a matter of notation, a logic expression can be written in the form of $F = \sum_i m_i$. This will be illustrated in example 2.5.

Example 2.5

Use a K-map to find the MSOP expression for (2.3). Calculate its complexity.

Solution: First, draw the K-map that corresponds to the logical expression in (2.3) repeated here for convenience.

$$F = \bar{A}BCD + A\bar{B}CD + AB\bar{C}D + ABC\bar{D} + ABCD \qquad (2.3)$$

The associated K-Map is shown in figure 2.8. The K-map is helpful in reducing complexity when adjacent cell groups of size 2^n can be identified. In figure 2.8, there are four groups of two 1's. Observe that adjacent cells differ by only a single bit. This is a requirement for the K-map to work. It is imperative to understand that the K-map wraps around both horizontally and vertically. That is, cells in the top row are adjacent to cells in the bottom row and cells in the far left column are adjacent to cells in the far right column. Notice that these adjacent cells differ by only one bit. Therefore, the cell in the top left corner is adjacent to the cell in the bottom left corner. The second K-map in figure 2.8 shows how the 1's can be grouped. A group of two adjacent cells eliminates one input variable. To see how absorption works, consider the 1 in the cell in the center (1111) and the 1 in the cell about it (1101). The SOP expression for the two cells is given by (2.4).

$$ABCD + AB\bar{C}D = (\bar{C} + C)ABD = ABD \tag{2.4}$$

In this example, the *C* term has been absorbed. Try to go directly from the K-map to the result. This can be accomplished by observing that the *C* term is the only term that has both a 1 and a 0 in the grouping. By this process, the entire map produces

$$F = ABD + BCD + ACD + ABC \tag{2.5}$$

The complexity can be obtained directly from (2.5): 4 AND gates, 1 OR gate, for a total of 5 gates. The number of inputs for each AND gate is 3, and 4 inputs for the OR gate, for a total of 16 inputs. This results in a complexity of 21. Thus, the minimization process to find the MSOP has resulted in a complexity reduction of 39 – 21 = 18.

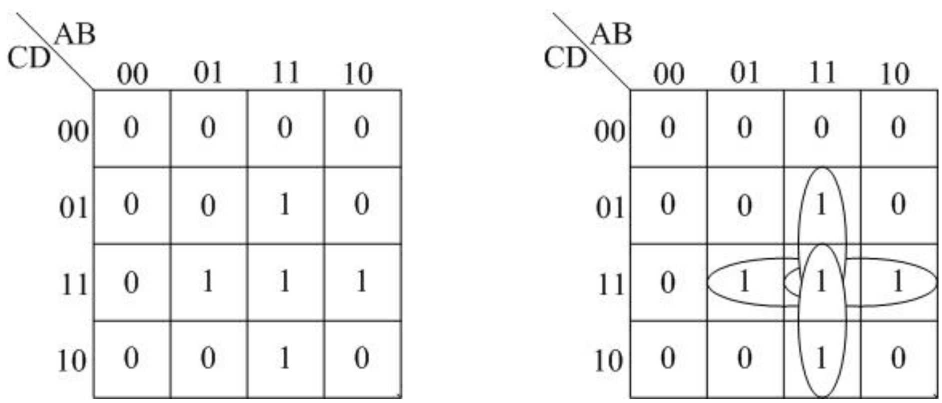


Figure 2.8 K-map for the SOP expression in (2.3) and cell groupings for minimization.

As an illustration of the mapping index notation, using example 2.3, the truth table is reproduced in figure 2.9 with the first column indicating the mapping indices.

m	A	B	C	D	F
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	1
8	1	0	0	0	0
9	1	0	0	1	0
10	1	0	1	0	0
11	1	0	1	1	1
12	1	1	0	0	0
13	1	1	0	1	1
14	1	1	1	0	1
15	1	1	1	1	1

Figure 2.9 The truth table in figure 2.4, which was used for example 2.3, including an additional column showing the mapping indices.

Using the mapping indices for the functional 1's in figure 2.9, the logic function *F* can be written

$$F = \sum m(7,11,13,14,15) \tag{2.6}$$

As an example of using the mapping notation, consider the following example.

Example 2.6

Given $F = \sum m(0,2,8,10)$, find the MSOP expression for *F* and calculate its complexity.

Solution: The mapping notation for F is equivalent to the K-map in figure 2.10. By carefully examining the K-map in figure 2.10 it is apparent that the four corners form a 2x2 grouping of cells. Thus, the four corners can be combined into one logic expression with two terms. The resulting expression is $F = \bar{B}\bar{D}$ and the complexity is 7. It is interesting to note that applying DeMorgan's theorem to this logic expression results in $F = \overline{B + D}$, which has a complexity of 3! The conclusion is that the minimum sum of products (MSOP) may not produce the expression with the least complexity.

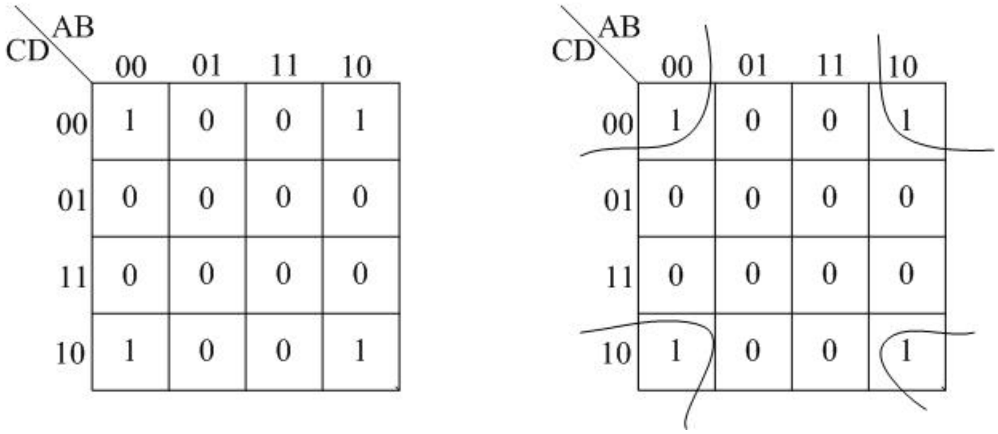


Figure 2.10 The K-map that is produced from the mapping summation of example 2.6 as well as the grouping of all four corners.

2.7 Implementation using Only One Type of Universal Gate

There may be situations in which your company has a large number of NOR gates or NAND gates, and it is desirable to learn to implement a given logic function using only one type of universal logic gate. As an example, consider the MSOP expression in (2.5) and you have to implement it using only NAND gates.

Example 2.7

Given the MSOP expression in (2.5), repeated here for convenience, write the logic expression necessary for implementation using only NAND gates.

$$F = ABD + BCD + ACD + ABC \tag{2.5}$$

Solution: Double inversion of a logic expression retains equality. Apply this operation on the logic expression followed by the application of DeMorgan's theorem.

$$F = \overline{\overline{ABD + BCD + ACD + ABC}} = \overline{(\overline{ABD})(\overline{BCD})(\overline{ACD})(\overline{ABC})} \tag{2.7}$$

(2.7) can be implemented using only NAND gates. Had there been any single input inversions, the inversion is accomplished by using a two-input NAND gate with its inputs connected together.

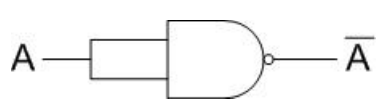


Figure 2.11 Creation of the inversion operation using a NAND gate.

It is left as an exercise to the reader to prove that an inverter results from a NAND gate with its inputs tied together. Double negating the entire SOP expression and invoking DeMorgan's theorem places F in a form that can be implemented using only NAND gates. Using 4 three input NAND gates and 1 five input NAND gate forms the function F and illustrates the general approach to implementation using all NAND gates. There is a similar approach for an all NOR gate implementation. Interestingly, inverters are created using a two input NOR gate with its inputs connected. It is left to the reader to confirm this. The general approach for all NOR gate implementation involves double negating each product in the MSOP form, using DeMorgan's theorem using the lower of the two inversions to get a sum under an inversion (NOR form) and double negating the entire expression so that the function can be implemented using all NOR gates.

Consider the MSOP expression of example 2.5 repeated here for convenience

$$F = ABD + BCD + ACD + ABC \tag{2.5}$$

$$F = \overline{\overline{ABD}} + \overline{\overline{BCD}} + \overline{\overline{ACD}} + \overline{\overline{ABC}} \tag{2.8}$$

$$F = \overline{\bar{B} + \bar{C} + \bar{D}} + \overline{\bar{A} + \bar{C} + \bar{D}} + \overline{\bar{A} + \bar{B} + \bar{D}} + \overline{\bar{A} + \bar{B} + \bar{C}} \tag{2.8}$$

There are four terms, each of which can be implemented by a NOR gate. However, the final function is an OR function. Therefore, we **must invert the function twice** to retain equality.

$$F = \overline{\overline{\overline{\bar{B} + \bar{C} + \bar{D}} + \overline{\bar{A} + \bar{C} + \bar{D}} + \overline{\bar{A} + \bar{B} + \bar{D}} + \overline{\bar{A} + \bar{B} + \bar{C}}}} \tag{2.8}$$

The bottom functional inversion is part of a NOR operation. The second inversion (on the top) is necessary for functional equality. We had to invert *F* twice to retain equality. (2.8) is therefore in "NOR form."

Chapter 3

Electric Current and Voltage

3.1 Electricity in the Broad Sense, Electric Fields, and Charge

Electricity is a generic term for electric charges, currents and voltages. These quantities have very specific meanings, and each term will be rigorously defined and exercised. We will be investigating the physical behavior of electric circuits, devices and systems. The most fundamental particle with electric charge is the electron (e^-), a negatively charged particle normally bound to an atom to maintain electrical neutrality. Every electric charge has an ***electric field*** around it. In fact, a primary mechanism for the creation of electric current is the interaction of charges through their electric fields. As will be examined later in the textbook, it will be found that a magnetic field can also establish electric current.

Electric **charge** is a property of some particles, molecules, and objects of virtually any size that possess an **electric field**. The unit of electric charge is the coulomb (C). Charge is considered one of the seven fundamental physical quantities.* The force between charges is conveyed by its electric field such that **like charges repel** and **opposite charges attract**.

An **Electric Field** ($\vec{E}$) is a directional, invisible force field that exerts a force on **charges** through the space of the field. Its direction is such that an infinitesimal positive **charge** would follow the direction of the electric field. All charges contribute their vector field to produce a composite electric field whose value and direction are influenced by nearby charges.

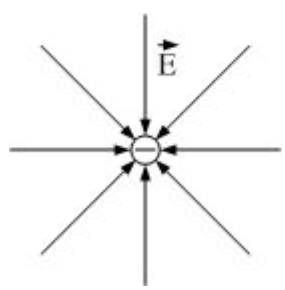


Figure 3.1 Electric field near an isolated negative charge.

Consider an isolated negative charge. Its electric field would be omnidirectional, equal in value at a given distance and directed radially inward as shown in figure 3.1.

*There are seven fundamental physical quantities: mass, length, time, angle, charge, cycles and temperature.
Now consider the electric field distribution for the case of two equal negative charges. The electric field would look like the structure shown in figure 3.2.

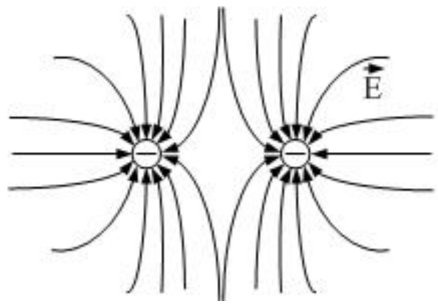


Figure 3.2 Electric field near two negative charges of equal magnitude

Remember that like charges repel. The negative charges interact through their electric fields forcing the charges apart. It is this repulsive interaction between electrons in a good conductor that is an important aspect of electric current. In a good conductor (e.g. silver (best conductor), copper, gold), at room temperature, a small mass has billions of free electrons. For example, copper has $8.5 \cdot 10^{28}$ free electrons per m^3 at room temperature. When an electron experiences an electromotive force along the length of the wire, a chain reaction occurs such that electrons far away from the initially disturbed electrons will experience a force to move along the length of the wire in the opposite direction of the electric field. Remember that electrons are negatively charged.

3.2 Electric Current

Electric current is the rate of flow of charge in a given direction. Its instantaneous value is defined by

$$i(t) \triangleq \frac{dq(t)}{dt} \tag{3.1}$$

where q(t) is instantaneous **charge** (in coulombs). The unit of electric current is the ampere or amp designated by the single letter "*A*". The mathematical symbol $\triangleq$ is to be read as, "by definition is equal to…" Therefore an $A \triangleq \frac{C}{s}$. Consider a copper wire with -5 C of electron charge moving to the right each second. This constitutes a current of -5 A flowing to the right. The direction of the current is designated by an arrow and the symbolic value of this current is *i*. This situation is depicted in the diagram of figure 3.3.

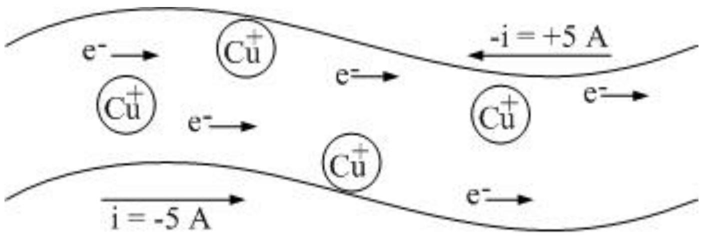


Figure 3.3 -5 A of electron current flowing in a wire to the right.

Now consider moving along with the electrons. The velocity of electrons in a good conductor is called the drift velocity and is very small, The drift velocity is dependent upon the amount of current flowing in the wire (denoted as i), the amount of free electrons per unit volume of the wire n, the cross-sectional area of the wire A, and the absolute value of the charge on an electron $|q|$. We will denote the drift velocity as v in this context. All of these quantities are related as expressed in (3.2).

$$i = nAv|q| \tag{3.2}$$

where $q = -1.6 \cdot 10^{-19}$ C is the charge on an electron and n $\approx 10^{29}$ free electrons per m^3 for a good conductor. If we solve this equation for the drift velocity v in a copper wire, and use reasonable values for i (2 A) and A (1 $mm^2 = 10^{-6}$ m^2), the drift velocity is computed in (3.3).

$$v = \frac{i}{nA|q|} = \frac{2}{(10^{29})(10^{-6})(|-1.6 \cdot 10^{-19}|)} = 0.125\ \frac{mm}{s} \tag{3.3}$$

This speed is slower than a wounded ant! If we imagine moving along with the electrons and observe the charge per time, we would observe +5 C of charge moving to the left each second. Hence, a current of –i = +5 A is flowing to the left. Current flowing in one direction is the negative of the current flowing in the other direction. A complete description of current requires two things: 1) value (positive or negative) and 2) direction (designated by an arrow). A value alone or an arrow alone is an incomplete designation of a current.

The device for measuring current is called an **ammeter**. It has a red probe and a black probe. An ammeter measures the amount of current flowing from the red probe to the black probe within the meter. You have to break the circuit and insert the ammeter where you want to measure the current. Consider a wire with the current i = -5A flowing to the right as depicted in the wire in figure 3.4.

Figure 3.4 Demonstration that current flowing in one direction is the negative of the current flowing in the opposite direction.

To measure the current flowing in the wire, we must break the wire and insert the ammeter. Consider the meter connected as follows.

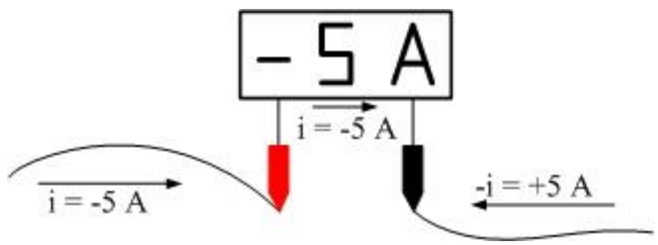


Figure 3.4 Using an ammeter to measure the current flowing in the wire to the right.

Remember, the current measured by the ammeter is from the red probe to the black probe within the meter. In this case the measured current is i = -5 A. There is only one other way to measure the current in the wire and that is to connect the red probe to the right-hand wire and the black probe to the left-hand wire. The ammeter always reads the current flowing from the red probe to the black probe within the meter. In this case the meter reads –i = +5 A.

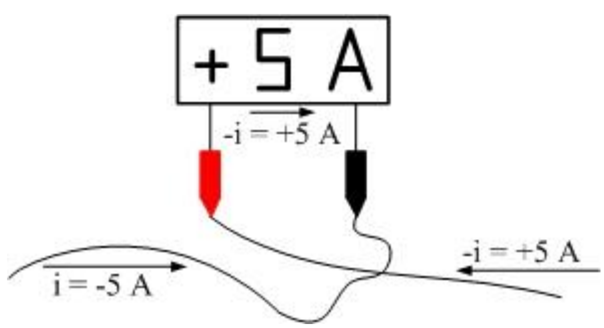


Figure 3.5 Physical measurement of current to demonstrate that current flowing in one direction is the negative of the current flowing in the opposite direction.

This shows once again that current flowing in one direction is the negative of the current flowing in the other direction, and all students should become comfortable with this concept.

Ampacity is the maximum current a wire or cable can carry safely. All wires and cables have an ampacity that is primarily determined by the cross-sectional area of the wire, and the wire's composition and temperature. As an example of a large value of ampacity, consider the aluminum conductor steel reinforced (ACSR) cable power transmission line shown in Figure 3.6. There are 75 aluminum conductors twisted around a band of 20 steel strength members. The ampacity of this cable is 6,000 A but it is operated at a typical current of 1,000 A. These cables are found on large steel towers, and each current carrying cable is called a **transmission line**. Transmission lines are typically operated at voltages exceeding 100,000 V. Power transmission towers typically have a lightning shield mounted above the current carrying transmission lines that is grounded to the steel tower so that if lightning strikes one of the conductors, hopefully it will be the lightning shield and not one of the power transmission lines. If lightning hits one of the transmission lines, it will cause a significant power surge that travels down the lines and can cause damage to equipment including transformers, switches and other equipment. If the surge is severe enough, it can literally exceed the fusing current of the cable and sever it. A pictorial diagram of a power transmission tower and lines are shown in Figure 3.7. Current normally does not flow in the lightning shield unless there is a fault. Birds land on these wires on a regular basis. However, the power transmission lines are too hot (temperatures of approximately 160° F are not unusual) and birds avoid them to prevent their feet from being burned. The reason the birds avoid the transmission lines is not because they will be shocked. The birds avoid the transmission lines because of high thermal temperature. As we will see in the next chapter, a simple analysis will show that the bird would have no current flowing through it were it to land on the transmission line.

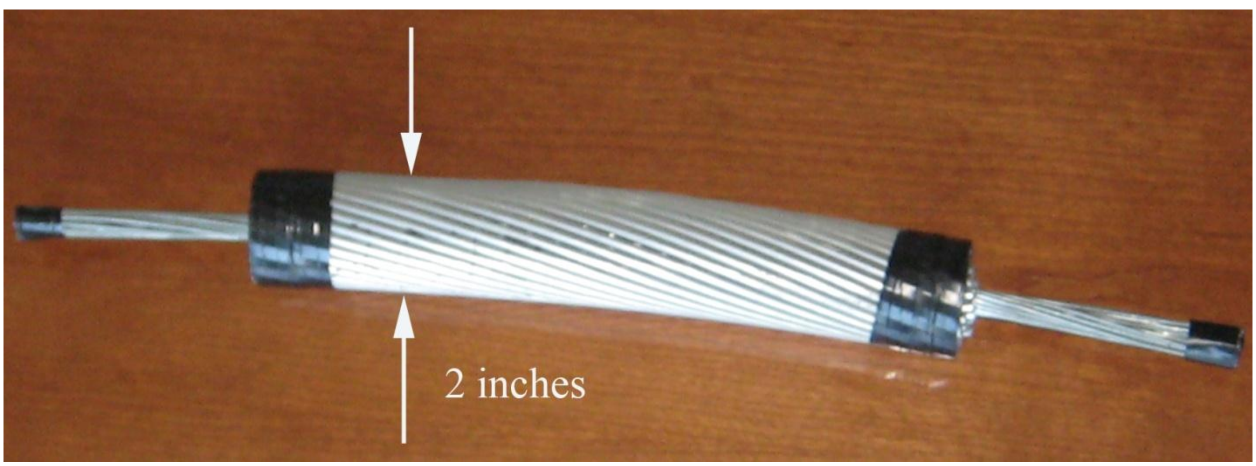


Figure 3.6 A segment of aluminum conductor steel reinforced (ACSR) power transmission line (courtesy of Duke Energy, Charlotte, NC) with an ampacity of 6,000 A.

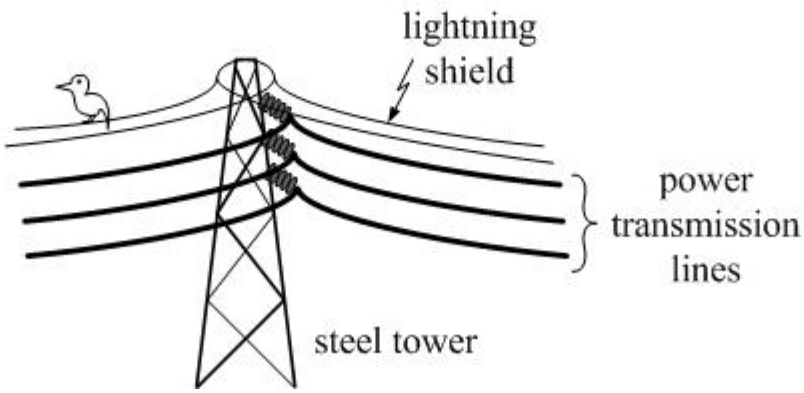


Figure 3.7 Power transmission line showing steel tower and bird on lightning shield

The transmission lines are insulated by air as they have no insulating jacket. In the U.S. and Canada, wire sizes are measured by their **American wire gauge (AWG)**. In general, as the gauge value is reduced, the conductor diameter is larger. The following picture shows several wires with various gauges.

Figure 3.8 Wires of various gauges. From left to right are 1/0 AWG, 200 A ground wire, 8 AWG stove cable, 12 AWG, 14 AWG two conductor with ground cable, 14 AWG, 18 AWG and 24 AWG twisted pair. All of the conductors are copper except for the 1/0 cable, which is made of aluminum and used to provide power to buildings.

Associated with each wire gauge is an **ampacity**. From the **National Electrical Code (NEC)**, the ampacities of different gauges of copper wire as well as their fusing currents (current that will melt the wire) are shown in figure 3.9.

AWG	Diameter (mm)	Ampacity	Fusing Current
1/0	8.25	125 A	1.9 kA
2	6.5	95 A	1.3 kA
4	5.2	70 A	946 A
6	4.1	55 A	668 A
8	3.3	40 A	472 A
10	2.6	30 A	333 A
12	2.1	25 A	235 A
14	1.6	20 A	166 A
16	1.3	13 A	117 A
18	1.0	10 A	83 A
20	0.8	7.5 A	59 A
22	0.6	5.0 A	41 A
24	0.5	2.1 A	29 A

Figure 3.9 Wire ampacity and fusing current chart as a function of AWG and the equivalent wire diameter.

3.4 Voltage

Voltage is the energy per charge required to move an infinitesimal positive charge from – to +. Instantaneous voltage is defined by

$$v(t) \triangleq \frac{dw(t)}{dq(t)} \tag{3.4}$$

The unit of voltage is the volt, abbreviated by V. $V \triangleq \frac{J}{C}$ where *w(t)* is the instantaneous energy measured in joules (J). Students are typically much more familiar with voltage than with current because many of myriad of consumer electronic devicess use batteries, and batteries are specified in part by their voltage. For example, the typical car battery is specified as delivering 12 V. Just as current has both a value and a direction, voltage has a value (positive or negative) and a direction. It's direction is given by its polarities (+,-). A **voltage drop** is from + to -, and a **voltage rise** is from – to +. Consider a 12 V car battery. One of the terminals is designated as the positive terminal and the other is designated as the negative terminal as shown in the figure 3.10.

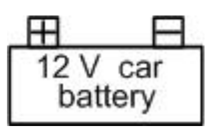


Figure 3.10 Car battery with lead plates and diluted sulfuric acid electrolyte inside.

Inside a car battery are 6 cells of 2 V each, connected in series, providing 12 V at the battery's terminals. In each cell are a set of plates. The positive plate is composed of porous lead and the negative plates are composed of either lead oxide or lead dioxide. The plates are bathed in the **electrolyte** composed of water and sulfuric acid. This electrolyte reacts with the lead plates through an oxidation reduction reaction that liberates electrons on the negative terminal leaving positive ions on the positive electrode. The negatively charged electrons on the negative terminal experience a force driving them toward the positive terminal external to the battery.

Voltmeters measure the voltage drop from the red probe to the black probe. Consider the measurement of voltage across the terminals of a 12 V car battery with the red probe connected to the positive terminal of the battery as shown in figure 3.11.

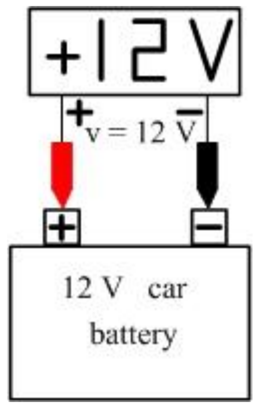


Figure 3.11 Car battery with its terminal voltage being measured.

The measurement observed by the voltmeter is always the voltage drop from the red probe to the black probe. In this case the voltage measured is v = +12 V. The only other way to measure the voltage across the battery terminals is to swap the voltmeter probes such that the red probe is connected to the negative terminal of the battery and the black probe of the voltmeter is connected to the positive terminal of the battery as shown in figure 3.12.

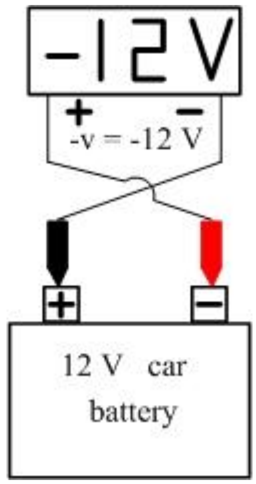


Figure 3.12 Car battery with its terminal voltage being measured with the measurement probes swapped.

The meter reads –v = -12 V because the voltage drop from the red probe to the black probe is –v = -12 v. This example shows that changing the polarities of the measurement changes the sign of the voltage measured. Another way of saying this is that the voltage drop to the right is the negative of the voltage drop to the left.

A complete description of voltage requires both a value (positive or negative) and a direction (given by the polarities +,-). If either the value or the polarities are missing, the voltage is ambiguous and incompletely specified.

3.5 Batteries

The first battery was created originally by Alessandro Volta (1745-1827). It was a single cell of zinc and copper as electrodes immersed in an electrolyte of sulfuric acid. A simple version of a single cell can be created with any two dissimilar metals immersed in a lemon, which can then be used to power an electronic clock! In the case of the lemon battery, it is the juice of the lemon that acts as the electrolyte with citric acid acting as the oxidizing agent. Volta also created a "Voltaic pile" or series of cells to create a higher voltage. A battery may be a single cell or it can be several cells connected in series (end to end – to +) to produce NV volts, where N is the number of cells and V is the voltage produced per cell. The most common primary (un-rechargeable, disposable) cell is the alkaline cell made from zinc and manganese dioxide with an electrolyte of potassium chloride and generates a voltage of 1.5 V per cell. Rechargeable batteries have become widely used. The most common types of rechargeable batteries are Lead-acid, NiCd (nickel cadmium), NiMH (nickel metal hydride), and (Li-ion) Lithium Ion. Rechargeable batteries typically cost more than primary batteries but can be recharged many times making their value much greater than primary batteries. The rechargeable batteries typically have a lower voltage than alkaline cells (e.g. 1.25 V). A battery's voltage is dependent on the electrode metals and electrolyte, not on the physical size of the battery. Several common examples of primary batteries are shown in figure 3.13.

Figure 3.13 Common primary batteries. In order from left to right are a 6 V (lantern) battery, 9 V battery, D battery (1.5 V), C battery (1.5 V), AA battery (1.5 V), AAA battery(1.5 V). Each battery has a positive and a negative terminal, and markings on the battery indicate the polarities. For example, the D cell clearly shows that the top of the battery is +.

Batteries are a source of DC voltage. DC is an acronym for "direct current," which implies that the flow of current is in one direction only. However, the term "DC" has evolved into a synonym for "constant." With this understanding, it makes sense to use the phrase DC voltage, which means "constant" voltage and is not the literally contradictory phrase "direct current voltage" that would arise if we were to utilize the literal acronym for DC.

What is different between the batteries in figure 3.13? This should be a topic of discussion with your classmates. Engineers provide "specifications" of elements and devices, which characterize the ability of the element or device to operate under normal or extreme conditions. What specifications are needed to characterize a battery? In addition to voltage, battery specifications also include Amp-Hrs (or total charge), but the correct specification is Watt-Hrs (energy). When a battery is specified in Amp-Hrs, it is in the context of the battery's terminal voltage, which varies from battery to battery. Next time you are shopping for a battery for your car, see what specifications are provided for car batteries.

Chapter 4

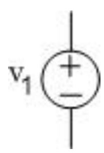
Circuit Elements and Basic Current and Voltage Laws

In this chapter the reader is introduced to ideal independent sources, resistance, the current-voltage relationship for a resistor known as **Ohm's law**, and **Kirchhoff's laws**, which are essentially conservation of energy and charge laws. Ideal independent sources do not physically exist, but that does not mean they are not useful. By connecting an ideal source to a resistor, a physical source can be modeled accurately.

4.1 Ideal Independent Sources

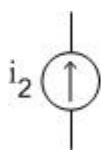
An ideal source has a circle as its symbol with either a set of polarities indicating a voltage source or an arrow depicting a current source.

Ideal Independent Voltage Sources



The voltage v_1 is **independent** of the rest of the circuit.

Ideal Independent Current Sources



The current i_2 is **independent** of the rest of the circuit.

Figure 4.1 Ideal independent voltage and current sources.

The subscripts for the sources shown in figure 4.1 were chosen arbitrarily. The key feature of an independent source is that its value and direction do not depend on what the source is connected to. If the voltage source $v_1 = 10$ V, then no matter what the voltage source is connected to, its voltage will always be a 10 V drop downward. If the current i_2 is 4 A, then this current is 4 A flowing upward no matter what the current source is connected to. It should be noted that it is possible to draw contradictory circuit configurations that produce impossible situations. For example, two ideal independent current sources could be drawn in series, meaning that they share a common current. If the current sources are of differing values, then the current flowing through the pair of current sources is ambiguous and the model fails. In a similar fashion, two ideal independent voltage sources could be drawn in parallel, meaning that they share the same voltage. However, if the two voltage sources have different values, the voltage is ambiguous and once again the model fails. It is important to recognize appropriate circuit diagrams that do not contain contradictory voltages or currents.

4.2 Resistors

A **resistor** is a two terminal device that converts electrical energy into heat. Its symbol is shown in figure 4.2.

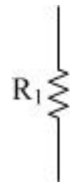


Figure 4.2 Resistor symbol.

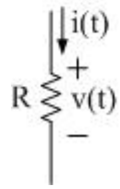
The electrical elements on an electric cooking range are designed to produce heat in order to cook. In this case the production of heat is the desired outcome. In electric circuits resistors are elements used to control the values of current and voltage in various locations within a given circuit. The most common resistor has a carbon filament encased in an epoxy jacket with a sequence of colored bands on the outside of the jacket to indicate the resistor's nominal value. There are typically four bands. The first two bands form the mantissa of the value, the third band designates the power of ten and the fourth band indicates the tolerance. A universal resistor color code is translated to the ten digits as follows:

Black	0
Brown	1
Red	2
Orange	3
Yellow	4
Green	5
Blue	6
Violet	7
Gray	8
White	9

Figure 4.3 Resistor color code.

As an example consider the first three bands of a resistor to be Brown Red Orange. Then the nominal value of that resistor is $12 \cdot 10^3$ Ω or 12 kΩ. The fourth band is the tolerance. Typical tolerances are 5% (Gold) and 10% (Silver). If you had a bag of a thousand 12 kΩ resistors with a 5% tolerance, then the resistors in that bag would have a value between 12 kΩ – 0.05(12 kΩ) and 12 kΩ + 0.05(12 kΩ) or between 11.4 kΩ and 12.6 kΩ. It is possible to obtain more precisely specified resistances (e.g. 1% tolerance), and the cost of these precision resistors is not that much greater than the standard 5% tolerance resistors. Many applications do not require a tight tolerance on the resistance value while other applications require very precise values. Other applications may require closely matched resistances in which multiple resistors with identical values are necessary. It is important to understand the meaning of and difference between accuracy and precision. Many bench top multimeters can measure the value of a resistance to the micro-Ohm. This is a specification of precision. Unless the meter is calibrated, it is possible to precisely measure an inaccurate value of resistance! The resistance of a resistor (or any other objective quantity) has an exact value. The ability of an ohmmeter to accurately measure a resistance is determined by the difference between the exact value and the measured value. The number of significant digits measurable by the meter determines the precision. Ideally, the meter should be both accurate and precise. The resolution of a meter is the smallest difference between the most precise measurements. Can you explain the meaning of and relationship between accuracy, precision and resolution? Try explaining these concepts to a non-engineering student.

The current-voltage relationship for a resistor is given by Ohm's law.
Ohm's law states that the voltage drop across resistance R is R times the current in the same direction.

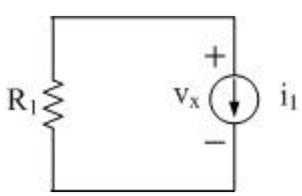


$v(t) = Ri(t)$

Figure 4.4 Resistor labeled with current in the direction of voltage drop to illustrate Ohm's law.

The unit of resistance is the Ohm and is denoted by an upper case Greek letter omega Ω. Resistance is the opposition of current. A large value of resistance means that a large voltage is required to obtain a moderate value of current. A resistor is a bidirectional device meaning that it has no polarity. A resistor in a circuit can be removed and reinserted with its terminals swapped and no difference in any voltage or current in the circuit will have changed in value from before the resistor was removed. As an example of Ohm's law consider example 4.1.

Example 4.1



Given the values for i_1 and R_1 calculate the value of the voltage v_x in terms of the given symbols.

Solution: There is no law that will produce the value of the voltage across a current source by knowing its current alone. One must examine what the current source is connected to. In this case, the current source is connected to a resistor. Therefore Ohm's law is used to obtain the voltage (since v_x is the voltage across both i_1 and R_1): $v_x = R_1(-i_1)$ Note that the current used in this expression is $-i_1$. Remember, Ohm's law relates the voltage across a resistor to the current through it in the direction of the voltage drop. The voltage drop across the resistor is directed downward. Therefore, the voltage v_x, the voltage drop across R_1 is the product of R_1 and the current flowing downward ($-i_1$). Ohm's law does not apply to current sources.

Physical Resistors

A physical resistor is a two terminal device that converts electrical energy into heat. The most common types of resistors are carbon film and carbon composition. Resistors are not only rated by their nominal value and tolerance, but also by their ability to dissipate heat. Resistors have a power rating that specifies the ability of the resistor to dissipate heat. Figure 4.5 illustrates several ¼ Watt resistors and one 10 Watt resistor.

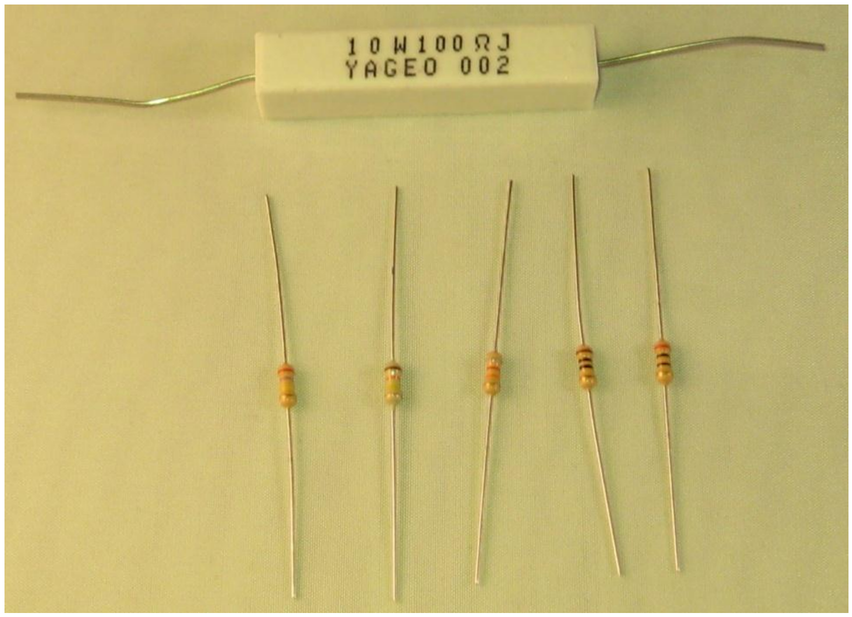


Figure 4.5 Various ¼ Watt carbon film resistors and one 100 Ω, 10 Watt power resistor

Potentiometers

A **potentiometer,** or more familiarly, a “pot,” is a variable resistor. Potentiometers have three terminals with a fixed resistance between the outer terminals and a variable resistance between the center terminal, or wiper, and the end terminals. A diagram of the two most common pots (rotary dial and linear slide) is shown in figure 4.6.

As the stem of the rotary dial pot is rotated clockwise, the resistance between the center terminal and the left end terminal increases while the resistance between the center terminal and the right end terminal decreases and approaches a short circuit. The resistance between the two outer terminals remains fixed, independent of the location of the wiper, and this resistance is the value of the potentiometer’s resistance specification. As the stem of the linear slide pot is pushed down from the top, the resistance between the top terminal and the center terminal increases and the resistance between the center terminal and the bottom terminal decreases and approaches a short circuit.

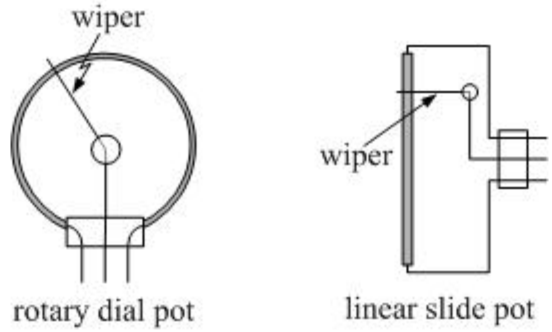


Figure 4.6 Diagrams of a Rotary dial potentiometer and a linear slide potentiometer.

The resistance between the top and bottom terminal remains fixed, independent of the location of the wiper, and this resistance is the value of the potentiometer's resistance specification.

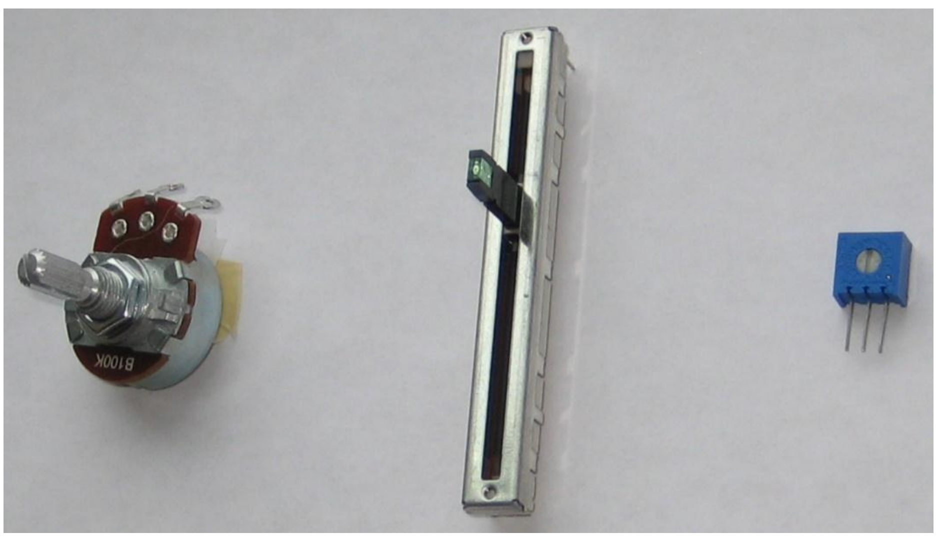
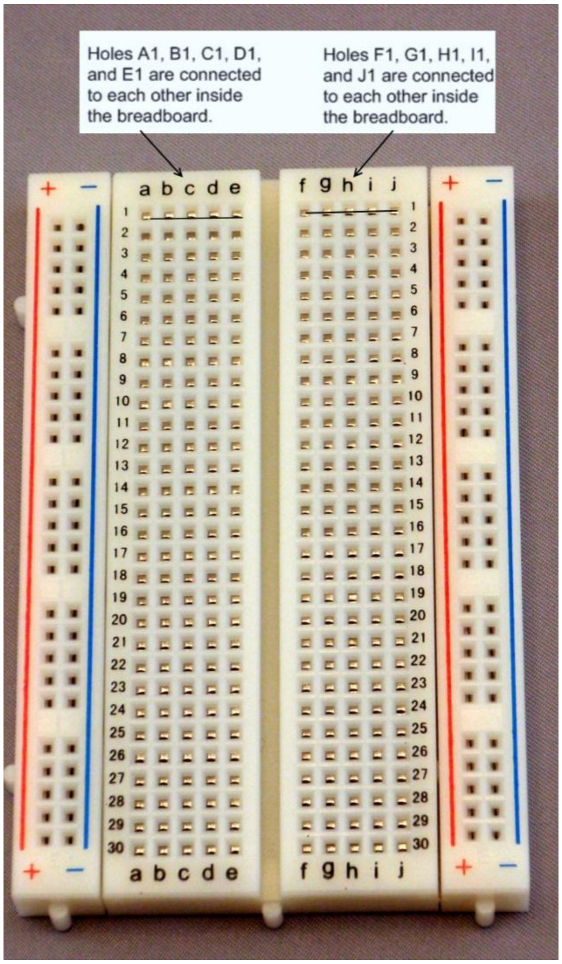


Figure 4.7 Two rotary dial pots (far left and right) and a linear slide pot (center).

The three terminals are clearly visible on the two rotary dial pots. The pot on the right is small and designed to fit into either a breadboard or a printed circuit board (PCB), and a slotted screw driver is used to adjust the wiper position. Many applications of the PCB pot use the pot to "trim" a voltage or current to a specified value. This is why the literature often refers to this particular pot as a "trim pot."

A breadboard is a platform for rapid prototyping of circuits. Components can be plugged in and taken out rapidly forming a variety of circuits and without the use of solder. You will learn to use a breadboard in lab and will quickly realize its utility for circuit building. Two pictures of a breadboard are shown in figure 4.8 that illustrate how the breadboard's internal connections provide a way to connect elements together. Individual holes in the breadboard are referenced in the form "column # row #." For example, the first hole in row 1 is "a1" or "A1."



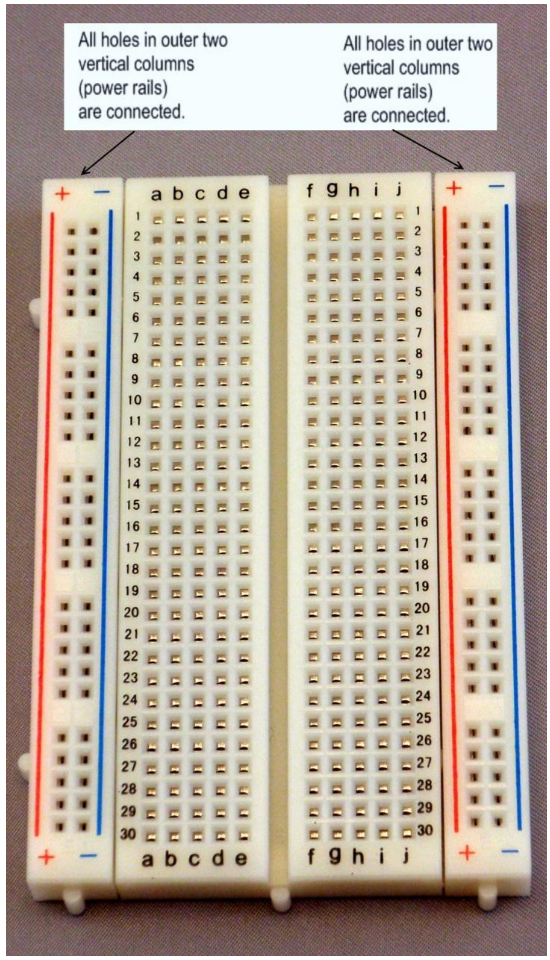


Figure 4.8 Circuit prototyping breadboard illustrating connections internal to the breadboard.

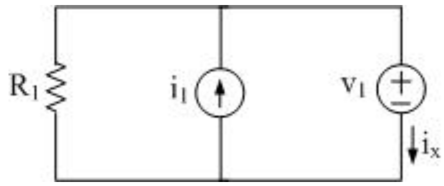
4.3 Kirchhoff's Laws

Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL) are conservation laws. KCL is a conservation of charge law, and KVL is a conservation of energy law.

Kirchhoff's Current law (KCL) states that the algebraic sum of the currents leaving any node must be zero.

Definition: A **node** is connected conductors

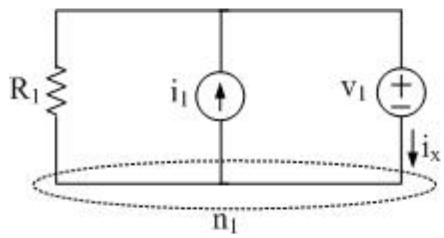
Example 4.2



v_1 = 10 V
i_1 = 2 A
R_1 = 4 Ω

Find i_x.

Solution:



In order to find i_x we will apply KCL at node n_1 because the resulting equation will relate the unknown current i_x to the element values given. Node n_1 has been outlined by a dotted line. There are three currents involved in this KCL.

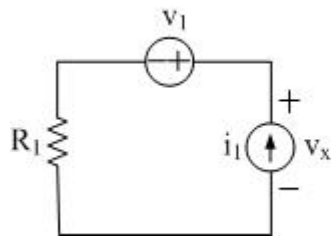
$$\text{KCL } n_1\text{: } \frac{-v_1}{R_1} + i_1 - i_x = 0 \tag{4.1}$$

Solving this equation for i_x leads to

$$i_x = i_1 - \frac{v_1}{R_1} = 2 - \frac{10}{4} = \boxed{-0.5\ A} \tag{4.2}$$

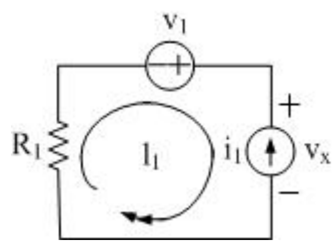
Kirchhoff's Voltage Law (KVL) states that the algebraic sum of the voltage drops around any loop must equal zero. A **loop** is any closed path. As an example of KVL consider example 4.3.

Example 4.3



$i_1 = 4$ A
$v_1 = 8$V
$R_1 = 3\ \Omega$

Find v_x. In order to find the unknown voltage v_x across the current source we will apply KVL around the one and only loop in the circuit.



Any node can be chosen as the beginning node (hence the ending node as well). Another choice is whether to add voltage drops going clockwise or counter clockwise. This book will always begin KVL equations at the lower left node and add voltage drops going clockwise. The double headed arrow with the label l_1, denoting loop 1, is an aid to indicate where one begins the loop and having two arrowheads it is distinct from currents. Adding voltage drops going clockwise produces.

$$R_1(-i_1) - v_1 + v_x = 0 \tag{4.3}$$

Solving for the unknown voltage v_x results in:

$$v_x = v_1 + R_1 i_1 = 8 + 3(4) = \boxed{20\text{ V}} \tag{4.4}$$

All numerical results should always be boxed in and accompanied by the appropriate units.

4.4 Open Circuits, Short Circuits and Switches

The simplest type of switch is called a **single pole single throw (SPST) switch**. The switch can only be in one of two states, open or closed. When the switch is closed, it forms a **short circuit** and when it is open, it forms an open **circuit**. This is illustrated in figure 4.9.

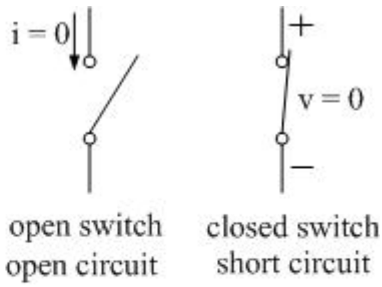


Figure 4.9 Diagram of open and closed switches

Definition: An open circuit means <u>zero current</u>
Definition: A short circuit means <u>zero voltage</u>

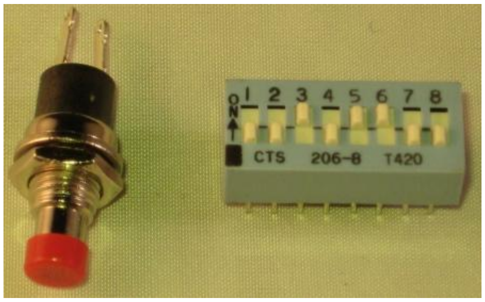
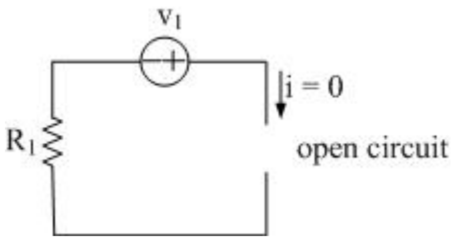


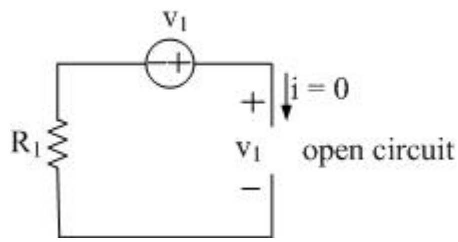
Figure 4.10 Push button switch and a dual in-line package (DIP) switch array.

Example: 4.4



This circuit has an open circuit on its right side. By definition, an open circuit has zero current as shown in the schematic. Think about the voltages across the elements in the circuit. What do you think the voltage across R_1 is? Since the current through the open circuit is zero, and R_1 is in

series with the open circuit, the current through R_1 is zero, Ohm's law states that the voltage across R_1 is zero. Elements in **series** share a common current. Kirchhoff's voltage law states that the algebraic sum of the voltage drops around the loop must be zero. Therefore the voltage across the voltage source, v_1, must appear somewhere else in the loop. The only other place v_1 could appear is across the open circuit. But what are the polarities of this voltage? Think about Kirchhoff's voltage law. If the polarities are a voltage drop directed upward, then the sum of the voltage drops around the loop is $2v_1$! The polarities of the voltage across the open circuit with value v_1 must be a drop directed downward as shown in the following circuit.



Application of Kirchhoff's voltage law verifies that the polarities of v_1 across the open circuit are correct. Now consider the following circuit.

Example 4.5

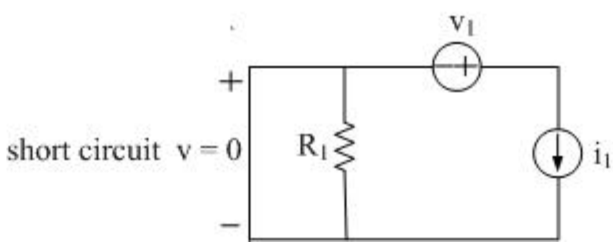


Figure 4.11 Circuit illustrating a short circuit across a resistor resulting in zero voltage across the resistor and therefore zero current through R_1.

A short circuit has been placed across the resistor R_1. The short circuit, by definition, has 0 V across it. Therefore the voltage across R_1 is zero. Ohm's law states that if the voltage across R_1 is zero, then the current through R_1 is zero. The result is that v_1 and i_1 are now in parallel and in series! and all of v_1 is across the current source i_1 and the current through the short circuit and through the voltage source is i_1 as shown in figure 4.12.

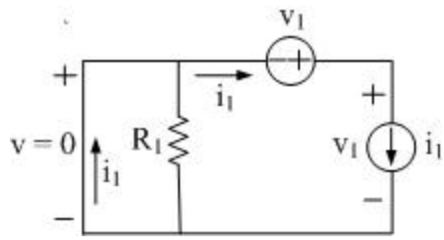


Figure 4.12 Circuit with R_1 shorted labeled for analysis.

It should be noted that **short circuits can be dangerous**. When jump starting a car using a good battery from another car and a pair of jumper cables, the set of jumper cables connect the bad (low charge) battery to a good (charged) battery, and the order that the connections are made is important for your safety. When this connection is made, it is extremely important that the red and black terminals on the "other" side of the jumper cables are not in contact with anything, especially each other. It is a common mistake to allow the unconnected terminals to touch each other, forming a **short circuit** across the good battery. A huge spark follows, and hopefully the cable terminals will be disconnected from each other immediately. During the short circuit duration, it is possible for the battery to explode sending hot sulfuric acid out of the destroyed battery. The proper procedure for connecting a set of jumper cables to a battery begins with separating and maintaining separation of the jumper cable terminals that will be connected to the "dead" battery. Then the other set of terminals are connected to the "good" battery carefully noting which terminal is positive (+) and which terminal is negative (-). Jumper cables usually have colored (red and black) plastic insulators on both ends of the cables, and it is intended for the red cables to be connected to the positive terminals of each battery. One of the black terminals is connected to the negative terminal of the "good" battery. The last step of the connection process is to clamp the black (negative) terminal of the cable to the chassis of the car with the "bad" battery. It is dangerous to connect the final black terminal to the negative terminal of the "dead" battery because the battery could explode, sending sulfuric acid onto the body and possibly into the eyes of the person making the connections. The negative terminal of a car battery is connected to the chassis of a car making the whole metal chassis one large ground node of the car's electrical system. The **final** connection of the jumper cables is to clamp the black terminal of the jumper cable to the chassis of the car with the dead battery. This connection should be well away from the battery to keep you out of harm's way. It should be noted that a car battery is capable of delivering an enormous amount of current (thousands of Amps) for a short period of time, and represents a potentially dangerous source, and care must be taken when jump starting a vehicle.

Circuit breakers are used to protect humans and circuits in the case of a short circuit. A circuit breaker is a switch that can be turned off manually or electrically when a specific current value is exceeded. For example, most wall socket circuits in the U.S. utilize 15 A circuit breakers. If the device plugged into a 15 A circuit attempts to draw more than 15 A, the circuit breaker trips and the circuit is broken (an open circuit is imposed). The fault in the circuit should be identified before the circuit is re-energized.

Exercise: You walk into a room and as you bump into a table with a lamp, the bulb flickers. A friend says, “There is a short in it.” What do you say in response?

Solution: You state that a short circuit would trip the circuit breaker. What is actually happening is that there is a fault in the lamp system, probably the bulb isn’t all the way in or the switch is broken. The problem is that an **open circuit** is causing the bulb to turn off and the circuit is changing states back and forth from a closed circuit to an open circuit.

4.5 Meters

Any ideal meter does not interfere with the circuit as measurements are taken. For example, when a multimeter is configured as a voltmeter and used to measure the voltage across two nodes, the meter’s presence should not change any of the voltages or currents in the circuit. Think about the model that would be appropriate for a voltmeter. The answer is to have the meter with a huge (as large as possible) resistance. Ideally, a voltmeter should appear as an open circuit to the circuit under test. Prior to making a voltage test, make sure the multimeter is configured to measure voltage and touch the probes together. The meter should read zero volts for the short circuit being measured,].

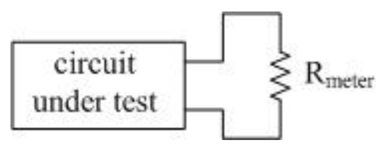


Figure 4.13 Circuit configuration for a voltmeter modeled by its internal resistance connected to a circuit whose voltage is to be measured.

Practical voltmeters have a finite internal resistance. The measurement reading indicated is hopefully only slightly affected by the voltmeter. However, if the circuit under test has large valued resistors, the measurements may be very inaccurate. Consider the following circuit.

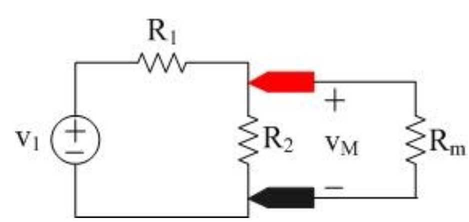


Figure 4.14 Voltmeter connected across large resistance R_2.

It is left as an exercise for the reader to verify the voltage that is measured by the meter is given by the expression in (4.5)

$$v_M = (v_1)\frac{\dfrac{1}{\dfrac{1}{R_2}+\dfrac{1}{R_M}}}{R_1 + \dfrac{1}{\dfrac{1}{R_2}+\dfrac{1}{R_M}}} \tag{4.5}$$

where the correct voltage, which is the voltage without the presence of the meter is given by the expression in (4.6).

$$v_M = (v_1)\frac{R_2}{R_1 + R_2} \tag{4.6}$$

The only case in which the meter will not affect the voltage it is measuring is the case in which R_M is large compared to R_2. It is left to the reader to take the limit of (4.5) as $R_M \rightarrow \infty$. This example illustrates that care that must be taken when making a measurement. Many modern bench top multimeters have a high impedance setting for measuring voltage in conditions where the unknown voltage is across a high valued resistance.

In a similar manner, care must be taken when measuring electric current through a very small resistance. Ideally, an ammeter has zero resistance (is a short circuit), but all ammeters have a very small resistance. Consider the following circuit where the current i_M is being measured.

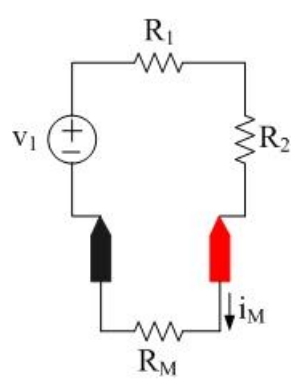


Figure 4.15 Circuit configured to measure the current through R_1 and R_2.

With the ammeter present, the current measured is given by (4.7)

$$i_M = \frac{v_1}{R_1 + R_2 + R_M} \tag{4.7}$$

The accurate or correct value of i_M is given by (4.8)

$$i_M = \frac{v_1}{R_1 + R_2} \tag{4.8}$$

which is achieved only if the ammeter's internal resistance R_M is small compare to R_1 and R_2.

Show that the current measured by the meter is correct if the limit of i_M in the expression in (4.7) is taken as $R_M \rightarrow 0$.

The purpose of these two measurement examples is to show that it is a mistake to assume that if a multimeter is set to measure voltage or current, that the value displayed is always correct. Care must be taken when measuring voltage across a circuit with large valued resistors and when the current is measured in circuits with very small valued resistors. The technical manual for the multimeter you are using is very likely to be online. When you are unsure about a measurement, consult the specifications of the meter to find its resistance when measuring voltage or current. Similar caution must be observed when measuring the resistance of large valued and small valued resistors. For example, your body has a finite resistance between any two points on your body. If you hold the probe on one end of a resistor and hold the other probe on the other end of the resistor, the meter reads the resistance of the parallel combination of your body and the resistor being tested. It is good practice to measure the resistance of a resistor without the use of your fingers to hold the probes of the meter on the leads of the resistor under test.

Chapter 5

Power

5.1 Power Definition and Calculations of Power Delivered and Power Absorbed

Power is the rate of flow of energy in a given direction. Mathematically, the definition of power is given by

$$p(t) \triangleq \frac{dw(t)}{dt} \tag{5.1}$$

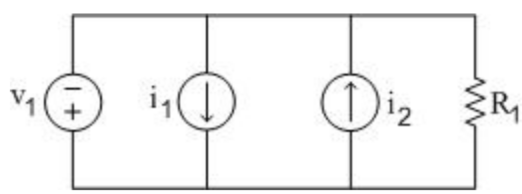
where *w(t)* is the instantaneous energy. The unit of power is the watt ($W = \frac{J}{s}$).

Electrically, the derivative of energy with respect to time can be expressed as the product of voltage and current.

$$p(t) \triangleq \frac{dw(t)}{dt} = \frac{dw(t)}{dq(t)} \cdot \frac{dq(t)}{dt} = v(t)i(t) \tag{5.2}$$

The direction of power is obtained from the directions of voltage and current. Power is said to be delivered if current i(t) is in the direction of the voltage rise v(t). Power is said to be absorbed if current i(t) is in the direction of the voltage drop v(t). It follows that the power absorbed by an element is the negative of the power delivered by that element.

Example 5.1



$v_1 = 8$ V
$i_1 = 3$ A
$i_2 = 5$ A
$R_1 = 2\ \Omega$

Given the element values, calculate the power absorbed by each element.

Solution: In order to calculate power absorbed you need both the voltage and the current in the direction of the voltage **drop**. Examining the circuit, we observe that all elements are in parallel meaning they all share the same voltage, or equivalently they all share the same pair of nodes. Therefore, there is only one voltage in the circuit, and that is v_1 (voltage drop directed upward).

$$p_{abs\ by\ R1} = v_1\left(\frac{v_1}{R_1}\right) = 8\left(\frac{8}{2}\right) = 32\ W \tag{5.3}$$

$$p_{abs\ by\ i1} = v_1(-i_1) = 8(-3) = -24\ W \tag{5.4}$$

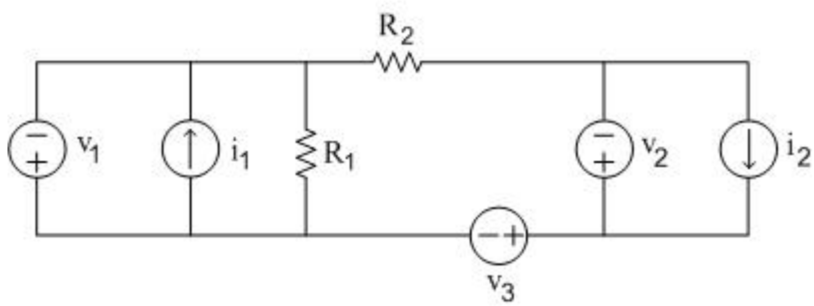
$$p_{abs\ by\ i2} = v_1(i_2) = 8(5) = 40\ W \tag{5.5}$$

$$p_{abs\ by\ v1} = v_1\left(i_1 - i_2 - \frac{v_1}{R_1}\right) = 8\left(3 - 5 - \frac{8}{2}\right) = -48\ W \tag{5.6}$$

It should be noted that power is conserved:

$$\sum_{\substack{all\ elements\\ in\ a\ circuit}} p_{abs} = \sum_{\substack{all\ elements\\ in\ a\ circuit}} p_{del} = 0$$

Example 5.2



$v_1 = 10$ V
$v_2 = 15$ V
$v_3 = 12$ V
$i_1 = 2$ A
$i_2 = 1$ A
$R_1 = 2\ \Omega$
$R_2 = 7\ \Omega$

Given the element values, find the power delivered by each element and confirm the conservation of power.

Solution:

When calculating power delivered by an element, you need the voltage across the element and the **current in the direction of the voltage rise**.

$$p_{deli1} = v_1(-i_1) = 10(-2) = -20\ W \tag{5.7}$$

$$p_{delR1} = v_1\left(\frac{-v_1}{R_1}\right) = 10(\frac{-10}{2}) = -50\ W \tag{5.8}$$

$$p_{delR2} = (v_1 + v_3 - v_2)\left(\frac{-v_1 - v_3 + v_2}{R_2}\right) \tag{5.9}$$

$$p_{delR2} = (10 + 12 - 15)(\frac{-10 - 12 + 15}{7}) = -7\,W \tag{5.10}$$

$$p_{deli2} = v_2(i_2) = 15(1) = 15\,W \tag{5.11}$$

$$p_{delv1} = v_1(i_1 + \frac{v_1}{R_1} + \frac{v_1 + v_3 - v_2}{R_2}) = 10\left(2 + \frac{10}{2} + \frac{10+12-15}{7}\right) = 80\,W \tag{5.12}$$

$$p_{delv2} = v_2\left(-i_2 + \frac{-v_1 - v_3 + v_2}{R_2}\right) = 15\left(-1 + \frac{-10 - 12 + 15}{7}\right) = -30\,W \tag{5.13}$$

$$p_{delv3} = v_3\left(\frac{v_1 + v_3 - v_2}{R_2}\right) = 12\left(\frac{10 + 12 - 15}{7}\right) = 12\,W \tag{5.14}$$

Now, to test for conservation of power, we add together all of the powers delivered by each element:

$$\sum_{\substack{all\ elements \\ in\ the\ circuit}} p_{del} = -20 - 50 - 7 + 15 + 80 - 30 + 12 = 0$$

Thus, we have confirmed that power is conserved. This provides some confirmation that the individual calculations are correct. It is highly unlikely that the check for power conservation turns out correct and one of the individual power calculations is incorrect, but it is possible.

5.2 AC Power Systems

Power calculations are important at all levels. An example of power at the “low level” is the power dissipated by a transistor, an electronic component that we have not studied yet but will toward the end of this book. It was the invention of the transistor in 1947 that truly began the beginning of digital computers. Have you ever noticed that your laptop gets hot sometimes? That is because the microprocessor is performing so many computations per second that the power dissipated by each of the transistors is not insignificant. It is actually of great importance. A microprocessor typically contains millions of transistors. Hence, the power delivered per transistor becomes critical. If you look at a computer motherboard, you will see that the microprocessor is connected to a heat sink with vertical fins and a cooling fan. These thermodynamic measures are necessary to move the heat generated by the transistors away from the microprocessor to avoid melting the device. Power at the high level is also important. Thunderstorms generate a large charge difference between the bottom of a cloud and ground that can generate thousands of volts. If the voltage becomes large enough, an electrical discharge occurs called lightning. The interested reader is encouraged to read other sources for information about lightning and the heat dissipation problem in microprocessors. We use electric power every day. Can you imagine a world where electricity is not available? The AC electrical power grid is critical to our daily lives. A portion of the AC power grid is examined to illustrate how power is generated and made available in buildings. Figures 5.1a,b,c show the chain of power delivery from generation to a wall socket at the end user.

Power is usually generated by fuel burned to heat water to create steam, and under great pressure the steam is used to turn the rotor of large generators at a rate of 60 cycles per second (60 Hz). The output voltage of the generator is sinusoidal at a frequency of 60 Hz. The voltage is immediately stepped up to values of 300,000 to 1 MV in order to reduce the current that must be transmitted (typically a few kA, recall that power is the product of voltage and current), sometimes over long distances, by power transmission lines. Power transmission lines are often accompanied by a grounded lightning shield mounted above the transmission lines (see figure 4.1a). Under normal operating conditions, there is no current through the lightning shield, and birds often land on the grounded wire. However, the transmission lines are too hot (e.g. 160 °F) for birds feet and they avoid them. Once the transmission lines are near the user, step-down transformers, located at a power sub-station, are used to reduce the voltage to approx. 12-30 kV on what are called distribution lines. Distribution lines typically operate at temperatures near 100 °F, and these lines are attractive to birds and squirrels, especially in the winter.

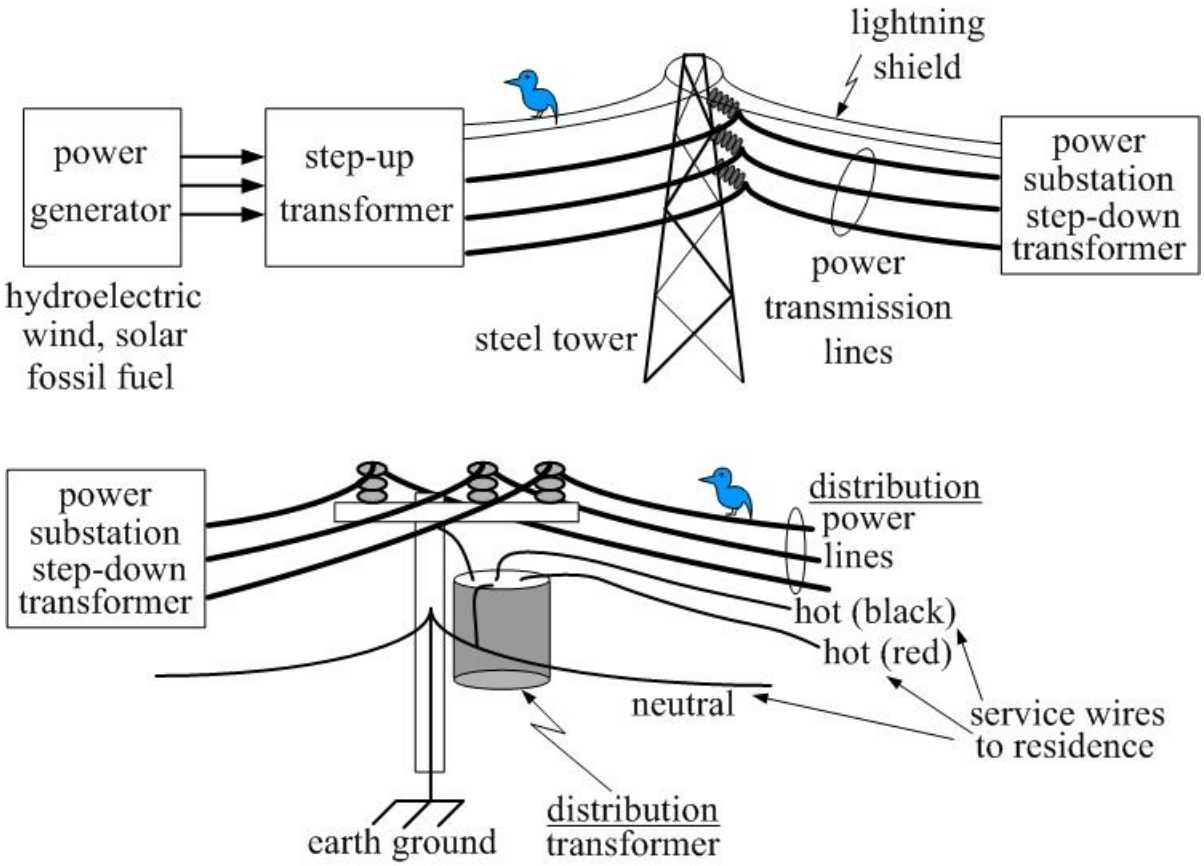


Figure 5.1a AC Power system diagram from generation to service wires entering residence.

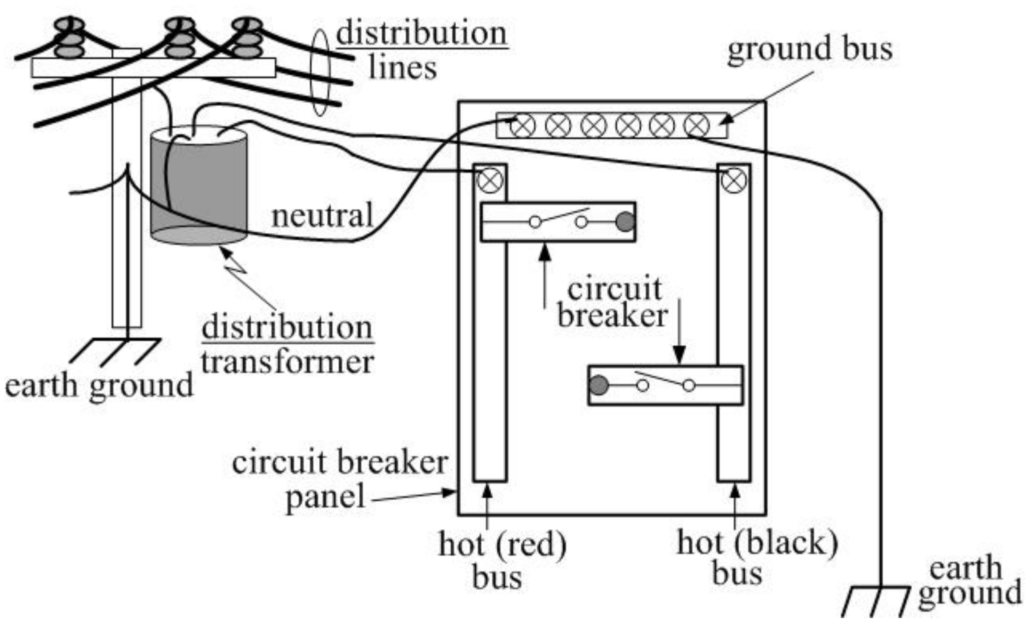


Figure 5.1b AC power system from the distribution line to the circuit breaker panel.

To obtain AC power at amplitudes useful and relatively safe for humans, a distribution transformer, either mounted up on the power pole or in one of the “green boxes” that are typically seen with decals reading, “Danger, High Voltage,” and show a human falling backwards as a lightning bolt strikes the head. The distribution transformer provides the service wires that enter a building or residence. Typically two (electrically) hot cables and a grounded neutral cable are connected to the building’s circuit breaker panel. At the circuit breaker panel, ground and neutral are both connected to ground. However, once ground and neutral leave the circuit breaker panel, their usage is very different. Neutral is used as a current carrying conductor, usually with white insulation and ground is used as a safety conductor, usually with green insulation or bare wire. The two hot lines from the circuit breaker panel typically have black and red insulation. This color scheme is part of the national electrical code. It is counter-intuitive for one of the “hot” wires has black colored insulation, and this has led to many wiring errors. Wiring errors in a building’s power system wiring can be extremely dangerous. A concept often difficult for students to understand is the difference between neutral and ground. Both neutral and ground are connected to the same node at the circuit breaker panel, which, in turn, is connected to a buried conductor in the earth and to the grounded neutral wire coming from the power pole. The grounded neutral wire at the power pole is connected to a buried conductor at the base of every power pole all the way back to the substation where all the equipment is connected to the grounded neutral wire and to a large buried conductor. Returning to the question at hand. What is the difference between neutral and ground? Neutral is a current carrying conductor through which the current flowing from the hot wire(s) flows back to the panel through the appliance being powered. Recall also that the voltage being provided by the hot wires is sinusoidal (AC), meaning that current flows in one direction for $\frac{1}{120}$ s and then reverses direction for the next $\frac{1}{120}$ s for a total time per cycle of $\frac{1}{60}$ s. The only time when there is current through the ground wire is when there is a fault in the system. Ground is typically connected to the metal housing of an appliance such as a dryer, washing machine, or refrigerator. If one of the hot wires becomes loose and touches the metal chassis of the appliance (electrifying the appliance housing), a short circuit is created and the circuit breaker trips, de-energizing the circuit until the fault is corrected.

A typical 120 V wall socket is wired as shown in figure 5.1c. Lines connected to both neutral and ground are connected to the ground bus back at the circuit breaker panel. A circuit breaker is a device that under normal operating conditions is a closed switch allowing current to pass through it. However, if a certain value of current is exceeded, the circuit breaker is tripped and becomes an open circuit and power to the outlet is cut off. A typical wall socket circuit breaker is rated at 15 A, and 14 AWG wire is used for the connection. Some special electrical outlets are designed for 20 A, and 12 AWG wire is used to feed these outlets.

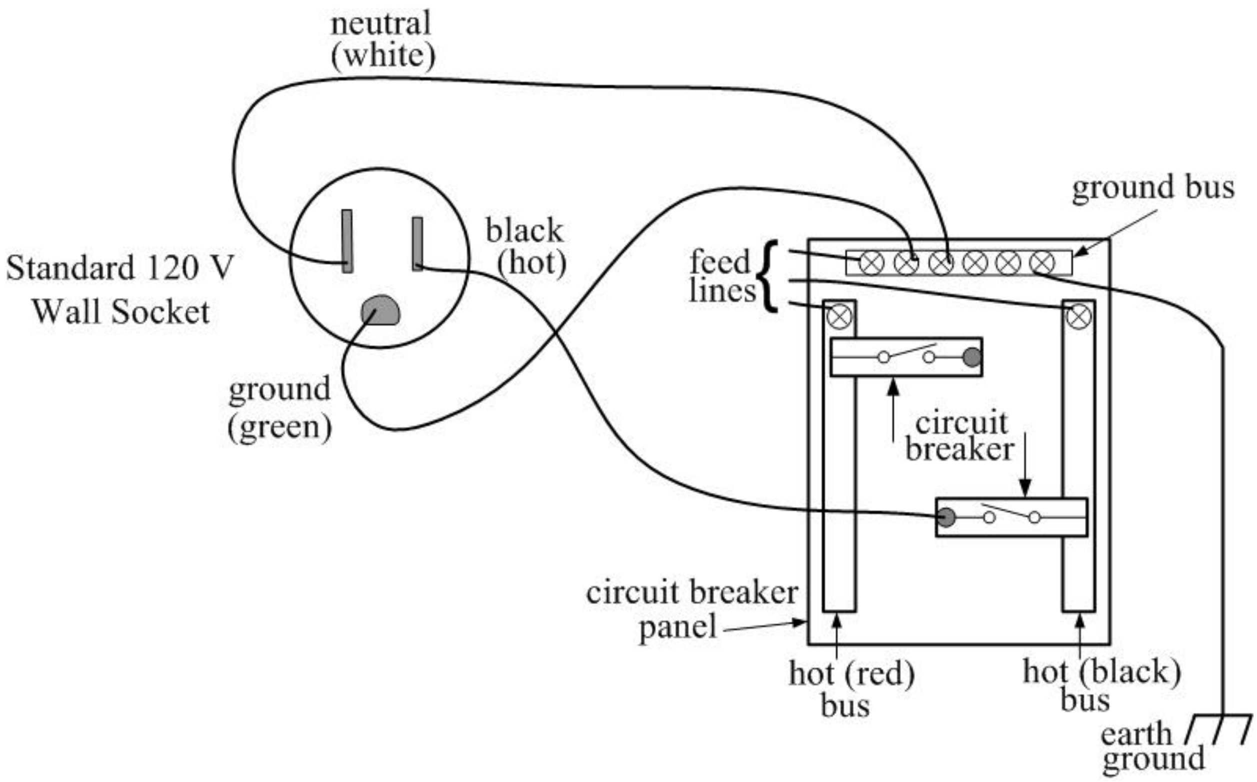


Figure 5.1c AC power system from circuit breaker panel to 120 V wall socket.

Some appliances would need more than 20 A at 120 V to operate. These appliances are fed with 240 V, which is obtained by powering the appliance with both the red and the black 120 V lines. You might wonder how these wires work together to produce 240 V. The answer is that the red and black lines are 180° out of phase with each other. A graph of the black AC voltage is shown along with the red AC voltage to show the reader how it is possible to obtain 240 V from the two lines in figure 5.2.

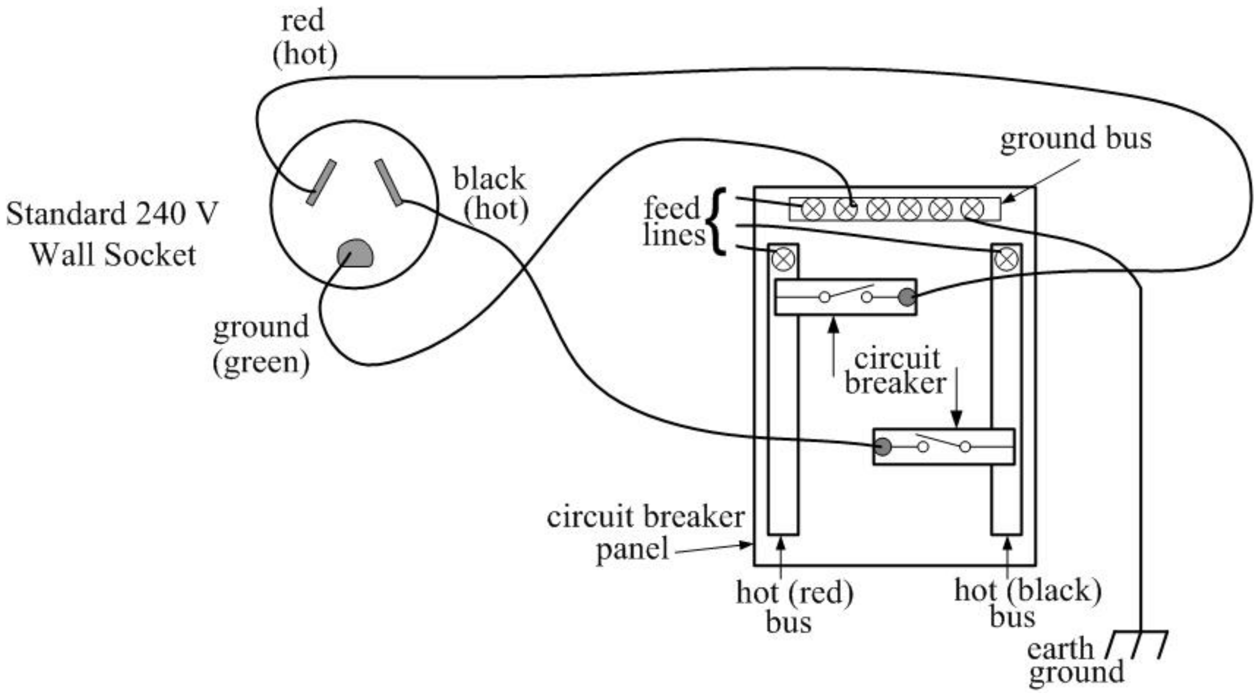


Figure 5.1d Circuit breaker panel feeding a 240 V outlet

5.2 Root Mean Square Values (RMS)

AC is an acronym for "Alternating Current," but the term has become equivalent to the term "sinusoidal." The phrase "AC voltage or AC current" makes sense if it is understood that "AC" is interpreted to mean "sinusoidal." A single number is used when talking about AC such as 120 V or 240 V, but what does that single number mean when the voltage or current is actually sinusoidal? 120 V is obtained by taking the potential difference (voltage) between only a single "hot" wire and neutral, while 240 V is obtained by taking the potential difference between the two "hot" wires (black and red). To visualize this, the sinusoidal voltage associated with the black wire relative to neutral and the red wire relative to neutral are graphed together in figure 5.2.

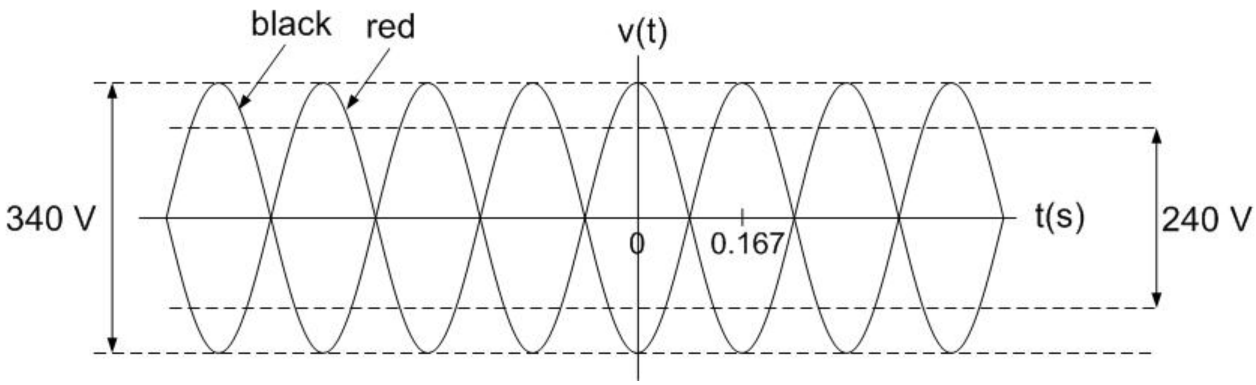


Figure 5.2 The voltage waveforms produced by the difference between each of the red and the black wires and neutral, showing the 180° phase shift between the two voltages and the 16.7 ms period (f = 60 Hz).

Larger appliances (water heater, central heating and air conditioning, stove) utilize 240 V, which is obtained by taking the voltage difference between the black and red wires, each of which has an RMS* value of 120 V. The resulting potential difference is shown in figure 5.3.

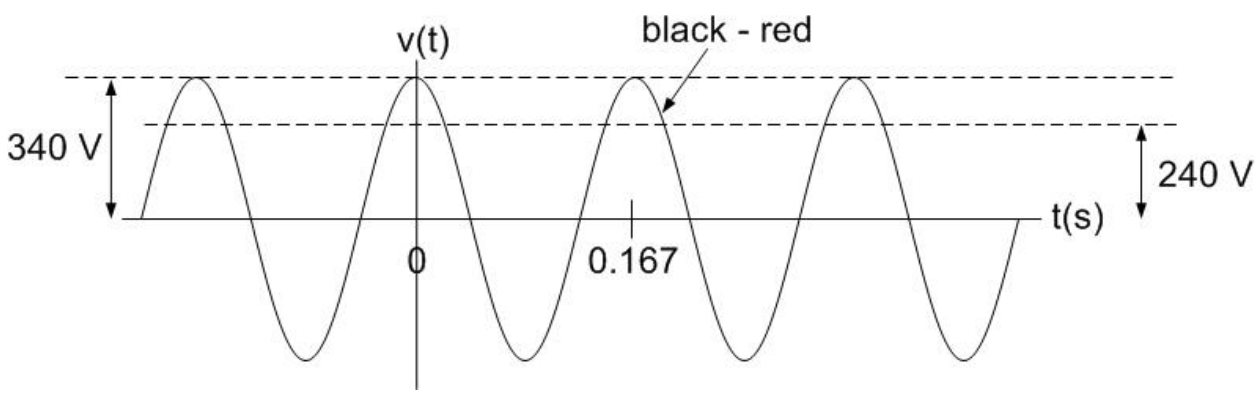


Figure 5.3 240 V RMS* obtained by taking the potential difference between the black and the red wires.

This sinusoidal waveform has an amplitude of 340 V. The question remains as to what the 240 V value means. Consider two resistors of the same value. One with the sinusoidal voltage and one with a DC value called v_{RMS} as shown in figure 5.4.

*RMS is an acronym for Root Mean Square

Figure 5.4 Two resistors of the same value. One resistor with sinusoidal voltage v(t) across it and one with a DC voltage v_{RMS} across it.

The DC voltage v_{RMS} is the equivalent DC voltage that will dissipate the same average power in a resistor as the AC waveform. The average power dissipated for the sinusoidal voltage v(t) is calculated in (5.15a) and (5.15b) and the average power dissipated for the DC voltage v_{RMS} is calculated in (5.16) and finally these values are equated in (5.17) with the final result presented in (5.18).

$$P_{ave} = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{v^2(t)}{R}\, dt = \frac{1}{TR}\int_{t_0}^{t_0+T} V_{max}^2 cos^2\,(\omega t)\, dt \tag{5.15a}$$

$$= \frac{1}{TR} V_{max}^2 \int_{t_0}^{t_0+T} cos^2\,(\omega t)\, dt = \frac{1}{TR} V_{max}^2 \int_{t_0}^{t_0+T} \frac{1}{2} + \frac{1}{2}\cos(2\omega t)\ dt$$

where the trigonometric identity $cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$ has been invoked.

Utilizing the fact that the integral of a sum is the sum of integrals

$$P_{ave} = \frac{1}{TR} V_{max}^2 \int_{t_0}^{t_0+T} \frac{1}{2}\, dt + \frac{1}{TR} V_{max}^2 \int_{t_0}^{t_0+T} \frac{1}{2}\cos(2\omega t)\ dt \tag{5.15b}$$
$$= \frac{1}{TR} V_{max}^2 \int_{t_0}^{t_0+T} \frac{1}{2}\, dt = \frac{v_{max}^2}{2R}$$

The average power dissipated in the same resistance for the DC value v_{RMS} is

$$P_{ave} = \frac{v_{RMS}^2}{R} \tag{5.16}$$

Equating (5.16) with (5.17) and solving for v_{RMS}

$$P_{ave} = \frac{v_{RMS}^2}{R} = \frac{v_{max}^2}{2R} \tag{5.17}$$

$$v_{RMS} = \frac{v_{max}}{\sqrt{2}} \tag{5.18}$$

Thus, the equivalent DC value that will dissipate the same average power in a resistor as the sinusoidal voltage is called the RMS (Root Mean Square) value because of the mathematical operations that were necessary to calculate it. Applying this result to the waveform in figure 5.3 in which the amplitude is 340 V yields 240 V (RMS). The same calculation holds for the waveforms in figure 5.2. Each of the sinusoids has an amplitude of 170 V yielding an RMS value of 120 V. RMS values are a convenient way to express AC (time-varying sinusoidal) voltages with a single value. Its physical interpretation is very important. The RMS value is the equivalent DC value that will dissipate the same average power in a resistor as the sinusoidal voltage.

5.3 AC Plug and Socket Nomenclature

Figure 5.5 presents an illustrated extension cord plug and socket along with the associated blades and slots, and ground pin and hole. All students studying electrical and computer engineering should be familiar with each of these terms. This set of plug and socket is polarized. The neutral slot and blade is wider than the hot slot and blade. This asymmetry makes it impossible to insert a neutral blade into a hot slot. It is left to the student to explore the reason for making the blade/slot pairs of different width.

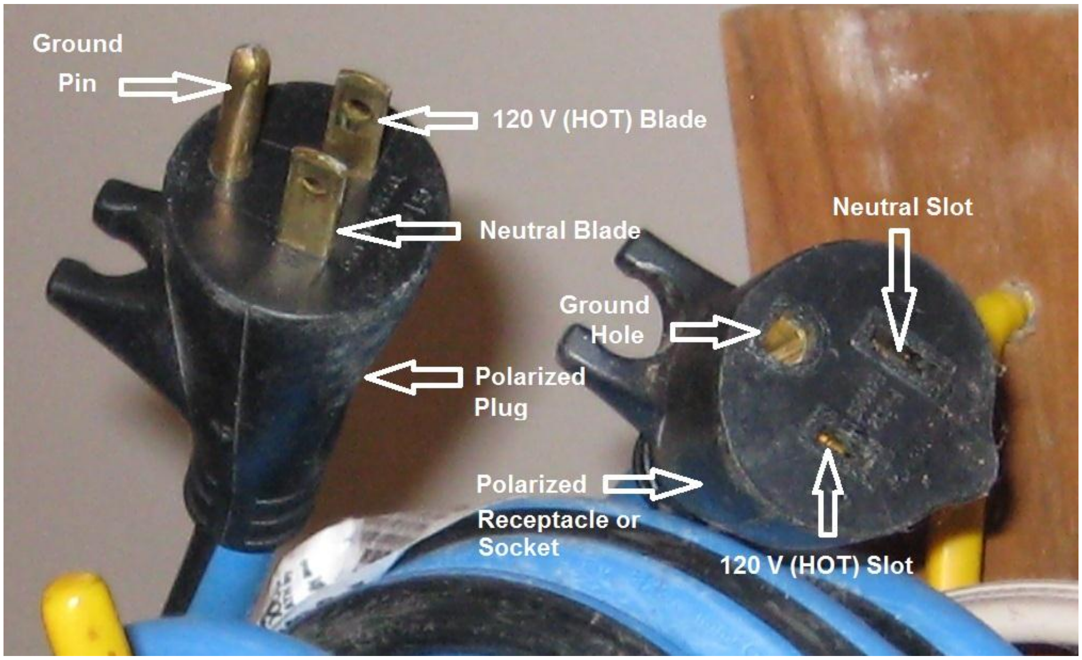


Figure 5.5 Labeled three terminal 120 V extension cord plug and receptacle (or socket) at the other end of the cord. It is important to know which blade/slot pair is “hot” and which blade/slot pair is neutral, because this information enables safe usage.

5.4 Maximum Power Condition for Resistive Circuits

As will be shown in Chapter 6, it is possible to reduce any resistive circuit to a voltage source in series with a resistance as seen at a pair of terminals. This simplified equivalent circuit is called the Thevenin equivalent circuit. If the Thevenin equivalent circuit elements are represented by v_{Th} and R_{Th}, the Thevenin equivalent circuit schematic is shown in figure 5.6.

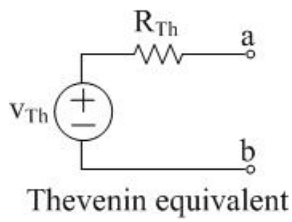


Figure 5.6 Thevenin equivalent circuit as seen at terminals ab.

Given the Thevenin equivalent circuit of a circuit, what load resistance R_L will result in the maximum power absorbed by the load? With the load resistance present, the circuit is shown in figure 5.7.

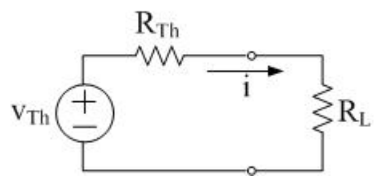


Figure 5.7 Thevenin equivalent circuit with a load resistor R_L attached.

The objective is to find the value of R_L that will result in the maximum power absorbed by the load. The approach to this problem will be to find the expression for the current *i*, then the power absorbed by the load resistance R_L, and finally the expression for the power absorbed by the load. Using calculus, we will find the derivative of the power absorbed by the load with respect to the load resistance R_L and set it equal to zero and solve for R_L.

$$i = \frac{v_{Th}}{R_{Th} + R_L} \tag{5.19}$$

$$p_L = i^2 R_L = \frac{v_{Th}^2}{(R_{Th} + R_L)^2} R_L \tag{5.20}$$

$$\frac{dp_L}{R_{Th}} = \frac{(R_{Th} + R_L)^2 v_{Th}^2 - v_{Th}^2 R_L 2(R_{Th} + R_L)}{(R_{Th} + R_L)^4} = 0 \tag{5.21}$$

$$(R_{Th} + \ R_L)^2 = R_L 2(R_{Th} + \ R_L) \tag{5.22}$$

$$R_{Th}^2 + \ R_L^2 + 2R_{Th}R_L = 2R_{Th}R_L + 2R_L^2 \tag{5.23}$$

$$R_{Th}^2 + \ R_L^2 = \ 2R_L^2 \tag{5.25}$$

$$\boldsymbol{R_L = \ R_{Th}} \tag{5.25}$$

This simple result is extremely important. It tells us the load resistance that will absorb maximum power is the **same** as the Thevenin resistance.

Chapter 6

Circuit Analysis Techniques

In this chapter the reader is introduced to two independent methods for solving circuits in which a simple application of Ohm's law or Kirchhoff's laws is not applicable. These two circuit analysis methods are the **node voltage method (NVM)** and the **branch current method (BCM)**. Both methods are completely general and may be used on circuits of any complexity and topology. Being independent methods, students may use both methods to solve the same circuit to ensure that their solutions are correct. NVM and BCM are utilized when a simple application of KVL, KCL or Ohm's law will not produce the desired unknown.

6.1 The Node Voltage Method (NVM)

The node voltage method (NVM) relies on understanding what a node is and what a node voltage is. **A node is connected conductors**, and never includes an element. A node voltage is always a voltage drop from a node to ground. Ground is a designated node that serves as the reference node for all node voltages in a circuit. Imagine a voltmeter with its red and black probes. To measure a node voltage, the black probe is always kept on the ground node and the red probe is connected to the node whose node voltage is to be measured. Consider the circuit in figure 6.1

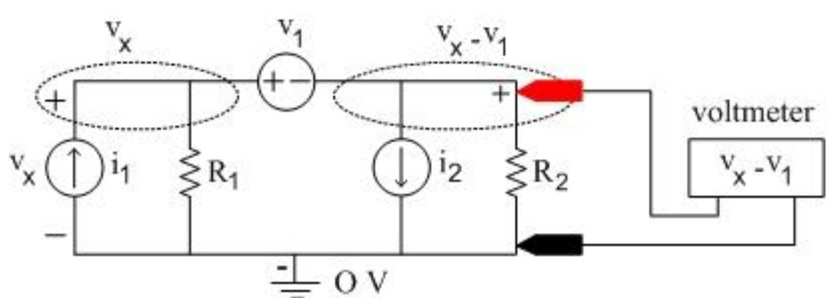


Figure 6.1 Illustration of ground, node voltages and a voltmeter measuring the node voltage at the top right node.

In this circuit, the ground node is labeled. Remember, a node is connected conductors. Therefore, the ground node extends from below R_2 all the way across the bottom of the circuit to the bottom of i_1. The ground node is the reference node. As an analogy, the node voltage at ground can be thought of as sea level. What is the value of the node voltage at ground? If you take a voltmeter and put the red probe and the black probe both at the ground node, what do you think the meter will read? It will read 0 V because the voltage at any node relative to itself is 0 V. As long as the black probe is kept connected to the ground node, wherever the red probe is connected, the meter measures the node voltage at the node where the red probe is connected. The circuit in figure 6.1 has a total of three nodes. Can you identify them? The ground node extends all the way across the bottom of the circuit. There is a top left node and a top right node, and they are outlined by the dotted contour lines. A node voltage is a voltage and therefore has a set of polarities. The black probe is kept on the ground node, and hence the **ground node has a negative polarity for its node voltage and all other node voltages in the circuit have a positive polarity**. This does not mean the voltage at the other nodes are all positive, only their polarity. The value of the node voltages depend on the circuit elements and the circuit topology. Beginning from ground, going through a voltage rise is analogous to walking up some stairs, and going through a voltage drop is like going down some stairs. In this analogy, the altitude, relative to sea level, at a particular node is equivalent to its node voltage. In the circuit in figure 6.1, to get to the top left node voltage from ground, you have to go through a voltage rise of v_x volts. To get to the top right node, if we go across i_2 and R_2, the voltage is unknown, and we would have to introduce another unknown, but if we go across the voltage rise of v_x across R_1 and i_1 and then the voltage drop of v_1, across the voltage source, the node voltage at the top right node is the sum of the voltage rise of v_x volts and the voltage drop of v_1 volts to obtain a value of $v_x – v_1$. The rest of the problem involves a KCL at the top left node, the algebraic manipulation necessary to solve for v_x and the arithmetic to calculate the numerical value of v_x. This procedure will all be presented in example 6.2 that follows.

The NVM procedure is independent of the circuit to which it is applied. The approach is general and its steps are as presented in figure 6.2,

1) Label a **node*** with a ground symbol The ground symbol designates the reference node.
2) Label all other node voltages without introducing unnecessary unknowns.
3) Apply KCL at the node where the unknown node voltage appears. KCL will have to be applied the same number of times as there are unknown voltages.

Figure 6.2 The node voltage method (NVM) procedure.

***node**: a node is connected conductors (in our circuit diagrams, called schematics, the wires are assumed to have zero resistance. In practical circuits, the wires have such a small resistance compared to the resistors that the wire resistance can be neglected.)

There are two ways to specify a current in a **branch****; the value of an independent current source and the use of Ohm's law with a resistor. The key to expressing current flow in a given direction in terms of node voltages is to remember the directedness of Ohm's law: The current through a resistance is the voltage drop in the **same direction** divided by the resistance. The diagram in figure 6.3 shows expressions for the current in both directions through resistance R_1 in terms of v_a and v_b, the node voltages on either side of the resistance.

****branch**: a branch is a single wire or string of elements all of which share a common current.

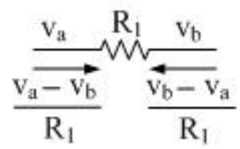
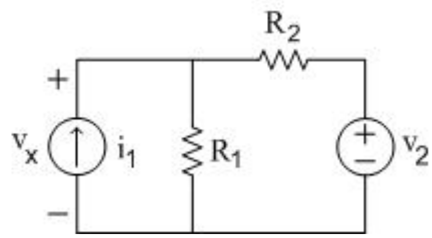


Figure 6.3 Illustration of expressing the current in either direction through a resistor using node voltages and Ohm's law.

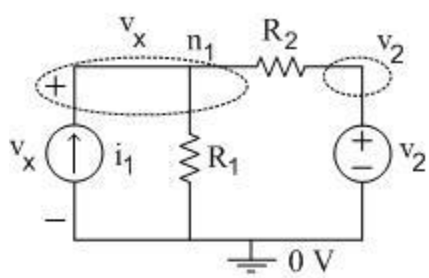
Example 6.1 Consider the following circuit.



Given the element values i_1, v_2, R_1 and R_2, the problem is to find the voltage v_x.

Solution: There is no law that provides the voltage across a current source by knowing the current source value alone. We observe however that the current source is in parallel* with R_1 and therefore v_x is the voltage across R_1. If we knew the current through R_1, a simple application of Ohm's law would provide the solution for v_x. However, it is not evident what the current through R_1 is. If we knew the voltage across R_2, we could use KVL around the outside loop and write v_x in terms of the known voltage across R_2 and the voltage source value v_2. However, we do not know the voltage across R_2. So what do we do?

We will apply a circuit analysis technique known as the node voltage method (NVM). The first step is to label a node with a ground symbol indicating the reference node. This circuit has three nodes and any one of these nodes can be selected as the ground node. There is a convention of selecting the bottom node as ground, and we will follow convention here, but the reader should be aware that there is nothing special about the bottom node for its selection as the ground node. The following schematic shows the ground node labeled and the other nodes labeled with their node voltages. It should be noted that all node voltages are voltages, and hence they have a set of polarities. When measuring a node voltage, the black terminal of the voltmeter is kept at the ground node and the red probe is used to measure the node voltage at each of the other nodes, all with respect to ground. All node voltages are always measured with respect to ground, the reference node.



*__parallel__: Elements are in parallel when they share the same pair of nodes.

Applying KCL at node n_1 we obtain (6.1)

$$-i_1 + \frac{v_x - 0}{R_1} + \frac{v_x - v_2}{R_2} = 0 \tag{6.1}$$

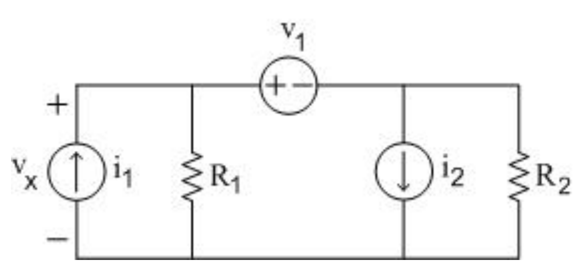
From which the single unknown v_x may be solved for in terms of the given symbols

$$v_x\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = i_1 + \frac{v_2}{R_2} \tag{6.2}$$

$$\boxed{v_x = \frac{i_1 + \frac{v_2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}} \tag{6.3}$$

No specific element values were given in this example because the emphasis is on the symbolic solution - all problems are to be solved in terms of the given symbols completely before numbers are inserted. There are two main reasons for solving entirely in terms of symbols before inserting numbers. First, once numbers are substituted for the symbols, the information regarding what impact each element has on the unknown is completely lost, and second, when the solution is obtained in terms of symbols, a quick dimensional check will determine the suitability of substituting numbers. For example, if in the numerator a current is added to a resistance or anything other than a current, an error is immediately known to have occurred, and the process of troubleshooting the solution is required. Troubleshooting is a matter of tracking backwards in the symbolic solution process until the error is found and corrected. As we delve into circuits with additional elements, the algebraic solutions will become more complex, and the value of solving in symbols becomes even more important. This topic of solving in symbols will be revisited on multiple occasions throughout the text.

Example 6.2



Given the element values v_1, i_1, i_2, R_1 and R_2, calculate the voltage v_x.

Solution: As in Example 6.1, we seek the voltage across a current source and observe that R_1 is in parallel with the current source. However, we do not know the current through R_1 and therefore cannot simply apply Ohm's law to find v_x. The voltage across the parallel combination of the current source i_2 and resistance R_2 is not known. Therefore we cannot apply a simple KVL around the outside loop to obtain v_x. We will therefore apply the NVM. Labeling the ground node and the other node voltages result in the following schematic.

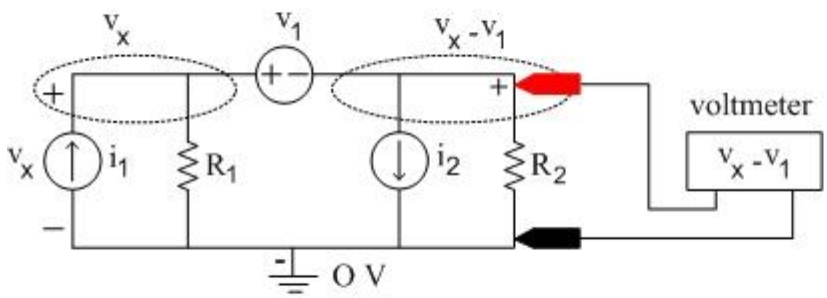
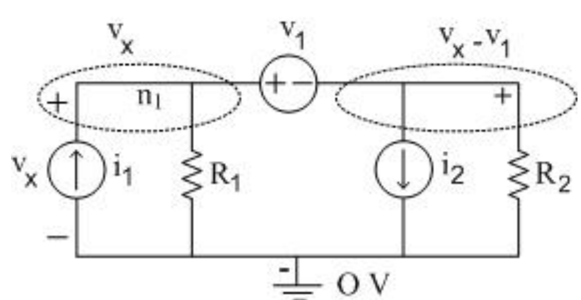


Figure 6.4 Voltmeter placement for measuring the node voltage $v_x - v_1$.

There are several things to observe in this schematic. First, node voltages, being voltages, have a set of polarities. Since node voltages are always measured with respect to ground, the black probe of a voltmeter measuring a node voltage is **always connected to ground** making the polarity of the node voltage negative at ground and positive at the node whose node voltage is being measured. If the voltage at ground is measured, the red probe is also connected to ground and a value of 0 V will result. Second, node voltages are always obtained by starting from ground and progressing through voltage rises and drops until each node voltage has been identified. For example, to progress from ground (0 V) to the top-left node, we go through a voltage rise of v_x volts making the node voltage $+V_x$. In order to progress from ground to the top-right node, we could try to go directly from ground to the top-right node across i_2 and R_2. However, the voltage across these elements is not labeled. We could introduce a new unknown voltage, say v_y, but the node voltage procedure requires us to avoid introducing unnecessary unknown voltages. We therefore try an alternative path. If we first go from ground to the top-left node (and the associated voltage rise of v_x volts) and then move across the voltage drop of v_1 volts, the net voltage change from ground is $v_x - v_1$. Thus, this circuit has only one unknown voltage. Once v_x is known, all voltages will be known. To introduce unnecessarily a new unknown voltage (say v_y) at the top-right node falsely implies that there are two unknown voltages (which there are not) and increases the complexity of the solution unnecessarily. We must now apply KCL to one of the nodes, and it turns out that it does not matter which node. If a KCL is applied to the top left node, it should be labeled (such as node n_1). The completely labeled circuit is shown below.



Applying KCL at node n_1 involves three currents. However, the current leaving node n_1 to the right through the voltage source is not determinable by knowing the voltage of the voltage source alone. There are only two ways to define a current: the value of an independent current source and the use of Ohm's law with a resistor. Therefore, when seeking the current through a voltage source, recognize that the current that enters one end of v_1 is the same as the current that leaves the other end of v_1 and continue. The current to the right through v_1 splits. Part of it goes down through the current source i_2 and the other part goes down through the resistor R_2. If we add these two currents together, we will have the expression for the current leaving node n_1 flowing to the right through v_1. The KCL equation can now be written at node n_1:

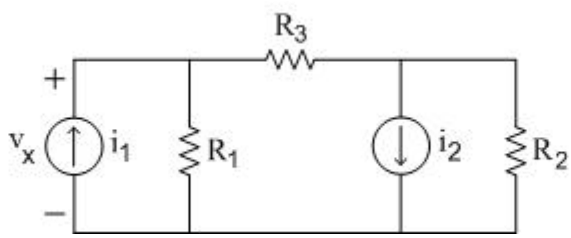
$$-i_1 + \frac{v_x}{R_1} + i_2 + \frac{v_x - v_1}{R_2} = 0 \tag{6.4}$$

With v_x as the single unknown, its solution is easily obtained as

$$v_x = \frac{i_1 - i_2 + \frac{v_1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \tag{6.5}$$

The units of the symbolic solution for v_x are all consistent and result in a voltage. Therefore, if numerical values were given for the elements it would be appropriate to make the substitution of values for the symbols at this point.

Example 6.3



Given the element values i_1, i_2, R_1, R_2, and R_3, calculate the voltage v_x.

Solution: As in Examples 1 and 2, we seek the voltage across a current source and observe that R_1 is in parallel with the current source. However, we do not know the current through R_1 and therefore cannot simply apply Ohm's law to find v_x. Using the node voltage method (NVM), we need to designate ground and label as many of the other node voltages as possible without introducing a new unknown voltage. However, examination of the following labeled schematic shows that we are unable to label all of the node voltages without introducing a new unknown node voltage (in this case v_y is used as the second unknown node voltage). We were forced to introduce a second unknown voltage. Therefore, this circuit has two unknown voltages (v_x and v_y).

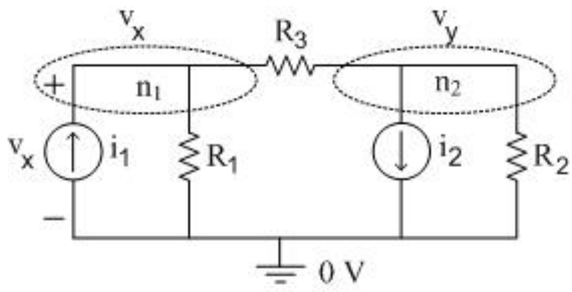


Figure 6.5 Circuit of example 6.3 with node voltages labeled.

Since there are two unknown voltages, we will have to apply KCL twice.

KCL n_1:

$$-i_1 + \frac{v_x}{R_1} + \frac{v_x - v_y}{R_3} = 0 \tag{6.6}$$

KCL n_2:

$$\frac{v_y - v_x}{R_3} + i_2 + \frac{v_y - 0}{R_2} = 0 \tag{6.7}$$

At this point we have two equations in two unknowns (v_x and v_y). Many students will be tempted to solve for v_y in the first equation and substitute it into the second equation and then solve for v_x. The method of substitution is a very inefficient way to solve for v_x. The most direct approach to solve two equations in two unknowns is to use Cramer's Rule.

The first thing to do is to put the two equations into standard form.

$$[1] \quad v_x\left(\frac{1}{R_1} + \frac{1}{R_3}\right) + v_y\left(\frac{-1}{R_3}\right) = i_1 \tag{6.8}$$

$$[2] \; v_x\left(\frac{-1}{R_3}\right) + v_y\left(\frac{1}{R_2} + \frac{1}{R_3}\right) = -i_2 \tag{6.9}$$

v_x can now be expressed as the ratio of two determinants

$$v_x = \frac{\begin{vmatrix} i_1 & \frac{-1}{R_3} \\ -i_2 & \frac{1}{R_2} + \frac{1}{R_3} \end{vmatrix}}{\begin{vmatrix} \frac{1}{R_1} + \frac{1}{R_3} & \frac{-1}{R_3} \\ \frac{-1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{vmatrix}} = \boxed{\frac{(i_1)\left(\frac{1}{R_2} + \frac{1}{R_3}\right) - (-i_2)(\frac{-1}{R_3})}{\left(\frac{1}{R_1} + \frac{1}{R_3}\right)\left(\frac{1}{R_2} + \frac{1}{R_3}\right) - (\frac{-1}{R_3})(\frac{-1}{R_3})}} \tag{6.10}$$

As you can see Cramer's Rule is very efficient in the solution of two equations in two unknowns. Cramer's Rule is also the most efficient method of solving for three unknowns in three equations. If more than three unknowns are present, then other matrix methods become most efficient, and generally a computer is used to solve for the unknowns. In Cramer's Rule the bottom matrix is the matrix of coefficients of the unknowns in the same order as they appear in the equations. The numerator matrix is a modified version of the coefficient matrix. The modification is the replacement of the column corresponding to v_x with the contents to the right of the equal signs in the two equations. Evaluating the two determinants yields the solution for v_x and is shown in (6.10)

6.2 The Branch Current Method (BCM)

The branch current method (BCM) procedure is independent of the circuit to which it is applied. Being independent of the node voltage method (NVM), both the NVM and the BCM may be used to confirm the result from the use of the other method. The approach utilized in the BCM is general and can be applied to any circuit regardless of its complexity or topology. Its steps are as follows:

1) Label the Currents through each resistor without introducing unnecessary unknown currents.

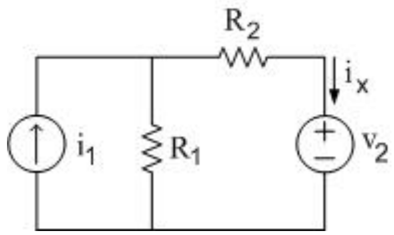
2) Apply KVL around loops* with no current source in them. You will need to apply KVL to as many loops as there are unknown currents. Label the loops to which KVL will be applied with a loop arrow with two heads (to differentiate it from a current arrow) and label the loop with a lower case "l" with a subscript. This allows reference to the loop.

Figure 6.5 The branch current method procedure.

The **branch current method (BCM) is not the same as mesh current analysis**, which utilizes fictitious loop currents. The BCM utilizes only physical currents.

A *loop is any closed path. KVL can certainly be applied to loops with current sources. However, in the context of the branch current method (BCM), applying KVL to loops with current sources does not help in the determination of an unknown current using the BCM. There are only two ways to define a current; the value of a current source and the use of Ohm's law with a resistor (this is the reason for labeling the currents through the resistors).

Example 6.4



Given the element values i_1, v_2, R_1 and R_2, the problem is to find the current i_x.

Solution: There is no law that provides the current through a voltage source by knowing the voltage source value alone. We observe however that the voltage source is in series* with R_2 and therefore i_x is the current through R_2. If we knew the voltage across R_2, a simple application of Ohm's law would provide the solution for i_x. However, it is not evident what the voltage across R_1 is. If we knew the current through R_1, we could use KCL at the top left node and write i_x in terms of the known current through R_1 and the current source value i_1. However, we do not know the current through R_1. So what do we do?

We will apply a circuit analysis technique known as the **branch current method (BCM)**. The first step is to label the currents through the resistors. There are only two resistors in the circuit, R_1 and R_2, and we already have a label for the current through R_2 (it is the unknown current i_x that we seek). But it remains to label the current through R_1. We can select the current arrow directed upward or downward. As long as the correct value for the current is labeled, the direction selected is completely arbitrary. Suppose we choose the direction of the current through R1 to be flowing up. The value of this current is $i_x - i_1$ obtained by a KCL at either the top left node or the bottom node. The right hand loop is the only loop of the three loops present in the circuit that contains no current sources. You may choose to add voltage rises or voltage drops and to go clockwise or counter clockwise, the choice is completely arbitrary. The final result will be exactly the same in any combination of voltage rise/drop summation and clockwise/counter clockwise traversing of the loop as long as the application of KVL is performed correctly. In this example, we will begin the KVL at the lower left corner of the right-hand loop and add voltage drops going clockwise. The circuit labeled properly resulting from these selections is shown below.

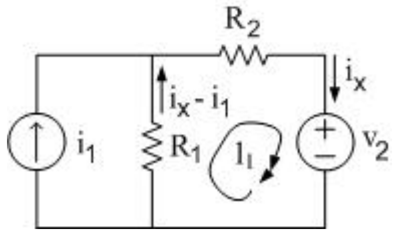


Figure 6.6 Circuit from example 6.4 labeled appropriately for applying KVL as part of the branch current method.

The loop labeled l_1 contains three elements, so we need to be certain that all three voltage drops are included.

$$\text{KVL } l_1\text{:} \quad R_1(i_x - i_1) + R_2(i_x) + v_2 = 0 \tag{6.11}$$

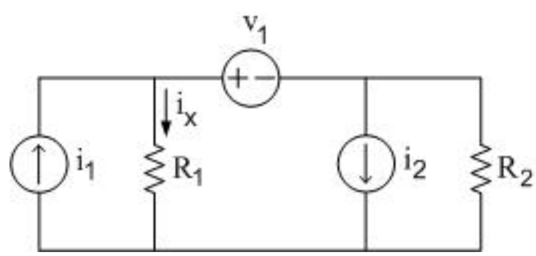
***series** elements share a common current

Solution of this equation for the unknown current i_x is

$$\boxed{i_x = \frac{R_1 i_1 - v_2}{R_1 + R_2}} \tag{6.12}$$

This is the complete solution for i_x in terms of the symbols given. It is important to check the units of your final symbolic solution. If the dimensions of one term are inconsistent with the dimensions of other terms in the numerator or denominator, an error is present and the process of troubleshooting the error is required. Solving the problem in terms of the symbols given allows the most efficient troubleshooting process. Simply look back at each step until you find where the incompatibility of units occurs. The expression obtained for i_x has units of volts consistently in the numerator and ohms in the denominator. The ratio produces Amps, the proper unit for current. If we wanted to be sure our answer is correct, we could label voltage drop to the right across R_2 as v_x and use the node voltage method to obtain vx and then take the ratio of v_x to R_2 to obtain i_x. If the NVM produces the same result as the BCM, the chances are extremely good that your answer is correct.

Example 6.5



Given the element values v_1, i_1, i_2, R_1 and R_2, calculate the current i_x.
Solution: i_x is the current flowing down through R_1. The voltage across R_1 is unknown, and therefore a simple application of Ohm's law cannot be used to find i_x. The current through the voltage source i_1 is unknown and therefore a simple KVL around the outside loop cannot be used to obtain v_x. The BCM will be used to find i_x. Labeling the currents through all the resistors reduces to labeling the current through R_2. Note that there is only a single loop that contains no current sources. This means that there is only one unknown current. The current through R_2 is expressible in terms of already labeled currents. Once again the choice must be made for the direction of the current through R_2. However, prior knowledge that the KVL will be applied to the right-center loop, adding voltage drops going clockwise, the current is direction is selected downward. There is nothing "wrong" with labeling the current such that it is directed upward. The critical factor is in the correct labeling of the value of the current once the direction has been selected. The labeled circuit is shown below. The current through R_2 is obtained by KCL.

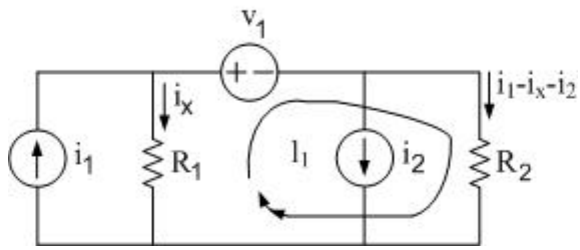


Figure 6.7 Circuit from example 6.5 labeled and ready for applying KVL as part of the BCM.
Applying KVL to l_1 produces the following equation.

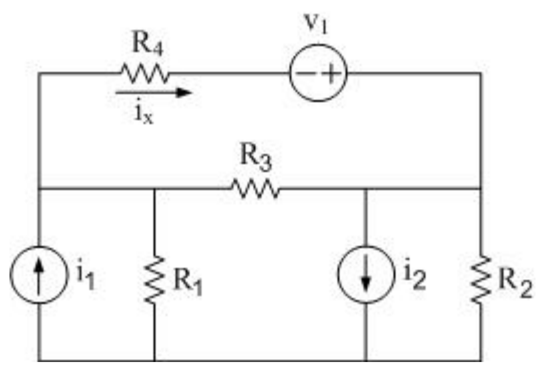
$$KVL\ l_1:\ R_1(-i_x) + v_1 + R_2(i_1 - i_x - i_2) = 0 \tag{6.13}$$

Solving this single equation in the single unknown i_x produces:

$$i_x = \frac{v_1 + R_2(i_1 - i_2)}{R_1 + R_2} \tag{6.14}$$

Checking units shows consistent voltages in the numerator and resistances in the denominator, producing i_x in amps.

Example 6.6



Given the element values v_1, i_1, i_2, R_1, R_2, R_3 and R_4, calculate the current i_x.
Solution: i_x is the current flowing to the right through R_4 and the voltage source v_1. There is no simple application of KCK to find i_x. The BCM will be used to find i_x. Labeling the currents through all the resistors requires the introduction of another unknown current, say i_y flowing up through R_1. This selection is completely arbitrary. The current through any of the other resistors can be labeled i_y. Once i_y has been labeled, the current through every other resistor may be expressed in terms of already labeled currents. There is no need to introduce any additional unknowns. It is important to keep the analysis as simple as possible. It makes the solution most efficient (less work) and it provides insight into the true number of unknown currents. Once i_x and i_y are determined, every current in the circuit becomes known. Our goal however, is only to find i_x. Once all the currents through the resistors are labeled, KVL must be applied the same number of times as there are unknown currents. In this circuit KVL must be applied twice. There are three loops containing no current sources and it does not matter which two of these three loops are chosen to apply KVL. The top loop and bottom-right-center loop are arbitrarily selected. The circuit with the currents and loops labeled is shown below.

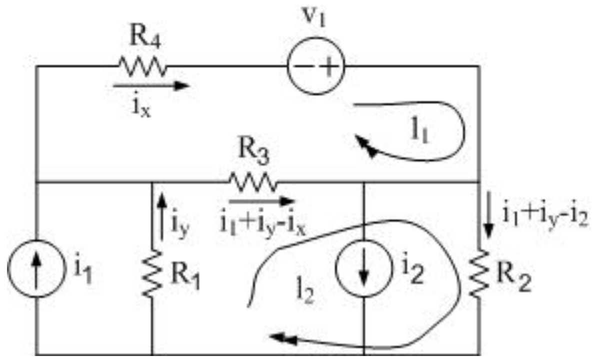


Figure 6.8 Circuit of example 6.8 labeled and ready for applying KVL as part of the BCM.

Applying KVL around loops l_1 and l_2 produce the following two equations.

$$KVL\ l_1:\ \ R_4(i_x) - v_1 + R_3(-i_1 - i_y + i_x) = 0 \tag{6.15}$$

$$KVL\ l_2:\ \ R_1(i_y) + R_3(i_1 + i_y - i_x) + R_2(i_1 + i_y - i_2) = 0 \tag{6.16}$$

Using Cramer's Rule to find i_x first requires the two equations to be factored as follows.

$$1)\ \ i_x(R_4 + R_3) + i_y(-R_3) = v_1 + R_3(i_1)$$

$$2)\ \ i_x(-R_3) + i_y(R_1 + R_3 + R_2) = -(R_3 + R_2)(i_1) + R_2(i_2)$$

Now ix can be expressed as the ratio of two determinants (Cramer's Rule)

$$i_x = \frac{\begin{vmatrix} v_1 + R_3(i_1) & -R_3 \\ -(R_3 + R_2)(i_1) + R_2(i_2) & R_1 + R_3 + R_2 \end{vmatrix}}{\begin{vmatrix} R_4 + R_3 & -R_3 \\ -R_3 & R_1 + R_3 + R_2 \end{vmatrix}}$$

$$i_x = \boxed{\frac{(v_1 + R_3(i_1))(R_1 + R_3 + R_2) - (-(R_3 + R_2)(i_1) + R_2(i_2)(-R_3)}{(R_4 + R_3)(R_1 + R_3 + R_2) - (-R_3)(-R_3)}} \tag{6.17}$$

As a review of Cramer's Rule, i_x is the ratio of two determinants. The denominator matrix elements are the coefficients of the unknown currents, i_x and i_y in equations 1) and 2) above. The numerator matrix elements are the same as the denominator matrix elements except the column associated with i_x is replaced by the contents to the right of the equal signs. Finally, i_x is expanded by taking the ration of the two determinants. It is important to check the units of the numerator and denominator to make sure they are each consistent and lead to the correct units of i_x (amps). In this particular case, the units of the numerator are consistently VΩ and the units of the denominator are consistently Ω^2, whose ratio is Amps as expected. If numerical values for the elements had been given, they would be substituted in exactly the same location as the elements appear. There is no need to algebraically manipulate the equation.

Chapter 7

Equivalent Circuits

What does the phrase "equivalent circuit" mean? In what way are two circuits considered to be equivalent? Certainly identical circuits that have identical values for the elements and the same connections of those elements are equivalent. The question of equivalence in electric circuits has an implication of viewing the circuit at a pair of terminals.

Circuits are equivalent when they have the same current-voltage relationship at a pair of terminals. In this case the two circuits are equivalent at that pair of terminals. Another way to think about equivalence between two circuits is that if the same valued resistor is placed between the terminals of each equivalent circuit, the same voltage will appear across the resistor and consequently the same current will flow through that resistance. Furthermore, this is true for **any** value of resistance, as long as it is the same value being placed across the terminals of both circuits. The whole concept of equivalent circuit is based on the assumption of a **linear circuit**. For information regarding what it means for a circuit to be linear, see Appendix 1.

The **simplest** equivalent circuit is one that contains the least number of elements and has the same current-voltage relationship at a given pair of terminals.

7.1 Resistors in Series and Parallel

Given any purely resistive circuit, there is an equivalent resistance as seen at a pair of terminals. Simple combinations of resistances are termed **series** and **parallel**.

Resistors in series share a common current

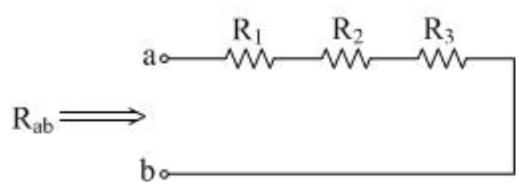


Figure 7.1 Resistors connected in series.

There is an **equivalent resistance** between terminals ab that is the **input resistance** to the terminals R_{ab}. These resistors are connected in **series** because they share a **common current**. An ohmmeter measures resistance and does so by applying a known voltage to a pair of terminals and measuring the resulting current. The ratio of the applied voltage to the resulting current is the equivalent resistance as seen from the pair of terminals where these measurements are made. If a voltage source v is applied to terminals ab and the resulting current i is obtained, the equivalent resistance R_{ab} will be the ratio $\frac{v}{i}$. The direction of the current is very important – it must be in the direction of the voltage drop for the equivalent resistance.

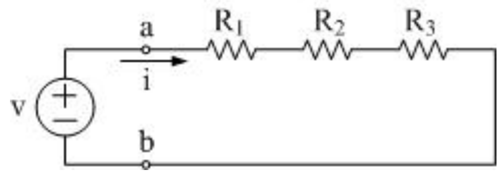


Figure 7.2 Resistors in series driven by an external voltage source.

Applying KVL to the loop yields

$$-v + R_1 i + R_2 i + R_3 i = 0 \tag{7.1}$$

Then solving for the ratio $\frac{v}{i}$ gives

$$R_{ab} = \frac{v}{i} = R_1 + R_2 + R_3 \tag{7.2}$$

This shows that **resistances in series add**. Some students find it difficult to see that the current *i* is in the direction of the voltage drop *v* for the equivalent resistance. If all three series resistances are combined into a single resistance R_{ab} it is easier to see that the current *i* is in fact in the direction of the voltage drop *v* across R_{ab}.

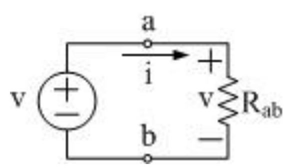


Figure 7.3 Illustrating the concept of equivalent resistance.

Some students look at Figure 7.3 and say, "But isn't the current in the direction of the voltage rise?" The answer to which is "yes" if you are talking about the voltage source. But Ohm's law says nothing about sources. Ohm's law is a relationship between current and voltage for a resistance: The voltage drop across resistance R_{ab} is R_{ab} times the current *i*, which is in the direction of the voltage drop *v*.

Resistors in parallel share a common voltage

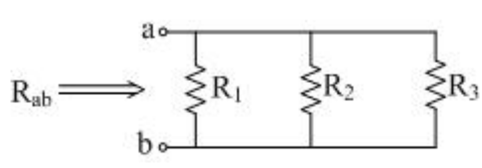


Figure 7.4 Resistors in parallel.

Elements are in parallel when they share the same voltage, or equivalently, they share the same pair of nodes. An ohmmeter measures resistance by applying a known voltage and measuring the resulting current. The ratio of the applied voltage to the resulting current is the equivalent resistance. Applying this concept to the parallel resistive circuit in Figure 7.4 produces the diagram in Figure 7.5.

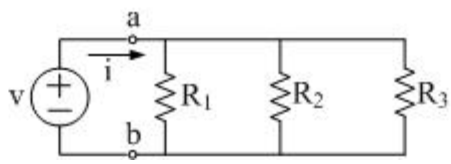


Figure 7.5 Resistors in parallel being driven by an external voltage source

Applying KCL at the top node produces

$$-i + \frac{v}{R_1} + \frac{v}{R_2} + \frac{v}{R_3} = 0 \tag{7.3}$$

Solving for the ratio of *v* to *i* produces R_{ab}.

$$R_{ab} = \frac{v}{i} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \tag{7.4}$$

The equivalent resistance of resistances in parallel is the reciprocal of the sum of reciprocals of each resistance in parallel.

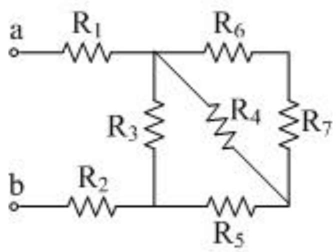
Definition: Conductance is the inverse of resistance and has the symbol G. The units of conductance are mhos (plural of Ohm spelled backwards) ℧ or siemans S.

Using the definition of conductance, we can write the equivalent conductance at terminals ab of the parallel circuit in Figure 7.4 as

$$G_{ab} = G_1 + G_2 + G_3 \tag{7.5}$$

Conductances in parallel add.

Example 7.1



$R_1 = 6\ \Omega$
$R_2 = 5\ \Omega$
$R_3 = 10\ \Omega$
$R_4 = 3\ \Omega$
$R_5 = 8\ \Omega$
$R_6 = 4\ \Omega$
$R_7 = 2\ \Omega$

Given all of the resistance values, calculate the equivalent resistance at terminals ab.

Solution: Generally, you begin reducing resistors at the furthest point away from the input terminals and work your way back until you have reduced the entire network of resistances to a single resistance. For example, in this circuit R_6 and R_7 are in series. Their equivalent resistance is $R_{67} = R_6 + R_7$. Then R_{67} is in parallel with R_4, and so forth until you reach the input terminals ab. At the highest level, looking into the input terminals ab there are three resistances in series: R_1, R_2 and the parallel combination of R_3 with the equivalent resistance to its right. The complete expression for the equivalent resistance at terminals ab is given by

$$R_{ab} = R_1 + R_2 + \frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{1}{R_4} + \dfrac{1}{R_6 + R_7}} + R_5} + \dfrac{1}{R_3}} \tag{7.6}$$

$$R_{ab} = 6 + 5 + \frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{1}{3} + \dfrac{1}{4+2}} + 8} + \dfrac{1}{10}} = 16\ \Omega \tag{7.7}$$

7.2 Resistive Circuits with Independent Sources

Consider a circuit that is comprised of resistors and independent sources. What is the simplest equivalent circuit? It turns out that there are two simplest equivalent circuits. They are known as the Thevenin and Norton equivalent circuits. These two equivalent circuits have the schematics shown in figure 7.6.

Figure 7.6 Thevenin and Norton equivalent circuits.

The subscript on v_{oc} indicates it is the **open circuit voltage** drop from terminal a to terminal b. The subscript on i_{sc} indicates it is the **short circuit current** flowing from terminal a to terminal b under short circuit conditions. Just as v_{oc} is the voltage drop from terminal a to terminal b, the short circuit current is the current that flows under short circuit conditions from terminal a to terminal b. Both of these circuits can represent any resistive circuit with any number of independent sources as seen at a pair of terminals.

Solving for either the Thevenin or the Norton Equivalent Circuit

Under open circuit conditions, you must solve for the voltage drop in the original circuit from terminal a to terminal b. This is the open circuit voltage v_{oc}. Examination of the Thevenin equivalent indicates that if there is nothing connected to the terminals ab, current cannot flow due to the open circuit. Therefore the voltage drop across resistance R_{th} is zero. This is why v_{oc} is called the open circuit voltage. In order for the Norton equivalent to be equivalent to both the original circuit and to the Thevenin equivalent circuit, the open circuit voltage must be the same for all three circuits. Therefore,

$$v_{oc} = R_{Th} i_{sc} \tag{7.8}$$

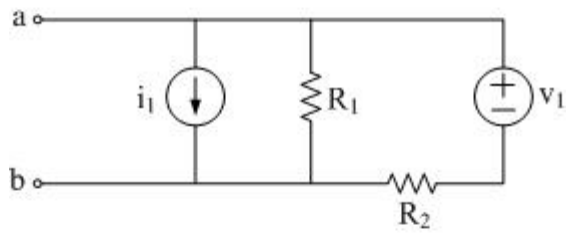
Under short circuit conditions, all three circuits must produce the same short circuit current from terminal a to terminal b. Redraw the circuit with a short circuit across its terminals. It is important to redraw the circuit because the circuit has changed when the short circuit is inserted, and we do not want to confuse the short circuit with the open circuit. Once the circuit is redrawn, draw a wire from terminal a to terminal b and show an arrow from a to b indicating the short current i_{sc} direction is from a to b. This directedness of both the open circuit voltage drop and short circuit current is very important. In order for the Thevenin equivalent circuit to have the same short circuit current as the Norton equivalent circuit (and therefore the original circuit)

$$i_{sc} = \frac{v_{oc}}{R_{Th}} \tag{7.9}$$

Where R_{Th} is called the Thevenin resistance and has the same value in both the Thevenin and Norton equivalent circuits.

In summary, to find either the Thevenin or the Norton equivalent circuit, you must find the open circuit voltage v_{oc} and the short circuit current i_{sc}. Finally, to obtain the Thevenin resistance R_{Th} that is the same in both circuits, you take the ratio of the open circuit voltage to the short circuit current.

Example 7.2



$v_1 = 10$ V
$i_1 = 1.5$ mA
$R_1 = 6$ kΩ
$R_2 = 4$ kΩ

Figure 7.7 Example 7.2 in which the objective is to obtain both the Thevenin and Norton equivalent circuits.

Given the element values, obtain both the Thevenin and Norton equivalent circuits.

Solution: We need to obtain v_{oc} and i_{sc} and then take their ratio to obtain the Thevenin resistance R_{Th}. The order in which you find v_{oc} or i_{sc} is arbitrary. We will solve for v_{oc} first. Take the time to redraw the circuit with the open circuit voltage clearly labeled as shown in Figure 7.8. The solution for v_{oc} is readily obtained by the NVM with KCL applied to the top node.

$$KCl\, n_1{:}\;\; i_1 + \frac{v_{oc}}{R_1} + \frac{v_{oc} - v_1}{R_2} = 0 \tag{7.10}$$

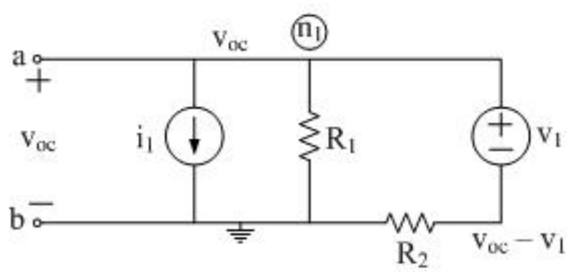


Figure 7.8 Example 7.2 circuit labeled in preparation for using the NVM to find v_{oc}.

Equation (7.10) has a single unknown, v_{oc}. Factoring v_{oc} and solving results in the expression

$$v_{oc} = \frac{\frac{v_1}{R_2} - i_1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{\frac{10}{4 \cdot 10^3} - 1.5 \cdot 10^{-3}}{\frac{1}{6 \cdot 10^3} + \frac{1}{4 \cdot 10^3}} = \frac{1 \cdot 10^{-3}}{0.4167^{-3}} = \boxed{2.40 \text{ V}} \tag{7.11}$$

Now, redrawing the circuit with the terminals ab shorted results in the schematic

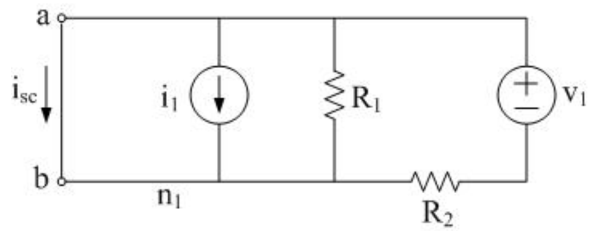


Figure 7.9 Circuit of example 7.2 with its terminals short circuited.

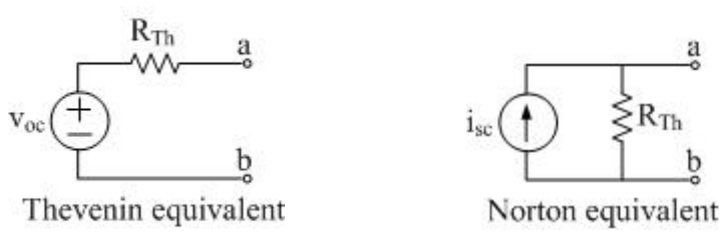
i_{sc} may be found by applying KCL at the bottom left node labeled n_1. The current i_{sc}, entering n_1, must equal the sum of all the other currents leaving n_1.

$$i_{sc} = -i_1 + \frac{v_1}{R_2} = -1.5 \cdot 10^{-3} + \frac{10}{4 \cdot 10^3} = 1.0 \cdot 10^{-3} = \boxed{1\text{ mA}} \tag{7.12}$$

It is important to note that the current through R_1 is zero due to the short circuit from a to b. The short circuit has significantly changed the circuit. That is the reason for redrawing the circuit. The Thevenin resistance R_{Th} is obtained by taking the ratio of v_{oc} to i_{sc}.

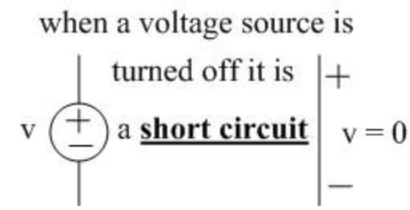
$$R_{Th} = \frac{v_{oc}}{i_{sc}} = \frac{\dfrac{\frac{v_1}{R_2} - i_1}{\frac{1}{R_1} + \frac{1}{R_2}}}{-i_1 + \frac{v_1}{R_2}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{\frac{1}{6 \cdot 10^3} + \frac{1}{4 \cdot 10^3}} = \boxed{2.4\text{ k}\Omega} \tag{7.13}$$

Now that v_{oc}, i_{sc} and R_{Th} are all known, both the Thevenin and the Norton equivalent circuits are known.

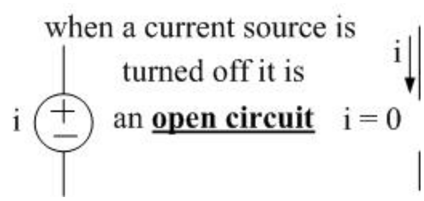


7.3 Turning Sources Off

It is very useful to be able to look into a pair of terminals with all of the sources turned off. This allows the Thevenin resistance to be found by determining the equivalent resistance looking into the pair of terminals when all the sources are turned off.

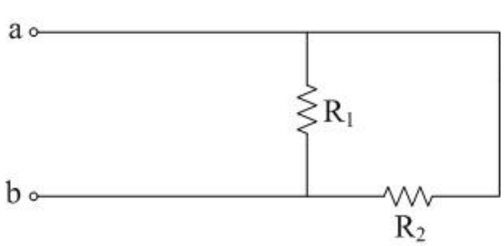


A voltage source turned off is a short circuit.



A current source turned off is an open circuit.

The ability to understand how a circuit behaves when its sources are turned off enables one to find the equivalent resistance or Thevenin resistance (same thing) as seen at a pair of terminals. For example, consider the circuit in example 7.2 shown in figure 7.7. Looking into terminals ab with the sources turned off produces the circuit shown in figure 7.9.

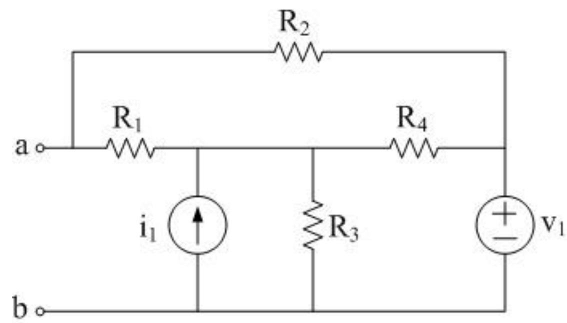


When all of the sources are turned off, that is when v_1 is replaced with a wire and i_1 is taken out of the circuit, the circuit in figure 7.9 results.

Figure 7.9 Circuit of example 7.2 from figure 7.7 with all of its sources turned off.

Find the equivalent resistance as seen from terminals ab. The equivalent resistance at terminals ab is R_1 in parallel with R_2, exactly the Thevenin resistance found in (7.13). This is no coincidence. The Thevenin or equivalent resistance of any circuit with independent sources and resistors may be found in this manner.

Example 7.3



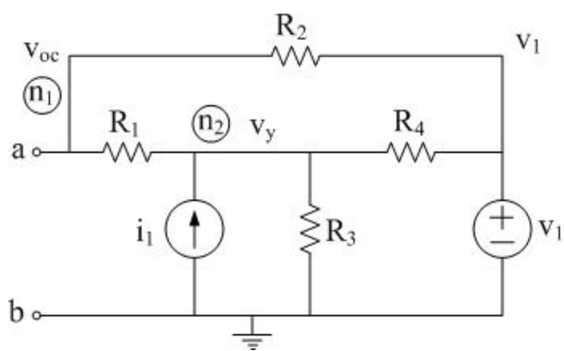
$v_1 = 4$ V
$i_1 = 0.5$ A
$R_1 = 10\ \Omega$
$R_2 = 5\ \Omega$
$R_3 = 8\ \Omega$
$R_4 = 12\ \Omega$

Figure 7.10 Example 2 of finding Thevenin and Norton equivalent circuits

Given the element values, determine both the Thevenin and Norton equivalent circuits.

Solution: In order to find both the Thevenin and Norton equivalent circuits. We must calculate both the open circuit voltage and the short circuit current.

<u>Open Circuit Voltage</u>: We will use the NVM with the bottom node as ground and the node above v_1 and R_3 as another unknown node voltage v_y along with v_{oc}. The properly labeled circuit is shown in figure 7.11.



$v_1 = 4$ V
$i_1 = 0.5$ A
$R_1 = 10\ \Omega$
$R_2 = 5\ \Omega$
$R_3 = 8\ \Omega$
$R_4 = 12\ \Omega$

Figure 7.11 Second example of finding the Thevenin and Norton equivalent circuits labeled for finding v_{oc}.

Since there are two unknown node voltages, we will need to apply KCL twice and factor the unknown node voltages to prepare for the use of Cramer's rule.

KCL n_1: $\frac{v_{oc}-v_1}{R_2} + \frac{v_{oc}-v_y}{R_1} = 0$ $\qquad v_{oc}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + v_y\left(\frac{-1}{R_1}\right) = \frac{v_1}{R_2}$

KCL n_2: $\frac{v_y-v_{oc}}{R_1} - i_1 + \frac{v_y}{R_3} + \frac{v_y-v_1}{R_4} = 0$ $\qquad v_{oc}\left(\frac{-1}{R_1}\right) + v_y\left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) = i_1 + \frac{v_1}{R_4}$

Using Cramer's rule, we can express v_{oc} as the ratio of two determinants.

$$v_{oc}=\frac{\begin{vmatrix}\frac{v_1}{R_2} & \frac{-1}{R_1}\\ i_1+\frac{v_1}{R_4} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\end{vmatrix}}{\begin{vmatrix}\frac{1}{R_1}+\frac{1}{R_2} & \frac{-1}{R_1}\\ \frac{-1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\end{vmatrix}}=\frac{\left(\frac{v_1}{R_2}\right)\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\right)-\left(\frac{-1}{R_1}\right)\left(i_1+\frac{v_1}{R_4}\right)}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\right)-\left(\frac{-1}{R_1}\right)\left(\frac{-1}{R_1}\right)} \tag{7.14}$$

Finally, after a careful unit check to ensure all terms are compatible, we substitute numerical values for the symbols.

$$v_{oc}=\frac{\left(\frac{4}{5}\right)\left(\frac{1}{10}+\frac{1}{8}+\frac{1}{12}\right)-\left(\frac{-1}{10}\right)\left(0.5+\frac{4}{12}\right)}{\left(\frac{1}{10}+\frac{1}{5}\right)\left(\frac{1}{10}+\frac{1}{8}+\frac{1}{12}\right)-\left(\frac{-1}{10}\right)\left(\frac{-1}{10}\right)}=\boxed{4.0\text{ V}} \tag{7.15}$$

<u>Short Circuit Current</u>: As is the case with circuits in general, there is no one right way to solve a problem. It is good to develop the ability to solve a circuit using the most direct approach that will lead to the least amount of work. It keeps equations as simple as possible, and this reduces the likelihood of error. The short circuit is shown in figure 7.12.

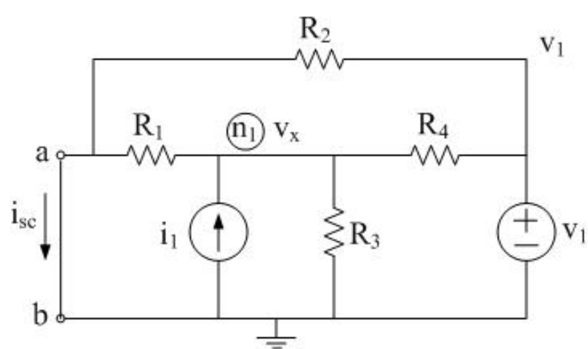


Figure 7.12 Circuit of example 7.3 labeled for solving for i_{sc} by first solving for v_x.

Solution: For this particular circuit, it is easier to solve for the single unknown node voltage v_x and then express i_{sc} in terms of v_x, than it is to solve for i_{sc} directly using the BCM. The circuit labeled for solving for v_x using the NVM is shown in figure 7.12.

$$\text{KCL n}_1:\ \frac{v_x}{R_1}-i_1+\frac{v_x}{R_3}+\frac{v_x-v_1}{R_4}=0 \tag{7.16}$$

Factoring out and solving for v_x gives

$$v_x\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}\right) = i_1 + \frac{v_1}{R_4} \tag{7.17}$$

$$v_x = \frac{i_1 + \frac{v_1}{R_4}}{\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4}} = \frac{0.5 + \frac{4}{12}}{\frac{1}{10}+\frac{1}{8}+\frac{1}{12}} = \boxed{2.7027\,V} \tag{7.18}$$

Now i_{sc} can be expressed in terms of v_x giving

$$i_{sc} = \frac{v_1}{R_2}+\frac{v_x}{R_1} = \frac{4}{5}+\frac{2.7027}{10} = \boxed{1.0703\text{ A}} \tag{7.19}$$

Taking the ratio of v_{oc} to i_{sc} provides R_{Th}: $R_{Th} = \frac{v_{oc}}{i_{sc}} = \frac{4.0}{1.0703} = \boxed{3.74\ \Omega}$

Let us examine the circuit that exists at terminals ab with all the sources turned off.

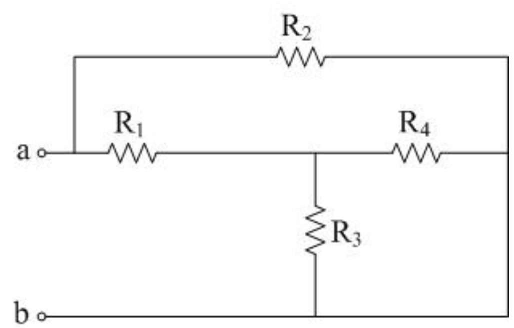


Figure 7.13 Circuit of example 7.3 with all of the sources turned off.

The equivalent resistance between terminals ab is

$$R_{Th} = R_{ab} = \frac{1}{\frac{1}{R_2}+\frac{1}{R_1+\frac{1}{\frac{1}{R_3}+\frac{1}{R_4}}}} \tag{7.20}$$

$$R_{Th} = R_{ab} = \frac{1}{\frac{1}{5} + \frac{1}{10 + \frac{1}{\frac{1}{8} + \frac{1}{12}}}} = \boxed{3.74\ \Omega} \tag{7.21}$$

This value of R_{th} is exactly the same value that is obtained by solving for v_{oc} and i_{sc}, and taking their ratio. Hence, this technique is valuable depending on whether you are seeking the Thevenin equivalent, the Norton equivalent, or simply using an independent method of confirming your result. In this example, the problem was to find both the Thevenin and the Norton equivalent circuits, which are shown below.

Figure 7.14 The Thevenin and Norton equivalent circuits that result from example 7.3 (and any other circuit).

7.4 Source Transformation

Given any Thevenin equivalent circuit, it is easy to find the Norton equivalent circuit and vice versa. The Thevenin resistance R_{Th} is the same for both circuits and the relationship $v_{oc} = R_{Th}i_{sc}$ allows one to easily compute the third quantity from the other two. This relationship was derived in the development of both (7.8) and (7.9). Thus, it is possible to identify the Thevenin and Norton equivalent circuits in a larger circuit and transform them as desired to make the analysis of the unknown voltage or current more easily than using the original circuit.

Reconsider example 7.2, the circuit of which is reproduced here for convenience.

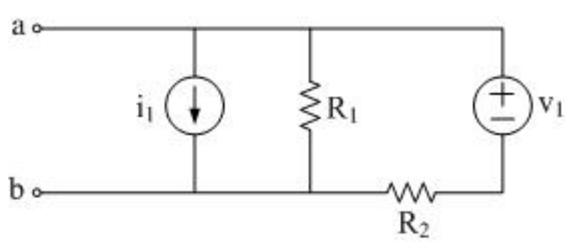


Figure 7.16 Circuit of example 7.2 reproduced here to demonstrate source transformation

Suppose the Thevenin equivalent circuit on the right hand side of the circuit in figure 7.2 is replaced by its Norton equivalent circuit. The result is shown in Figure 7.17.

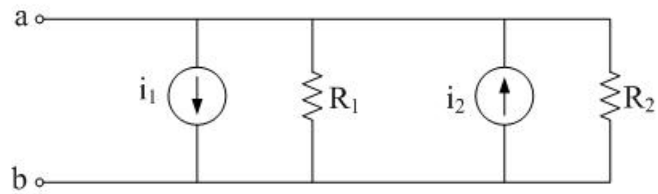


Figure 7.17 Circuit of example 7.2 with the Thevenin source on the right replaced by it Norton equivalent.

The idea is that now the open circuit voltage drop from a to b and the short circuit current flow from a to b is simpler to determine. By inspection of the circuit, the open circuit voltage drop from a to b is

$$v_{oc} = (i_2 - i_1)\left(\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}\right) = \left(\frac{v_1}{R_2} - i_1\right)\left(\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}\right) \tag{7.22}$$
$$= \left(\frac{10}{4 \cdot 10^3} - 1.5 \cdot 10^{-3}\right)\left(\frac{1}{\frac{1}{6 \cdot 10^3} + \frac{1}{4 \cdot 10^3}}\right) = \boxed{2.4\ \text{V}}$$

and the short circuit current from a to be is simply

$$i_{sc} = i_2 - i_1 = \frac{v_1}{R_2} - i_1 = \frac{10}{4 \cdot 10^3} - 1.5 \cdot 10^{-3} = \boxed{1.0\ \text{mA}} \tag{7.23}$$

The Thevenin resistance R_{Th} may be computed by either taking the ratio of v_{oc} to i_{sc}

$$i_{sc} = \frac{v_{oc}}{i_{sc}} = \frac{2.4}{1 \cdot 10^{-3}} = \boxed{2.4\ \text{kW}} \tag{7.24}$$

$$R_{Th} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{\frac{1}{6 \cdot 10^3} + \frac{1}{4 \cdot 10^3}} = \boxed{2.4\ \text{k}\Omega} \tag{7.25}$$

or by turning off the independent sources and finding the equivalent resistance from a to b.

However, approaching circuit analysis by hunting for opportunities to apply source transformations diminishes students' ability to solve any circuit of any complexity using the exact circuit topology given. It is therefore recommended that students learn to solve the circuits using the exact topology given. Both the node voltage method and the branch current method, which are introduced in the next chapter, can be used to solve any circuit of any topology, and students should become comfortable solving circuits in the configuration in which the problem is given.

7.5 Voltage and Current Division

Voltage Division

Consider an independent voltage source applied to three resistors in series as shown in figure 7.17.

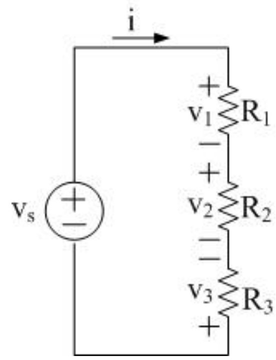


Figure 7.17 Voltage source driving three resistors in series.

Given the element values, the voltage v_1 can be obtained by applying Ohm's law to find the current i and Ohm's law again to find the voltage v_1 as shown in (7.26,27).

$$i = \frac{v_s}{R_1 + R_2 + R_3} \tag{7.26}$$

$$v_1 = R_1 i = (R_1)\frac{v_s}{R_1 + R_2 + R_3} = v_s \frac{R_1}{R_1 + R_2 + R_3} \tag{7.27}$$

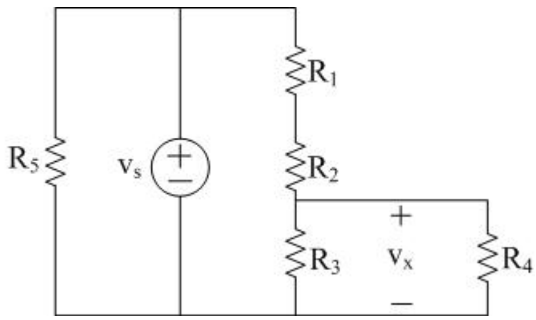
The expression for v_1 has been algebraically rearranged so that the voltage v_1 can be interpreted as a fraction of v_s. The fraction of v_s is a ratio of the resistance R_1 to the total series resistance $R_1 + R_2 + R_3$. This is called **voltage division**. **Voltage divides in proportion to resistance**. Can you figure out the expression for v_2 using voltage division? That is, using voltage division directly, without the double use of Ohm's law. V2 is a fraction of v_s, the fraction is a ratio of the resistance R_2 to the total series resistance $R_1 + R_2 + R_3$ as expressed in (7.28).

$$v_2 = v_s \frac{R_2}{R_1 + R_2 + R_3} \tag{7.28}$$

To obtain the expression for v_3 using voltage division, you must be very attentive to the signs of all the circuit quantities because v_3 is a **negative** fraction of v_s as shown in (7.17).

$$v_3 = -v_s \frac{R_2}{R_1 + R_2 + R_3} \tag{7.29}$$

Example 7.4



Given all of the element values, find the expression for v_x.

Solution: Voltage divides among series resistances. Therefore, R_5 has no effect on v_x. Recall that the ratio in voltage division is the resistance you want the voltage across divided by the total series resistance. The resistance we want the voltage across is the parallel combination of R_3 and R_4. v_x is a positive fraction of v_s, the fraction is a ratio of R_3 in parallel with R_4 divided by the series resistance comprised of the sum of R_3 in parallel with R_4 added to R_1 and R_2 as shown in (7.30).

$$v_x = v_s \frac{\frac{1}{\frac{1}{R_3} + \frac{1}{R_4}}}{R_1 + R_2 + \frac{1}{\frac{1}{R_3} + \frac{1}{R_4}}} \tag{7.30}$$

Current Division

Consider a current source in parallel with three resistances as shown in figure 7.18.

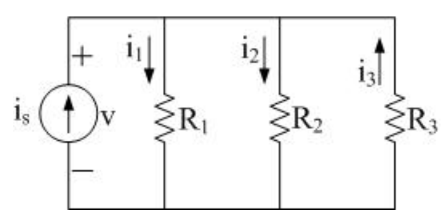


Figure 7.18 Current source driving three resistors all in parallel.

Given all of the element values and solving for i_1 can be performed using Ohm's law twice. First to obtain the voltage v and then to obtain the current i_1 as presented in (7.31,32).

$$v = i_s \left(\frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \right) \tag{7.31}$$

$$i_1 = \frac{v}{R_1} = i_s \left(\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \right) \tag{7.32}$$

Defining conductance as the inverse of resistance $\triangleq \frac{1}{R}$, (7.20) can be interpreted as follows. The current i_1 is a fraction of the current i_s. The fraction is a ratio of the conductance $\frac{1}{R_1}$ to the total parallel conductance $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. **In general, current divides in proportion to conductance.** Are you able to write the expression for i_2 directly using current division instead of Ohm's law twice? The value of i_2 is given by (7.33).

$$i_2 = i_s \left(\frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \right) \tag{7.33}$$

To use current division to express i_3, you must pay very close attention to the signs of the circuit variables.

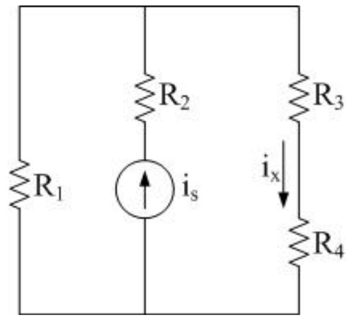
$$i_3 = -i_s \left(\frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \right) \tag{7.34}$$

The current i_3 is a fraction of the current $-i_s$. The fraction of $-i_s$ is a ratio of the conductance $\frac{1}{R_2}$ to the total parallel conductance $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

Conductance, denoted by the symbol $G \triangleq \frac{1}{R}$ has units of mhos (Ω^{-1}) or Siements (S).

Students generally find current division significantly more abstract and difficult than voltage division. **The key to understanding current division is that current divides in proportion to conductance**. To find the current through a resistance when only a current source is present and driving resistances in parallel, use current division. The current through one of the parallel resistances is the conductance of that branch divided by the total parallel conductance times the source current. Consider the following example.

Example 7.5



Given the element values, find the expression of the current i_x.

Solution: The current i_x is a positive fraction of the current i_s. The fraction is a ratio of the conductance $\frac{1}{R_3+R_4}$ to the total parallel conductance $\frac{1}{R_3+R_4}+\frac{1}{R_1}$. Note that R_2 does not affect the current i_x at all. R_2 is in series with the current source i_s. **Current divides among parallel conductances**.

$$i_x = (i_s)\frac{\frac{1}{R_3+R_4}}{\frac{1}{R_3+R_4}+\frac{1}{R_1}} \tag{7.35}$$

The analytical equation for i_x (7.35) shows that the resistance R_2 does not affect the value of i_x. One way to step through a current division problem is to ask, "What is the resistance of the branch with the unknown current?" Answer that question and take the inverse of the branch resistance to get the conductance through which the unknown current flows. That term goes in the numerator of the fraction. Then, the denominator is the total parallel conductance. You must be able to identify resistances in parallel. The parallel conductance is the sum of each parallel branch conductance. Remember, **current divides in proportion to conductance**.

Chapter 8

Capacitance and Electromagnetics

8.1 Capacitance

The focus of this chapter is the current-voltage behavior of a capacitor and its context in the general set of Maxwell's equations and basic principles of electromagnetic waves. A capacitor's current-voltage relationship arises from its ability to store energy. An examination of a capacitor from an electromagnetic perspective provides insight into a capacitor's ability to store energy along with electrical phenomena well beyond the ability of a capacitor to store electrical energy. Maxwell's equations form the basis for the study of electromagnetics. The chapter begins with the physical structure of a capacitor followed by the current-voltage relationship for a capacitor and a brief examination of general electromagnetics to provide a deeper level of physical understanding of capacitance. In the process electromagnetic waves are introduced along with antennas.

A capacitor is a pair of parallel conducting plates separated by a **dielectric** (electrical insulator). This physical structure is shown in figure 8.1.

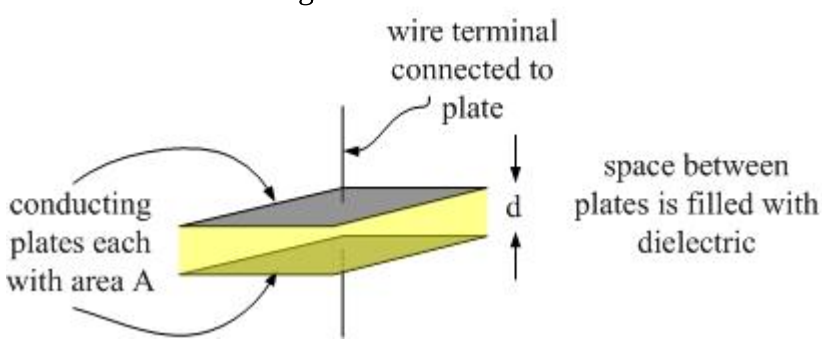


Figure 8.1 Physical structure of a parallel plate capacitor showing the conducting plates, each of area *A*, separated by a distance *d*, and sandwiching a **dielectric**.

A **dielectric** medium is characterized by its **permittivity** ε. In general $\varepsilon = \varepsilon_0\varepsilon_r$ where ε_0 is the permittivity of free space and ε_r is the relative permittivity or **dielectric constant**. The permittivity of a material is a measure of how easily a capacitor can store electric charge on its plates for a given voltage applied. For example, titania (TiO_2) has a dielectric constant of 100 and free space (or air) has a dielectric constant of 1. Therefore, it is 100 times easier for an [titania to store charge than air. The value of the capacitance of a parallel plate capacitor in the general case is

$$C = \frac{\epsilon A}{d} \tag{8.1}$$

where A is the area of each conducting plate and d is the distance separating the plates. Its symbol is a pair of parallel lines and its value is denoted by its capacitance C measured in units called Farads (F).

A picture of several different values and types of capacitors is shown in figure 8.2.

Figure 8.2 Several different values and types of capacitors including electrolytic, mylar, tantalum, ceramic and adjustable plate overlap air dielectric (on right).

The current-voltage relationship for a capacitor is given by **Maxwell's law**.

Maxwell's law states that the current through a capacitor is the capacitance times the time rate of change of the voltage drop in the same direction.

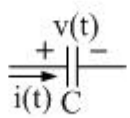


Figure 8.3 Capacitor symbol labeled with its voltage drop in the direction of current flow.

It is worth stating Maxwell's law a few times to get a feel for the relationship between the directions of current flow and voltage drop. There are two immediate consequences of Maxwell's law. First, a capacitor is an open circuit for DC. The term "DC" was originally used as an acronym for "direct current" implying the flow of charge in one direction only. However, with time the term "DC" became synonymous with the term "constant." A DC circuit is one in which all the voltages and currents in the circuit are constant. Hence, if the voltage across a capacitor is constant, the current through it is zero, and the capacitor behaves as an open circuit. Second, it is impossible for the voltage across a capacitor to be discontinuous. If the voltage were discontinuous, the current would be infinite at the time of the discontinuity in the voltage, which is impossible. Therefore, the voltage across a capacitor must be continuous. The argument form used here is known as *reductio ad absurdum*. A short review of the continuity of a function may be useful. A function f(t) is continuous in time if and only if the limit from the left as t $\rightarrow$ t_0 equals the limit from the right as $t_0 \leftarrow$ t, which equals the value of the function at the point t_0, $f(t_0)$, for all t_0. This can be written compactly as: if $\lim_{t\to t_0} f(t) = \lim_{t_0 \leftarrow t} f(t) = f(t_0)\ \forall_{t_0}$, then the function $f(t)$ is continuous. Thus, there can be no sudden jumps or abrupt changes in the value of f(t). For example, a triangle wave is continuous while a square wave is discontinuous as shown in the figure 8.4.

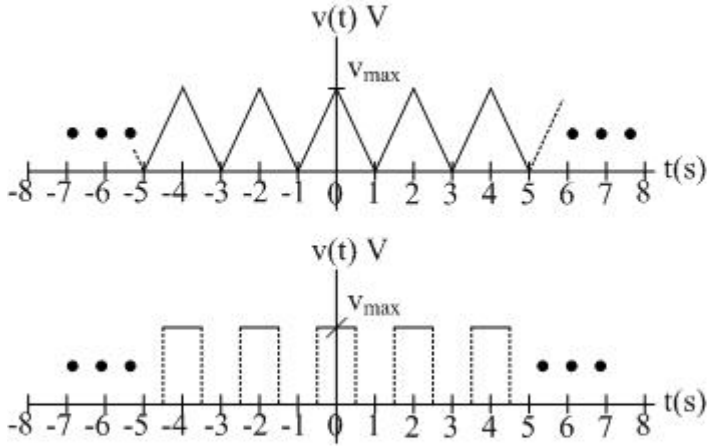


Figure 8.4 Triangle wave and rectangular pulse train illustrating a continuous signal and a discontinuous signal.

Thus, it is perfectly acceptable for a triangle wave to be the voltage waveform across a capacitor, but it is impossible for the voltage across a capacitor to be a square wave due to the waveform's discontinuities. A sinusoidal or AC voltage across a capacitor produces a sinusoidal or AC current. AC is an acronym for "Alternating Current," but has evolved into meaning "sinusoidal." When one speaks of an AC voltage or current, the acronym may appear inappropriate. But when it is understood to mean "sinusoidal," the phrases "AC voltage" or "AC current" make sense.

Example 8.1

Consider the voltage $v(t) = 10\cos(2\pi \cdot 10^6 t)$ V across the capacitor $C_1 = 2.2\ \mu F$ below, and calculate the current $i(t)$.

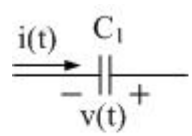


Figure 8.5 A capacitor C_1 with its current labeled in the direction of its voltage rise.

Solution: Using Maxwell's law, the current is

$$i(t) = C_1 \frac{d(-v(t))}{dt} = (2.2 \cdot 10^{-6})(10)(\ 2\pi \cdot 10^6)(sin(2\pi \cdot 10^6 t) \tag{8.2}$$

$$= 138.2\, sin(2\pi \cdot 10^6 t)\ \text{V} = 138.2\, cos(2\pi \cdot 10^6 t - 90°)\ \text{V}$$

in which the *sin*(·) function has been written as a *cos*(·) function by adjusting the phase of the *cos*(·) function. A question should be arising regarding the derivative relationship between current and voltage for a capacitor, and this leads us to the examination of Maxwell's modification of Ampere's law and the examination of Maxwell's four electromagnetic equations.

8.2 Maxwell's Equations

Hermann von Helmholtz (1821-1894) formulated a theorem stating that if the curl and divergence of a vector field are known everywhere, the vector field is uniquely defined. This is why there are four fundamental equations in electromagnetics, two curl equations and two divergence equations (one pair for each field: electric and magnetic). Different scientists discovered the four relationships, but it was James Maxwell (1831-1879) who completed Ampere's law and is attributed with the recognition that it is this set of four equations that governs all electromagnetic phenomena. Figure 8.6 presents the four equations as originally formulated and by whom.

Faraday's law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Gauss' law	$\nabla \cdot \vec{D} = \rho$
Amperere's law	$\nabla \times \vec{H} = \vec{J}$
Unclaimed	$\nabla \cdot \vec{B} = 0$

$\vec{E}$ Electric Field Intensity $\left(\frac{V}{m}\right)$

$\vec{D}$ Electric Flux Density $\left(\frac{C}{m^2}\right)$

ρ Electric Charge Density $\left(\frac{C}{m^3}\right)$

$\vec{H}$ Magnetic Field Intensity $\left(\frac{A}{m}\right)$

$\vec{J}$ Electric Current Density $\left(\frac{A}{m^2}\right)$

$\vec{B}$ Magnetic Flux Density $\left(\frac{Wb}{m^2}\right)$

Figure 8.6 Electromagnetic field quantities and their associated units.

In order to understand the basic concepts of these equations, one must understand the difference between a scalar (e.g. charge density ρ) that has only a magnitude and a vector (e.g. the electric field intensity $\vec{E}$) that has both a magnitude and a direction. The nabla symbol or del operator in Cartesian coordinates is defined by $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{y}\frac{\partial}{\partial z}$ and represents a vector of partial directional derivatives. Since vector calculus is not a prerequisite for this course, a qualitative description of curl and divergence is provided. The **curl** of a vector $\vec{F}$, defined by the cross product of ∇ with the vector to produce $\nabla \times \vec{F}$, is the twisting or circulation of the vector $\vec{F}$. For analogies, consider an eddy current in a river or a tornado or simply water in a river going around a bend. The curl is calculated mathematically by taking the determinate of a 3x3 matrix with the first row being the Cartesian directions $(\hat{x}, \hat{y}, \hat{z})$, the second row is the set of three partial directional derivatives of the del operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, and the last row contains the set of directional components of the vector field (F_x, F_y, F_z), such that

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Figure 8.7 The curl of a function $\vec{F}$ defined in terms of the determinant of a matrix.

Divergence is **flux** per volume. **Flux** is the **amount** of something **crossing** a **surface** such as water, electric field, threads in a bed sheet, pine needles). Some pine trees have very dense needles (higher flux density) and some have very sparse needles (lower flux density). The divergence is a measure of a vector's increase in the direction it points. Because there are no magnetic monopoles, all magnetic field lines must end on themselves. For any given closed surface, the magnetic flux that leaves the surface must equal the magnetic flux that enters the surface. That is what $\nabla \cdot \vec{B} = 0$ tells us. It also tells us that there is no such thing as magnetic charge. There does exist electric charge, and it is known that if there is a net charge inside of a closed surface, the net electric flux density passing through the surface is equal to the net charge density enclosed by the surface. This is Gauss' law. Divergence is calculated by taking the dot product of the del operator with the vector field thereby producing a scalar quantity.

Faraday's law tells us that a circulation in the electric field is equal to the negative time variation of the magnetic flux density. Ampere's law states that a circulation of a magnetic field is always accompanied by conduction current, and Maxwell found this to be erroneous. Maxwell modified Ampere's law to include displacement current, which is a time varying electric flux density. The time variation of an electric field constitutes an electric current, and it is this displacement current that flows through a capacitor. This new equation has been named after Maxwell, and the complete set of four equations are commonly known as Maxwell's equations.

Faraday's law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Gauss' law $\nabla \cdot \vec{D} = \rho$

Maxwells's law $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Unclaimed $\nabla \cdot \vec{B} = 0$

Figure 8.8 Maxwell's four laws of electromagnetism

There are three fundamental relationships that depend on the material through which the fields penetrate. They are called **constitutive** relationships and are given by

$$\vec{D} = \epsilon\vec{E} \quad \vec{B} = \mu\vec{H} \quad and \quad \vec{J} = \sigma\vec{E}$$

Figure 8.9 Electromagnetic constitutive relationships

Permittivity $\epsilon = \epsilon_0\epsilon_r$ where ϵ_0 is the permittivity of free space (or air) and is equal to

$\epsilon_0 = 8.854 \cdot 10^{-12}\ \frac{F}{m}$ and $\epsilon_r$is the relative permittivity or dielectric constant. Permeability $\mu = \mu_0\mu_r$ where μ_0 is the permeability of free space and is equal to $\mu_0 = 4\pi \cdot 10^{-7}\ \frac{H}{m}$ and $\mu_r$is the relative permeability. The last constitutive relationship is Ohm's law $\vec{J} = \sigma\vec{E}$. Both ϵ_r and μ_r are dimensionless constants, and σ has units of inverse resistivity (called conductivity $\sigma = \frac{1}{\rho}$). It is useful to think of permittivity ϵ to be a measure of how easily a material can store energy in the form of an electric field. Permeability is a measure of how easily a substance can be magnetized. The conductivity σ is a measure of how easily an electric current can flow through the material.

It is important to realize that these equations were **discovered**, **not derived** from some set of mathematical axioms. Physical measurements were made (physical observations), and these relationships (both Maxwell's laws and the constitutive relationships) were found to hold true in general.

Now that the concepts of permeability and permittivity have been introduced, basic principles of electromagnetic waves are introduced.

8.3 Electromagnetic Waves

Electromagnetic waves travel in air at their free space velocity, denoted by c in 8.3.

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3 \cdot 10^8\ \frac{m}{s} \tag{8.3}$$

The electric field and magnetic field are mutually orthogonal with each other and with the Poynting vector that is in the direction of the wave's propagation. Such a wave is called a **transverse electromagnetic wave** (TEM). All electromagnetic (EM) waves in a substance travel with velocity equal to the group velocity in that medium, and experimental data has verified this velocity depends only on the permittivity and permeability of the medium that the wave is propagating through.

$$v_g = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} \tag{8.4}$$

(8.4) becomes (8.3) when the wave is in free space or air. If the medium is magnetizable beyond the magnetizability of free space or the medium can support an electric field beyond free space, then the correct formulation is (8.4). As an example, glass has a dielectric constant of approximately 4. Glass is not magnetizable more than air. As a consequence, light travels 0.5 times the free space velocity of light when it passes through glass.

A TEM wave can be visualized as shown in figure 8.10.

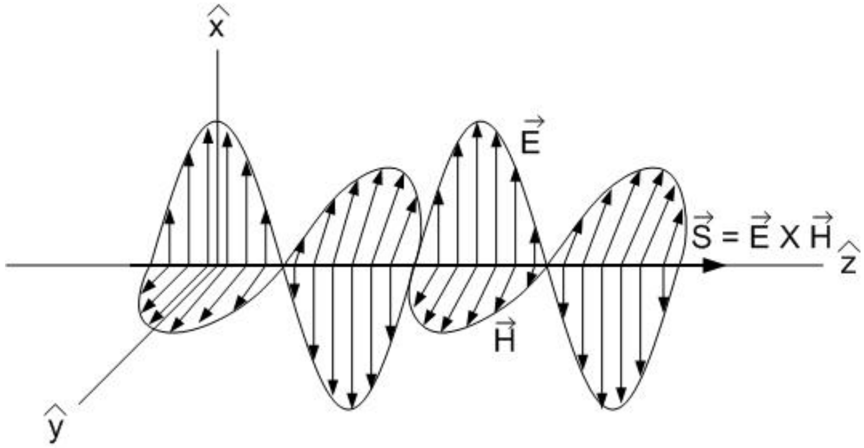


Figure 8.10 A vertically polarized TEM wave propagating to the right. The poynting vector $\vec{S} = \vec{E} \times \vec{H}$ is shown and is equal to the power density of the wave in units of $\frac{w}{m^2}$.

The Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ is the power density in $\frac{w}{m^2}$, and points in the direction of wave propagation. If the Poynting vector is integrated over the effective aperture of a receiving antenna, the integration yields the power incident on the receiving antenna.

Polarization is a term that provides the orientation of the TEM wave. The **polarization direction is defined to be the direction of the electric field**. Polarization is important because in many applications a receiving antenna must be oriented with the same polarization as the transmitting antenna.

A TEM wave has a wavelength and frequency that are functions of the permeability and permittivity of the medium through which it is traveling, and is quantified by 8.5.

$$v_g = f\lambda \tag{8.5}$$

where λ is the wavelength in $\frac{m}{cycle}$ (commonly expressed only in length, meters, with the implied understanding that this length is the length of one cycle), f is frequency in $\frac{cycles}{s} \triangleq Hz$, and the units of group velocity are $\frac{m}{s}$. Assuming TEM waves in air, some example physical phenomena and associated wavelength and frequency are presented in figure 8.11. DC is an acronym for direct current. However, the term "DC" has become synonymous with the term "constant." Thus, it makes sense to talk about DC voltages or DC currents. If we were to utilize the literal acronym for DC, it would not even make sense because a "Direct Current current" is redundant and a "Direct Current voltage" would be a contradiction. However if we use the term "constant" in place of DC, then both the phrases, "constant voltage" and "constant current" make sense.

It is useful to think of DC as a special case of AC in which the frequency is zero. All batteries produce DC voltages and currents (when they are connected to a circuit).

Physical Phenomenon	Wavelength in meters	Frequency in Hz
Direct Current	∞	0
Power Waves	6 Mm	50/60 Hz
AM Broadcast	~300 m	520 kHz – 1.61 MHz
FM Broadcast	~3 m	88 MHz – 108 MHz
Analog Cell Phone	$\sim \frac{1}{3}$ m	850 MHz – 900 MHz
Digital Cell Phone	16 cm	1.9 GHz
X-Band Police Radar	2.9 cm	10.5 GHz
Infrared Laser Light	0.75 μm – 1 mm	300 GHz (at 1 mm)
Visible Light	0.4 – 0.7 μm	600 THz (at 0.5 μm)
Ultraviolet Light	10 – 400 nm	30 PHz (at 10 nm)
X-Rays	10 pm – 10 nm	30 EHz – 30 PHz

Figure 8.11 Physical phenomena and associated frequency and wavelength.

8.4 Basic Principles of antennas

There are two primary functions of an antenna. One is to launch or receive EM waves and the other is to direct radiated (or received) EM waves in a desired direction while suppressing radiation (or reception) in undesired directions. There are two fundamental types of antennas: a dipole antenna and a monopole antenna.

Dipole Antennas

The length of a dipole antenna is typically an integer multiple of $\frac{\lambda}{2}$. A dipole of length $\frac{\lambda}{2}$ is called a half-wave dipole. If the dipole is oriented vertically, it is termed “vertically polarized,” and if it is oriented horizontally, it is termed “horizontally polarized.” This is because the electric field launched by the dipole is along the line of the antenna. The radiated magnetic field is geometrically circles around the line of the antenna. If you imagine the EM wave a significant distance from the antenna, the EM wavefront is a plane. The **radiated wave is called a plane wave**. The direction of the radiated wave is such that the electric field lines are in the same direction as the line of the antenna. Hence, the **antenna radiates an EM wave whose polarization is in the same direction as the line of the antenna**. When a dipole antenna is of length $\frac{\lambda}{2}$ and fed at its center, it presents a load to the transmitter that is equivalent to a resistance of 73 Ω. If the antenna is tuned to a frequency f_0 and is stimulated with a frequency less than f_0, it presents a resistive plus capacitive load to the transmitter, and part of the power that leaves the transmitter is reflected back to the transmitter instead of being transmitted by the antenna. Consider a horizontally polarized, half-wave dipole antenna tuned to a frequency of 300 MHz. At 300 MHz in air, the wavelength is 1 meter. So the half-wave dipole should be made 0.5 meters long. It is horizontally polarized, so the line of the antenna is oriented horizontally as shown in figure 8.12.

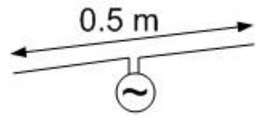


Figure 8.12 Center fed, horizontally polarized, half-wave dipole tuned to a frequency of 300 MHz.

Example 8.2

Given a vertically polarized half-wavelength dipole tuned to a frequency of 100 MHz, which is in the middle of the FM broadcast band, is used to receive an FM radio station whose frequency is 99.7 MHz. Draw the antenna showing its orientation and length, and comment on how the antenna appears to the transmitter.

A half-wave dipole tuned to 100 MHz will need to be 1.5 meters long. Unless otherwise specified, all dipole antennas are assumed to be center fed. The antenna is presented in figure 8.13. If it is receiving a wave at a frequency of 99.7 MHz, the wavelength is 3.01 m. So the antenna is a little too short. The current wave will lead the voltage wave and consequently the antenna acts capacitively in addition to resistively $(R_{ant} \cong 73\ \Omega)$ and a small fraction of the power leaving the transmitter is reflected back to the transmitter.

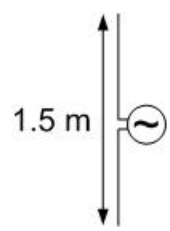


Figure 8.13 Vertically polarized, half-wave dipole, tuned to 100 MHz.

Can you figure out why the antenna is resonant at a length of $\frac{\lambda}{2}$? That is, what is so special about this dipole antenna length? The first thing that you must realize is that the circuits you build in lab have components and connecting wire that are very small compared to a wavelength. What wavelengths have you been working with in the laboratory? Next time you are in the lab, see what the maximum frequency is that is producible by the function generator and calculate the wavelength in air. Compare this wavelength with the dimensions of your circuit board and components. For a half-wave dipole, the antenna is half of the wavelength in length. What happens when the component, in this case an antenna, is on the order of a wavelength? It may appear to you that looking into the antenna from the feedpoint, the antenna is an open circuit. But that is not the case. There is a current wave, and a half-wave dipole is the shortest length the antenna can be and resonance can be achieved. If the antenna is $\frac{\lambda}{2}$ in length, the current wave will be perfectly in phase with the voltage wave, and under these conditions, the antenna will radiate much better than if the current wave is not in phase with the voltage wave. This is something for you to think about.

Monopole Antennas

A monopole has a length that is typically an integer multiple of $\frac{\lambda}{4}$. The most common length is $\frac{\lambda}{4}$ and is called a quarter-wave monopole. Unless otherwise specified, the quarter-wave monopole is fed at its base, and there is always a ground plane perpendicular to the line of the monopole. Metal acts as an excellent reflector just as a mirror is a reflector for visible light. Consequently, the monopole behaves like a dipole because the monopole sees its image in the ground plane. The earth may serve as a ground plane. The trunk of a car or even your body can also serve as ground planes. Whenever you see a monopole, you should think about what is serving as the ground plane. A diagram of a vertically polarized monopole is shown in figure 8.14.

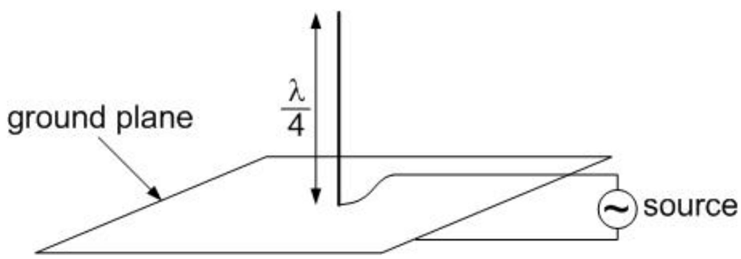


Figure 8.14 A vertically polarized monopole antenna. There is a layer of insulating material between the monopole and the ground plane.

The radiation pattern (radiation energy as a function of spatial direction) of a monopole is the same as the radiation pattern of a dipole but only above the ground plane. There is no radiation below the ground plane. The reception pattern of an antenna is the same as its radiation pattern. This is due to the principle of **reciprocity**. The radiation pattern is typically presented by showing two slices of the three dimensional pattern. The slices are a vertical slice and a horizontal slice. Take, for example, a vertically polarized dipole. The horizontal slice is a circle. The radiation pattern in the horizontal direction is uniformly **omnidirectional**. The radiation pattern of a vertically polarized half-wave dipole in the vertical direction is shown in figure 8.15. In three dimensions, it is a torus (doughnut shaped). But the radiation pattern shown in figure 8.15 is a slice of that doughnut shape.

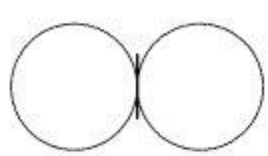


Figure 8.15 The vertical slice of the radiation pattern of a half-wave dipole.

This radiation pattern indicates that there is no radiation along the line of the antenna and the direction of maximum radiation, or reception by reciprocity, is perpendicular to the line of the antenna. The radiation pattern of a vertically polarized monopole is exactly the same as the radiation pattern of a vertically polarized dipole antenna, but only above the ground plane. No radiation occurs below the ground plane as shown in figure 8.16.

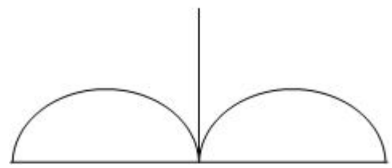


Figure 8.16 Radiation for a vertically polarized monopole antenna. The antenna is shown only for the purpose of providing orientation. This radiation pattern indicates that there is zero radiation along the line of the antenna and maximum radiation perpendicular to the antenna.

Example 8.3

A vertically polarized monopole antenna is tuned 1 MHz (middle of the AM broadcast band). How long should a quarter-wave monopole be for this frequency?

Solution: The wavelength at a frequency of 1 MHz is 300 m, and consequently a quarter-wave monopole should be 75 meters in length. This corresponds to about 82 feet. Have you ever seen an AM radio station antenna? What structure is approximately 82 feet in length? The answer is the steel tower. The tower itself may have many other types of antennas mounted on it such as cell antenna arrays, terrestrial and satellite dish antennas and ham radio repeater antennas, but the tower itself is the AM radio antenna. Take a closer look at an AM radio antenna tower next time you pass an AM radio station. There may be some other towers nearby, but those towers are not actively driven by a source of radio waves. These passive antennas are intended to absorb radio frequency energy and re-emit it, providing some directivity to the radio station's transmitted wave. That is, the presence of the passive tower serves to direct the energy of the transmitted wave toward it or it may be acting as a reflector so that no radiation extends in that direction.

The only antenna that will radiate equally in all directions is called an **isotropic** radiator, and it generates waves that that are uniformly equal in strength in all directions. An **isotropic** radiator is also referred to as a **point source**. **Isotropic** radiators are fictitious and used in this context as a standard for comparing the true radiation pattern of an antenna.

We have examined some of the properties of the two most common antenna types, the dipole and the monopole. This analysis provided the context for exploration of electromagnetic waves. Maxwell was responsible for completing Ampere's law and recognized that the set of four equations, now known as Maxwell's equations, provides the mathematical relationships that provide the analytic framework for electromagnetic phenomena. When a person refers to the singular "Maxwell's equation," they are referring to displacement current, which explains the current-voltage behavior of capacitors. We now return our focus to capacitors with an examination of the electric energy stored in the dielectric medium between the plates of a capacitor.

8.5 Energy Stored in a Capacitor

Capacitors store energy in the form of an electric field. The electric field exists in the dielectric medium between the conducting plates and is due to the difference in charge present on the conducting plates. By integrating the power absorbed by a capacitor over an arbitrary interval of time, the expression for energy absorbed over that same time interval will result. Beginning with the definition of power as the time rate of change of energy and equating this to the power absorbed for a capacitor allows differential energy to be expressed in terms of differential voltage by separation of variables

$$p(t) \triangleq \frac{dw(t)}{dt} = v(t)i(t) = v(t)C\frac{dv(t)}{dt} \tag{8.6}$$

$$dw(t) = v(t)Cdv(t)$$

This expression is now integrated from an arbitrary time t_1 to an arbitrary time t_2 recognizing that *C* is a constant that can be moved outside of the integrand.

$$\int_{t_1}^{t_2} w(t) = C \int_{t_1}^{t_2} v(t)dv(t)$$

Performing the anti-derivatives and substituting the integration limits results in the energy absorbed by the capacitor from t_1 to t_2 where $t_2 > t_1$ in terms of the difference between the energy stored in C at time t_2 and the energy stored in *C* at time t_1.

$$w_{abs}(\mathrm{t}_1 < t < \mathrm{t}_2) = w(\mathrm{t}_2) - w(\mathrm{t}_1) = \frac{1}{2}\,\mathrm{C}\,\mathrm{v}^2(\mathrm{t}_2) - \frac{1}{2}\,\mathrm{C}\,\mathrm{v}^2(\mathrm{t}_1)$$

The general expression for the **energy stored in *C* at any arbitrary time t** is given by

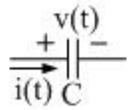
$$w(\mathrm{t}) = \frac{1}{2}\,\mathrm{C}\,\mathrm{v}^2(\mathrm{t}) \tag{8.7}$$

Thus, the energy absorbed by a capacitor over an interval of time is the energy stored in the capacitor at the end of the interval of time minus the energy stored in the capacitor at the beginning of the interval of time. If the derivation had begun with power delivered instead of power absorbed, the result would be energy delivered over the same interval of time. Mathematically this means that

$$w_{del}(t_1 < t < t_2) = -w_{abs}(t_1 < t < t_2) = w(t_1) - w(t_2) = \frac{1}{2}\,C\,v^2(t_1) - \frac{1}{2}\,C\,v^2(t_2)$$

Example 8.4

Given the voltage $v(t) = 10\cos(2\pi \cdot 10^6 t)\,V$ across the capacitor C = 5 μF below, calculate and graph the energy stored for at least two cycles.



The energy stored in *C* as a function of time is

$$w_C(t) = \frac{1}{2}\,C\,v^2(t) = \frac{1}{2}(5 \cdot 10^{-6})100cos^2(2\pi \cdot 10^6 t) = 250cos^2(2\pi \cdot 10^6 t)\;\mu J$$

Note that the proper units (Joules) accompany the numerical value of the stored energy. In order to make this function easier to graph, the trigonometric identity

$$cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

is invoked to obtain $w_C(t) = 125 + 125\cos(4\pi \cdot 10^6 t)\;\mu J$, which is fairly easy to graph. This analytical expression is a sinusoid shifted up by the DC value of 125 μJ. The sinusoid has no phase shift, and the cosine function has a maximum when its argument is zero. Therefore there is a maximum at $t = 0$. The coefficient of time *t* is called the **angular frequency** whose symbol is **ω** and has units of radians per second. The next maximum will occur one period (***T***) later in time. The relationship between angular **frequency** and **period** is given by

$$\omega = 2\pi f = \frac{2\pi}{T}$$

where the frequency *f* is measured in Hertz (Hz) or cycles/second, and the period (time to complete one cycle) is measured in seconds. Solving for the period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi \cdot 10^6} = 0.5\;\mu s$. Graphs of *v(t)* and *w(t)* are shown in figure 8.10. Note the following important points.

1) The energy stored in a capacitor is an instantaneous function of time

2) The energy stored in a capacitor is a non-negative function.

3) An AC voltage produces an energy waveform with twice the frequency of the voltage (and offset such that the minimum of $w_C(t)$ = 0 and that this occurs when the voltage across *C* is zero.

Exercise: Given the voltage and energy stored waveforms for the 5 μF capacitor C in example 8.4, calculate the energy absorbed between $t = 130\ ns$ and $t = 220\ ns$.

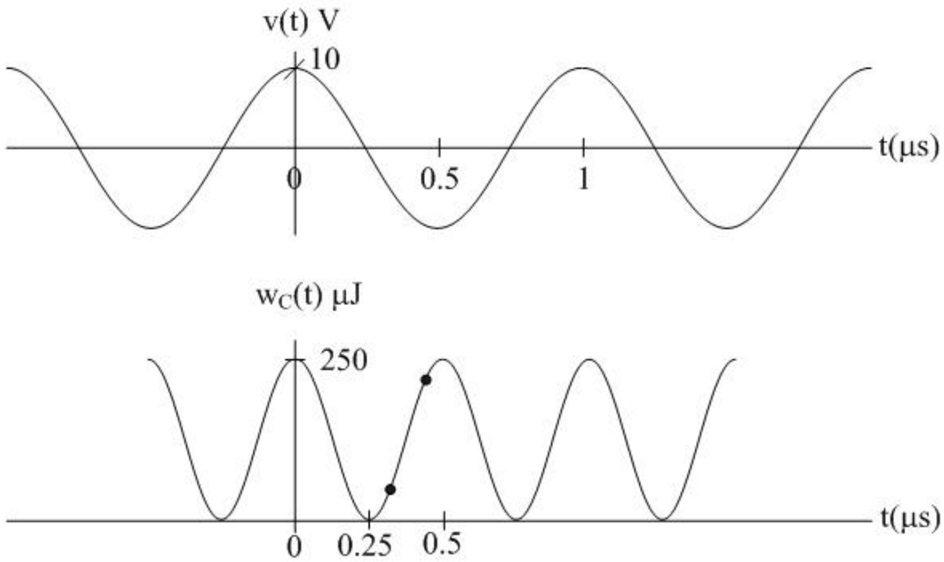


Figure 8.17 The voltage across a 5 μF capacitor and the energy stored in its electric field. Markers are also present that approximate the calculation points for calculating the energy absorbed from 255 ns to 450 ns

Solution: The energy absorbed by the capacitor C from $t_1 = 300\ ns$ to $t_2 = 450\ ns$ is calculated by taking the difference of the energy stored in C at time t_2 and at time t_1. Specifically,

$$w_{abs_C}(300\ ns < t < 450\ ns) = w(450\ ns) - w(300\ ns) \tag{8.8}$$

$$= 250 \cdot 10^{-6}\left[\cos^2\left((2\pi \cdot 10^6)(450 \cdot 10^{-9})\right) - \cos^2\left((2\pi \cdot 10^6)(300 \cdot 10^{-9})\right)\right] = \boxed{202.3\ \mu J}$$

It is critically important to keep track of the units of the components in an equation. For example, the arguments of the cosine functions above have units of radians, and students often mistakenly perform the calculations with their calculators in degree mode.

Polarized vs. Unpolarized Capacitors

Quite often schematics will show symbols of polarized capacitors. The symbol for a polarized capacity looks like the symbol for a general capacitor but one of the plates is shown curved as in Figure 8.19.

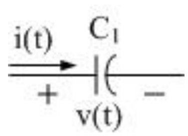


Figure 8.19 Polarized capacitor labeled with the current flowing in the direction of the voltage drop.

Polarized capacitors require that the node voltage at the positive terminal is greater than the node voltage at the negative terminal in order for the capacitor to perform properly. The most common type of polarized capacitor is the electrolytic capacitor. Most capacitors are non-polarized and they can be inserted with the leads in any orientation with no change in behavior.

8.6 Physical Relationship between Conduction Current and Displacement Current

Consider an independent voltage source connected across the terminals of a capacitor as shown in figure 8.20.

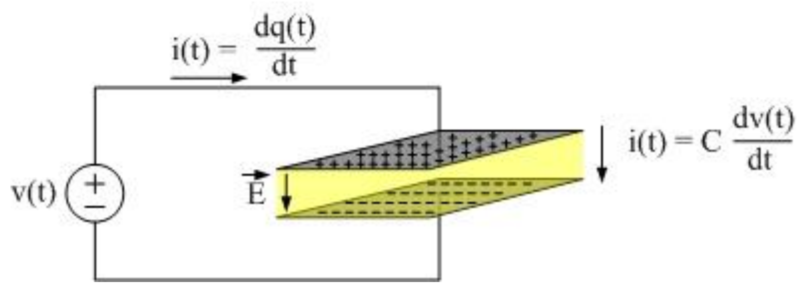


Figure 8.20 Independent voltage source connected across the terminals of a capacitor.

If the voltage v(t) across the capacitor is some constant positive value, the current through the circuit is zero. There is no charge moving in the wire and there is no change taking place in the charge stored on the plates of the capacitor. However, if the voltage v(t) is changing with time, the charge stored on the plates of the capacitor is changing with time. Therefore the electric field between the plates is changing with time because it is the difference in charge on the plates that causes the electric field. This time-varying electric field constitutes displacement current flowing between the plates of the capacitor. This displacement current must equal the conduction current flowing in the wire. Therefore, $i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$. Some people may argue that there is no current actually flowing through the capacitor. Displacement current is a time-varying electric field and is just as physically real as conduction current. Characterizing displacement current and completing Ampere's law was a significant contribution made by James Maxwell. Maxwell is also responsible for recognizing the set of four equations that govern all electromagnetic phenomena, and as a consequence, the set of four equations have collectively become known as Maxwell's equations. However, whenever someone refers to the singular equation, Maxwell's equation, they are referring to the current-voltage relationship for a capacitor.

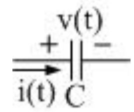
Chapter 9

RC Switched Circuits (Transient Analysis)

RC switched circuits may contain any number of switches, sources (possibly time-varying), resistors and capacitors. We restrict analysis to circuits with DC (constant) sources only and a single capacitor. The circuit to be analyzed will always be shown in its position prior to the first switch throwing, and it will be assumed that the circuit has been in that initial position long enough to assume steady-state conditions. Since we will be dealing with capacitance, it is appropriate to review Maxwell's law.

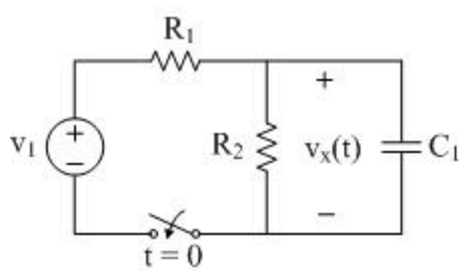
9.1 Maxwell's Law

Maxwell's law states that the current through a capacitor is the time rate of change of the voltage drop in the same direction.



As previously discussed, there are two immediate consequences of Maxwell's law. First, a capacitor is an open circuit for DC. If the voltage across a capacitor is constant, the current through it is zero, and the capacitor is an open circuit. Second, the voltage across a capacitor must be continuous. If the voltage were discontinuous, the current would be infinite at the time of the discontinuity in the voltage, which is impossible. Therefore, the voltage across a capacitor must be continuous. Since all sources are DC, and assuming the circuit is in steady-state prior to the first switch throwing, all capacitors in the circuit are open circuits. With all capacitors open circuits, the circuits reduce to DC resistive circuits, which have already been analyzed. Solving for the capacitor voltage gives the initial voltage across the capacitor. Then one or more switches are thrown and eventually the circuit will reach steady-state after the last switch is thrown. Once this state has been reached, the capacitor once again is an open circuit.

Example 9.1



$v_1 = 10$ V
$R_1 = 80\ \Omega$
$R_2 = 20\ \Omega$
$C_1 = 50\ \mu$F

Calculate and graph $v_x(t)$ for all time.

Solution: The way to begin all RC switched circuits is to state that **the circuit is a DC circuit in steady-state**, **and therefore the capacitor is an open circuit**. **The current through the capacitor is zero**. When solving RC switched circuits, this is always the way to begin. Write down the bolded phrases every time.

Returning to the example: The unknown voltage $v_x(t)$ is across both R_2 and C_1. With the switch open, the current through the capacitor is the current through R_2. If the current through R_2 is zero, $v_x(t) = 0$ by Ohm's law. This is the initial condition: $v_x(t \leq 0) = 0$. Notice that equality must hold at $t = 0$ because the **voltage across the capacitor must be continuous**.

Now draw the circuit that exists after the switch is thrown (take the time to do this!).

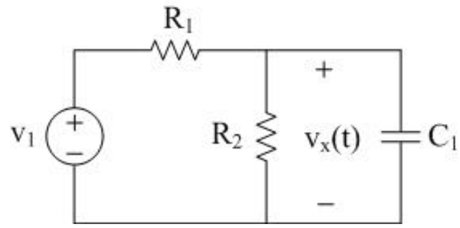


Figure 9.1 Circuit of example 9.1 that exists for $t \geq 0$.

This is a DC circuit, but it is not in steady-state initially. By the NVM and applying KCL at the top right node, the following equation results:

$$\frac{v_x(t) - v_1}{R_1} + \frac{v_x(t)}{R_2} + C_1 \frac{dv_x(t)}{dt} = 0 \tag{9.1}$$

(9.1) is a first order differential equation with constant coefficients. We will depart from example 9.1 temporarily and examine the solution procedure for a general first order DE. Then we will return to the example problem.

9.2 General Solution to 1st Order Differential Equations with Constant Coefficients

The general, standard form of a first order differential equation (DE) with constant coefficients is

$$y'(t) + ay(t) = b \tag{9.2}$$

where $y'(t) \triangleq \frac{dy(t)}{dt}$ is short hand notation for time differentiation, and a and b are both constants. The constant b is the forcing function. Therefore a constant will solve the DE. What constant will solve the DE? We will call this solution the **particular solution** or **steady-state solution** and use the symbol A to represent it. To find A, plug A into the DE.

$$A' + aA = b \tag{9.3}$$

The derivative of a constant is zero. Therefore the first term is zero. Solving for A produces

$$A = \frac{b}{a} \tag{9.4}$$

Thus, the particular or steady-state solution can be obtained from inspection of the DE in standard form. A is not the only solution to the DE, however. What is the only function that can be differentiated and added to a scaled version of itself and yield a value of zero? The answer is a scaled **exponential** function, which will be denoted by

$$Be^{st} \tag{9.5}$$

where B and s are constants to be determined. (9.5) is referred to as the **complementary solution** or **transient solution**. Mathematicians call (9.5) the complementary solution because it complements the particular solution to produce the complete solution

$$y(t) = A + Be^{st} \tag{9.6}$$

Engineers call (9.5) the transient solution because it decays to zero at a rate determined by coefficient s of time t in the exponent. The word "transient" means "short lived." Thus, the complete solution for ***y(t)* is the sum of the steady-state solution and the transient solution**. To constructively find the complete solution of the DE, plug in complete solution form (9.6) into the DE (9.2).

$$(A + Be^{st})' + a(A + Be^{st}) = b \tag{9.7}$$

Performing the differentiation of the solution and grouping the exponential terms yields

$$Be^{st}(s + a) + aA = b \tag{9.8}$$

There are no exponential terms on the right side of the equation. Therefore, either $B = 0$ or $s = -a$. B cannot be equal to zero in the general case as this would mean that the transient solution is always zero. Therefore, $\boxed{\boldsymbol{s = -a}}$, another coefficient of the equation that can be written by inspection of the DE. This leaves B. B is always found from the initial condition

$$y(t_0^-) = y(t_0^+) = A + Be^{st_0^+} \tag{9.9}$$

The initial value of *y(t)* is assumed to be known. Therefore

$$B = \frac{y(t_0^-) - A}{e^{st_0^+}} \tag{9.10}$$

Putting this all together results in the complete solution for $y(t) = A + Be^{st}$.

9.3 Solving RC Switched Circuits

Returning to example 9.1 where we obtained the DE (9.1), which in standard form is

DE
$$v_x'(t) + \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_x(t) = \frac{v_1}{C_1 R_1} \tag{9.11}$$

which has the solution

$$v_x(t) = A + Be^{st} \tag{9.12}$$

By inspection of (9.11): $A = \frac{v_1}{1+\frac{R_1}{R_2}} = \frac{10}{1+\frac{80}{20}} = \boxed{2\,V}$ and

$$s = -a = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -\frac{1}{50^{-6}}\left(\frac{1}{80} + \frac{1}{20}\right) = \boxed{-1{,}250}$$

B is always found from the initial condition, which for this circuit is

$$v_x(0^-) = v_x(0^+) = A + Be^0 = 0$$

The only unknown in this equation is *B*. $B = v_x(0^-) - A = -A = \boxed{-2\,V}$
Putting this all together the complete solution is obtained.

$$v_x(t \geq 0) = A + Be^{st} = \boxed{2 - 2e^{-1{,}250t}\,V}$$

This result is valid for $t \geq 0$. The problem statement requested the calculation of $v_x(t)$ and a graph of the result for **all** time. $v_x(t \leq 0) = 0$, $v_x(t \geq 0) = 2 - 2e^{-1{,}250t}\,V$. The graph of $v_x(t)$ is shown in figure 9.2. Be sure to include the value of $v_x(t \leq 0)$.

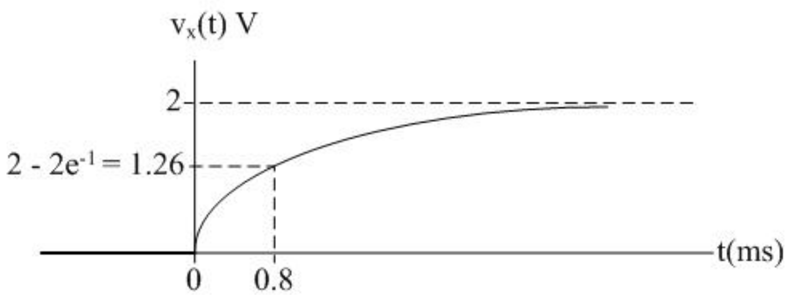


Figure 9.2 Graph of the solution to example 9.1 including the time constant coordinates.

9.4 Time Constant

The time constant is a convenient way for indicating how quickly an exponential decays. The **time constant** is the amount of time required for the exponential to drop to e^{-1} times its initial value. $e^{-1} \cong 0.368$. There is nothing particularly special about the time constant. It is simply a convenient way of indicating how rapidly, or slowly, the exponential decays. It is similar to the concept of half-life used in nuclear engineering. To obtain the value of the time constant, solve for the time required for the exponential to drop to a value of e^{-1} of its initial value. If you look at the transient part of the solution, it contains a decaying exponential. Its initial value is the coefficient of the exponential in the transient solution. In example 9.1, the initial value of the exponential is – 2 V. The time required for the exponential to drop to $-2e^{-1}$ is $\frac{1}{1{,}250} = 0.8$ ms. The **time constant coordinates must be shown on all graphs of transient solutions**. In general, any graph of a function must have enough information shown on the graph to be able to write the analytical expression. With knowledge of the time constant and the value of the voltage across the capacitor at that time, it is possible to write the exponential analytically.

9.5 Bridging Analytical Solutions with Physical Understanding of Circuits

There are several observations to be made from the solution of example 9.1. First of all, you can look at the circuit that exists for $t \geq 0$. Think about the state of the circuit as $t \to \infty$. The capacitor is an open circuit, so no current flows through it. The steady-state circuit reduces to a simple voltage division problem.

$$v_x\,(t \to \infty) = v_2 \frac{R_2}{R_1 + R_2} = 10\left(\frac{20}{100}\right) = \boxed{2\,V} \tag{9.13}$$

Therefore, you can check your analytical result by examining the final solution by basic circuit analysis (and your physical understanding of the circuit).

RC Time Constant

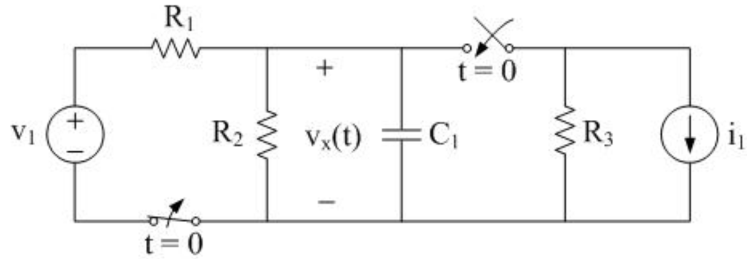
The RC time constant is a very important concept. It allows you to find the time constant of the circuit immediately by calculating the equivalent resistance seen by the capacitor when all the sources are turned off. The time constant is the product of the capacitance and the equivalent resistance seen by the capacitor. Turning off sources is not a new concept. We have used it in several contexts. Consider the capacitor in example 9.1. What is the resistance seen by the capacitor if all sources are turned off? The answer is R_1 in parallel with R_2. The RC time constant is calculated in (9.14).

$$\tau = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}\, C_1 = \frac{1}{\frac{1}{80} + \frac{1}{20}}\, 50 \cdot 10^{-6} = \ 0.8\, ms \tag{9.14}$$

precisely what we obtained from the analytical solution of the DE. This is further evidence that your physical understanding of the circuit being analyzed is extremely useful. In the next example problem, the Thevenin equivalent circuit on the left end of the circuit is removed from the circuit containing the capacitor at t = 0, while a Norton equivalent circuit is switched into the circuit at the same instant. While several aspects of the circuit undergo significant changes at t = 0, you can be confident that the voltage across the capacitor remains continuous through all the switching at t = 0. This is what allows you to solve the problem. If the problem were to solve for any other quantity than the voltage across the capacitor, you would have to **solve for the voltage across the capacitor first. This is because the voltage across the capacitor must be continuous.** Then, once you have the expression for the voltage across the capacitor, any quantity in the circuit can be obtained in terms of the voltage across the capacitor. If the objective is to solve for some other voltage or any current in the circuit, you may be able to get a DE in that unknown. But when you solve the DE, you will have to enforce a continuity constraint to obtain the coefficient of the exponential. If you assume the quantity is continuous and it is actually discontinuous, a significant error will occur. But you know from Maxwell's law that the voltage across the capacitor must be continuous. That is why you must solve for the voltage across the capacitor first, no matter what the unknown is.

9.6 Continuation of Solving Examples of RC Switched Circuits

Example 9.2



$v_1 = 12$ V
$i_1 = 2$ mA
$R_1 = 4$ kΩ
$R_2 = 6$ kΩ
$R_3 = 12$ kΩ
$C_1 = 2$ μF

Calculate and graph v_x(t) for all time.

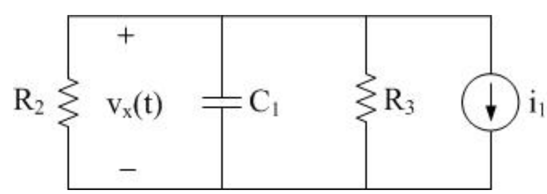
The first thing to do when solving RC switched circuits is to write down the following:

t < 0: This is a DC circuit in steady-state. Therefore, the capacitor is an open circuit, and no current flows through it. Then, treating the capacitor as an open circuit, solve for the voltage across it by solving the DC circuit for t < 0. For example 9.2 use voltage division,

$$v_x(t \leq 0) = v_1 \frac{R_2}{R_1 + R_2} = 12 \frac{6}{10} = \boxed{7.2\,V}$$

This is the initial condition for solving the DE for $t \geq 0$.

$t \geq 0$: Redraw the circuit that exists after the switches have both been thrown at t = 0.



Using the NVM and applying KCL at the top node produces

$$\frac{v_X(t)}{R_2} + C_1 v_x'(t) + \frac{v_X(t)}{R_3} + i_1 = 0 \tag{9.15}$$

The DE in standard form is

DE
$$v_x'(t) + \frac{1}{C_1}\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_x(t) = \frac{-i_1}{C_1} \tag{9.16}$$

Solution $$v_x(t) = A + Be^{st} \tag{9.17}$$

$$A = -i_1 \frac{1}{\frac{1}{R_2}+\frac{1}{R_3}} = -2\cdot 10^{-3}\left(\frac{1}{\frac{1}{6\cdot 10^3}+\frac{1}{12\cdot 10^3}}\right) = \boxed{-8\,V}$$

$$s = -a = -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = -\frac{1}{2\cdot 10^{-6}}\left(\frac{1}{6\cdot 10^3}+\frac{1}{12\cdot 10^3}\right) = \boxed{-125}$$

B is always obtained from the initial condition:

$$v_x(0^-) = v_x(0^+) = A + Be^0 \quad B = v_x(0^-) - A = 7.2 + 8 = \boxed{15.2\,V}$$

Putting it all together:

$$v_x(t \geq 0) = A + Be^{st} = \boxed{-8 + 15.2e^{-125t}\,V}$$

To check your work, the steady-state solution of the DE is the Ohm's law expression for $v_x(t \to \infty)$. The solution for $v_x(t \geq 0)$ at $t = 0^+$ is $v_x(t = 0^+) = -8 + 15.2e^{-(125)0^+} = \boxed{7.2\,V}$, which is precisely the initial condition. The RC time constant is the resistance seen by the capacitor with the source(s) turned off times the capacitance, which is equal to $(R_2||R_3)C_1 = \tau = \left(\frac{1}{\frac{1}{R_2}+\frac{1}{R_3}}\right)C_1 = \left(\frac{1}{\frac{1}{6\cdot 10^3}+\frac{1}{12\cdot 10^3}}\right)2\cdot 10^{-6} = \boxed{8\,ms}$, which is precisely $\frac{1}{125} = \boxed{8\,ms}$, the time constant by definition, the value of time required for the exponential to drop to e^{-1} times its initial value (15.2 V). This process of checking the analytical solution to the DE with the physical results of the initial and final values along with the RC time constant is extremely useful in developing confidence in your analysis.

The problem was to calculate and graph $v_x(t)$ for all time.

$$v_x(t \leq 0) = 7.2\,V \quad v_x(t \geq 0) = -8 + 15.2e^{-125t}\,V \tag{9.18}$$

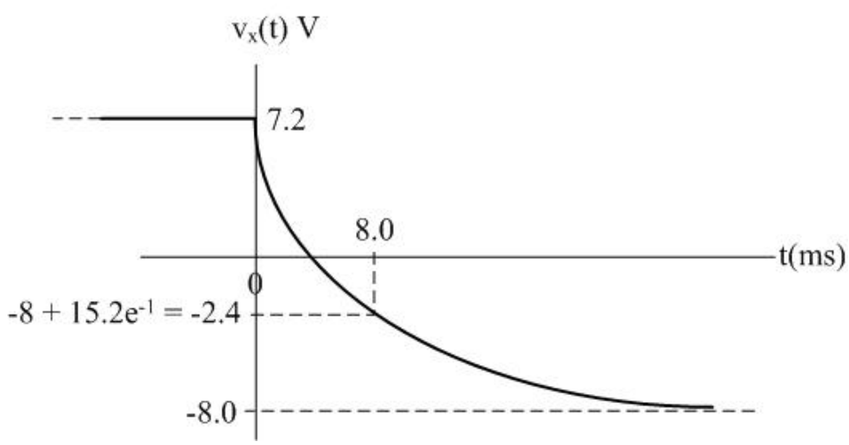
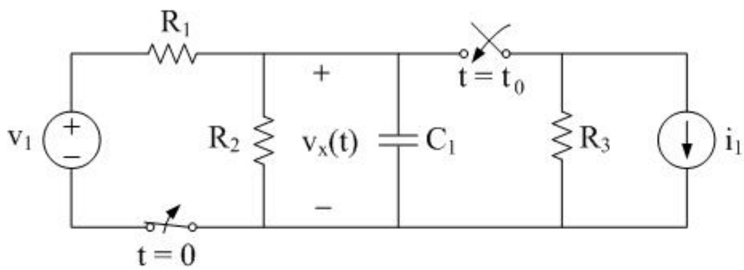


Figure 9.3 Graph of the solution to example problem 9.2 including the time constant coordinates.

Sequential Switching

An RC sequentially switched circuit has either a single switch that is thrown at more than one time or two or more switches that are thrown at different times. In example 9.3, there are two switches, one that is thrown at t = 0, and the other switch is thrown at

Example 9.3



$v_1 = 12$ V
$i_1 = 2$ mA
$R_1 = 4$ kΩ
$R_2 = 6$ kΩ
$R_3 = 12$ kΩ
$C_1 = 2$ μF

Given that the switching time is $t_0 = 20$ ms, calculate and graph v_x(t) for all time.

Solution: There are three regions of time to consider in this problem: $t \leq 0, 0 \leq t \leq t_0,$ and $t \geq t_0$.

$t \leq 0$ This is a DC circuit in steady-state. Therefore, the capacitor is an open circuit. The current through C_1 is 0. The circuit that exists is shown in figure 9.4.

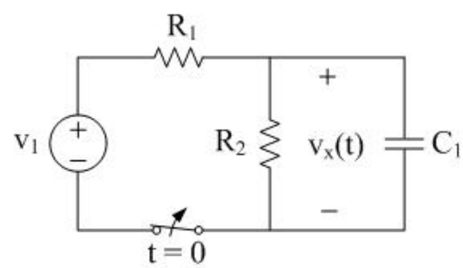


Figure 9.4 Circuit of example 9.3 that exists for t < 0.

$$v_x(t \leq 0) = v_1 \frac{R_2}{R_1 + R_2} = 12 \frac{6}{10} = \boxed{7.2\ V}$$

This is the initial condition for the next region of time.

$0 \leq t \leq t_0$ The circuit is not in steady state. The RC time constant is

$\tau = R_2C_1 = (6 \cdot 10^3)(2 \cdot 10^{-6}) = 12$ ms. The circuit will remain in this state for 20 ms, not quite two time constants. The circuit that exists that contains the capacitor in this time interval is simply R_2 in parallel in parallel with C_1 as shown in figure 9.5.

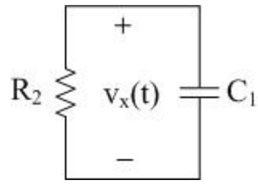


Figure 9.5 Circuit that exists in example 9.3 for the time interval $0 \leq t \leq t_0$.

The DE for this circuit is obtained by combining Maxwell's law with Ohm's law.

$$v_x(t) = -C_1 v_x'(t) R_2$$

When put into standard form, this DE has no forcing function, the equation is homogeneous.

DE
$$v_x'(t) + \frac{1}{R_2C_1} v_x(t) = 0 \tag{9.20}$$

Solution
$$v_x(t) = A + Be^{st} \tag{9.21}$$

$$A = 0 \qquad s = -\frac{1}{R_2C_1} = -\frac{1}{(6 \cdot 10^3)(2 \cdot 10^{-6})} = \boxed{-83.\dot{3}}$$

Note that the time constant, τ, is the negative inverse of s: $\tau = -\frac{1}{s} = R_2C_1$, which we already knew and calculated τ to be 12 ms.

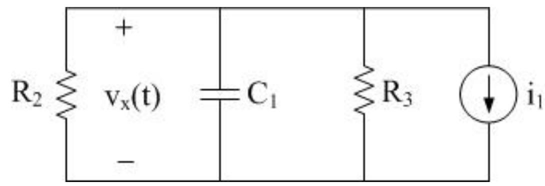
B: $v_x(0^-) = v_x(0^+) = A + Be^0 \quad B = v_x(0^-) - A = v_x(0^-) = \boxed{7.2V}$

solution
$$v_x(t) = Be^{st} = \boxed{7.2e^{-83.\dot{3}t}\ V} \tag{9.22}$$

Remember, this solution is valid only for 20 ms, when the second switch is thrown, which brings us to the third and last time interval.

$t \geq t_0 = 20\,ms$: In this region of time, the Norton equivalent circuit on the right side of the original circuit is suddenly switch into the capacitive circuit. Initially, the circuit is not in steady state. The solution procedure is nearly identical to example 9.2, but the initial condition is different and the switching time is not at $t = 0$.

Redraw the circuit that exists after the switches have both been thrown at t = 20 ms.



Using the NVM and applying KCL at the top node produces

$$\frac{v_X(t)}{R_2} + C_1 v_x'(t) + \frac{v_X(t)}{R_3} + i_1 = 0 \tag{9.23}$$

The DE in standard form is

DE
$$v_x'(t) + \frac{1}{C_1}\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_x(t) = \frac{-i_1}{C_1} \tag{9.24}$$

Solution
$$v_x(t) = A + Be^{st} \tag{9.25}$$

$$A = -i_1 \frac{1}{\frac{1}{R_2} + \frac{1}{R_3}} = -2 \cdot 10^{-3} \left(\frac{1}{\frac{1}{6 \cdot 10^3} + \frac{1}{12 \cdot 10^3}} \right) = \boxed{-8\, V}$$

$$s = -a = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -\frac{1}{2 \cdot 10^{-6}}\left(\frac{1}{6 \cdot 10^3} + \frac{1}{12 \cdot 10^3}\right) = \boxed{-125}$$

B is always obtained from the initial condition:

$$v_x(t_0^-) = v_x(t_0^+) = v_x(t = 20\, ms) = ?$$

We have to go back to the solution to the previous time interval and insert *t* = 20 ms in order to obtain the initial condition for this interval of time.

$$v_x(t = 20\, ms) = 7.2e^{-83.\dot{3}(20 \cdot 10^{-3})} = \boxed{1.36\, V}$$

This is the initial condition we need.

$$v_x(t_0^-) = v_x(t_0^+) = A + Be^{st_0^+} \quad B = \frac{v_x(t_0^-) - A}{e^{st_0^+}} = \frac{1.36+8}{e^{-125(0.02)}} = \boxed{114\, V}$$

Putting it all together for this interval of time:

solution $$v_x(t \geq 20\,ms) = A + Be^{st} = \boxed{-8 + 114e^{-125t}\ V} \qquad (9.26)$$

To test this result, substitute 20 ms to t and see if you get the initial condition of 1.36 V.

The problem was to calculate and graph $v_x(t)$.

$v_x(t \leq 0) = \boxed{7.2\ V}$

$v_x(0 \leq t \leq 20\,ms) = \boxed{7.2e^{-83.\dot{3}t}\ V}$

$v_x(t \geq 20\,ms) = \boxed{-8 + 114e^{-125t}\ V}$

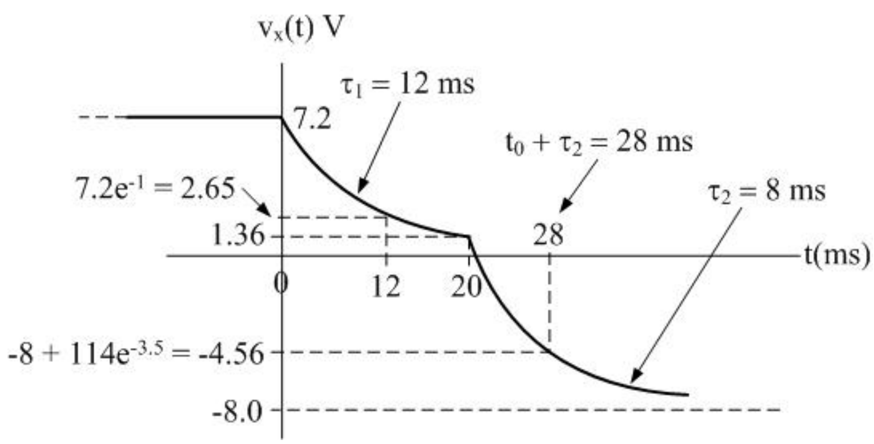


Figure 9.6 The graph of the complete solution to example 9.3. Note that the coordinates that must be shown in the third interval of time correspond to a time value of $t_0 + \tau_2$.

The coordinates associated with the time constant in the third time interval are not quite as simple to find as those in the second interval. The reason is that the switch is not thrown at t = 0. The time constant in the third time interval may still be obtained by inverting 125 (negative coefficient of t in the exponent) to get τ = 8 ms, but the time that must be used is $t = t_0 + \tau_2 =$ 28 ms. Substituting 28 ms into the equation $v_x(t \geq 20\,ms) = \boxed{-8 + 114e^{-125t}\ V}$ gives $-8 + 114e^{-3.5} = -4.56\,V$.

Solving the sequentially switched circuit is more complicated than solving a single switch throwing time at t = 0. The solution for the second time interval has to be evaluated at the second switching time to get the initial condition needed to solve the DE for the third time interval. Furthermore, when solving for the coefficient B in the DE, the exponential is not simply equal to one, but is the value e^{st_0}.

Chapter 10

Complex Numbers

In the next set of chapters, complex numbers are going to be used to represent AC voltages and currents as well as impedance. We will be transforming circuits from the time domain to the frequency domain, solve the circuit in the frequency domain, and then transform the result back to the time domain. Using complex numbers makes the solution of AC circuits much easier, but it will take practice to become proficient in using your calculator to do what are called rectangular to polar and polar to rectangular conversions.

This chapter deals with the definition and manipulation of complex numbers in preparation for solving AC circuits in the frequency domain

Complex numbers contain both a real part and an imaginary part. In mathematics the unit imaginary number is defined using the symbol $i = \sqrt{-1}$. However, since we use i in ECE to represent electric current, we use $j = \sqrt{-1}$ as the unit imaginary number. There are three forms that can be used to represent a complex number*, and fluency in conversion from one form to another is necessary for proceeding to the following chapters.

Complex Number Forms

One form for a complex number is **rectangular or Cartesian form**. An example is $\tilde{A} = 3 + j4$. The general rectangular form is $\tilde{A} = a_r + ja_i$. In the example the real part of $\tilde{A}$, $a_r = 3$ and the imaginary part of $\tilde{A}$, $a_i = 4$. Complex numbers can be interpreted geometrically as **vectors in the complex plane**. The complex plane has two orthogonal (perpendicular) axes. The horizontal is the real axis and the vertical axis is the imaginary axis. If we were to graph $\tilde{A} = 3 + j4$ in the complex plane, it would look like the vector shown in figure 10.1.

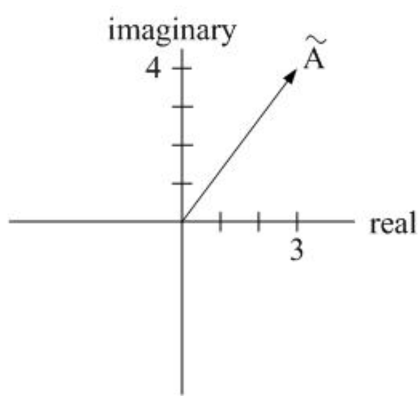


Figure 10.1 Vector interpretation of $\tilde{A} = 3 + j4$ in the complex plane.

*Complex numbers will be denoted by a letter with a tilde (~) above it.

Another form of a complex number is **polar form**. In polar form $\tilde{A} = 5\angle 53.1°$ as shown in figure 10.2. Note that the **angle is measured counter clockwise from the real axis**.

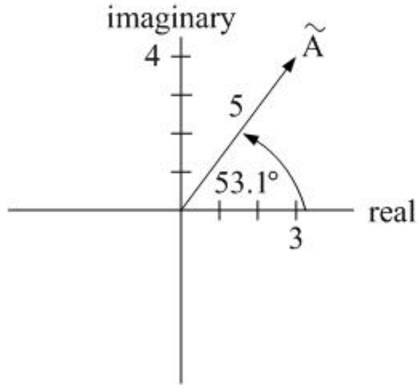


Figure 10.2 Vector interpretation of $\tilde{A} = 5\angle 53.1°$ in the complex plane.

Changing the form of $\tilde{A}$ from rectangular form to polar form is called a **rectangular to polar conversion** and is denoted by R→P. Can you think of the general mathematical operations that must be performed in order to go from $\tilde{A} = a_r + ja_i$ to $\tilde{A} = |\tilde{A}|\angle\theta_a$? $|\tilde{A}|$ is referred to the magnitude of $\tilde{A}$ and is interpreted as the length of the vector in the complex plane. The term "**magnitude**" is very important and is not to be confused with absolute value, which is only the case for real numbers. The mathematical operations that are necessary for R→P are given in (10.1).

R→P
$$|\tilde{A}| = \sqrt{(a_r)^2 + (a_i)^2} \qquad \theta_a = tan^{-1}\left(\frac{a_i}{a_r}\right) \tag{10.1}$$

Referring back to our example $\tilde{A} = 3 + j4 = 5\angle 53.1°$ is a R→P, and can be accomplished using the mathematical operations in (10.1). But there is a much easier way to perform this calculation on your calculator. Every calculator is different, so a procedure cannot be presented. You will have to look at your calculator's User's Guide, which is most likely online. You enter the real part and the imaginary part and hit the R→P key and the output is the polar form. The correct way to read 5∠53.1° is "5 through an angle of 53.1°.

What about going in the opposite direction? That is, from polar to rectangular P→R. Using the general symbols a_r, a_i, $|\tilde{A}|$, θ_a, try to write the mathematical operations necessary to perform a P→R. The operations are shown in (10.2).

P→R
$$a_r = |\tilde{A}|cos(\theta_a), \;\; a_i = |\tilde{A}|sin(\theta_a) \tag{10.2}$$

While the mathematical operations in (10.2) are correct, there is an easier way to perform a P→R and that is to use your calculator's conversion keys. If you use the operations of (10.1) and (10.2), you will be spending a lot of unnecessary time doing math calculations instead of solving circuits. This cannot be emphasized enough – learn how to use your calculator to perform conversions (R→P and P→R).

Polar form of a complex number $|\tilde{A}|\angle\theta_a$ is "short-hand" notation for the complex exponential $\tilde{A} = |\tilde{A}|\angle\theta_a = |\tilde{A}|e^{j\theta_a}$. To make this point clear, think of Euler's Identity. Have you ever seen Euler's Identity?

Euler's Identity is $\boldsymbol{e^{j\alpha} = cos(\alpha) + jsin(\alpha)}$. This identity can be proven easily by performing a Taylor series expansion of $cos(\alpha)$, and then the expansion for $jsin(\alpha)$ and add them together. You will find that the sum is exactly the Taylor series expansion of $e^{j\alpha}$. Euler's identity does two things for us. It provides a polar to rectangular conversion and shows that polar form is a short-hand for the complex exponential.

Arithmetic Operation of Complex Numbers

When adding or subtracting complex numbers, use rectangular form. Consider the addition/subtraction of $\tilde{A}$ and $\tilde{B}$ as shown in (10.3).

$$\tilde{A} \mp \tilde{B} = (a_r \mp b_r) + j(a_i \mp b_i) \tag{10.3}$$

The operation of addition or subtraction takes place on the real part and imaginary part separately.

Multiplication and division are most easily accomplished in polar form.

$$\tilde{A}\tilde{B} = (|\tilde{A}|\angle\theta_a)(|\tilde{B}|\angle\theta_b) = |\tilde{A}||\tilde{B}|\angle\theta_a + \theta_b \tag{10.4}$$

As shown in (10.4) the result of multiplying two complex numbers is the product of magnitudes through an angle that is the sum of the two angles $\theta_a + \theta_b$. Remember, polar form is a short-hand notation for the complex exponential. When you multiply two exponentials, you multiply the coefficients of the exponentials and add the exponents. Division of two complex numbers is shown in (10.5).

$$\frac{\tilde{A}}{\tilde{B}} = \frac{|\tilde{A}|\angle\theta_a}{|\tilde{B}|\angle\theta_b} = \frac{|\tilde{A}|}{|\tilde{B}|}\angle\theta_a - \theta_b \tag{10.5}$$

As shown in (10.5) the ratio of two complex numbers is the ratio of magnitudes through an angle of the numerator minus the angle of the denominator. Remember, polar form is a short-hand notation for the complex exponential. When taking the ratio of two exponentials, you take the ratio of the coefficients of the exponentials and take the angle of the numerator exponential and subtract the angle of the denominator exponential to get the angle of the resulting ratio.

Example 10.1

Given: $\tilde{A} = 3 + j4$ and $\tilde{B} = 4\angle - 120°$, Calculate a) $\tilde{C} = 2\tilde{A} - 3\tilde{B}$ b) $\widetilde{D} = \frac{3\tilde{A}}{2\tilde{B}}$

Solution: a) $\tilde{C} = 2\tilde{A} - 3\tilde{B} = 2(3 + j4) - 3(4\angle - 120°) = 6 + j8 - 12\angle - 120° =$
$6 + j8 - (-6 - j10.4) = \boxed{12 + j18.4}$

b) $\tilde{D} = \frac{3\tilde{A}}{2\tilde{B}} = \frac{3(3+j4)}{2(4\angle-120°)} = \frac{9+j12}{8\angle-120°} = \frac{15\angle53.1°}{8\angle-120°} = 1.875\angle173.1° = \boxed{-1.86 + j0.225}$

A **principal value angle** is any angle that satisfies either $-180° \leq \theta \leq 180°$ or $0 \leq \theta \leq 360°$. Some calculators use the first interval for principal value angles and some use the second interval. It really does not matter which angle interval you use, but an angle of $\theta = -624°$ is **not acceptable**. You should make a decision about which principal value you will use and be consistent in your work.

Once you learn how to use your calculator properly, specifically to use the R→P and P→R functions on your calculator, you will be able to do these calculations all internally in your calculator and go directly to the result without writing down the intermediate steps.

Example 10.2

Find all of the unique cubed roots of –*j*.

Solution: -j is the same quantity as $1\angle - 90°$. Think about a complex number such that when it is cubed its value is $1\angle - 90°$. For example, $1\angle - 30°$, which is one of the solutions. What about the other direction? The answer is j, which is the same thing as $1\angle90°$, which when cubed is $1\angle270° = 1\angle - 90° = -j$. What is the third root? The answer is $1\angle - 150°$. If you cube this number, you get $1\angle - 450°$, which is equal to $1\angle - 450° + 360° = 1\angle - 90° = -j$.

Example 10.3

What does multiplication by *j* do to a vector?

Solution: A general expression for a complex vector is $\tilde{A} = |\tilde{A}|\angle\theta_a$. When this vector is multiplied by $j = 1\angle90°$, the result is $|\tilde{A}|\angle\theta_a(1\angle90°)$ = $\left|\tilde{A}\right|\angle\theta_a + 90°$. Thus, multiplication by j rotates a vector counterclockwise by 90°.

Complex Conjugation

The complex conjugate of a complex number is obtained by replacing j by –j and is denoted by an asterisk superscript.

Example 10.4

Given $\tilde{A} = 4 + j3 = 5\angle36.9°$ Find $\tilde{A}^*$. Solution: $\tilde{A}^* = 4 - j3 = 5\angle - 36.9°$

Complex conjugation reflects the vector about the real axis.

Chapter 11

Sinusoids, Phasors and Impedance

Sinusoids

The term "AC" was originally an acronym for "alternating current," which referred to the sinusoidal current that was produced by electric generators that supplied electric power to the immense power grid that provided electric power to factories and residences. Until the transistor was invented in 1947, the only prominent waveforms were DC, the direct current supplied by batteries, and AC, the alternating current that was supplied by electrical generators whose rotors were spun at a rate of 60 revolutions per second, producing a 60 Hz AC power wave. Since the transistor was invented, many different waveforms became common such as square waves, triangle waves, saw tooth waves, and essentially all types of waves. The term AC evolved into a word synonymous with the word "sinusoidal." We often speak of AC voltages and currents. If the term "AC" were taken literally as an abbreviation for "alternating current," it would appear either contradictory or redundant. But if the term "AC" is recognized as its evolved definition, "sinusoidal," the contradiction and redundancy evaporate. In the context of this textbook, the term "AC" is used synonymously with the term "sinusoidal."

Any AC voltage or current can be written in the form:

$$v(t) = V_{max}\cos(\omega t + \varphi_v)$$
$$i(t) = I_{max}\cos(\omega t + \varphi_i)$$

where V_{max} and I_{max} are called the **amplitude** of the sinusoids. The amplitude is the maximum value of the sinusoid and the negative of the minimum. Every sinusoid has an **angular frequency** ω, which is the coefficient of time t in the argument of the cosine function. ω is a lower case omega in the Greek alphabet, and should never be referred to as "w" in the English alphabet. The angular frequency is the number of **radians per second** the sinusoidal wave undergoes each second. Electrical and computer engineers much more frequently communicate the frequency of the wave f in Hz, or cycles per second. The relationship between the angular frequency ω and the frequency f is given by (13.1).

$$f = \frac{\omega}{2\pi} = \frac{1}{T} \tag{11.1}$$

(13.1) also introduces the period T, which is the amount of time required for the sinusoid to complete one cycle. You should practice looking at expressions of sinusoids and asking yourself, "What is the frequency and period of that waveform?" It will not be long before you can look at a mathematical expression for a sinusoid and immediately tell what the frequency and period are. This is a very useful ability, and you should focus your attention on developing the ability to identify the frequency and period of a sinusoid by inspection until you are fluent in ability. The third parameter in a sinusoid is the **phase angle**, usually given in degrees, that identifies when the maximum near t = 0 occurs. A cosine function is maximum when the argument is equal to zero. By setting the argument equal to one and solving for the time *t* when this occurs yields the time of a maximum. If you wanted to find the time of all maxima of the sinusoid, you could set the argument of the cosine function equal to $n2\pi$, n integer, and solve for the times that make this true. One must be careful, however, since the product of ω and *t* gives an angle measured in radians. There is an incompatibility of units if the phase angle is written in degrees, which it often is. When setting the argument of the cosine function equal to zero and solving for time t when this occurs, it will be necessary to convert the radian term to degrees or the degree term to radians. When graphing a sinusoid, you must show enough information on your graph that is necessary to write the analytical expression for the sinusoid. In general, this means showing the time of two maxima and the value of the sinusoid at t = 0. Knowing the time of two maxima enables one to observe the period and write the angular frequency, and knowing the value of the function at t = 0 coupled with the time of the maximum nearest t = 0 enables one to write the phase angle.

Example 13.1 Graphing a Sinusoid

Accurately graph the AC voltage $v(t) = 100\cos(2\pi \cdot 10^6 t + 120°)$ V.

Solution: By inspection, the angular frequency is $\omega = 2\pi \cdot 10^6\ \frac{r}{s}$ and therefore the frequency $f = \frac{\omega}{2\pi} = 1$ MHz and the period $T = \frac{1}{f} = 1\ \mu$s. The time of the maximum near *t* = 0 is obtained by equating the argument of the cosine function with zero and solving for time *t*.

$$2\pi \cdot 10^6 t = -120°\left(\frac{\pi}{180°}\right) \quad t = -\frac{12}{36 \cdot 10^6} = -0.33\ \mu s \tag{11.2}$$

Thus, the maximum nearest t = 0 occurs at t = -0.33 μs. The phase shift is 120°, which is $\frac{1}{3}$ of a cycle. Since each cycle is $1\ \mu$s in duration, it should make sense that the time of the primary maximum occurs at -0.33 μs. You might wonder where the minus sign comes from. Remember that the cosine function is maximum when the argument of the cosine function is zero. With a positive phase shift, a **negative** value of time is required to make the argument equal to zero. Putting all of this information together produces the waveform in figure 11.1.

It is absolutely critical that all students of ECE are able to graph any AC voltage or current. It is often the case that students assume that they are able to accurately graph an AC voltage or current. However, it has been the author's experience that many students leave the course unable to produce an accurate AC waveform. The ability to produce accurate graphs of AC waveforms can only result from practice.

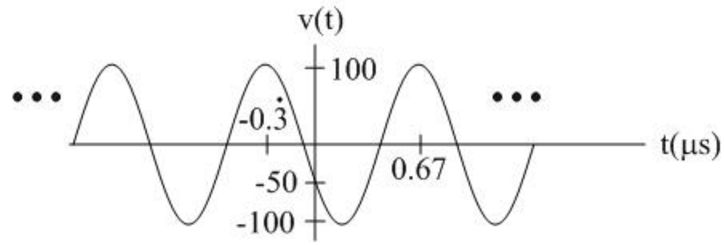


Figure 11.1 Graph of $v(t) = 100\cos(2\pi \cdot 10^6 t + 120°)$ V with all of the necessary information for writing the analytical expression for *v(t)* from the graph.

Phasors

Recalling the fact that any AC voltage or current can be written in the form:

$$v(t) = V_{max}\cos(\omega t + \varphi_v)$$
$$i(t) = I_{max}\cos(\omega t + \varphi_i)$$

Using Euler's identity the expressions can be written as the real part of a complex exponential.

$$v(t) = V_{max}\cos(\omega t + \varphi_v) = Re(V_{max}e^{j(\omega t + \varphi_v)})$$
$$i(t) = I_{max}\cos(\omega t + \varphi_i) = Re(I_{max}e^{j(\omega t + \varphi_i)})$$

Where the real part of a complex quantity is written as $Re\{\cdot\}$. Using the laws of exponentials, we can write

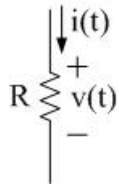
$$v(t) = Re\left(V_{max}e^{j(\omega t + \varphi_v)}\right) = Re\left(V_{max}e^{j\varphi_v}e^{j\omega t}\right) = Re\left(\tilde{V}e^{j\omega t}\right)$$
$$i(t) = Re\left(I_{max}e^{j(\omega t + \varphi_i)}\right) = Re\left(I_{max}e^{j\varphi_i}e^{j\omega t}\right) = Re\left(\tilde{I}e^{j\omega t}\right)$$

where the phasor voltage and phasor current are defined by

$\tilde{V} \triangleq V_{max}e^{j\varphi_v}$ and $\tilde{I} \triangleq I_{max}e^{j\varphi_i}$ or in polar coordinates $\tilde{V} \triangleq V_{max}\angle\varphi_v$ and $\tilde{I} \triangleq I_{max}\angle\varphi_i$. It is important to recognize that phasors are complex and independent of time. A phasor is the coefficient of $e^{j\omega t}$ in an AC voltage or current expressed as the real part of a complex function of time. Phasors are non-physical. They are a mathematical construct that will make calculations easier. Now that AC voltages and currents have been expressed in terms of their phasors, we need to relate the phasor current to the phasor voltage for a resistor and a capacitor. This will enable the solution of AC circuits in the frequency domain, and as we will see, all calculations for voltages and currents will require the solution of algebraic equations instead of differential equations, thereby making the solution process easier.

Current-Voltage Relationships in the Frequency Domain

Resistor:



Ohm's law: $v(t) = Ri(t)$ By moving the term on the right side to the left side we have $v(t) - Ri(t) \equiv_t 0$, which is to be read, "$v(t) - Ri(t)$ is identically equal to 0 for all time."

The objective here is to write Ohm's law in terms of Phasors by utilizing the relationships in (11.3).

$$v(t) = Re\left(\tilde{V}e^{j\omega t}\right) \quad (11.3)$$
$$i(t) = Re\left(\tilde{I}e^{j\omega t}\right)$$

By using the relationships in (13.3) and Ohm's law as written in (11.4)

$$v(t) - Ri(t) \equiv_t 0 \quad (11.4)$$

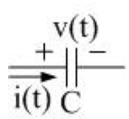
Ohm's law can be rewritten as

$$Re\left\{\left(\tilde{V} - R\tilde{I}\right)e^{j\omega t}\right\} \equiv_t 0 \quad (11.5)$$

Since the left side of (11.5) must be identically equal to zero for all time, the only possible way the equation can be true is for

$$\tilde{V} - R\tilde{I} = 0 \quad or \quad \tilde{V} = R\tilde{I} \quad (11.6)$$

The development of the equations leading to (11.6) lead to the conclusion that Ohm's law in the time domain is transforms to Ohm's law in the frequency domain. If this were the extent of using complex phasors, there would be no reason to use frequency domain analysis. We now examine the current-voltage behavior for a capacitor in the frequency domain.



Maxwell's law states that $i(t) = C\frac{dv(t)}{dt}$. Moving the right hand side to the left side of the equation, we have $i(t) - C\frac{dv(t)}{dt} \equiv_t 0$. Writing this equation using the relationships in (11.3) produces (11.7).

$$Re\left\{\left(\tilde{I} - j\omega C\tilde{V}\right)e^{j\omega t}\right\} \equiv_t 0 \quad (11.7)$$

The real part of $\left\{\left(\tilde{I} - j\omega C\tilde{V}\right)e^{j\omega t}\right\}$ is equal to zero at certain moments in time, when the vector is pointing straight up or straight down in the complex plane, but (11.7) requires the real part of

$\{(\tilde{I} - j\omega C\tilde{V})e^{j\omega t}\}$ to be identically equal to zero for all time. The only way that can happen is if the length of the vector coefficient of $e^{j\omega t}$ is zero for all time. The only way that can happen is if

$$\tilde{I} - j\omega C\tilde{V} = 0 \quad or \quad \tilde{V} = \frac{1}{j\omega C}\tilde{I} \tag{11.8}$$

(11.8) is an "Ohm's law" type of relationship between phasor voltage and phasor current. Specifically, the phasor voltage and current have an algebraic relation instead of a derivative relationship. As a consequence, solving circuits containing multiple capacitors in the frequency domain require the solution of linear algebraic equations instead of high order differential equations with sinusoidal forcing functions. It is useful to summarize what has been developed in this chapter.

Time Domain	**Frequency Domain**
$v(t) = V_{max}\cos(\omega t + \phi_v)$	$\tilde{V} = V_{max}\angle\phi_v$
$i(t) = I_{max}\cos(\omega t + \phi_i)$	$\tilde{I} = I_{max}\angle\phi_i$

Figure 11.2 AC voltages expressed in the time domain as sinusoidal functions of time and in the frequency domain as complex phasors. The time domain sinusoidal voltages and currents expressed as a function of time are physical and real. The phasors are complex mathematical constructs.

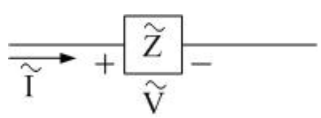


Figure 11.3 General notation for an element in the frequency domain expressed as its impedance. The general relationship between phasor voltage and phasor current is an algebraic "Ohm's law" type of relation: $\tilde{V} = \tilde{Z}\tilde{I}$.

Time Domain Relationship	**Frequency Domain Relationship**
Resistor R: $v(t) = Ri(t)$	Impedance R: $\tilde{V} = R\tilde{I}$
Capacitor C: $i(t) = C\,\frac{dv(t)}{dt}$	Impedance $\frac{1}{j\omega C}$: $\tilde{V} = \frac{1}{j\omega C}\tilde{I}$

Figure 11.3 Current-voltage relationships for resistors and capacitors in both the time domain and frequency domains.

Impedance is a generalization of resistance. It is a measure of how difficult it is for current to flow in addition to the phase shift between the voltage across and current through an element.

The **impedance** of a resistor is its resistance $\tilde{Z}_R = R$, which is purely real and independent of frequency.

The **impedance** of a capacitor is $\tilde{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$, which is purely imaginary (negative) and inversely proportional to frequency. We already know that a capacitor is an open circuit for DC ($f = 0$). The impedance of a capacitor at $f = 0$ is infinite, which implies that no current can pass through it. The expression for the impedance of a capacitor at DC is consistent with what we already knew about its behavior at DC ($f = 0$), but it also tells us how a capacitor behaves for ***all*** frequencies. A capacitor is a frequency dependent electric valve. As the frequency increases, its impedance decreases, allowing more current to pass through it.

Impedance is complex. It is not a phasor, but the ratio of two phasors (ratio of phasor voltage to phasor current).

$$\tilde{Z} \triangleq \frac{\tilde{V}}{\tilde{I}} = \frac{V_{max}}{I_{max}} \angle \phi_v - \phi_i$$

The impedance of a capacitor is $\tilde{Z}_C = -j\frac{1}{\omega C} = \frac{1}{\omega C}\angle -90°$

What this tells us about a capacitor is that the magnitude of its impedance $\frac{1}{\omega C} = \frac{V_{max}}{I_{max}}$ and the phase of its impedance $-90° = \phi_v - \phi_i$. Now there is no mystery about the meaning of capacitive impedance being purely imaginary and negative. That simply means that there is a $-90°$ phase shift between the voltage across it and the current through it, independent of frequency. The magnitude of its impedance $\frac{1}{\omega C}$ tells us that the ratio $\frac{V_{max}}{I_{max}}$ is inversely proportional to frequency. At DC (f = 0), the impedance is infinite. As the frequency gets larger and larger, the impedance magnitude gets smaller and smaller, allowing more current to flow through it. As the f→∞, a capacitor becomes a short circuit and will allow as much current as the circuit can supply (until the capacitor fails and becomes an open circuit). As we shall see, impedance not only makes the solution of circuits easier, but it also provides insight into the physical behavior of an element or collection of elements.

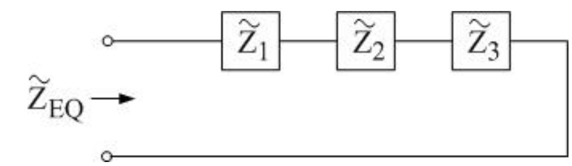
The expression of the impedance of a capacitor $\tilde{Z}_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$ implies that $\frac{1}{j} = -j$. This is easily proven. $\frac{1}{j} = \frac{1}{j}\left(\frac{-j}{-j}\right) = -j$.

What does it mean for a capacitance to have a purely imaginary impedance? What does the negative sign mean in its expression for impedance? These are questions that you should have and be able to answer now, but as we use phasors and impedance to solve circuits in the frequency domain, it will become even clearer.

Impedance Reduction and Admittance

Impedances combine in the same general way that resistors do. They add in series and their inverses add in parallel. The inverse of impedance is called **admittance** and its symbol is $\tilde{Y} \triangleq \frac{1}{\tilde{Z}}$.

Series
(share a common current)

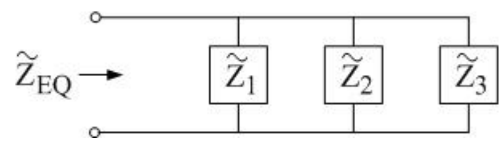


The equivalent impedance of any number of impedances in series is the sum of impedances. The only difference between adding impedances in series and adding resistances in series is that impedances are complex.

$$\tilde{Z}_{EQ} = \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3 \tag{11.9}$$

Impedances in series add.

Parallel
(share a common voltage)



$$\tilde{Z}_{EQ} = \frac{1}{\frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \frac{1}{\tilde{Z}_3}} \tag{11.10}$$

The equivalent **admittance** is the reciprocal of the equivalent impedance.

$$\tilde{Y}_{EQ} = \frac{1}{\tilde{Z}_{EQ}} = \tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3$$

Admittances in parallel add.

<u>Reactance</u>

In rectangular form, the real part of impedance is called resistance and the imaginary part of impedance is called **reactance**. The symbol for reactance is “X.”

$$\tilde{Z} = R + jX \tag{11.11}$$

In general, both the resistance R and reactance X are real functions of frequency.

Example 11.1

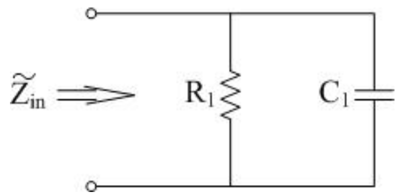


Figure 11.4 Parallel RC circuit to illustrate resistance and reactance.

Consider the circuit in figure 11.4. Suppose we want to know the input resistance as a function of frequency. We begin by finding the input impedance and then find the real part of the input impedance. Input impedance is the same thing as "equivalent impedance" or "Thevenin Impedance." All of these terms can be used interchangeably.

The first thing to do is to transform the circuit to the frequency domain.

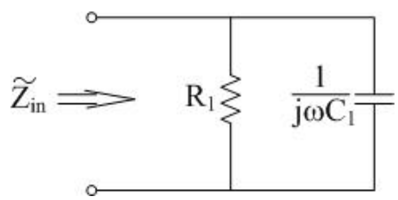


Figure 11.5 Circuit of example 11.1 in the frequency domain.

The input impedance is given by

$$\tilde{Z}_{in} = \frac{1}{\frac{1}{R_1} + j\omega C_1} \tag{11.12}$$

In order to separate this expression into its real and imaginary parts, the **denominator is rationalized**. To rationalize the denominator, multiply the numerator and denominator by the **complex conjugate** of the denominator.

$$\tilde{Z}_{in} = \frac{1}{\frac{1}{R_1} + j\omega C_1} = \frac{1}{\frac{1}{R_1} + j\omega C_1}\left(\frac{\frac{1}{R_1} - j\omega C_1}{\frac{1}{R_1} - j\omega C_1}\right) = \frac{\frac{1}{R_1}}{\frac{1}{R_1^2} + (\omega C_1)^2} + j\left(\frac{-\omega C_1}{\frac{1}{R_1^2} + (\omega C_1)^2}\right) \tag{11.13}$$

The input impedance has now been put into the form of $R_{in} + jX_{in}$ where

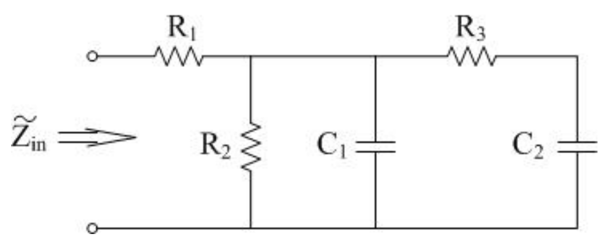
$$R_{in} = \frac{\frac{1}{R_1}}{\frac{1}{R_1^2} + (\omega C_1)^2} \qquad \text{and} \qquad X_{in} = \frac{-\omega C_1}{\frac{1}{R_1^2} + (\omega C_1)^2} \tag{11.14}$$

This example illustrates that resistance and reactance are, in general, both functions of frequency. The input resistance R_{in} is equal to R_1 at f = 0 (DC) because the capacitor is an open circuit. This is clearly the result when f = 0 is inserted into (11.14). As the frequency is increased, the input resistance monotonically and asymptotically approaches zero. This is consistent with our knowledge that the capacitor becomes a short circuit as f →∞.

The next example is somewhat more difficult. The key thing for you to remember is that impedances combine in the same manner that resistances combine in purely resistive circuits, except now you must deal with complex quantities called **impedances**.

Example 11.2

Consider the following circuit.



Can you determine the DC resistance of this circuit? It is $R_1 + R_2$ because the capacitors are open circuits at f = 0. Obtain the expression for the impedance of this circuit without manipulating the expression into its real and imaginary parts.

Solution: The circuit in the frequency domain is shown in figure (11.6).

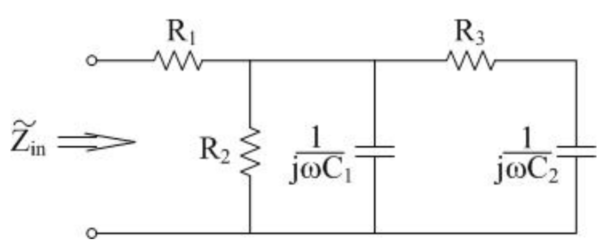


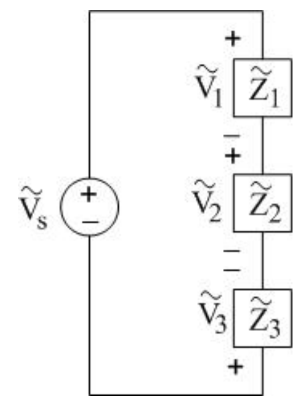
Figure 11.6 Circuit of example 11.2 in the frequency domain.

The way to think about this circuit is $\tilde{Z}_{in} = R_1 + \tilde{Z}_{eq}$, where $\tilde{Z}_{eq} = \frac{1}{\frac{1}{R_2}+j\omega C_1+\frac{1}{R_3+\frac{1}{j\omega C_2}}}$

$$\tilde{Z}_{in} = R_1 + \frac{1}{\frac{1}{R_2} + j\omega C_1 + \frac{1}{R_3 + \frac{1}{j\omega C_2}}} \tag{11.15}$$

One way to check your result is to substitute f = 0 into (11.15) and see if you get $\tilde{Z}_{in} = R_1 + R_2$. Do you? Looking back at the circuit, what do you think the impedance should be as $\omega \rightarrow \infty$? The answer is R_1. To obtain this result, recognize that as f,$\omega \rightarrow \infty$, the capacitors become short circuits. In your mind (or you can sketch on paper) imagine the capacitors being replaced with wires. When you look at the asymptotic behavior of the circuit as f,$\omega \rightarrow \infty$, you can see that C_1 shorts out R_2, leaving only R_1 in the circuit as seen from the input terminals. When you look at the analytical solution for $\tilde{Z}_{in}$ and take the limit as $\omega \rightarrow \infty$, the denominator of the second term goes to infinity, and hence the entire fraction goes to zero, which leaves only R_1. These types of checks, comparing the physical circuit to the analytical solution under asymptotic conditions helps identify errors, which you can correct as you observe inconsistencies between the physical circuit behavior and analytical expressions.

Voltage Division



Phasor voltage divides in proportion to impedance.

The phasor voltage $\tilde{V}_1$ is a fraction of the voltage $\tilde{V}_s$. The fraction is a ratio of the impedance $\tilde{Z}_1$to the total series impedance $\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3$.

$$\tilde{V}_1 = \tilde{V}_s\left(\frac{\tilde{Z}_1}{\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3}\right)$$

The phasor voltage $\tilde{V}_2$ is a fraction of the voltage $\tilde{V}_s$. The fraction is a ratio of the impedance $\tilde{Z}_2$ to the total series impedance $\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3$.

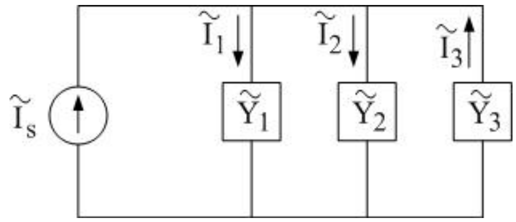
$$\tilde{V}_2 = \tilde{V}_s\left(\frac{\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3}\right)$$

The phasor voltage $\tilde{V}_2$ is a **negative** fraction of the voltage $\tilde{V}_s$. The fraction is a ratio of the impedance $\tilde{Z}_3$ to the total series impedance $\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3$.

$$\tilde{V}_3 = -\tilde{V}_s\left(\frac{\tilde{Z}_3}{\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3}\right)$$

Current Division

Phasor current divides in proportion to **admittance**.



The phasor current $\tilde{I}_1$ is a fraction of the current $\tilde{I}_s$. The fraction is a ratio of the admittance $\tilde{Y}_1$ to the total parallel admittance $\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3$.

$$\tilde{I}_1 = \tilde{I}_s \left(\frac{\tilde{Y}_1}{\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3} \right)$$

The phasor current $\tilde{I}_2$ is a fraction of the current $\tilde{I}_s$. The fraction is a ratio of the admittance $\tilde{Y}_2$ to the total parallel admittance $\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3$.

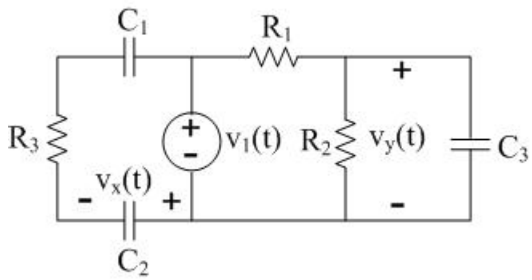
$$\tilde{I}_2 = \tilde{I}_s \left(\frac{\tilde{Y}_2}{\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3} \right)$$

The phasor current $\tilde{I}_3$ is a **negative** fraction of the current $\tilde{I}_s$. The fraction is a ratio of the admittance $\tilde{Y}_3$ to the total parallel admittance $\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3$.

$$\tilde{I}_3 = -\tilde{I}_s \left(\frac{\tilde{Y}_3}{\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3} \right)$$

The key thing to remember when using current division is that current divides in proportion to admittance. It applies only in the case when a current splits among pure conductances. **There can be no sources in the parallel branches**.

Example 11.3



$v_1(t) = 12\cos(200\pi t)\ V$
$R_1 = 100\ \Omega$
$R_2 = 200\ \Omega$
$R_3 = 75\ \Omega$
$C_1 = 15\ \mu F$
$C_2 = 10\ \mu F$
$C_3 = 12\ \mu F$

Use voltage division to find $v_x(t)$ and $v_y(t)$.

Solution: Voltage divides among series impedances. So the left-hand branch does not affect the currents or voltages in the portion of the circuit to the right of the voltage source and vice-versa. Transforming the circuit to the frequency domain yields

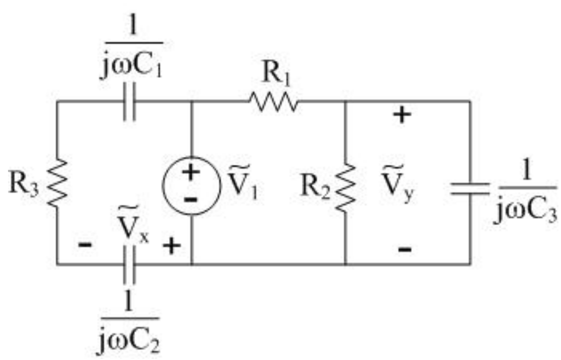


Figure 11.7 Circuit of example 11.3 in the frequency domain.

The voltage $\tilde{V}_x$ is a **negative** fraction of the voltage $\tilde{V}_1$. The fraction is a ratio of the impedance $\frac{1}{j\omega C_2}$ to the series impedance $\frac{1}{j\omega C_1} + R_3 + \frac{1}{j\omega C_2}$.

$$\tilde{V}_x = -(12)\frac{\frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + R_3 + \frac{1}{j\omega C_2}} = j\frac{12}{\omega C_2 R_3 - j\left(1 + \frac{C_2}{C_1}\right)} \tag{11.16}$$

Check units, then substitute values for the symbols if the units check,

$$\tilde{V}_x = j\frac{12}{200\pi(10^{-5})75 - j\left(1 + \frac{10}{15}\right)} = 6.9\angle 164^\circ\ V \tag{11.17}$$

Transforming $\tilde{V}_x$ back to the time domain: $v_x(t) = \boxed{6.9\cos(200\pi t + 164°)\ V}$.

$v_y(t)$: Using the schematic in the frequency domain in figure 11.7, $\tilde{V}_y$ is a positive fraction of $\tilde{V}_1$. The fraction is a ratio of the impedance $\frac{1}{\frac{1}{R_2}+j\omega C_3}$ to the series impedance $R_1 + \frac{1}{\frac{1}{R_2}+j\omega C_3}$. The reason why the numerator is the parallel combination of R_3 and the impedance of C_3 is because the voltage $\tilde{V}_Y$ is across the parallel impedance, **not just one impedance or the other**. When you use voltage division, be careful to ensure that you have the complete impedance the voltage is across in the numerator and the complete series impedance in the denominator and that you do **not** include impedances that are not part of the series impedance. The complete expression for $\tilde{V}_y$ is

$$\tilde{V}_y = \left(+\tilde{V}_1\right)\frac{\frac{1}{\frac{1}{R_2}+j\omega C_3}}{\frac{1}{\frac{1}{R_2}+j\omega C_3}+R_1} = \frac{V_1}{1+\frac{R_1}{R_2}+j\omega R_1 C_3} \tag{11.18}$$

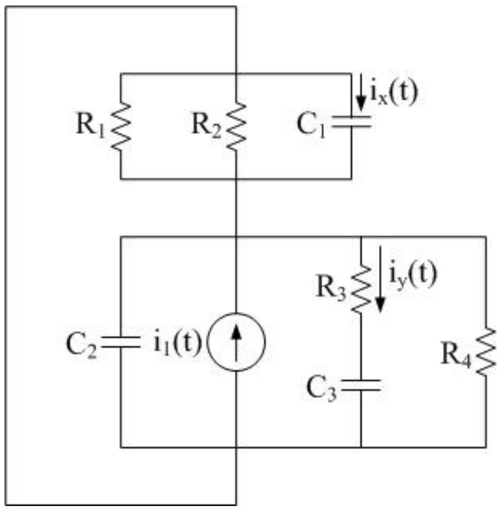
After checking units for consistency and correctness, substitute numerical values for the symbols and perform the complex arithmetic.

$$\tilde{V}_y = \frac{12}{1+\frac{100}{200}+j200\pi(100)12\cdot 10^{-6}} = 7.15\angle -26.7°\ V \tag{11.19}$$

Transforming back into the time domain: $\boxed{v_y(t) = 7.15\cos(200\pi t - 26.7°)\ \ V}$

In Example 11.4, current division is illustrated.

Example 11.4



$i_1(t) = 10\cos(400\pi t)\ \ mA$
$R_1 = 200\ \Omega$
$R_2 = 500\ \Omega$
$R_3 = 150\ \Omega$
$R_4 = 300\ \Omega$
$C_1 = 2.5\ \mu F$
$C_2 = 3\ \mu F$
$C_3 = 1.75\ \mu F$

Given the element values, use current division to find $i_x(t)$ and $i_y(t)$.

This is an AC circuit, therefore we will transform the circuit to the frequency domain and use current division to solve for both $\tilde{I}_x$ and $\tilde{I}_y$ since the current source $\tilde{I}_1$ is driving several admittances in parallel with it.

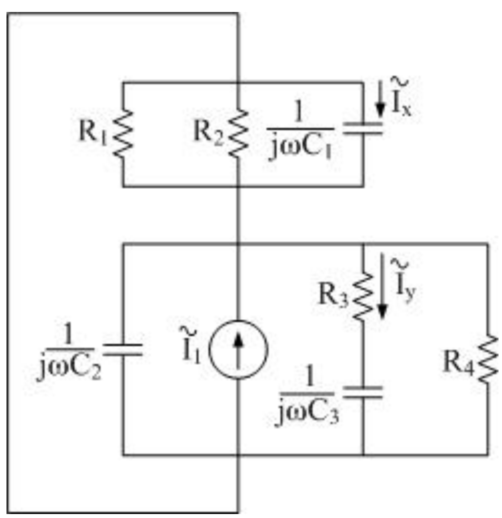


Figure 11.8 Circuit in example 11.4 in the frequency domain.

The first observation you need to make is every element in this circuit is in parallel. This circuit has only two nodes, therefore all of the elements **must** be in parallel. To see this more clearly, the circuit is redrawn in figure 11.9.

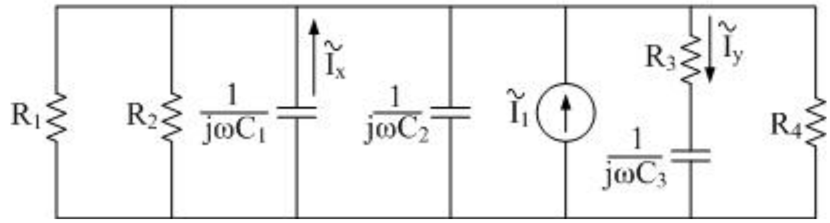


Figure 11.9 Circuit of example 11.4 in the frequency domain redrawn to visualize all branches of the circuit are in parallel.

The next observation to make is whether $\tilde{I}_x$ and $\tilde{I}_y$ are positive or negative fractions of $\tilde{I}_1$. At first glance of figure 11.8, it may appear that $\tilde{I}_x$ and $\tilde{I}_y$ are in the same direction, therefore they should have the same sign. This is an **error**. There are two distinct portions of the circuit, an "upper portion" containing R_1, R_2 and C_1, and a "lower portion" containing R_3, R_4, $C_2$and C_3 and the current source $\tilde{I}_1$. Imagine the upper portion being slid to the left and down to produce the circuit shown in figure 11.9. As you imagine this operation, consider the direction of $\tilde{I}_x$. $\tilde{I}_x$ points downward in figure 11.8 and as the upper portion of the circuit is slid to the left and down as performed in figure 11.9, for the direction of $\tilde{I}_x$ to remain the same as the original circuit, $\tilde{I}_x$ must point upward through the capacitive impedance of C_1.

Using the fundamental technique of current division, the following equations lead to the values of $\tilde{I}_x$ and $\tilde{I}_y$.

Referring to the circuit of 11.9, using current division to find $\tilde{I}_x$ and $\tilde{I}_y$ results in

$$\tilde{I}_x = -\tilde{I}_1 \frac{j\omega C_1}{\frac{1}{R_1} + \frac{1}{R_2} + j\omega C_1 + j\omega C_2 + \frac{1}{R_3 + \frac{1}{j\omega C_3}} + \frac{1}{R_4}} \tag{11.20}$$

$$= \frac{-j0.01(400\pi)(2.5 \cdot 10^{-6})}{\frac{1}{200} + \frac{1}{500} + \frac{1}{300} + j(400\pi)(5.5 \cdot 10^{-6}) + \frac{1}{150 + \frac{1}{j400\pi(1.75 \cdot 10^{-6})}}}$$

$$\tilde{I}_x = 2.2\angle - 129°\ mA \tag{11.21}$$

Transforming back to the time domain

$$\boxed{i_x(t) = 2.2\cos(400\pi t - 129°)\ \ mA} \tag{11.22}$$

Solving for $\tilde{I}_y$, it should be clear that $\tilde{I}_y$ is a positive fraction of the current $\tilde{I}_1$ and the fraction is a ratio of the admittance $\frac{1}{R_3+\frac{1}{j\omega C_3}}$ to the total parallel admittance $\frac{1}{R_1}+\frac{1}{R_2}+j\omega C_1+j\omega C_2+\frac{1}{R_3+\frac{1}{j\omega C_3}}+\frac{1}{R_4}$. Sometimes it helps to think about the admittance of a branch as being the inverse of the branch's impedance. For example, in the numerator of the current division for $\tilde{I}_y$ is the admittance $\frac{1}{R_3+\frac{1}{j\omega C_3}}$. That expression is nothing more than the inverse of the impedance of that branch. The impedance is obtained by adding the two impedances in series $R_3+\frac{1}{j\omega C_3}$. Then, to obtain the admittance, the impedance is inverted. Using current division to find $\tilde{I}_y$ is shown in (11.22).

$$\tilde{I}_y = \tilde{I}_1 \frac{\frac{1}{R_3+\frac{1}{j\omega C_3}}}{\frac{1}{R_1}+\frac{1}{R_2}+j\omega C_1+j\omega C_2+\frac{1}{R_3+\frac{1}{j\omega C_3}}+\frac{1}{R_4}} \tag{11.22}$$

$$= (0.01)\frac{\frac{1}{150+\frac{1}{j400\pi(1.75\cdot 10^{-6})}}}{\frac{1}{200}+\frac{1}{500}+\frac{1}{300}+j(400\pi)(5.5\cdot 10^{-6})+\frac{1}{150+\frac{1}{j400\pi(1.75\cdot 10^{-6})}}} \tag{11.23}$$

$$\tilde{I}_y = 1.48\angle 33^\circ \ \ mA \tag{11.24}$$

Transforming back to the time domain:

$$\boxed{i_y(t) = 1.48\cos(400\pi t + 33^\circ) \ \ mA} \tag{11.25}$$

The complex arithmetic is not difficult to perform once you learn how to use your calculator's complex number form change denoted by $R \leftrightarrow P$ and keeping track of calculated values using your calculator's memory. A beginner will take about 15 minutes to calculate (11.23) and find themselves writing down several intermediate calculations. A person fluent in the use of their calculator can calculate (11.23) in a couple of minutes and will do so without writing anything down until the final numerical answer is obtained. The benefit of keeping all intermediate calculation results in the calculator is that the **full precision of the calculator is utilized**. Writing down intermediate calculation results and using those values to make a calculation limits the accuracy of the final result to the least accurate value written down.

Chapter 12

Frequency Response and Filters

Frequency Response

The frequency response of a circuit is the ratio of its phasor output divided by its phasor input. Frequency response is complex, but it is not a phasor. A phasor can be transformed back from the frequency domain to the time domain as a sinusoid. Frequency response has both a magnitude and a phase. The magnitude response tells you how large the amplitude of the output is relative to the input. The phase response tells you the phase of the output relative to the phase of the input.

Frequency response $\triangleq \frac{phasor\ output}{phasor\ input} = \widetilde{H}(\omega)$

The phasor output can be a voltage or a current as can the phasor input. There are four combinations of frequency response ratios all of which will be denoted $\widetilde{H}(\omega)$.

Filters

A filter is a frequency dependent circuit, usually characterized by its magnitude response (magnitude of the frequency response). The four most common types of filters are:

1. Low Pass Filter (LPF)
2. High Pass Filter (HPF)
3. Band Pass Filter (BPF)
4. Band Reject Filter (BRF)

Initially, we will concentrate on the first two filter types: LPF and HPF. An ideal LPF will have a perfectly flat passband response and zero response in the stopband as illustrated in figure 12.1. This means that sinusoids with frequencies below ω_{HP} are multiplied by a non-zero constant H_{max} and sinusoids with frequencies above ω_{HP} are multiplied by zero. It is impossible to create an ideal low pass filter (LPF), but a circuit can be constructed whose frequency response magnitude is arbitrarily close to the ideal LPF. Generally, cost goes up as well as circuit complexity as a filter's magnitude response is made closer to the ideal LPF. The angular frequency ω_{HP} is called the "half power frequency."

At "**half power**" the frequency response magnitude drops to $\frac{1}{\sqrt{2}}$ times the maximum response. In a resistor, power is proportional to the square of the voltage across it or the square of the current through it. If the power is reduced to $\frac{1}{2}$ of its maximum value, the output voltage or current is reduced to $\frac{1}{\sqrt{2}}$ times its maximum value. Using the **half power frequency** to delineate band edges is arbitrary. However, it is convenient and universally recognized as the metric for measuring band edges.

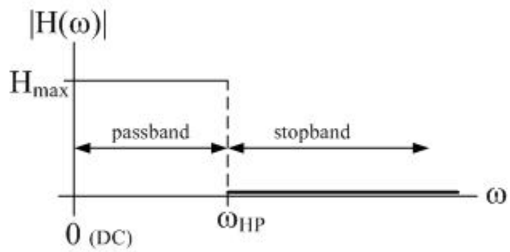
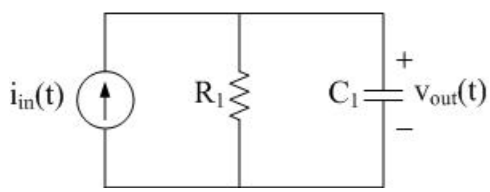


Figure 12.1 Frequency response magnitude for an ideal low pass filter (LPF).

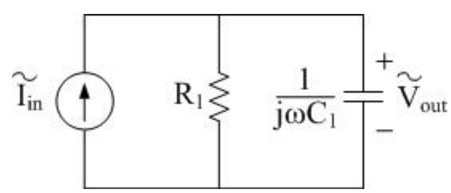
Example 12.1



$R_1 = 1\ k\Omega$
$C_1 = 80\ nF$

Calculate and graph the magnitude and phase of the frequency response.

Solution: The first thing to do is **transform the circuit into the frequency domain**.



Transforming the circuit to the frequency domain requires the circuit to be **redrawn** with phasors replacing the time domain voltages and currents and the impedances of the elements to be labeled on the circuit diagram. Notice that there are no “t’s” anywhere in the frequency domain circuit. Nothing in this circuit varies with time, and it is an egregious error to write any of the quantities as a function of time. Do **not** do this!

Using impedance reduction and “Ohm’s law” in the frequency domain, $\tilde{V}_{out}$ can be expressed in terms of $\tilde{I}_{in}$ and the frequency response can be written.

$$\tilde{V}_{out} = \tilde{I}_{in} \frac{1}{\frac{1}{R_1} + j\omega C_1} \tag{12.1}$$

Once the algebraic relationship between the output phasor quantity and the input phasor quantity has been established in terms of the symbols whose values are given, the frequency response may be expressed.

$$\widetilde{H}(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{I}_{in}} = \frac{1}{\frac{1}{R_1} + j\omega C_1} = \frac{R_1}{1 + j\omega R_1 C_1} \tag{12.2}$$

This is a complex function of frequency that determines the operation on the input phasor $\widetilde{I}_{in}$ to produce the output phasor $\widetilde{V}_{out}$.

Magnitude Response:

$$|\widetilde{H}(\omega)| = \frac{R_1}{\sqrt{1 + (\omega R_1 C_1)^2}} \tag{12.3}$$

This is a monotonically decreasing function of ω. When taking the magnitude of a ratio as in (12.2) – (12.3) the magnitude of the ratio is a ratio of magnitudes. The magnitude of the numerator is just R_1. The magnitude of the denominator is the square root of the sum of the squares of the real and imaginary parts. The "j" does not get squared!

Since (12.3) is a monotonically decreasing function of frequency, it is the magnitude response of a **low pass filter** (LPF).

At "half power" the frequency response magnitude drops to $\frac{1}{\sqrt{2}}$ times the maximum response. Can you look at (12.3) and determine the maximum response? It is R_1, which occurs at DC (f = 0). To find the half power angular frequency, we set the magnitude of the frequency response equal to $\frac{R_1}{\sqrt{2}}$ and solve for ω.

$$\frac{R_1}{\sqrt{1 + (\omega R_1 C_1)^2}} = \frac{R_1}{\sqrt{2}} \tag{12.4}$$

Squaring both sides and inverting gives

$$1 + (\omega R_1 C_1)^2 = 2 \tag{12.5}$$

$$\omega_{HP} = \boxed{\frac{1}{R_1 C_1}} \tag{12.6}$$

If you are interested in the half power frequency f_{HP}, just divide ω_{HP} by 2π.

$$f_{HP} = \frac{\omega_{HP}}{2\pi} = \boxed{\frac{1}{2\pi R_1 C_1}} = \frac{1}{2\pi \cdot 10^3 (80^{-9})} = \boxed{2\ kHz} \tag{12.7}$$

The numerical expression for the magnitude response (also called the gain) is presented in (12.8).

$$\left|\widetilde{H}(\omega)\right| = \frac{R_1}{\sqrt{1+(\omega R_1 C_1)^2}} = \frac{10^3}{\sqrt{1+\left(\omega 10^3(80\cdot 10^{-9})\right)^2}} \tag{12.8a}$$

$$\left|\widetilde{H}(\omega)\right| = \frac{10^3}{\sqrt{1+6.4\cdot 10^{-9}\omega^2}} \tag{12.8b}$$

Engineers generally think in terms of frequency in (Hz) rather that angular frequency in $\left(\frac{r}{s}\right)$. Changing variables from ω to f is easily performed by replacing ω to 2πf to obtain

$$\left|\widetilde{H}(f)\right| = \frac{10^3}{\sqrt{1+6.4\cdot 10^{-9}(2\pi f)^2}} = \frac{10^3}{\sqrt{1+6.4\cdot 10^{-9}(2\pi f)^2}} \tag{12.9a}$$

$$\left|\widetilde{H}(f)\right| = \frac{10^3}{\sqrt{1+6.4\cdot 10^{-9}(2\pi f)^2}} = \boxed{\frac{10^3}{\sqrt{1+253\cdot 10^{-9}f^2}}} \tag{12.9b}$$

A graph of the magnitude response is shown below in figure 12.2.

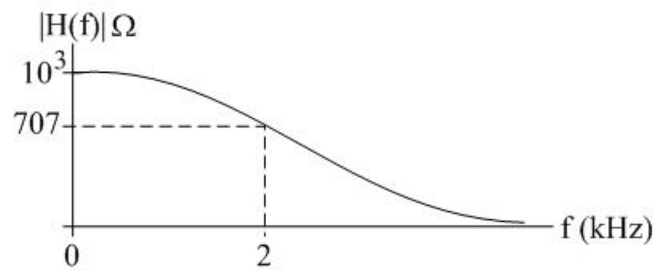


Figure 12.2 Frequency response magnitude (or gain) of the circuit in example 12.1.

The magnitude response (or gain) indicates how large the amplitude of the output will be relative to the input amplitude. For example, at DC (f = 0), if the input amplitude is 2 mA, then the output voltage will be 2 V. If the amplitude is kept the same (2 mA), but the frequency is changed to 2 kHz, the output voltage will be 1.41 V. As the frequency is increased, the output voltage amplitude decreases. This circuit passes low frequencies much better than high frequencies. That is why it is called a low pass filter (LPF).

Phase Response

The frequency response (12.2) has both a magnitude and a phase. We have already analyzed the magnitude. The phase response is the angle associated with the frequency response. Can you determine that angle of (12.2)? The angle of a fraction is the angle of the numerator minus the angle of the denominator. What is the angle of the numerator? The answer is 0. How do you determine the angle of the denominator? The angle of the denominator is the $tan^{-1}\frac{imag}{real}$. The phase of the frequency response is $\theta_H(\omega) = 0 - tan^{-1}(\omega R_1 C_1)$. The phase can also be expressed in terms of frequency $\theta_H(f) = 0 - tan^{-1}(2\pi f R_1 C_1) = -tan^{-1}(2\pi f 10^3 80 \cdot 10^{-9})$.

$$\theta_H(f) = \boxed{-tan^{-1}(503 \cdot 10^{-6} f)} \tag{12.10}$$

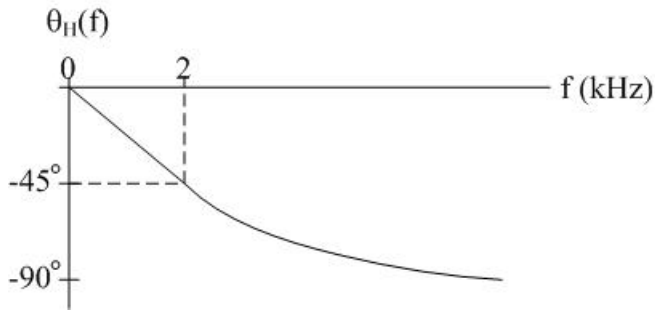
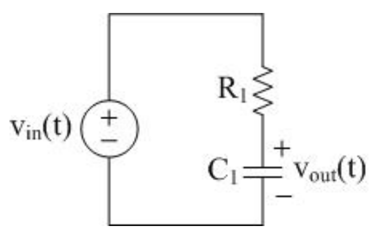


Figure 12.3 Phase response (12.10) of example 12.1.

The phase response indicates the phase of the output relative to the phase of the input as a function of frequency. If the phase of the input current at a frequency of 2 kHz is 22°, the output phase will be 22° - 45° = -23°.

Example 12.2 Series RC LPF

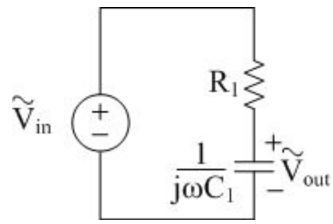


R1 = 100 Ω
C1 = 5 μF

Calculate and graph the frequency response magnitude and phase as a function of frequency f.

Solution: It is useful to be able to look at a circuit and be able to determine how the circuit behaves as a function of frequency purely by inspection. This is performed by understanding Maxwell's law in both the time domain and frequency domain. In the frequency domain, we know that the capacitor behaves as a frequency dependent valve. At DC (f = 0), the capacitor is an open circuit. As the frequency increases, the impedance of the capacitor drops and eventually as f →∞, the capacitor becomes a short circuit, and $v_{out}(t)$ → 0. If you piece all this information together, you will determine that this is a LPF, purely by inspection of the circuit. You know that a capacitor is an open circuit for DC. Therefore, the current flowing through all three elements is zero at f = 0. Therefore the voltage drop across the resistor R_1 is zero. Therefore $v_{out}(t) = v_{in}(t)$, making the gain 1 at DC. As the frequency increases, the capacitor allows more current to flow and the voltage across the resistor grows as the frequency is increased. The output voltage is the input voltage minus the voltage drop across the resistor. As f→∞, the output voltage continues to decrease until all of the input voltage drops across the resistor, leaving the output voltage zero.

We will now analyze the circuit in detail. The first step is to transform the circuit into the frequency domain. All time domain voltages and currents become their phasor counter parts, and all of the passive elements are represented by their impedances. Remember, there is no time variation in the frequency domain. None of the quantities shown in the transformed circuit should have the variable “t” in them.



By voltage division in the frequency domain:

$$\tilde{V}_{out} = \tilde{V}_{in}\left(\frac{\frac{1}{j\omega C_1}}{\frac{1}{j\omega C_1} + R_1}\right)$$

$$\tilde{H}(\omega) = \boxed{\frac{1}{1 + j\omega R_1 C_1}} \tag{12.11}$$

(12.11) is the simplified expression for the frequency response. We could write (12.11) in terms of frequency f, but will wait until we have the magnitude and phase expressions.

<u>Magnitude</u>

$$\left|\tilde{H}(\omega)\right| = \frac{1}{\sqrt{1 + (\omega R_1 C_1)^2}} = \frac{1}{\sqrt{1 + (2\pi f R_1 C_1)^2}} \tag{12.12}$$

At “half power” $\left|\tilde{H}(\omega)\right|$ drops to $\frac{1}{\sqrt{2}}$ times the maximum gain, which in this case is 1. Solving for the half power frequency:

$$\frac{1}{\sqrt{1 + (2\pi f R_1 C_1)^2}} = \frac{1}{\sqrt{2}}$$

Inverting both sides and squaring:

$$(2\pi f R_1 C_1) = 1$$

$$f_{HP} = \frac{1}{2\pi R_1 C_1} = \frac{1}{2\pi(100)5 \cdot 10^{-6}} = \boxed{318\,Hz} \tag{12.13}$$

Compare the symbolic result for half power frequency in (12.13) with the symbolic result from example 12.1 in (12.7). They are identical. Why would this be?

The magnitude response is given by (12.12), which is repeated here for convenience.

$$\left|\tilde{H}(f)\right| = \frac{1}{\sqrt{1+(2\pi f R_1 C_1)^2}} = \frac{1}{\sqrt{1+(2\pi f(100)5 \cdot 10^{-6})^2}} = \boxed{\frac{1}{\sqrt{1+9.87 \cdot 10^{-6} f^2}}}$$

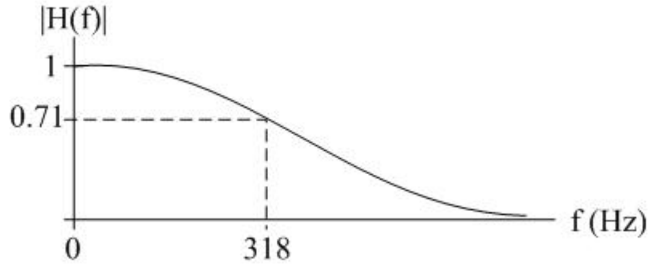


Figure 12.4 Magnitude response of the series RC LPF of example 12.2.

Phase Response

$$\theta_H(\omega) = 0 - tan^{-1}(\omega R_1 C_1) \quad \theta_H(f) = 0 - tan^{-1}(2\pi f R_1 C_1) \tag{12.14}$$

$$\theta_H(f) = -tan^{-1}(2\pi f(100)5 \cdot 10^{-6}) = \boxed{-tan^{-1}(3.14 \cdot 10^{-3} f)} \tag{12.15}$$

The graph of the phase is shown in figure 12.5.

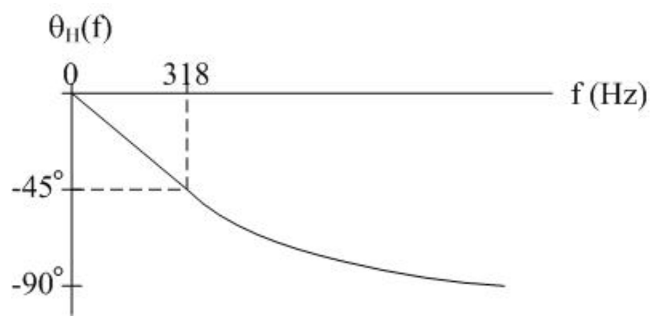
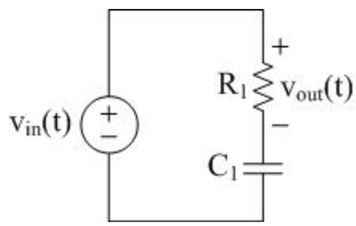


Figure 12.5 The phase response of the circuit in example 12.2. Note that it has the same shape and behavior as the phase response of the circuit in example 12.1.

In the third and final example of this chapter, we will again analyze the series RC circuit, but this time the output voltage is across the resistor instead of the capacitor. The circuit is shown in figure 12.6

Example 12.3 Series RC HPF



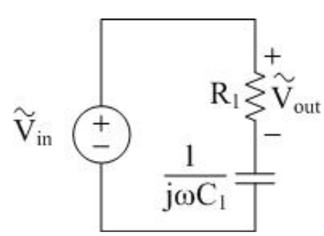
$R_1 = 100\ \Omega$
$C_1 = 50\ \mu F$

Figure 12.6 Time domain circuit of example 12.3 The same circuit as example 12.2, but the output voltage is taken across the resistor instead of the capacitor.

It is important that you are able to determine the behavior of a circuit by inspection. This provides insight into where the detailed analysis should lead you, and it develops your understanding of circuit elements and their behavior individually and collectively when connected in various ways with other circuit elements. When inspecting a circuit for its frequency response, identify the element whose behavior varies with frequency. In this circuit, it is the capacitor C_1. C_1 goes from being an open circuit (zero current) at DC and its impedance diminishes as frequency is increased. In the limiting case in which f → ∞, the capacitor becomes a short circuit (zero voltage) and $v_{out}(t) = v_{in}(t)$, and the gain is 1.

Go through the analysis in your head at you imagine $v_{in}(t)$ as a function generator where you can vary the frequency to whatever frequency you wish. Imagine keeping the amplitude of the input voltage fixed at 1 V. Can you piece all of the information together to determine that the circuit is a high pass filter? If not, that is OK. The ability to understand how a circuit behaves purely by inspection is not a trivial task. It requires critical, abstract thinking. But, at this point you should be able to analyze the frequency response of the circuit in detail. By the behavior of the gain, you should be able to determine the filter type.

We begin, as usual, by **transforming the circuit to the frequency domain**.



By voltage division in the frequency domain:

$$\tilde{V}_{out} = \tilde{V}_{in}\frac{R_1}{R_1 + \frac{1}{j\omega C_1}}$$

The frequency response as a function of frequency f is given by (12.16)

$$H(f) = \frac{1}{1 - j\frac{1}{2\pi f R_1 C_1}} \tag{12.16}$$

Magnitude Response

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{1}{2\pi f R_1 C_1}\right)^2}} \tag{12.17}$$

This is a monotonically increasing function of frequency.

At "half power" $|H(f)| = \frac{H_{max}}{\sqrt{2}} \rightarrow \frac{1}{\sqrt{1+\left(\frac{1}{2\pi f R_1 C_1}\right)^2}} = \frac{1}{\sqrt{2}}$ (since the maximum gain is 1)

Solving for f_{HP} (squaring and inverting):

$$1 + \left(\frac{1}{2\pi f R_1 C_1}\right)^2 = 2 \tag{12.18}$$

Continuing to solve for f_{HP}.

$$f_{HP} = \boxed{\frac{1}{2\pi R_1 C_1}} = \frac{1}{2\pi(100)5 \cdot 10^{-6}} = \boxed{318\ Hz} \tag{12.19}$$

The magnitude response is:

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{1}{2\pi f R_1 C_1}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{1}{2\pi f(100)5 \cdot 10^{-6}}\right)^2}} \tag{12.20}$$

Performing the numerical calculations results in (12.21).

$$|H(f)| = \frac{1}{\sqrt{1 + \frac{101 \cdot 10^{-3}}{f^2}}} \tag{12.21}$$

You can, of course, check your numerical result by substituting $f = 318\ Hz$ and seeing if you get the half power gain $\frac{1}{\sqrt{2}}$ in this case. Go ahead, plug in the half power frequency and check the result. A graph of the gain is presented in figure 12.7.

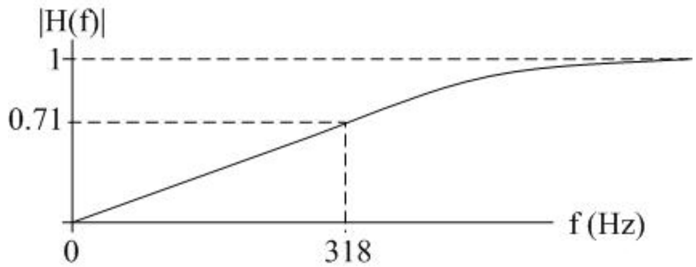


Figure 12.7 Magnitude of the frequency response of example 3. Notice that the function is almost linear from DC to the half power frequency. $|\widetilde{H}$(f)| asymptotically approaches unity as f → ∞. There is no inflection point in $|\widetilde{H}$(f)|.

Phase Response

The phase of the frequency response is the angle of (12.16). The angle of the numerator is zero. The negative of the angle of the denominator in (12.16) is the phase response and is given by

$$\theta_H(f) = tan^{-1}\left(\frac{1}{2\pi f R_1 C_1}\right) = tan^{-1}\left(\frac{1}{2\pi f (100) 5 \cdot 10^{-6}}\right) = tan^{-1}\left(\frac{318}{f}\right) \tag{12.22}$$

A graph of the phase response is shown in figure 12.8.

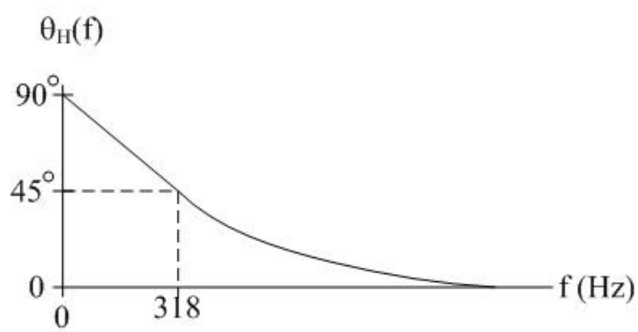


Figure 12.18 Phase response of the circuit in example 12.3.

Chapter 13

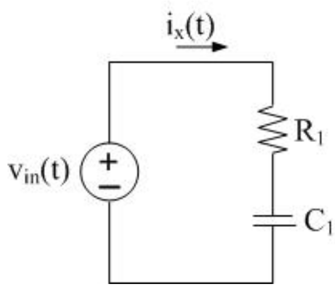
AC Circuit Response

Frequency Response vs. Response to a Single Frequency

In the last chapter we examined the behavior of circuits as a function of frequency. Analytically, such an examination is in the form of frequency response. In this chapter, we are going to look at circuits at a single frequency. All of the frequency dependent impedances will then have specific numerical values determined by the frequency of the sources.

We begin with a simple series connection of a resistor and a capacitor. Does the circuit look familiar?

Example 13.1 AC Response of a Series RC Circuit



$v_{in}(t) = 10\cos(400\pi t)$ V
$R_1 = 100\ \Omega$
$C_1 = 5\ \mu F$

Calculate and graph $i_x(t)$.

Solution: One way to solve this problem is to perform KVL in the time domain and solve the first order differential equation with sinusoidal forcing function. Everyone should try this approach to see if you are able to solve the problem this way and to provide a benchmark for the amount of effort required to solve this simple circuit. The solution approach that will be taken here is called frequency domain analysis. Recognizing that this is an AC circuit (source is sinusoidal), the circuit is transformed into the frequency domain as shown in figure 13.1.

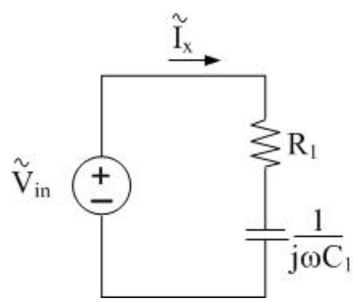


Figure 13.1 Circuit of example 13.1 in the frequency domain.

Note that this is exactly the same circuit that we analyzed in the last chapter. But instead of looking at the high-level frequency response of the circuit, we are interested in the circuit's specific response to a single frequency. Using "Ohm's law" in the frequency domain, $\tilde{I}_x$ can be expressed as

$$\tilde{I}_x = \frac{\tilde{V}_{in}}{R_1 + \frac{1}{j\omega C_1}} = \frac{10}{100 - j\frac{1}{(400\pi)5 \cdot 10^{-6}}} = \frac{10}{188\angle -58} = 53.2\angle 58°\ mA \tag{13.1}$$

Transforming the result back to the time domain: $i_x(t) = \boxed{53.2\cos(400\pi t + 58°)\ mA}$

Graphing $i_x(t)$ should be second nature to students at this point. Readers of this textbook should be quite comfortable with graphing a sinusoid. The amplitude of $i_x(t)$ is 53.2 mA. The period of the wave is 5 ms. $i_x(t)$ is maximum at t = -0.8 ms. The waveform is shown in figure 13.2.

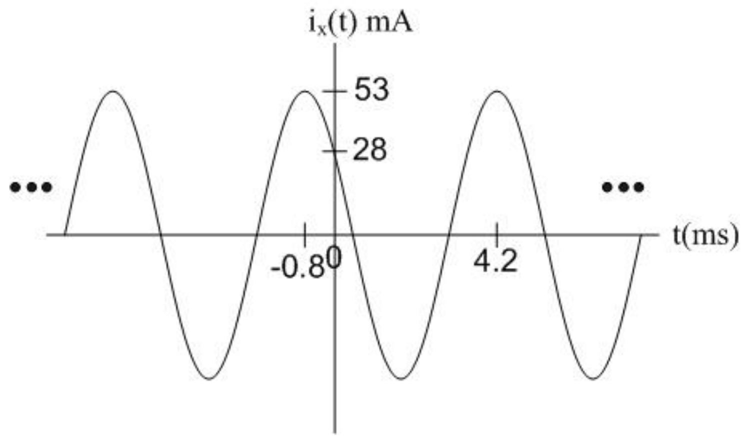
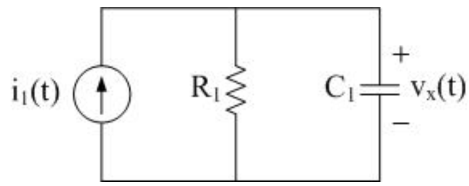


Figure 13.2 Graph of ix(t) in example 13.1

Interpretation of the current i_x(t)

Although this circuit is very simple, it offers an example that illustrates several very important relationships. Any finite, non-zero value of capacitance C_1 results in a finite, non-zero value of current $i_x(t)$. But as the value of C_1 is varied, what does this do to the current $i_x(t)$? If the value of C_1 is made very large, the impedance of the capacitor is negligible, and the current becomes $i_x(t) = 0.1\cos(400\pi t)$ mA. Thus, the current is larger in amplitude and the phase is zero, the same as the source phase. As the value of C_1 decreases, its impedance is no longer negligible, and the current begins to decrease in amplitude, and a phase shift in the current wave occurs moving the peak to the left (earlier in time). When the value of the capacitance is reduced such that the impedance of the capacitor is significantly greater than the impedance of the resistor, the amplitude of the current wave is further reduced, and phase shift of the current asymptotically approaches the value of -90°.

Example 13.2



$i_1(t) = 4\cos(4\pi \cdot 10^3 t)$ mA
$R_1 = 1\ k\Omega$
$C_1 = 80$ nF

Given the element values, find the voltage $v_x(t)$.

Solution: This is an AC circuit (the source is sinusoidal). Therefore, it will be solved in the frequency domain as shown in figure 13.3.

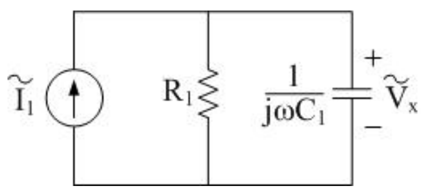


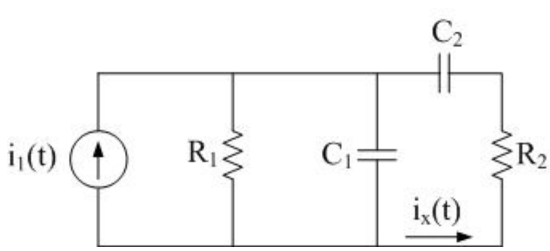
Figure 13.3 Circuit of example 13.2 in the frequency domain.

Using impedance reduction and "Ohm's law" in the frequency domain.

$$\tilde{V}_x = (\tilde{I}_1)\frac{1}{\frac{1}{R_1} + j\omega C_1} = (4 \cdot 10^{-3})\frac{1}{\frac{1}{10^3} + j4\pi \cdot 10^3(80 \cdot 10^{-9})} = 2.82\angle -45.2°\ V \tag{13.2}$$

Transforming back to the time domain: $\boxed{v_x(t) = 2.8\cos(4\pi \cdot 10^3 t - 45°)\ \ V}$

Example 13.3



$i_1(t) = 4\cos(2\pi \cdot 10^6 t)$ mA
$R_1 = 80\ \Omega$
$R_2 = 100\ \Omega$
$C_1 = 2$ nF
$C_2 = 1.5$ nF

Given all of the element values, calculate the current $i_x(t)$.

This is an AC circuit. Try solving this one in the time domain! It would require the solution of a second order differential equation with a sinusoidal forcing function. The problem will be solved here in the frequency domain. Transforming the circuit into the frequency domain (See figure 13.4):

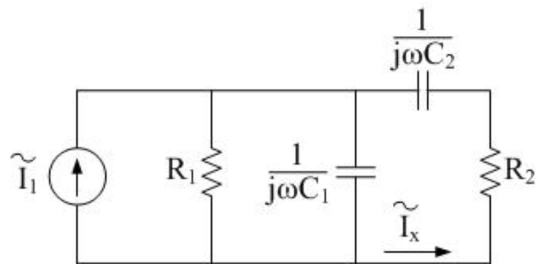


Figure 13.4 Circuit of example 13.3 in the frequency domain.

By current division in the frequency domain, (13.3 results)

$$\tilde{I}_x = (-\tilde{I}_1)\frac{\dfrac{1}{R_2+\dfrac{1}{j\omega C_2}}}{\dfrac{1}{R_2+\dfrac{1}{j\omega C_2}}+\dfrac{1}{R_1}+j\omega C_1} \tag{13.3}$$

$$\tilde{I}_x = (-\tilde{I}_1)\frac{1}{1+\dfrac{R_2}{R_1}+\dfrac{C_1}{C_2}+j\left(\omega C_1 R_2-\dfrac{1}{\omega C_2 R_1}\right)} \tag{13.4}$$

$$\tilde{I}_x = \frac{-4\cdot 10^{-3}}{1+\dfrac{100}{80}+\dfrac{2}{1.5}+j\left[2\pi\cdot 10^6(2\cdot 10^{-9})100-\dfrac{1}{2\pi\cdot 10^6(1.5\cdot 10^{-9})80}\right]}$$

$$\tilde{I}_x = -1.1\ mA \tag{13.5}$$

Transforming back to the time domain: $\boxed{i_x(t) = \ -1.1\cos(2\pi\cdot 10^6 t)\ \ mA}$

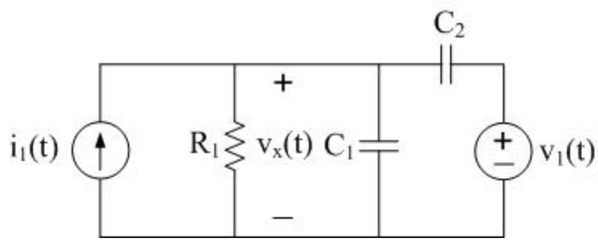
Chapter 14

AC Node Voltage and Branch Current Methods

Motivation for Circuit Analysis Methods

The node voltage method (NVM) and branch current method (BCM) techniques of solving circuits are completely general and can be used to solve any circuit of any complexity. They are necessary when the simple application of "Ohm's law" in the frequency domain and a simple KVL or KCL will not solve the problem. The procedure for utilizing these methods to solve AC circuits in the frequency domain are the same is they were when solving purely resistive circuits. Once the unknown voltage or current is determined in the frequency domain, it is converted back into the time domain. The best way to learn the methods is to solve some example problems.

Example 14.1



$v_1(t) = 12\cos(2\pi\cdot 10^4 t)$ V
$i_1(t) = 4\cos(2\pi\cdot 10^4 t)$ mA
$R_1 = 2$ kΩ
$C_1 = 8$ nF
$C_2 = 3$ nF

Given the element values, find the voltage $v_x(t)$.

Solution: This is an AC circuit and will be solved using frequency domain analysis. The solution is not a simple KCL or KVL or Ohm's law etc. The approach taken will be the NVM.

The circuit transformed to the frequency domain is shown in figure 14.1.

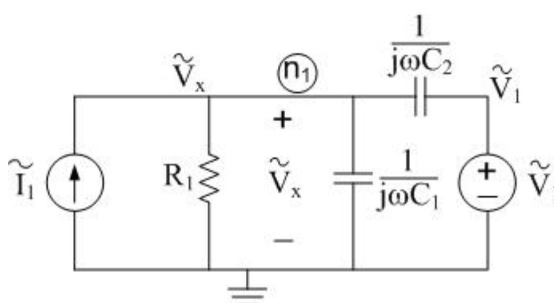


Figure 14.1 Circuit of example 14.1 transformed to the frequency domain and labeled in preparation of applying KCL as part of the NVM.

KCL at node 1:

$$-\tilde{I}_1 + \frac{\tilde{V}_x}{R_1} + \tilde{V}_x j\omega C_1 + \left(\tilde{V}_x - \tilde{V}_1\right) j\omega C_2 = 0 \tag{14.1}$$

$$\tilde{V}_x\left[\frac{1}{R_1}+j\omega(C_1+C_2)\right]=\ \tilde{I}_1+\tilde{V}_1 j\omega C_2 \tag{14.2}$$

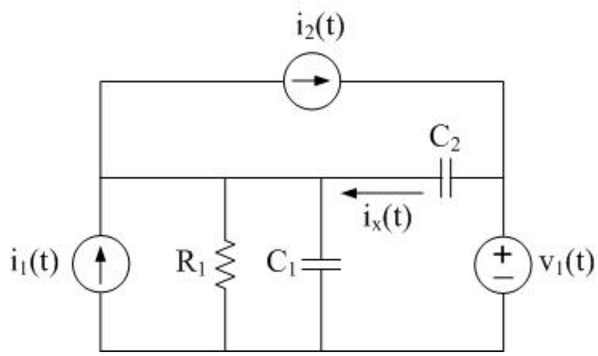
The symbolic solution is not complete until the unknown $\tilde{V}_x$ is on one side of the equation and only symbols whose values are given. Once the solution is obtained symbolically, check the units of your result before inserting numbers. **It is a waste of time to substituted numbers into an impossible symbolic result**. If you examine the symbolic solution shown in (14.3), you will find that the numerator is consistently current and the denominator is consistently admittance. The ratio is a voltage. It makes sense to substitute the numerical values.

$$\tilde{V}_x=\frac{\tilde{I}_1+\tilde{V}_1 j\omega C_2}{\frac{1}{R_1}+j\omega(C_1+C_2)}=\frac{4\cdot 10^{-3}+12j(2\pi\cdot 10^4)(3\cdot 10^{-9})}{\frac{1}{2\cdot 10^3}+\ j2\pi\cdot 10^4(8+3)\cdot 10^{-9}}=5.4\angle-25^\circ\ V \tag{14.3}$$

Transforming the phasor quantity $\tilde{V}_x$ back into the time domain gives

$$v_x(t)=5.4\cos(2\pi\cdot 10^4 t-25^\circ)\,V$$

Example 14.2



$v_1(t) = 2\cos(2\pi\cdot 10^5 t)$ V
$i_1(t) = 6\cos(2\pi\cdot 10^5 t)$ mA
$i_2(t) = 3\cos(2\pi\cdot 10^5 t\text{-}70^\circ)$ mA
$R_1 = 400\ \Omega$
$C_1 = 6$ nF
$C_2 = 5$ nF

Given the circuit element values, calculate the unknown current $i_x(t)$.

This is an AC circuit, and will therefore be solved in the frequency domain. But should we use NVM or BCM? You always have a choice to use either method, and either method is applicable. Ideally you would select a method that will require the least amount of work, reducing the likelihood of error. As time permits, you may choose to use **both** methods. If you use two completely independent methods to solve the same circuit, you can independently confirm your result. This is a very powerful technique to ensure a correct result. In research, we often take further steps of confirmation including computer simulation and physically building the circuit and measuring the voltages and currents.

As you look at the circuit and consider how many unknown node voltages there are, you should arrive at the conclusion that there is only one unknown node voltage. Once that unknown voltage becomes known, a simple application of “Ohm’s law” in the frequency domain enables $i_x(t)$ to be determined. It would seem, therefore, that the NVM should be used to solve this problem. However, the BCM is going to be used to demonstrate an important concept. If you look at the number of loops that do not contain a current source, you should find three and a total of two unknown currents. However, since we are not interested in the individual currents through R_1 and C_1, their impedances can be combined to reduce the problem to a single unknown current problem.

As is always the case, the circuit is **transformed into the frequency domain** as shown in figure 14.2. R_1 and C_1 are shown combined together in parallel, represented by their equivalent impedance. The current through this equivalent impedance is shown in the wire above the parallel combination. It is this current that flows up through the parallel combination of R_1 and C_1. By considering these two elements as a single impedance in the frequency domain, it reduces the complexity of the circuits when solving for the unknown current $i_x(t)$.

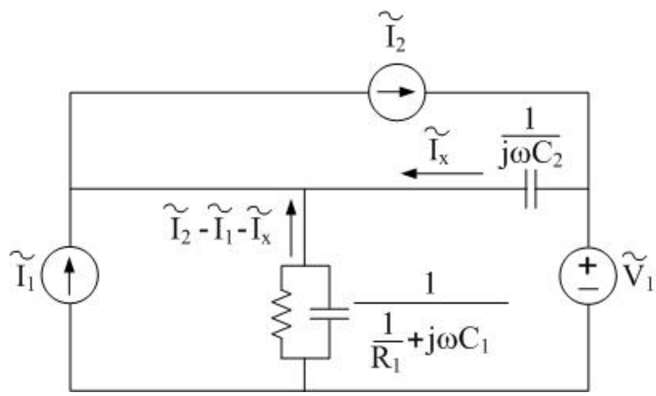


Figure 14.2 Circuit of example 14.2 with R1 and C1 combined in the frequency domain to simplify the circuit such that there is a single unknown current $\tilde{I}_x$.

Applying KVL around the bottom right loop (left unlabeled to keep the circuit as clear as possible).

$$\left(\frac{1}{\frac{1}{R_1}+j\omega C_1}\right)(\tilde{I}_2 - \tilde{I}_1 - \tilde{I}_x) + \frac{1}{j\omega C_2}(-\tilde{I}_x) + \tilde{V}_1 = 0 \tag{14.4}$$

$$\tilde{I}_x\left(\frac{1}{\frac{1}{R_1}+j\omega C_1} + \frac{1}{j\omega C_2}\right) = (\tilde{I}_2 - \tilde{I}_1)\left(\frac{1}{\frac{1}{R_1}+j\omega C_1}\right) + \tilde{V}_1 \tag{14.5}$$

$$\tilde{I}_x = \frac{(\tilde{I}_2 - \tilde{I}_1)\left(\dfrac{1}{\dfrac{1}{R_1} + j\omega C_1}\right) + \tilde{V}_1}{\dfrac{1}{\dfrac{1}{R_1} + j\omega C_1} + \dfrac{1}{j\omega C_2}} \tag{14.6}$$

Be absolutely sure that the units are all compatible before inserting numbers. If the units are not compatible, track your work backwards until you find where the error occurred. This is one reason for solving in symbols. **It is a complete waste of time to insert numbers into an impossible expression**. In (14.6), the numerator is consistently voltage and the denominator is consistently impedance. The ratio produces current. It makes sense to substitute numerical values. Another reason for solving problems entirely in terms of symbols whose values are given is to see how each circuit element affects the unknown, in this case, $\tilde{I}_x$. For example, you can see explicitly how R_1 affects $\tilde{I}_x$. If $\frac{1}{R_1}$ is large compared to $j\omega C_1$, (14.6) will reduce to a much simpler expression. Two exercises are left to the reader.

Exercise 14.1 Assuming $\frac{1}{R_1}$ is large compared to $j\omega C_1$in (14.6), reduce (14.6) to an appropriate expression. Hint: The numerator of (14.6) reduces to $(\tilde{I}_2 - \tilde{I}_1)R_1 + \tilde{V}_1$

Exercise 14.2 Use the NVM to solve example 14.2, and show that it is the exact same result as obtained using the BCM.

The final numerical answer to example 14.2 is obtained by substituting the element values for their symbols in (14.6) to obtain

$$\tilde{I}_x = \frac{(3 \cdot 10^{-3}\angle -70° - 6 \cdot 10^{-3})\left(\dfrac{1}{\dfrac{1}{400} + j2\pi \cdot 10^5(6 \cdot 10^{-9})}\right) + 2}{\dfrac{1}{\dfrac{1}{400} + j2\pi \cdot 10^5(6 \cdot 10^{-9})} + \dfrac{1}{j2\pi \cdot 10^5(5 \cdot 10^{-9})}} \tag{14.7}$$

$$\tilde{I}_x = 2.02\angle 110°\ mA \tag{14.8}$$

Transforming back to the time domain:

$$\boxed{i_x(t) = 2.02\cos(2\pi \cdot 10^5 \mathrm{t} + 110°)\ \mathrm{mA}} \tag{14.9}$$

Exercise 14.3 Resolve example 14.1 using BCM **without** introducing any unknowns other than $\tilde{I}_x$.

You will have to choose a location for $\tilde{I}_x$. If you choose to combine the impedances of R1 and C1 and choose the direction of $\tilde{I}_x$ to be downward, when you solve for the voltage $\tilde{V}_x$, a simple application of "Ohm's law" in the frequency domain will yield $\tilde{V}_x$.

Chapter 15

Fourier Series

15.1 Fourier Series Concept and Example of Voltage Square Wave

In this chapter, the Fourier series of a periodic signal is introduced along with its frequency spectrum. Periodic signals contain discrete frequency content. It will be shown that the spectrum of a periodic signal is a **line spectrum**, a discrete set of lines representing the amplitude and phase of each component of the periodic signal. Each of these terms will be defined and illustrated by examples.

Suppose a **periodic** voltage or current is given as the input to a **filter characterized by its frequency response** $\tilde{\boldsymbol{H}}(\boldsymbol{\omega})$. We already know how to use phasors and frequency domain analysis to find the output of the filter for a single sinusoidal input. In this chapter, a method for representing a periodic voltage or current as an infinite sum of harmonically related sinusoids is presented. This method of representing a periodic voltage or current as an infinite sum of harmonically related sinusoids is called a **Fourier series**. Although there are several forms of the Fourier series, the **cosine** form will be used exclusively in this textbook. The reason for using the cosine form of the Fourier series is that each term of the series is identical to the form of an AC voltage or current, and we are already familiar with the process of solving an AC circuit at a single frequency using the cosine form of the signal. In the next chapter, we will extend the process of solving an AC circuit from a source with a single frequency to a source that produces a periodic signal that can be represented by an infinite sum of harmonically related sinusoids. We will call that process ***response to a periodic input***.

A periodic signal obeys the relationship $f(t) = f(t+T)$ where the smallest value of T that satisfies this relationship is called the **period**. The period $\boldsymbol{T}$ is very a very special quantity because it defines the **fundamental frequency** of the signal $\boldsymbol{f_0} = \frac{\boldsymbol{1}}{\boldsymbol{T}}$. In a periodic signal every frequency present is harmonically related to the fundamental frequency. This means that every frequency present in the periodic signal is an integer multiple of the fundamental frequency $\boldsymbol{f_n} = \boldsymbol{n f_0}$ where the sinusoid whose frequency is $\boldsymbol{f_n} = \boldsymbol{n f_0}$ is called the n'th harmonic.

Any **periodic** voltage or current can be represented with zero mean-square error by its Fourier series:

$$v(t) = v_0 + \sum_{n=1}^{\infty} V_n \cos(n \frac{2\pi}{T} t + \phi_n) \tag{15.1}$$

$$i(t) = i_0 + \sum_{n=1}^{\infty} I_n \cos(n\frac{2\pi}{T}t + \phi_n) \tag{15.2}$$

where v_0 and i_0 are the average or DC voltage or current respectively, and ***T*** is the **period** of the waveform. The equations above are referred to as the **synthesis equations** as they enable one to synthesize either $v(t)$ or $i(t)$ from its Fourier series coefficients. This is a rather remarkable result as it provides the ability to find the output of any linear circuit for any periodic input. We know how to find the DC output of a circuit as well as the output for any sinusoidal (AC) input. By representing a periodic waveform by its Fourier series, we can use the **superposition principle** to obtain the output waveform Fourier series by summing the time domain outputs due to each one of the terms of the Fourier series of the input. Common periodic waveforms include sinusoids, rectangular pulse trains (including squarewaves), triangle waves, sawtooth waves, and rectified sinusoids. However, the Fourier series is completely general and applies to any physical, periodic voltage or current. It should be re-emphasized that there are multiple forms of the Fourier series including the complex exponential form and the sin-cosine form. We have selected the cosine form because the terms are harmonically related cosine functions, and we already know how to find the output of a linear circuit for any AC input. The **fundamental frequency** is

$$f_0 = \frac{1}{T} \tag{15.3}$$

and establishes the frequency of the **first harmonic** (AC) or **fundamental**. The frequency of the AC n'th harmonic is $nf_0 = \frac{n}{T}$. The Fourier series coefficients that must be determined are v_0 and $\tilde{V}_n = V_n \angle\phi_n$, or i_0 and $\tilde{I}_n = I_n \angle\phi_n$, for a periodic voltage or current respectively. To obtain the Fourier series coefficients, we use the Fourier series **analysis equations**:

$$v_0 = \frac{1}{T}\int_{t=t_0}^{t_0+T} v(t)\ dt \tag{15.4}$$

$$\tilde{V}_n = \frac{2}{T}\int_{t=t_0}^{t_0+T} v(t)\, e^{-jn\frac{2\pi}{T}t}\ dt = \frac{2}{T}\int_{t=t_0}^{t_0+T} v(t)\, e^{-jn\omega_0 t}\ dt \tag{15.5}$$

where identical, analogous equations hold for a periodic current. ω_0 is the **fundamental angular frequency** and is related to the period by $\boldsymbol{\omega_0 = \frac{2\pi}{T}}$.

Example 15.1 Voltage Square Wave

A voltage square wave with an amplitude of 8 V and period $T = 4\mu s$ is shown in figure 15.1.

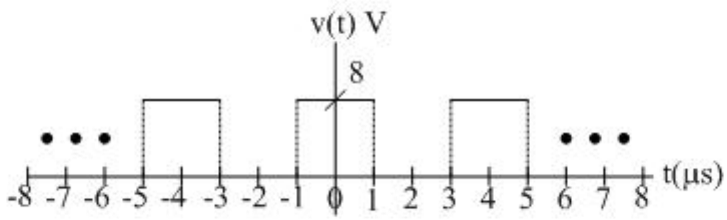


Figure 15.1 Voltage square wave with amplitude 8 V and period $T = 4\mu s$.

Given the voltage square wave in figure 15.1, determine its Fourier series and graph the DC and first five non-zero harmonics of its spectrum.

Solution: First, the DC term is calculated, which is the average value. By inspection of figure 15.1 the period is determined to be 4 μs (thus making the fundamental frequency $f_0 = \frac{1}{T} = 250\ kHz$) . If the period to be analyzed is selected to be centered at t = 0, the voltage is non-zero only during the interval -1 μs < t < 1 μs. Using the analysis equations 15.4 and 15.5, the DC and AC n'th harmonic coefficients are calculated as follows. The DC or average value is obtained by

$$v_0 = \frac{1}{4 \cdot 10^{-6}} \int_{-1\cdot10^{-6}}^{1\cdot10^{-6}} 8\, dt = 2 \cdot 10^6\ t\ \Big|_{-1\cdot10^{-6}}^{1\cdot10^{-6}} = \boxed{4\ V} \tag{15.6}$$

The Fourier series (FS) coefficients of the AC n'th harmonic are obtained by

$$\tilde{V}_n = \frac{2}{4 \cdot 10^{-6}} \int_{-1\cdot10^{-6}}^{1\cdot10^{-6}} 8 e^{-jn\frac{2\pi}{4\cdot10^{-6}}t}\, dt = 4 \cdot 10^6\ \frac{4 \cdot 10^{-6}}{-jn2\pi} e^{-jn\frac{2\pi}{4\cdot10^{-6}}t}\ \Big|_{-1\cdot10^{-6}}^{1\cdot10^{-6}} \tag{15.7}$$

$$\tilde{V}_n = \frac{8}{-jn\pi}\ \left[e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}}\right] \tag{15.8}$$

Invoking the trigonometric identity $2jsin\, \alpha = e^{j\alpha} + e^{-j\alpha}$, which arises from taking the difference between two versions of Euler's Identity

$$e^{j\alpha} = cos\alpha + jsin\alpha \tag{15.9}$$

and

$$e^{-j\alpha} = cos\alpha - jsin\alpha \tag{15.10}$$

to obtain

$$e^{j\alpha} - e^{-j\alpha} = 2jsin\alpha \tag{15.11}$$

The FS coefficients can be expressed as

$$\tilde{V}_n = \frac{8}{-jn\pi} \left[-2jsin(n\frac{\pi}{2})\right] = \boxed{\frac{16}{n\pi}\, sin\left(n\frac{\pi}{2}\right)\, V} \tag{15.12}$$

Although the FS coefficients are complex in general, these FS coefficients are purely real. This means that the phase of any term is either 0 or π depending on the sign of the coefficient. If the coefficient is positive, the phase is zero, and if the coefficient is negative, the phase is π. The first thing to observe is that the term $sin\left(n\frac{\pi}{2}\right)$ is zero for all even values of n. This means that all of the even harmonics are zero. $\frac{16}{n\pi}$ is always positive. The term $sin\left(n\frac{\pi}{2}\right)$ oscillates in value between ±1 for n odd. It is useful to form a table with columns $n, f, \left|\tilde{V}_n\right|$, and $\emptyset_n$ for the first few terms of the FS.

n	f	$\left\|\tilde{V}_n\right\|$ V	$\emptyset_n$
0	0 (DC)	4	0
1	250 kHz	$\frac{16}{\pi} = 5.1$	0
2	500 kHz	0	0
3	750 kHz	$\frac{16}{3\pi} = 1.7$	π
4	1 MHz	0	0
5	1.25 MHz	$\frac{16}{5\pi} = 1.0$	0
6	1. 5 MHz	0	0
7	1.75 MHz	$\frac{16}{7\pi} = 0.73$	π

8	2 MHz	0	0
9	2.25 MHz	$\frac{16}{9\pi} = 0.57$	0

Table 15.1 Fourier series coefficients as a function of coefficient index and frequency.

For the voltage square wave in which the **duty cycle** (ratio of the duration of the high value to the period × 100) is 50%, it is possible to express the phase in closed form (as a function of n) $\phi_n = \left(\frac{n-1}{2}\right)\pi$. This expression for the phase is not immediately evident. Substitute in a few odd values of *n* and see if this expression correctly produces the phase.

We can now write the Fourier series for the voltage square wave shown in figure 15.1

$$v(t) = 4 + \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{16}{n\pi} \cos\left(n0.5\pi \cdot 10^6 t + \left(\frac{n-1}{2}\right)\pi\right) \ V \tag{15.13}$$

15.2 Periodic Signal Synthesis using Fourier Series

To illustrate how a signal is synthesized by its Fourier series, consider the voltage square wave in figure 15.1 represented by its Fourier series in (15.13). The simplest approximation is obtained by using only a single term of the FS, the DC value of 4 V. Not a very good approximation, but remember, we are using only a single term of an infinite series. If we add the DC term to the fundamental and third harmonic, the approximation is better and is shown in figure 15.2. If 200 AC harmonics are used, the approximation is greatly improved as shown in figure 15.2. Notice the rising edges and falling edges of the Fourier series approximation. This overshoot is called the Gibb's Phenomenon [15.1] and converges to 18% of the square wave peak-to-peak value. The **spectrum** of a signal is its **frequency content**. **Each sinusoid contributes one pair of lines in its spectrum** expressed using magnitude and phase. The DC term is the average value and is always a real number. Therefore, its phase must always be either zero or π. The phase of the DC term for the voltage square wave is zero since the DC term is positive.

[15.1] A. Jerry, *The Gibbs Phenomenon in Fourier Analysis, Splines, and Wavelet Approximations*, Springer, 1998.

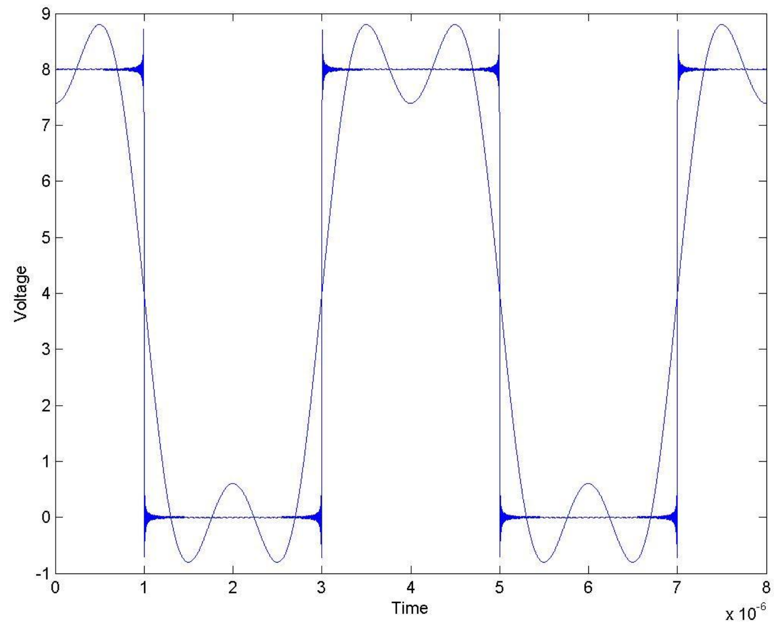


Figure 15.2 Three harmonic and 200 harmonic Fourier series approximations of the voltage square wave in figure 15.1.

The analytical expressions for the two approximations of the voltage square wave of figure 15.1 shown in figure 15.2 are

$$v(t) = 4 + \sum_{\substack{n=1 \\ n\ odd}}^{3} \frac{16}{n\pi} \cos\left(n0.5\pi \cdot 10^6 t + \left(\frac{n-1}{2}\right)\pi\right)\ V \tag{15.14}$$

$$v(t) = 4 + \sum_{\substack{n=1 \\ n\ odd}}^{200} \frac{16}{n\pi} \cos\left(n0.5\pi \cdot 10^6 t + \left(\frac{n-1}{2}\right)\pi\right)\ V \tag{15.15}$$

Both (15.14) and (15.15) are truncated FS. That is, the upper limit is the number of AC harmonics in the approximation of the square wave in figure 15.1. Only when the complete series is incorporated does the FS converge to the function *v(t).*

15.2 Signal Spectrum

The **spectrum of a signal is its frequency content**. Each sinusoid contributes one pair of lines in its spectrum expressed using magnitude and phase. The DC term is the average value and is always a real number. Therefore, its phase must always be either zero or π. The phase of the DC term is zero when the DC term is positive and the phase is π when the DC value is negative. It is important to remember that the values of magnitude for any complex number are always non-negative. The question arises, "What is the frequency content of a periodic signal?" The answer is that the spectrum of a periodic signal is the superposition of spectral line pairs, one pair for each sinusoid. If the Fourier series is truncated to 50 harmonics and the DC term, there will be 51 spectral line pairs in the magnitude and phase spectra of the truncated signal represented by the truncated Fourier series. If the Fourier series contains an infinite number of harmonics, there will be an infinite number of spectral lines in the magnitude and phase spectra. The spectrum of a periodic signal is a set of an infinite number of spectral line pairs, one pair per harmonic. As an example, consider the voltage square wave in figure 15.1. The spectrum of *v(t)* can be obtained readily from its Fourier series shown in eq. 15.13. The problem is to graph the first five non-zero terms of its spectrum. We know all of the even harmonics are zero. Therefore our graphs will contain the n = 0,1,3,5,7 terms. The DC magnitude is 4 V. The AC n'th harmonics have magnitude values of $\frac{16}{n\pi}$. The phase values of all the AC n'th harmonics are either zero or π. For the voltage square wave, the only terms in the n = 0,1,3,5,7 range that have a phase angle of π are the n = 3 and n = 7 terms. All the other terms have phases of 0. The following graphs illustrate the magnitude and phase spectra of the voltage square wave in figure 15.1.

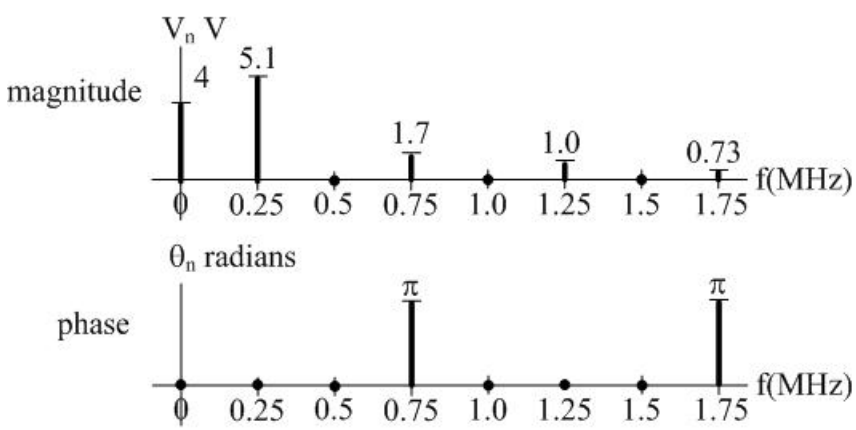


Figure 15.3 Magnitude and phase spectra of the first 8 terms of the FS for the voltage square wave in figure 15.1.

Example 15.2 Given the following line spectrum, write the Fourier series.

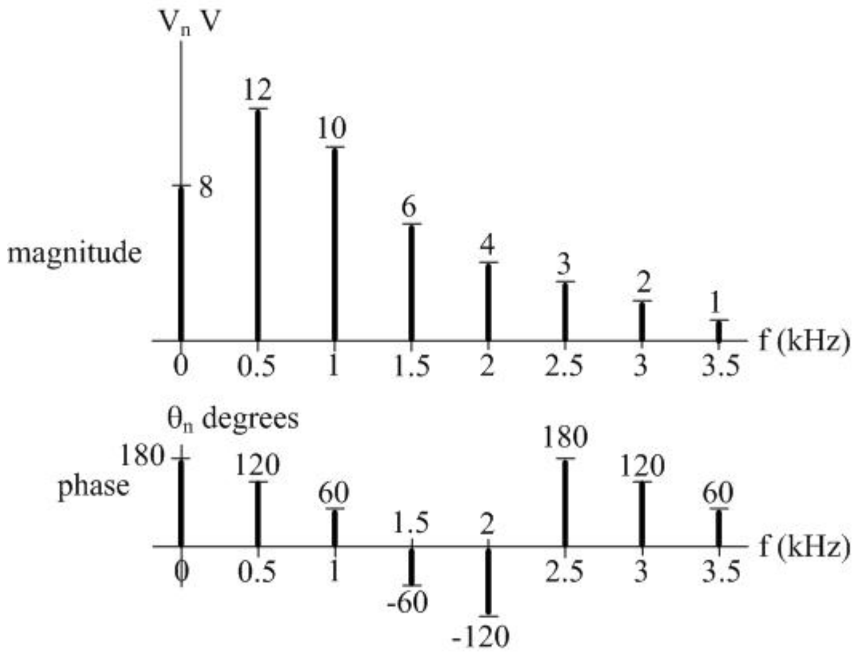


Figure 15.4 Magnitude and phase spectra for example 15.2.

Solution: The Fourier series has a DC term and 7 AC harmonics for a total of 8 terms. Every sinusoid is specified exactly be three properties: The frequency, the amplitude and the phase. This information is available from the graphs of the magnitude and phase spectra. The fundamental frequency is 500 Hz, the frequency of the first AC harmonic. The DC phase is 180°, and therefore the DC term is negative. The complete FS for *v(t)* is given by (15.16).

$$\begin{gathered} v(t) = -8 + 12\cos(\pi \cdot 10^3 t + 120°) + 10\cos(2\pi \cdot 10^3 t + 60°) + \\ 6\cos(3\pi \cdot 10^3 t - 60°) + 4\cos(4\pi \cdot 10^3 t - 120°) + \\ 3\cos(4\pi \cdot 10^3 t + 180°) + 2\cos(5\pi \cdot 10^3 t + 120°) + \cos(6\pi \cdot 10^3 t + 60°) \;\; V \end{gathered} \tag{15.16}$$

The $\tilde{V}_n$ for this waveform are:

$$v_0 = -8\,V, \tilde{V}_1 = 12\angle 120°\,V, \tilde{V}_2 = 10\angle 60°\,V, \tilde{V}_3 = 6\angle -60°\,V, \tilde{V}_4 = 4\angle -120°\,V,$$

$$\tilde{V}_5 = 3\angle 180°\,V, \tilde{V}_6 = 2\angle 120°\,V, \tilde{V}_7 = 1\angle 60°\,V$$

with a fundamental frequency of 500 Hz.

A sound synthesizer called an organ is a musical instrument with a keyboard and an array of controls for either reproducing the sound of a musical instrument or a totally unique sound. Organs "build" their sound output by combining sets of truncated Fourier series.

Fourier series is named after Joseph Fourier (1768-1830) who developed equations to solve problems associated with the heat flow in gun barrels. The temperature of a gun barrel increases when it is fired repeatedly. If it is fired too frequently, the temperature rises to the point where the barrel loses its hardness and begins to warp. At that point the gun is useless.

Chapter 16

Response to a Periodic Input

The output of a linear circuit to a periodic input voltage or current is often desired. Consider the case of determining the output of a linear circuit given its input represented by its Fourier series (FS), and the objective is to find the FS of the output. How would you go about attaining this objective? We already know how to find the output for each term of the FS at the input. Obtaining the output for a periodic input is a matter of using superposition to add together the time domain responses to each one of the terms of the FS. The FS of a periodic voltage and current are given by (16.1) and (16.2), which are repeated here for convenience.

$$v(t) = v_0 + \sum_{n=1}^{\infty} V_n \cos(n\frac{2\pi}{T}t + \phi_n) \tag{16.1}$$

$$i(t) = i_0 + \sum_{n=1}^{\infty} I_n \cos(n\frac{2\pi}{T}t + \phi_n) \tag{16.2}$$

where v_0 and i_0 are the average or DC voltage or current respectively, and T is the period of the waveform. The fundamental angular frequency $\omega_0 = \frac{2\pi}{T}$. A periodic voltage represented by its FS with a few of its terms expanded out can provide further understanding of the series.

$$v(t) = v_0 + V_1 \cos(\omega_0 t + \phi_1) + V_2 \cos(2\omega_0 t + \phi_2) + \cdots + V_n \cos(n\omega_0 t + \phi_n) \tag{16.3}$$

As stated previously, the FS is the sum of a DC term and an infinite sum of AC harmonics. v_0 is the DC or average value, $V_1\angle\phi_1$ is the phasor voltage of the fundamental or AC 1st harmonic, $V_2\angle\phi_2$ is the phasor voltage of the AC 2nd harmonic and $V_n\angle\phi_n$ is the phasor voltage of the AC nth harmonic. We know from previous chapters how to obtain the output of a linear circuit for each one of these terms individually. What we need to do here is to find the output for an infinite sum of a DC term plus and infinite sum of harmonically related sinusoids, and then add all their responses in the **time** domain.

Consider a linear circuit with periodic input voltage $v_{in}(t)$ and whose output voltage we seek $v_{out}(t)$. The circuit is characterized by its frequency response

$$\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} \tag{16.4}$$

This frequency response is valid for all angular frequencies from $\omega = 0$ to $\omega = \infty$. However, the circuit is being stimulated at only at discrete frequencies: DC ($\omega = 0$), ω_0, $2\omega_0$, $3\omega_0$, … , $n\omega_0$. Therefore, what we need is $H(n\omega_0)$ for $n = 0$ to $n = \infty$ (n integer).

The phasor output for all $\tilde{V}_{in}$ is given by

$$\tilde{V}_{out,n} = \tilde{V}_{in,n}\,\widetilde{H}(n\omega_0) \tag{16.5}$$

As an example, consider the periodic voltage square wave in figure 16.1 as the input to an RC low pass filter (LPF) as shown below.

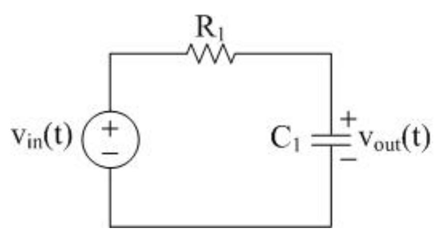


Figure 16.1 RC Low Pass Filter (LPF).

Using voltage division: $H(\omega) = \dfrac{\frac{1}{j\omega C_1}}{\frac{1}{j\omega C_1}+R_1} = \boxed{\dfrac{1}{1+j\omega R_1 C_1}}$ (16.6)

This gives us the frequency response for all frequencies. We are interested in the frequency response at the discrete frequencies of the input $n\omega_0$. Substituting $n\omega_0$ for ω gives

$$\widetilde{H}(n\omega_0) = \frac{1}{1 + jn\omega_0 R_1 C_1} \tag{16.7}$$

Referring back to figure 15.1 reproduced here as the input voltage to the RC LPF in figure 16.1.

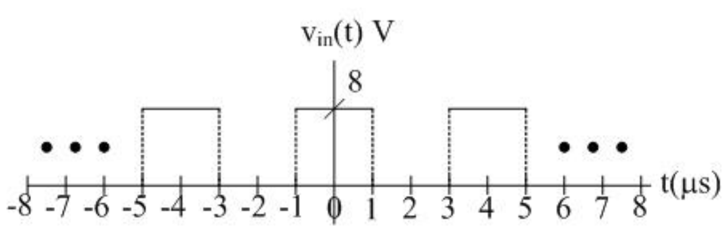


Figure 16.2 Input Voltage square wave

The period T is readily seen to be 4 μs, which produces a fundamental angular frequency of

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4 \cdot 10^{-6}} = 0.5\pi \cdot 10^6 \ \frac{r}{s} \tag{16.8}$$

The time constant of the RC LPF is R_1C_1. If we think about the voltage square wave of figure 16.2 as a DC source that is switched on and off every 4 μs (one rising edge and one falling edge every 4 μs with a 50% duty cycle), we can imagine the response of the RC LPF to this input voltage waveform. Immediately following a falling edge of the input voltage, the output voltage would begin to decal exponentially with a final, steady-state value of 0 V. The when input voltage rises abruptly to 8 V, the voltage rises towards a final steady state value of 8 V with a decaying exponential such that the final asymptotic voltage is 8 V. The rate that these waveforms approach their final, steady state value depends on the value of the RC time constant of the RC LPF. Suppose that the time constant R_1C_1 = 1 μs. Then, the time between each rising edge and each falling edge is two time constants. As an exercise, graph the approximate output voltage.

Exercise 16.1: Given the RC LPF in figure 16.1 and values of R_1 and C_1 such that the RC time constant is 1 μs, graph what you think is the approximate waveform for $v_{out}(t)$.

We will now compute the exact value for $v_{out}(t)$ and graph the result. This will show the accuracy of the estimation made in Exercise 16.1.

The frequency response evaluated at integer multiples of the fundamental frequency are

$$\widetilde{H}(n\omega_0) = \frac{1}{1 + jn(0.5\pi \cdot 10^6)(10^{-6})} = \frac{1}{1 + jn1.5708} \tag{16.9}$$

The phasor output voltages are given by (16.10)

$$\tilde{V}_{out,n} = \tilde{V}_{in,n}\widetilde{H}(n\omega_0) = \left(\frac{16}{n\pi}\angle\frac{(n-1)}{2}\pi\right)\left(\frac{1}{\sqrt{1+2.4674n^2}}\angle - tan^{-1}(n1.5708)\right) \tag{16.10}$$

The next step is to put the expression for $\tilde{V}_{out,n}$ in magnitude-phase form. Symbolically, this is written as

$$\tilde{V}_{out,n} = |\tilde{V}_{in,n}||\widetilde{H}(n\omega_0)|\angle\phi_{in,n} + \theta_n \tag{16.11}$$

Substituting values gives

$$\tilde{V}_{out,n} = \frac{16}{n\pi}\frac{1}{\sqrt{1+2.4674n^2}}\angle\frac{(n-1)}{2}\pi - tan^{-1}(n1.5708) \tag{16.12}$$

The DC term of the output is obtain by multiplying the DC term of the input by the DC gain H(0). The DC gain is obtained by substituting $n = 0$ into (16.5), which produces *H(0)* = 1.

$$v_{out,0} = v_{in,0}\, H(0) = 4(1)\text{ V} \tag{16.13}$$

We can now write the Fourier series of the output voltage. Symbolically, $v_{out}(t)$ is given by

$$v_{out}(t) = v_{out,0} + \sum_{n=1}^{\infty}\left|\tilde{V}_{out,n}\right| cos\,(n\omega_0 t + \phi_{in,n} + \theta_n) \tag{16.14}$$

Substituting numerical values results in the Fourier series of the output voltage.

$$v_{out}(t) = 4 + \sum_{\substack{n=1 \\ n\ odd}}^{\infty}\frac{16}{n\pi}\frac{1}{\sqrt{1+2.4674n^2}}\cos\left(0.5n\pi\cdot 10^6 t + \frac{(n-1)}{2}\pi - tan^{-1}(n1.5708)\right)\text{ V} \tag{18.15}$$

The following plot created using MATLAB illustrates the synthesized square wave and the wave resulting from passing the square wave through the RC low pass filter (LPF). 200 harmonics were used in each case. Discuss with your classmates what you think will happen if you decrease the RC time constant or increase it.

The process of finding the output of a linear circuit given a periodic input is obtained in the same manner independent of the circuit or the periodic input. The Fourier series of the periodic input is obtained. The frequency response of the circuit is obtained and evaluated at the discrete frequencies that are present in the input waveform. The phasor representation of each harmonic of the output is obtained by multiplying the phasor representation of each harmonic of the input by the frequency response evaluated at the frequency of each sinusoidal input. Then the phasor representation of each harmonic of the output is converted back to the time domain as a sinusoid and the output is expressed as a Fourier series.

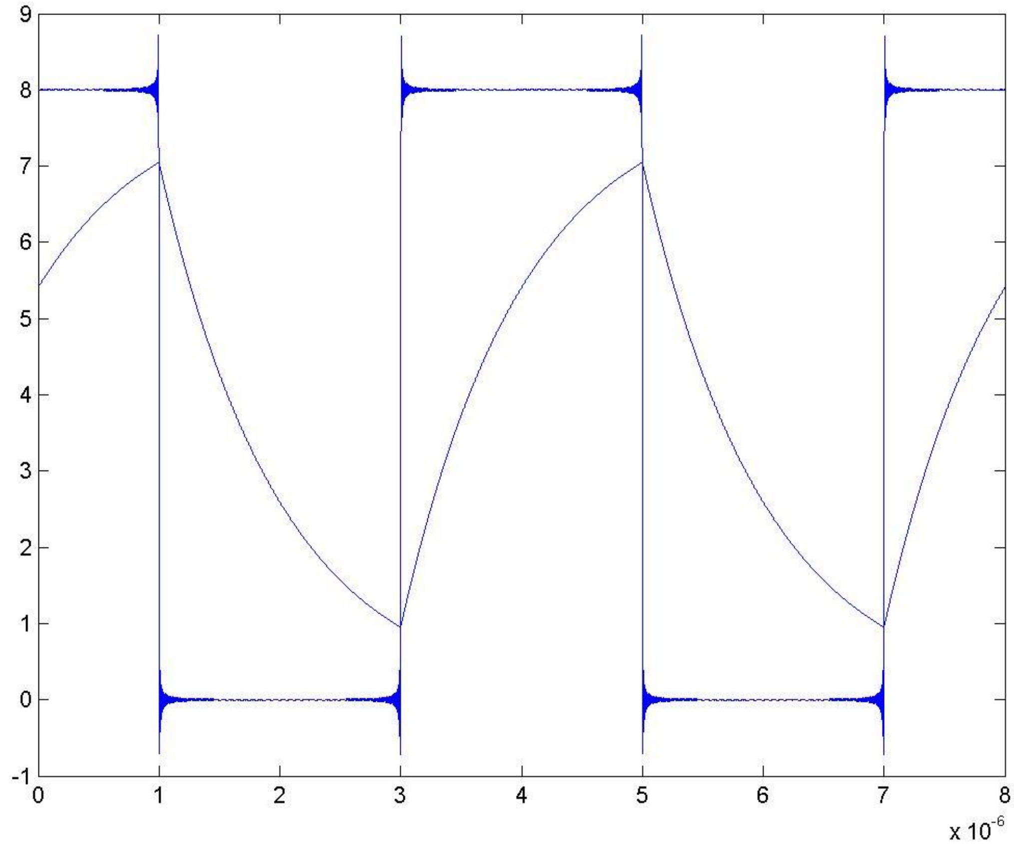


Figure 16.3 200 harmonics of the voltage square wave of figure 16.2 and 200 harmonics of the output of an RC low pass filter with a time constant of 1 μs.

Chapter 17

Inductance

Faraday's Law

When current flows through a wire a magnetic field is created around the wire. An inductor is a coil of wire that takes advantage of the magnetic flux linkage that occurs between the closely wrapped coil turns that occurs when current is passed through the coil. The symbol for an inductor is a coil of wire and its property, called inductance, is identified by the letter "L." The current-voltage relationship for an inductor is given by **Faraday's law**, named after Michael Faraday (1791-1867).

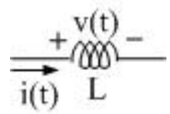


Figure 17.1 Inductor symbol showing the passive sign convention with current in the direction of voltage drop.

Faraday's law states that the voltage drop across inductance L is L times the time rate of change of the current in the same direction. Faraday's law is worth repeating several times until the directedness of the current-voltage relationship is understood.

Mathematically, **Faraday's law** is written as $v(t) = L\frac{di(t)}{dt}$. The unit of inductance is the henry $= \frac{Vs}{A}$, named after Joseph Henry (1797-1878). A picture of several inductors is shown in Figure 17.2.

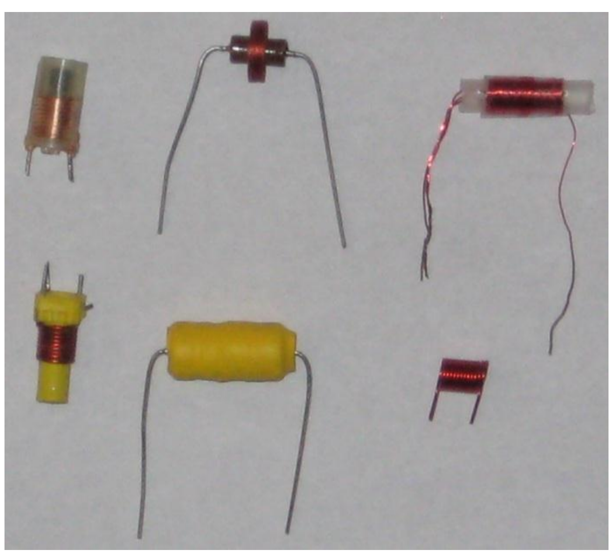
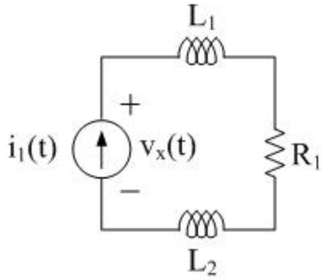


Figure 17.2 Photograph of several inductors.

Example 17.1



$i_1(t) = 3e^{-40t}$ A
$R_1 = 20\ \Omega$
$L_1 = 50$ mH
$L_2 = 30$ mH

Given the element values, calculate and graph $v_x(t)$.

Solution: There is no law that determines the voltage across a current source from the current source value alone. It is necessary to examine what the current source is connected to. The current source is connected to a series combination of two inductors and a resistor. We can use Ohm's law to obtain the voltage across R_1 and Faraday's law to obtain the voltage across each inductor. Finally, KVL around the loop will provide $v_x(t)$.

It is important to realize that this circuit is neither DC nor AC. The current is a decaying exponential, which is neither a constant nor a sinusoid. Hence, this circuit is not a candidate for solution in the frequency domain. Applying KVL in the time domain produces (17.1).

$$-v_x(t) + L_1 i_1'(t) + Ri(t) + L_2 i_1'(t) = 0 \tag{17.1}$$

$$v_x(t) = Ri(t) + (L_1 + L_2)\, i_1'(t) \tag{17.2}$$

$$v_x(t) = (20)3e^{-40t} + (0.08)3(-40)e^{-40t} = \boxed{50.4e^{-40t}\ V} \tag{17.3}$$

The problem statement required both the calculation and graph of the voltage $v_x(t)$. No initial time was given in the problem statement. We will just show the voltage as it approaches t = 0 for t < 0 and the decay to 0 for t > 0.

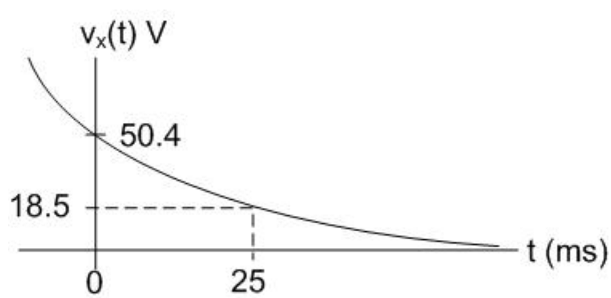


Figure 17.3 Graph of $v_x(t)$ in example 17.1. The coordinates associated with the time constant of the decaying exponential are shown to indicate the rate of decay of the exponential.

Energy Stored in an Inductor

Magnetic energy is stored within and around a coil as current passes through it. Starting with the definition of power and its value electrically and substituting Faraday's Law gives (17.4).

$$p(t) \triangleq w'(t) = v(t)i(t) = Li'(t)i(t) \tag{17.4}$$

Instantaneous power is the time derivative of energy. Electrically, the power absorbed by any element is the product of voltage and current in the direction of voltage drop. Substituting Faraday's law for v(t) and separating variables results in (17.5).

$$dw(t) = Li(t)di(t) \tag{17.5}$$

Anti-differentiating and calculating the definite integral from an arbitrary time t_1 to t_2 where $t_2 > t_1$ is set up in (17.6).

$$\int_{t_1}^{t_2} dw(t) = L\int_{t_1}^{t_2} i(t)di(t) \tag{17.6}$$

Performing the mathematical operations yields to **energy absorbed between t1 and t2**.

$$w_{abs}(t_1 < t < t_2) = w(t_2) - w(t_1) = \frac{1}{2}Li^2(t_2) - \frac{1}{2}Li^2(t_1) \tag{17.7}$$

The **instantaneous energy stored in L at any arbitrary time *t*** is given by

$$\boxed{w_L(t) = \frac{1}{2}Li^2(t)} \tag{17.8}$$

Example 17.2 A 5 μH inductor has an AC current passing through it of value

$$i(t) = 75\cos(2\pi \cdot 10^6 t - 37°) \quad mA \tag{17.9}$$

Calculate the energy stored in the inductor at t = 0.25 μs.

Solution: Inserting t = 0.25 μs into (17.8) produces

$$w_L(t = 0.25\,\mu s) = \frac{1}{2}(5 \cdot 10^{-6})(75 \cdot 10^{-3})^2 cos^2\left(\frac{\pi}{2} - 37°\right) = \boxed{5.1\text{ nJ}} \tag{17.10}$$

As was shown in figure 17.2, inductors come in a variety of shapes and sizes. The wire gauge is determined by the amount of current that the given application requires. Usually the coil is wrapped around a highly permeable core, typically some type of ferrite (iron alloy). Wrapping the coil around a ferrite core produces an inductor much smaller in size than one with an air core. When current flows through an inductor an electromagnet is produced. The current direction and coil orientation determined the poles of the electromagnet. Consider the current and coil orientation as shown in figure 17.4.

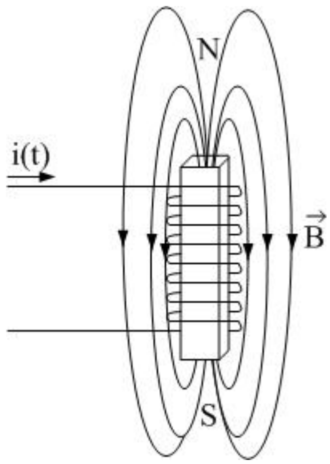


Figure 17.4 Diagram of a coil of wire wrapped around a ferrous core producing a magnetic field density $\vec{B}$ whose direction is given by the right-hand rule.

The right-hand rule determines the orientation of the magnetic field and the direction of the current in the coil. If you grab the wire anywhere on the coil (with your right hand), the direction your fingers are wrapped around the wire is the direction of the magnetic field. The core is typically made of a ferrite (iron alloy) with a relative permeability of several thousand. The orientation of the electromagnet poles are given by the direction of the magnetic field relative to the core. Magnetic flux flows from N to S outside the magnet and from S to N within the magnet.

Transformers

A transformer is an electromagnetic circuit with a primary coil and one or more secondary coils. A picture of a small audio transformer is shown in figure 17.5.

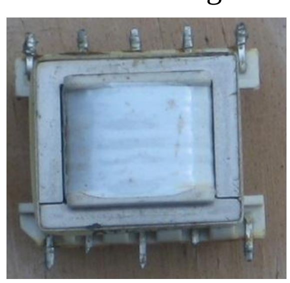


Figure 17.5 A small audio transformer.

The basis of transformer operation is inductance. A pictorial diagram of an electromagnetic circuit containing a transformer is shown in Figure 17.6.

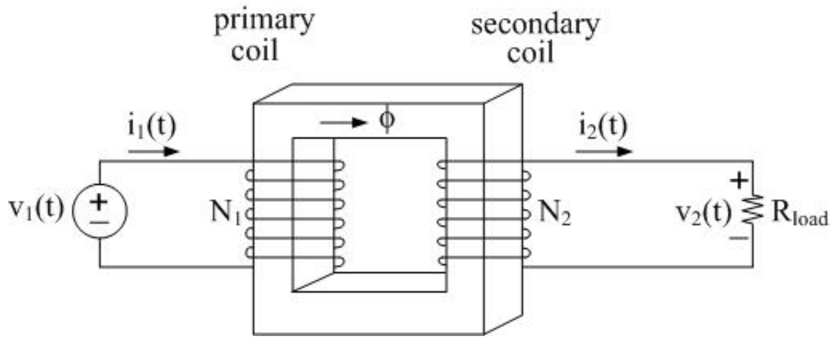


Figure 17.6 Circuit diagram of a voltage source driving the primary coil of a transformer with the secondary coil of the transformer driving a resistive load.

The diagram in figure 17.6 shows an AC voltage source driving the primary coil of a transformer. The primary coil contains N_1 turns. As the current $i_1(t)$ flows through the primary coil, it produces a magnetic flux in the ferrite core ϕ according to the relationship

$$v_1(t) = N_1 \frac{d\emptyset(t)}{dt} \tag{17.11}$$

This relationship assumes that the flux is the same throughout the coil. In reality, there is a small portion of the flux that links only a portion of the coil turns. Hence, (17.11) is an approximation. Assuming that 100 % of the flux generated in the primary coil passes through the magnetic core and then through the secondary coil, the relationship on the secondary side is given by (17.12).

$$v_2(t) = N_2 \frac{d\emptyset(t)}{dt} \tag{17.12}$$

This relationship assumes that the flux generated in the primary coil links the secondary coil. In reality, when the flux passes through the magnetic core, it generates **eddy currents** in the core that divert energy transfer from the primary coil to the secondary coil. The eddy currents are responsible for power loss in the transformer in the form of heat. However, there are techniques for minimizing the power loss in eddy currents in the core making the assumption that the flux in the magnetic core is the same everywhere in the core. In this case, combining (17.11) and (17.12) yields

$$\frac{N_2}{N_1} = \frac{v_2(t)}{v_1(t)} \tag{17.13}$$

This is an extremely important relationship because it states that the turns ratio of the transformer is equal to the voltage ratio. This relationship quantifies the ratio of the secondary voltage to the primary voltage in terms of the coil turns ratio. Thus, the voltage can be stepped up or down depending on the turns ratio. Assuming that that power into the transformer is the same as the power out,

$$v_1(t)i_1(t) = v_2(t)i_2(t) \tag{17.14}$$

Combining (17.13) and (17.14)

$$\frac{N_2}{N_1} = \frac{v_2}{v_1} = \frac{i_1}{i_2} \tag{17.15}$$

which are the governing relationships of the ideal transformer. The coupling of inductance at the primary coil and the secondary coil comprises *mutual inductance*. One last observation, use the right-hand rule on the current in the secondary coil of figure 17.6. Note that the flux produced by the secondary current opposes the flux that created $v_2(t)$. It is left to the reader to figure out why the flux produced by the secondary current opposes the flux that created that current.

In the next chapter, we will be solving RL switched circuits. Much of what you have learned in the context of solving RC switched circuits applies directly to RL switched circuits. The major difference between solving RC and RL switched circuits is the quantity that physically must be continuous. For RC switched circuits, it was the voltage across the capacitor. For RL switched circuits, it is the **current through the inductor** that must be continuous. No matter what the question asks for, you must solve for the current through the inductor first. Once you know the current through the inductor, you can solve for any quantity in the circuit. The continuity requirement of the current through an inductor is a direct consequence of Faraday's law repeated here for convenience. For current flowing in the direction of the voltage drop

$$v_L(t) = L\frac{di(t)}{dt} \tag{17.17}$$

There are two immediate **consequences that follow from Faraday's law**.

1) **a pure inductance is a short circuit for dc**.
2) it is **impossible for the current through an inductor to be discontinuous**. Suppose the current through an inductor were discontinuous. This would produce infinite current, which is physically impossible. Therefore, **the current through an inductor must be continuous**. The argument form utilized here is referred to as *reductio ad absurdum*.

These two consequences of Faraday's law are fundamentally important in the solution to RL switched circuits, which is the topic of the next chapter.

Chapter 18

RL Switched Circuits (Transient Analysis)

RL switched circuits contain resistance, inductance, one or more switches that are thrown (changed states) and in general one or more sources. Although the sources in RL switched circuits may be DC, AC or an arbitrary waveform, **only DC sources are considered in this textbook**. A review of switch behavior together with Faraday's law provides the starting point for examining and solving RL switched circuits.

The simplest type of switch is called a **single pole single throw switch (SPST)**. The switch can only be in one of two states, open or closed. When the switch is closed, it forms a **short circuit** and when it is open, it forms an open **circuit**. This is illustrated in figure 18.1 below.

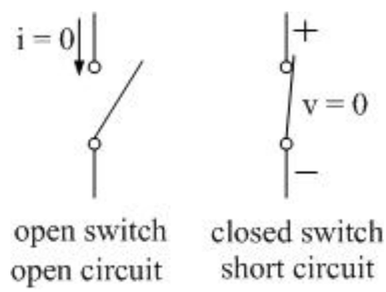
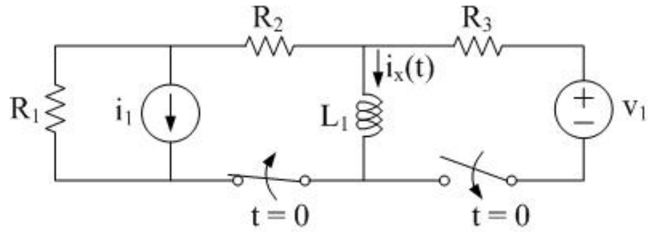


Figure 18.1 Single Pole Single Throw (SPST) switches and their behavior.

An open circuit means <u>zero current</u>. A short circuit means <u>zero voltage.</u> RL switched circuits will always be given with the switch shown in the state prior to being thrown (changed states) and a time will be given when the switch is thrown. The objective will be to solve for some unknown voltage or current for all time. Before the first switch is thrown, it is assumed that the circuit has been in its current state long enough to be in steady state. Therefore, **before any switch is thrown, the circuit is a DC circuit in steady state**. The relationship between the voltage across and current through an inductor is given by Faraday's law $v(t) = L\frac{di(t)}{dt}$. Therefore, before any switches are thrown, the inductor will behave as a short circuit because all voltages and currents in the circuit are constant and hence $\frac{di(t)}{dt} = 0$. The procedure for solving RL switched circuits is nearly identical to solving RC switched circuits. There are two main differences between the process of solving RL switched circuits and RC switched circuits. First, it is **the current through the inductor that must be continuous**. Therefore, the DE you will obtain and solve will be for the current through the inductor (as opposed to the voltage across the capacitor in RC switched circuits) and secondly, the time constant for an RL switched circuit is $\tau = \frac{L}{R_{eq}}$ where R_{eq} is the equivalent resistance seen by the inductor in the switching period being analyzed. We will now solve several example problems.

Example 18.1



$v_1 = 10$ V
$i_1 = 12$ mA
R1= 1 kΩ
R2 = 800 Ω
R3 = 750 Ω
L1 = 20 μH

Given the circuit elements, calculate and graph the unknown current $i_x(t)$ for all time.

Solution: For t < 0 this is a DC circuit in steady state. Therefore **the inductor is a short circuit** (imagine the inductor is a wire). The Thevenin equivalent to the right of the inductor is not a part of the circuit containing the inductor and can be omitted from the analysis. Remember, the switches are always shown in the position before any switch is thrown. The circuit that must be analyzed is shown in figure 18.2.

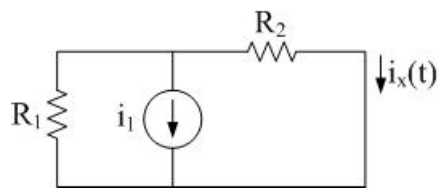


Figure 18.2 Circuit that exists for t < 0 in example 18.1

Current division produces

$$i_x(t) = (-i_1)\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}} = (-12\cdot 10^{-3})\frac{\frac{1}{800}}{\frac{1}{10^3}+\frac{1}{800}} = \boxed{-6.\dot{6}\,mA} \tag{18.1}$$

t > 0: For t > 0 the Norton equivalent on the left is removed and the Thevenin equivalent on the right is connected. The circuit that exists for t > 0 is shown in figure 18.3.

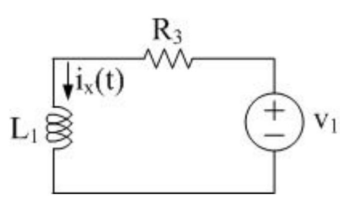


Figure 18.3 Circuit that exists for t > 0 in example 1.

This is a DC circuit, but not in steady state, at least not initially. You can use NVM or BCM, but before you begin the analysis, you should think through the steps and consider which technique will require less effort. In this particular circuit it turns out that either method requires only a single unknown. By the BCM, the KVL around the loop produces

$$L_1\left(-i_x'(t)\right) + R_3\left(-i_x(t)\right) + v_1 = 0 \tag{18.2}$$

DE
$$i_x'(t) + \frac{R_3}{L_1} i_x(t) = \frac{v_1}{L_1} \tag{18.3}$$

Solution
$$i_x(t) = A + Be^{st} \tag{18.4}$$

$$A = \frac{v_1}{R_3} = \frac{10}{750} = \boxed{13.\dot{3}\ mA} \qquad s = -\frac{R_3}{L_1} = -\frac{750}{20 \cdot 10^{-6}} = \boxed{-37.5 \cdot 10^6} \tag{18.5}$$

B is always found from the initial condition: $i_x(0^-) = i_x(0^+) = A + Be^{s(0)}$

$$B = i_x(0^-) - A = -6.\dot{6} \cdot 10^{-3} - 13.\dot{3} \cdot 10^{-3} = \boxed{-20\ mA} \tag{18.6}$$

Putting it all together: $i_x(t) = A + Be^{st} = \boxed{13.\dot{3} - 20e^{-37.5 \cdot 10^6 t}\ mA}$

A graph of $i_x(t)$ for all values of time is shown in figure 18.4.

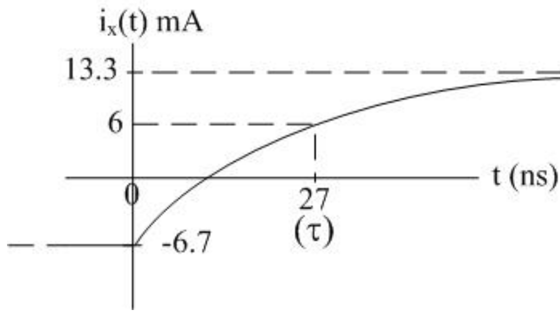
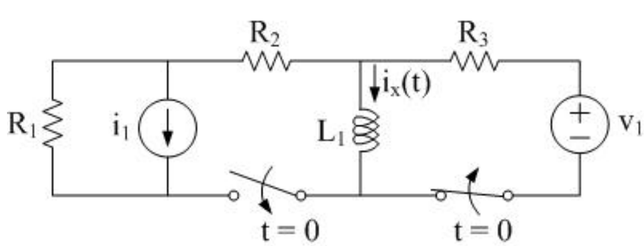


Figure 18.4 Graph of $i_x(t)$, the solution to example 18.1 for all time. As with all graphs of exponential functions, the coordinates associated with the time constant are shown to indicate the rate of decay of the exponential. The value of $i_x(t)$ for t < 0 is clearly shown to be -6.7 mA.

Example 18.2



$v_1 = 10$ V
$i_1 = 12$ mA
R1= 1 kΩ
R2 = 800 Ω
R3 = 750 Ω
L1 = 20 μH

Given the element values, calculate and graph $i_x(t)$. Notice that this is the same circuit as example 1. The difference is the switch positions. In this circuit the Norton equivalent on the left has no effect on the initial condition and the Thevenin equivalent on the right establishes the initial condition but has no effect on the inductive circuit after t = 0. Can you state in symbols what the initial condition is just by looking at the circuit? The circuit that exists for t < 0 is shown in figure 18.5.

t < 0 This is a DC circuit in SS. Therefore the inductor is a short circuit making $i_x(t) = \frac{v_1}{R_3} = \frac{10}{750} = \boxed{13.\dot{3}\ mA}$. This is the initial condition for solving the circuit for t > 0.

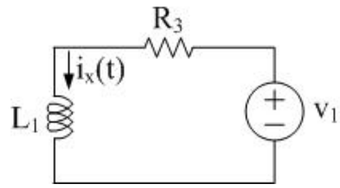


Figure 18.5 Circuit that exists for t < 0 for example 18.2.

At t = 0 both SPST switches are throw. The Thevenin equivalent on the right is removed from the inductive portion of the circuit and the Norton equivalent on the left is brought into action.

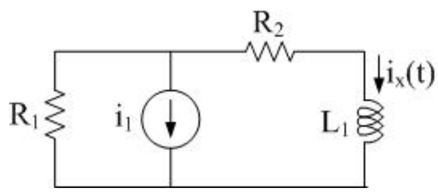


Figure 18.6 Circuit of example 18.2 for t > 0.

This is a DC circuit, but not in steady state, at least initially. We can use either NVM or BCM to obtain the DE. For comparison purposes, the NVM will be used and then the BCM and a comparison made. The reader can then decide which approach they would prefer.

NVM The circuit that exists for t > 0 labeled for applying the NVM is shown in figure 18.7.

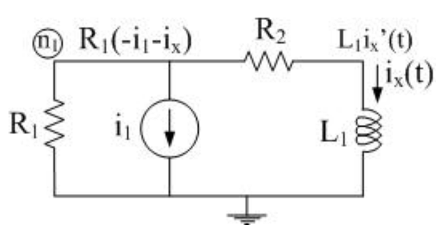


Figure 18.7 Circuit of example 18.2 for t > 0 labeled for using the NVM.

KCL at node n_1 produces:

$$-i_1 - i_x + i_1 + \frac{R_1(-i_1 - i_x) - L_1 i_x'(t)}{R_2} = 0 \qquad (18.7)$$

DE
$$i_x'(t) + \frac{R_1 + R_2}{L_1} i_x(t) = -i_1 \frac{R_1}{L_1} \tag{18.8}$$

We will now use the BCM to obtain the DE and compare the two techniques.

BCM Applying KVL to the outside loop in figure 18.6 produces the following equation

$$R_1\big(i_1 + i_x(t)\big) + R_2 i_x + L_1 i_x'(t) = 0 \tag{18.9}$$

DE
$$i_x'(t) + \frac{R_1 + R_2}{L_1} i_x(t) = -\frac{R_1 i_1}{L_1} \tag{18.10}$$

Solution
$$i_x(t) = A + Be^{st} \tag{18.11}$$

$$A = -\frac{R_1 i_1}{R_1 + R_2} = -\frac{(10^3)12 \cdot 10^{-3}}{1800} = \boxed{-6.\dot{6}\, mA} \tag{18.12}$$

$$s = -\frac{R_1 + R_2}{L_1} = -\frac{1800}{20 \cdot 10^{-6}} = \boxed{-90 \cdot 10^6} \tag{18.13}$$

B is always obtained from the initial condition: $i_x(0^-) = i_x(0^+) = A + Be^{s(0)}$

$$B = i_x(0^-) - A = 13.\dot{3} \cdot 10^{-3} + 6.\dot{6} \cdot 10^{-3} = \boxed{20\, mA} \tag{18.14}$$

Putting this all together:

$$i_x(t > 0) = A + Be^{st} = \boxed{-6.\dot{6} + 20e^{-90 \cdot 10^6 t}\, mA} \tag{18.15}$$

A graph of $i_x(t)$ for all time is presented in figure 18.6. Recall that the time constant is the time required for exponential to drop to e^{-1} times it initial value. In this case the time constant is calculated as follows: $\tau = \frac{1}{90 \cdot 10^6} = \boxed{11.\dot{1}\, ns}$

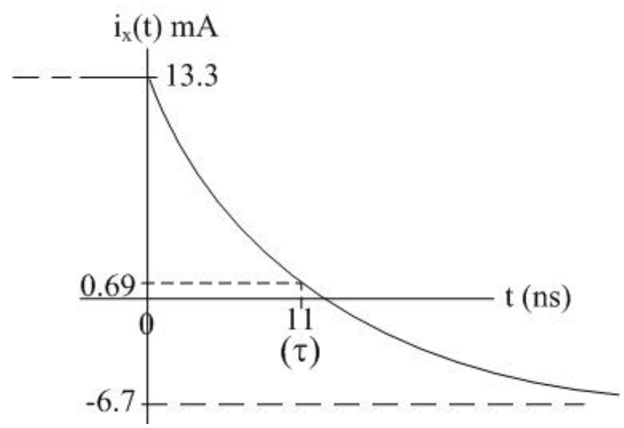


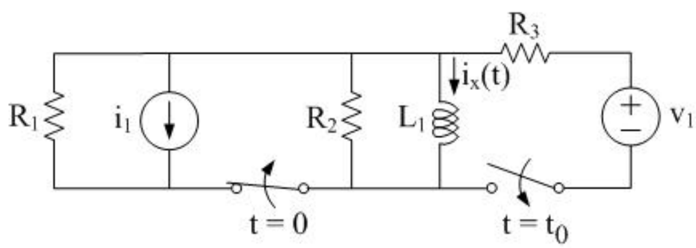
Figure 18.8 Graph of $i_x(t)$ for all time in example 18.2.

Note that the value of $i_x(t)$ for t < 0 is clearly 13.3 mA and that the coordinates associated with the time constant are present on the graph. It is possible to write the analytical expression for $i_x(t)$ from the graph, and this is the criterion for judging the completeness of a given graph.

Comparing the NVM and BCM techniques of solving this circuit, the BCM is somewhat more straight-forward. The complexity, however, is about equal (compare (18.7) with (18.9).

In this third and final example of RL switched circuits there are two switches that are thrown at different times. This is called **sequential switching**. Instead of having two durations of time in which to solve for the unknown, there will now be three regions of time: The time before any switch is thrown, the time between the first thrown switch and the time the second switch is thrown and finally the time after both switches have been thrown. There are several differences in the solution procedure due to the sequential switching process, and these differences will be highlighted throughout the solution process.

Example 3: Sequential Switching



v_1 = 10 V
i_1= 40 mA
R_1 = 300 Ω
R_2 = 250 Ω
R_3 = 250 Ω
L_1 = 2.5 mH

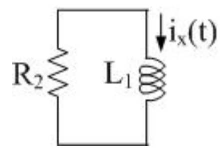
Given the element values and t_0 = 15 μs, calculate and graph $i_x(t)$ for all time.

<u>Solution</u>: There are three regions of time to consider 1) $t < 0$ 2) $0 < t < t_0$ 3) $t > t_0$

We will begin with t < 0, when the circuit is DC in SS. Get the initial condition for t > 0. Obtain and solve the DE for the region of time $0 < t < t_0$. Then calculate $i_x(t_0)$, which is the initial condition for $t > t_0$. Obtain the DE for $t > t_0$ and solve the DE subject to the initial condition, $i_x(t_0)$.

From a global view of the circuit, there are three parts: a) a Norton equivalent on the left, b) a parallel combination of a resistor and inductor in the middle, and c) a Thevenin equivalent on the right. In time sequence, the Norton is in the circuit for t < 0, and the Thevenin is out. During the period $0 < t < t_0$ both the Norton and the Thevenin are unconnected to the inductor. Then, for $t > t_0$, the Thevenin on the right is connected and the Norton remains unconnected.

t < 0: DC Circuit in SS. L_1 is a short circuit. $i_x(t \le 0) = -i_1 = \boxed{-40\ mA}$
This is the initial condition for the solution to the DE for $0 < t < t_0$.
<u>$0 < t < t_0$</u>: The circuit that exists during this region of time is only R_2 in parallel with L_1.



KVL: $R_2 i_x(t) + L_1 i_x'(t) = 0$ → DE $i_x'(t) + \frac{R_2}{L_1} i_X(t) = 0$ This is a homogeneous DE (the particular or steady state solution is zero because the forcing function is zero.) The solution to this homogenous DE is: $i_x(t) = Be^{st}$ $s = -\frac{R_2}{L_1} = -\frac{250}{2.5 \cdot 10^{-3}} = -10^5$

B is always obtained by the initial condition. For the transition at t = 0, the continuity requirement is $i_x(0-) = i_x(0+) = Be^{s(0)}$ → $B = i_x(0^-) = -40\ mA$.

$$i_x(0 \le t \le t_0) = Be^{st} = \boxed{-40e^{-10^5 t}\ mA}$$

To obtain the initial condition for t > t_0, we need to find $i_x(t_0^-)$.

$i_x(t_0^-) = -40e^{-10^5(15 \cdot 10^{-6})} = \boxed{-8.93\ mA}$

<u>t > 15 μs</u>

The circuit that is valid for the region of time t > 15 μs is shown in figure 18.9.

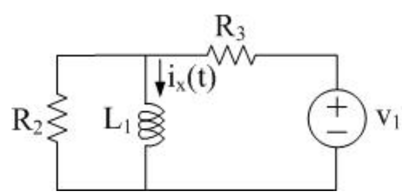


Figure 18.9 Circuit in example 18.3 for the time period t > t_0.

To obtain the DE, you may always use either the NVM or the BCM. To see this, both methods will be presented to obtain the DE.

NVM: $\frac{L_1 i_x'(t)}{R_2} + i_x(t) + \frac{L_1 i_x'(t) - v_1}{R_3} = 0$

BCM: $-L_1 i_x'(t) + R_3\left(\frac{-L_1 i_x'(t)}{R_2} - i_x(t)\right) + v_1 = 0$

Both of these independent methods produce the exact same DE:

DE
$$i_x'(t) + \frac{R_3}{L_1\left(1 + \frac{R_3}{R_2}\right)} i_x(t) = \frac{v_1}{L_1\left(1 + \frac{R_3}{R_2}\right)} \quad (18.16)$$

The solution to the DE is $i_x(t) = A + Be^{st}$

The particular or steady state solution is $A = \frac{v_1}{R_3} = \frac{10}{250} = \boxed{40\ mA}$

$$s = -\frac{R_3}{L_1\left(1+\frac{R_3}{R_2}\right)} = -\frac{250}{2.5\cdot 10^{-3}\left(1+\frac{250}{250}\right)} = \boxed{-50\cdot 10^3}$$

As always, B is found from the initial condition. But in this case, it is a little more complicated than the case when the switch is thrown at t = 0.

The initial condition in this case is $i_x(t_0^-) = i_x(t_0^+) = A + Be^{st_0^+}$

$$B = \frac{i_x(t_0^-) - A}{e^{st_0^+}} = \frac{-8.9\cdot 10^{-3} - 40\cdot 10^{-3}}{e^{-50\cdot 10^3(15\cdot 10^{-6})}} = \boxed{-103.52\ mA}$$

$$i_x(t \geq 15\mu s) = A + Be^{st} = \boxed{40 - 103.5e^{-(50\cdot 10^3)t}\ \ mA} \tag{18.17}$$

Restating previous solutions for convenience:

$$i_x(t \leq 0) = \boxed{-40\ mA}\quad i_x(0 \leq t \leq 15\ \mu s) = \boxed{-40e^{-10^5 t}\ \ mA} \tag{18.18}$$

The graph of $i_x(t)$ for all time is shown in figure 18.10.

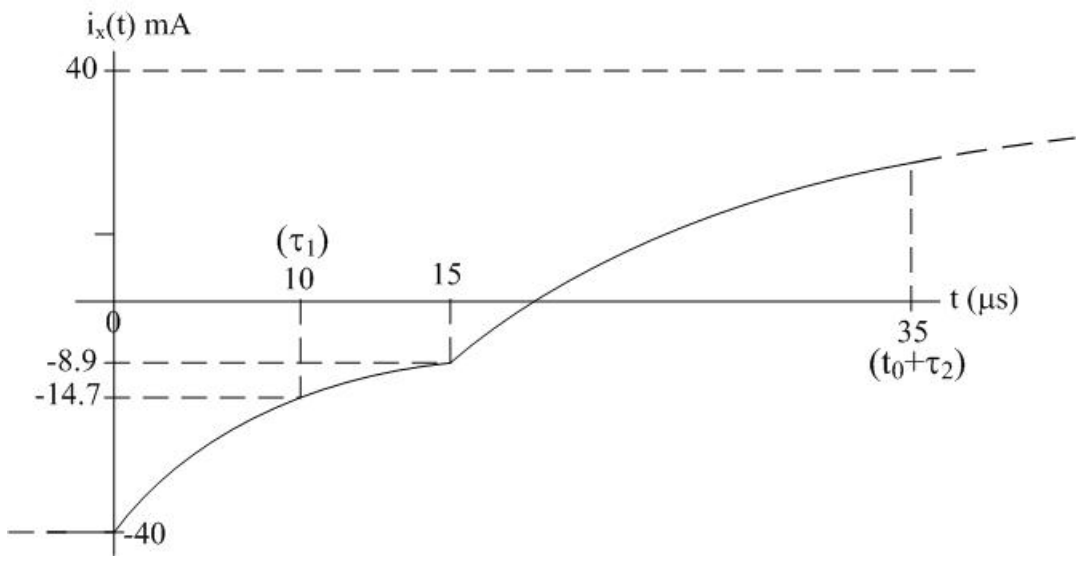


Figure 18.10 Graph of $i_x(t)$ in the sequentially switched circuit of example 18.3.

The time constant during the region $0 < t < 15$ μs is 10 μs, and the associated coordinates are shown on the graph. The second switch is thrown at 15 μs and the associated coordinates are shown. For the third region of time, $t > 15$ μs, the time constant is 20 μs. The proper coordinates to show are $t_o + \tau_2$, not τ_2. This is extremely important to understand.

It is important to be able to go through all of the calculations carefully and accurately. This can be assisted by checking your calculations as you go through the problem by using your physical understanding of the circuit. For $t < 0$, the inductor is a short circuit (acts like a wire). This short circuits the two parallel resistors R_1 and R_2 making the current through the resistors zero. Therefore, all of i_1 has to go through the inductor. A comparison of the directions of i_1 and i_x shows that there is a sign difference. That is how you determine i_x for $t < 0$ from a physical understanding of the circuit. When you solve for the current at $t = 15$ μs by substituting this time into the solution of the homogeneous DE governing the circuit that exists for $0 < t < 15$ μs, you can verify your solution for $t > 15$ μs by substituting $t = 15$ μs into the solution for $t > 15$ μs. This is true because the current through the inductor, $i_x(t)$, must be continuous. When you piece together all of the solutions for the three regions of time, be absolutely sure that the waveform is continuous.

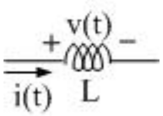
At this point, you have all of the theory and several examples of solving switched circuits. It is now a matter of practicing the solution process on enough switched circuits to generate the fluency needed to approach any switched circuit of any configuration.

Chapter 19

RLC Circuits in the Frequency Domain

Current-Voltage Relationship of Inductors in the Frequency Domain

The current-voltage relationship for an inductor in the time domain is Faraday's law.



Faraday's law states that the voltage drop across inductance L is L times the time rate of change of the current in the same direction.

$$v(t) = L\frac{di(t)}{dt}$$

For **AC circuits**, every voltage and current can be written in the form:

$$v(t) = Re\{\tilde{V}e^{j\omega t}\}$$
$$i(t) = Re\{\tilde{I}e^{j\omega t}\}$$

Faraday's law can be rewritten to emphasize that there is no time at which Faraday's law fails.

$$v(t) - L\frac{di(t)}{dt} \equiv_t 0$$

Rewriting the time domain voltage and current in terms of their phasor definitions gives

$$Re\{\tilde{V} - j\omega L\tilde{I}\} \equiv_t 0$$

where the symbol "$\equiv_t$" is to be read "is identically equal to for all time.

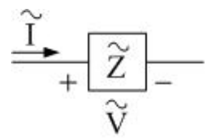
The only way that this equation can be satisfied is if $\tilde{V} - j\omega L\tilde{I} = 0$, which gives us the impedance of an inductor:

$$\tilde{V} = j\omega L\tilde{I} \tag{19.1}$$

Making the impedance of an inductor $\boldsymbol{\tilde{Z}_L = j\omega L}$. The impedance of an inductor is purely imaginary (positive) and proportional to frequency. At DC (f = 0) the impedance of an inductor is zero, or equivalently, the inductor is a short circuit. Then, as the frequency is increased, an inductor's impedance grows linearly with frequency. In the limiting case as f → ∞, the impedance goes to infinity.

Impedance: The General Case and Review of Phasors

In the general case $\tilde{V} = \tilde{Z}\tilde{I}$, a purely algebraic relation, and that is why it is referred to as "Ohm's law" in the frequency domain. It is worth summarizing all current-voltage relationships in the frequency domain as well as phasors. Impedance, $\tilde{Z}$, is, in general, a function of frequency. Impedance is not a phasor, but is the ratio of two phasors $\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}$. Phasors are representations of sinusoids in the frequency domain. Impedances represent the passive circuit elements (R, L, C) in the frequency domain. By transforming a circuit into the frequency domain, its analysis is greatly simplified, and significant insight into a circuit's behavior is also a very useful byproduct of frequency domain analysis.

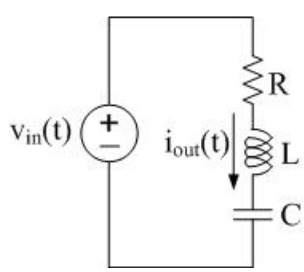


<u>Time Domain</u>	<u>Frequency Domain</u>
$v(t) = V_{max}\cos(\omega t + \phi_v)$	$\tilde{V} = V_{max}\angle\phi_v$
$i(t) = I_{max}\cos(\omega t + \phi_i)$	$\tilde{I} = I_{max}\angle\phi_i$
R	R
L	$j\omega L$
C	$\frac{1}{j\omega C}$

Figure 19.1 Time domain quantities and their transformations to the frequency domain. It is critically important that the reader understands that these **transformations apply to AC circuits only**.

All of the frequency domain quantities have been derived earlier in this textbook in Chapter 11, except the impedance of an inductor that has been derived in this chapter. Now that the impedance of an inductor has been derived, we can analyze circuits containing inductors in the frequency domain.

Example 19.1 Series Resonant RLC Bandpass Filter (BPF)



Given the elements (symbolically), solve for:

a) The frequency response
b) The magnitude and phase response
c) The half-power frequencies
d) Graph the magnitude and phase of the frequency response

Solution: The first step is to transform the circuit to the frequency domain. Then use your knowledge of circuit analysis to obtain the phasor output in terms of the phasor input. Then divide by the phasor input to obtain the frequency response, which is, in general, a function of frequency. Take the magnitude and determine the phase. Obtain the half-power frequencies by equating the maximum response to $\frac{1}{\sqrt{2}}$ times the maximum response and solve for ω.

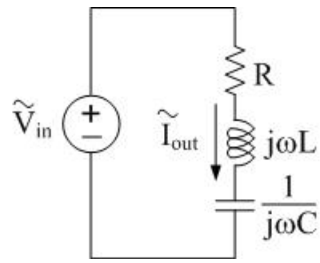


Figure 19.2 Circuit of example 19.1 in the frequency domain. It is important to understand that there are no functions of time *t* in the frequency domain. It is a serious error to show any quantity as a function of time.

$$\tilde{I}_{out} = \frac{\tilde{V}_{in}}{R + j\omega L + \frac{1}{j\omega C}} \tag{19.2}$$

$$\widetilde{H}(\omega) = \frac{\tilde{I}_{out}}{\tilde{V}_{in}} = \frac{1}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \tag{19.3}$$

where both sides of the equation have been divided by $\tilde{V}_{in}$ to obtain the frequency response $\widetilde{H}(\omega)$. Notice that $\widetilde{H}(\omega)$ is a complex function of frequency. The *j* has been factored out from the imaginary part in the denominator.

<u>Magnitude</u>

The magnitude of a fraction is a fraction of magnitudes. The magnitude of the numerator is 1 and the magnitude of the denominator is obtained by the Pythagorean theorem:

$$|\widetilde{H}(\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{19.4}$$

To calculate the half-power frequencies, equate $|\widetilde{H}(\omega)|$ to $\frac{1}{\sqrt{2}}|\widetilde{H}(\omega)|_{max} = \frac{1}{\sqrt{2}}\frac{1}{R}$. The maximum magnitude response occurs when $\omega L - \frac{1}{\omega C} = 0$, which occurs at the center angular frequency.

Center Angular Frequency

$$\boxed{\omega_C = \frac{1}{\sqrt{LC}}} \tag{19.5}$$

At this angular frequency, the magnitude response is maximized.

Half-Power Frequencies

$$\frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{1}{\sqrt{2}}\frac{1}{R} \tag{19.6}$$

Inverting both sides and squaring gives

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 \tag{19.7}$$

Subtracting R^2 from both sides and taking the square root of both sides (make sure you include both roots.

$$\omega L - \frac{1}{\omega C} = \pm R \tag{19.8}$$

This is a quadratic function. Put in standard form yields

$$\omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0 \tag{19.9}$$

$$\omega_{HP} = \pm\frac{R}{2L} \pm \sqrt{\frac{R}{2L} + \frac{1}{LC}} \tag{19.10}$$

Note that there are four solutions, not two. Two of these solutions result in negative frequencies. The non-physical solutions are associated with the minus sign in front of the radical.

The upper and lower half-power frequencies are given by

$$\omega_{HP_U} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad \omega_{HP_L} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{19.11}$$

Taking the difference between the two half-power frequencies yields the **bandwidth**.

$$\Delta\omega = \frac{R}{L} \qquad \Delta f = \frac{\Delta\omega}{2\pi} \tag{19.12}$$

A graph of the magnitude response is presented in figure 19.3.

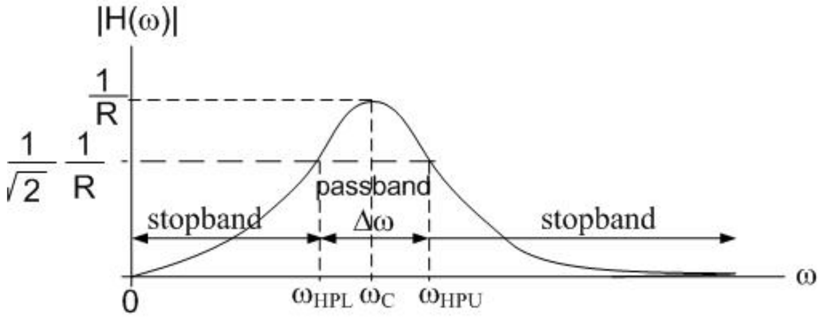


Figure 19.3 Magnitude of the frequency response of example 19.1.

Resonance

Resonance is a perfect harmony of exchange of energy between the inductor and the capacitor. The inductor stores energy in the form of a magnetic field. The capacitor stores energy in the form of an electric field. At resonance the magnetic energy reaches its maximum value at the very moment the energy stored in the capacitor is zero. At resonance the electric energy in the capacitor is maximum at the very moment the magnetic energy stored in the inductor is zero. At resonance, the frequency response is at its maximum value [for the series RLC bandpass filter]. The frequency at which this occurs is the resonance frequency that we have termed the "center frequency." Note that the frequency response magnitude is not symmetric. If it were, the graph would have a value of zero at a frequency of $2\omega_C$. But the Magnitude response only asymptotically approaches zero as the frequency goes to infinity.

Selectivity and Quality Factor

The **quality factor** Q is the standard metric for **selectivity**. What is selectivity? Imagine two BPF magnitude responses. One of the magnitude responses is very narrow and the other is much wider. The narrow response is more selective. The filter passes only a very narrow range of frequencies while the wider response passes a much wider range of frequencies. The quality factor Q is defined as the ratio of the center frequency to the bandwidth. A large value of Q indicates a highly selective filter while a low value of Q indicates low selectivity.

$$Q \triangleq \frac{\omega_C}{\Delta\omega} \tag{19.13}$$

For the series RLC BPF, $Q = \frac{\omega_C L}{R}$.

Phase Response

The phase response of a circuit is the angle of the frequency response (19.3). In general, the frequency response is a ratio of two complex functions of frequency. In the example series resonant bandpass filter, only the denominator is complex. The phase response is the phase of the numerator minus the phase of the denominator. The phase of the numerator is zero. The negative of the phase of the denominator is the phase response.

$$\theta_H(\omega) = -tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{19.14}$$

To graph this function, you must be familiar with the graph of the $tan^{-1}(x)$ function. The phase response is shown in figure 19.4.

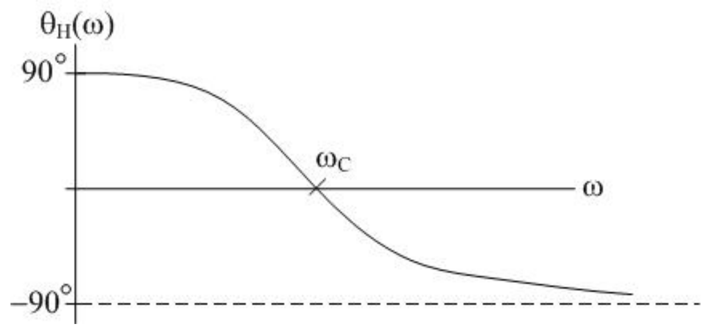
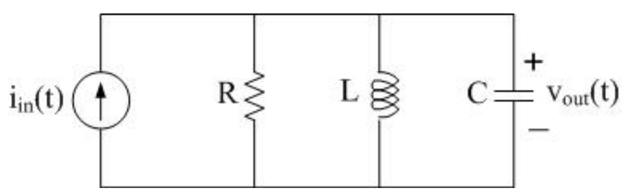


Figure 19.4 Phase response for the circuit in example 19.1

It should be noted that it is not recommended that students memorize the results of this analysis but rather the process. As will be shown in example 19.2, the frequency response results of another circuit can, and generally are, quite different from circuit to circuit.

Example 19.2 Parallel RLC BPF



Given the passive element values (R, L, C), analyze the frequency response of this circuit.

Solution: Analyzing the frequency response includes finding the frequency response, finding the magnitude and phase of the frequency response, calculation of half-power frequencies, the center frequency, the bandwidth and graphing the magnitude and phase of the frequency response with the appropriate coordinates. The first step, as in all frequency response problems, is to transform the circuit to the frequency domain.

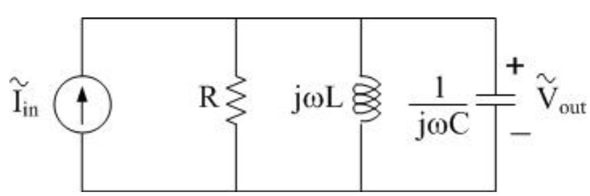


Figure 19.5 Circuit of example 19.2 in the frequency domain.

In order to find the frequency response, we need to relate the output phasor quantity to the input phasor quantity. The frequency response for this circuit is the impedance.

$$\tilde{V}_{out} = \tilde{I}_{in}\left(\frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C}\right) \tag{19.15}$$

The frequency response is obtained by dividing both sides by $\tilde{I}_{in}$.

$$\widetilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{I}_{in}} = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} \tag{19.16}$$

The magnitude of the frequency response is obtained by taking the magnitude of the numerator and dividing by the magnitude of the denominator. The magnitude of the numerator is 1, and the magnitude of the denominator is obtained by using Pythagorean's theorem.

$$\left|\widetilde{H}(\omega)\right| = \frac{1}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} \tag{19.17}$$

To find the half-power frequencies, equate $\left|\widetilde{H}(\omega)\right|$ with the maximum of the magnitude divided by $\sqrt{2}$, and solve for the frequencies that satisfy that equation. The maximum magnitude response occurs at resonance, which occurs when $\omega C - \frac{1}{\omega L} = 0$. The resonance or center angular frequency is easily solved

$$\omega_C = \frac{1}{\sqrt{LC}} \tag{19.18}$$

At resonance, the maximum magnitude response occurs and is equal to *R*. Solving for the half-power frequencies

$$\frac{1}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} = \frac{R}{\sqrt{2}} \tag{19.19}$$

Inverting both sides and squaring gives

$$\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2 = \frac{2}{R^2} \tag{19.20}$$

Subtracting $\frac{1}{R^2}$ from both sides of the equation and taking both square roots gives

$$\omega C - \frac{1}{\omega L} = \pm\frac{1}{R} \tag{19.21}$$

This is a second degree polynomial in ω, which can be solved for using the quadratic formula. The quadratic equation in standard form is given by

$$\omega^2 \pm \frac{1}{RC}\omega - \frac{1}{LC} = 0 \tag{19.22}$$

$$\omega = \pm\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{19.23}$$

There are four values of ω that satisfy this equation. Two are positive and two are negative (non-physical). The non-physical solutions are due to the minus sign in front of the radical. The way this can be known, is that the result of the radical must be greater than $\frac{1}{2RC}$ because the addition of $\frac{1}{LC}$ to $\left(\frac{1}{2RC}\right)^2$ produces a value greater than $\frac{1}{2RC}$. Therefore, the upper and lower half-power frequencies are

$$\omega_{HP_U} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \omega_{HP_L} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{19.24}$$

The bandwidth in either angular frequency or frequency is

$$\Delta\omega = \omega_{HP_U} - \omega_{HP_L} = \boxed{\frac{1}{RC}} \qquad \Delta f = \frac{\Delta\omega}{2\pi} = \boxed{\frac{1}{2\pi RC}} \tag{19.25}$$

A graph of the frequency response magnitude is presented in figure 19.5.

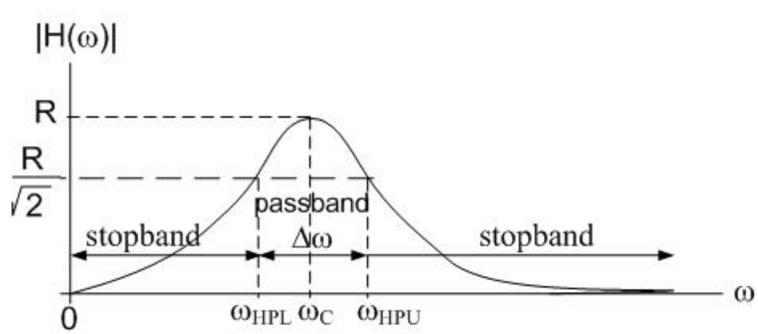


Figure 19.5 Magnitude response for the circuit in example 19.2

What do you think a graph of the magnitude response of an ideal BPF looks like? This is shown in figure 19.6

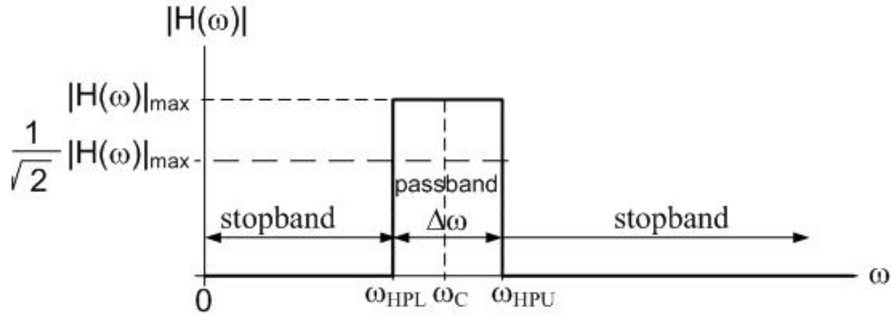


Figure 19.6 Graph of the magnitude response of the ideal BPF.

Phase Response

The phase response is the angle of the frequency response. The angle of the numerator is 0 since 1 is a positive, real number. Therefore the phase is the negative of the angle of the denominator.

$$\theta_H(\omega) = -tan^{-1}\left(\frac{\omega C - \frac{1}{\omega L}}{\frac{1}{R}}\right) \tag{19.26}$$

A graph of this phase function is shown in figure 19.7.

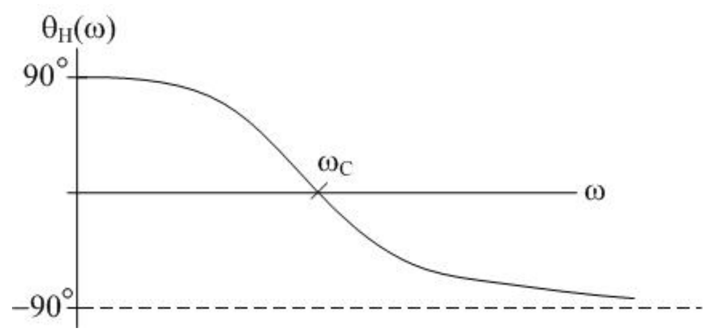


Figure 19.7 Graph of the phase response for the circuit in example 19.2

The quality factor Q for any circuit is, by definition, a ratio of the center frequency to the bandwidth. For this circuit, the quality factor Q is calculated as follows.

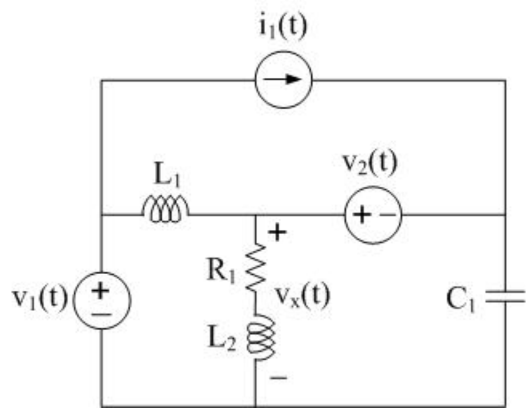
$$Q \triangleq \frac{\omega_C}{\Delta\omega} = \frac{RC}{\sqrt{LC}} \tag{19.27}$$

How do you think the phase response varies with Q?

Single Frequency RLC Circuit Analysis

When an AC circuit is analyzed at a single frequency, the energy storage elements (inductors and capacitors) have specific numerical impedances. If the frequencies of all sources are the same, every voltage and current in the circuit will have that same frequency. This assumes the circuit is linear. If the frequencies of the sources are different, then superposition must be applied. See Appendix 1 for information regarding superposition and linearity. Only circuits with a single frequency are examined in this textbook.

Example 19.3



$v_1(t) = 8\cos(2\pi \cdot 10^6 t)\, V$
$v_2(t) = 6\cos(2\pi \cdot 10^6 t)\, V$
$i_1(t) = 14\cos(2\pi \cdot 10^6 t)\, mA$
$R_1 = 500\ \Omega$
$L_1 = 80\ \mu H$
$L_2 = 100\ \mu H$
$C_1 = 300\ pF$

Given the element values, calculate and graph $v_x(t)$.

Solution: This is an AC circuit and will be solved in the frequency domain. The first thing to do is to transform the circuit into the frequency domain. Take the time to redraw the circuit labeling each element properly.

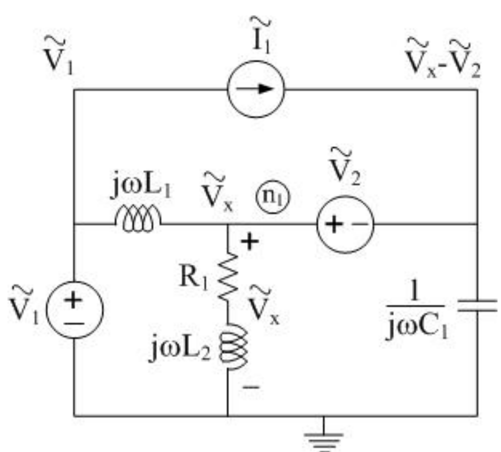


Figure 19.8 Circuit in example 19.3 in the frequency domain and labeled for solution using NVM.

The circuit in example 19.3 has been transformed into the frequency domain and labeled for applying the node voltage method in figure 19.8. Examination of the circuit should lead the reader to quickly ascertain that the BCM will require much more work than NVM.

Although the node voltage between R_1 and L_2 is unknown, we have no interest in its value. We will therefore treat the impedance of R_1 and L_2 as a single impedance. Applying KCL at node n_1 yields

$$\frac{\tilde{V}_x - \tilde{V}_1}{j\omega L_1} + \frac{\tilde{V}_x}{R_1 + j\omega L_2} + \left(\tilde{V}_x - \tilde{V}_2\right) j\omega C_1 - \tilde{I}_1 = 0 \tag{19.28}$$

It is obtaining this equation that is the engineering portion of the problem. What remains at this point is some algebra to isolated the unknown voltage $\tilde{V}_x$ and execute accurate complex arithmetic. If you are interested in knowing for sure if your numerical answer is correct, by all means, use the BCM. If you get the same numerical result from two independent methods, the chances of your answer being incorrect are extremely small. The following steps illustrate the steps that you need to show when solving (19.28). Be absolutely sure that you check units of your final symbolic result. If there is an inconsistency, trace your work backwards until you find your error. Make the change and follow it forward to the final result. Once you have consistent units, it is reasonable to substitute numerical values and calculate the numerical result.

$$\tilde{V}_x \left(\frac{1}{j\omega L_1} + \frac{1}{R_1 + j\omega L_2} + j\omega C_1 \right) = \frac{\tilde{V}_1}{j\omega L_1} + \tilde{V}_2 j\omega C_1 + \tilde{I}_1 \tag{19.29}$$

$$\tilde{V}_x = \frac{\frac{\tilde{V}_1}{j\omega L_1} + \tilde{V}_2 j\omega C_1 + \tilde{I}_1}{\frac{1}{j\omega L_1} + \frac{1}{R_1 + j\omega L_2} + j\omega C_1} \tag{19.30}$$

$\tilde{V}_x$ has been isolated and is expressed entirely in terms of the symbols whose values are given. There is no need to manipulate this expression. All of the numerator terms are consistently current and the denominator terms are consistently inverse ohms (or mhos). Check your work thoroughly for transcription errors. Sometimes subscripts are incorrectly placed. Therefore, it is reasonable to make the numerical substitution.

$$\frac{\frac{8}{j(2\pi \cdot 10^6)80 \cdot 10^{-6}} + 6j(2\pi \cdot 10^6)300 \cdot 10^{-12} + 14 \cdot 10^{-3}}{\frac{1}{j(2\pi \cdot 10^6)80 \cdot 10^{-6}} + \frac{1}{500 + j(2\pi \cdot 10^6)10^{-4}} + j(2\pi \cdot 10^6)300 \cdot 10^{-12}} \tag{19.31}$$

Check all of your numerical substitutions a second time to make sure you have the correct values. It is now a matter of using your calculator in an efficient manner to make the final calculation for $\tilde{V}_x$. The most precise method uses memory in the calculator for intermediate step results keeping all of the calculations internal to the calculator. However, if you find that you need to write down intermediate results, keep five significant figures and write your final result with three significant figures.

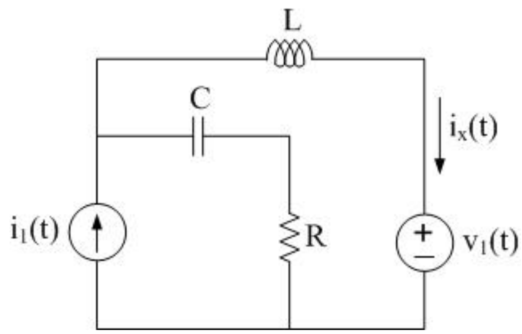
$$\tilde{V}_x = 11.1\angle 36° \tag{19.32}$$

Transforming this result back into the time domain results in

$$v_x(t) = \boxed{11.1\cos(2\pi \cdot 10^6 t + 36°)\ V} \tag{19.33}$$

The graph of $v_x(t)$ is left as an exercise.

Example 19.4



$v_1(t) = 40\cos(2\pi \cdot 10^3 t)\,V$
$i_1(t) = 250\cos(2\pi \cdot 10^3 t)\,mA$
$R = 150\,\Omega$
$L = 25\,mH$
$C = 1\,\mu F$

Calculate the current $i_x(t)$.

Solution: This is an AC circuit. Therefore it will be solved in the frequency domain.

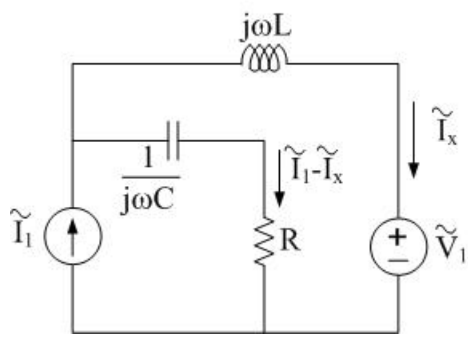


Figure 19.9 Circuit of example 19.4 in the frequency domain labeled for analysis using the BCM.

Given the element values, solve for $i_x(t)$.

Solution: The circuit of example 19.4 is shown in the frequency domain in figure 19.9 and labeled for using the BCM. Applying KVL around the only loop in the circuit with no current source results in the following equation.

$$\left(R + \frac{1}{j\omega C}\right)(\tilde{I}_x - \tilde{I}_1) + (j\omega L)(\tilde{I}_x) + \tilde{V}_1 = 0 \tag{19.34}$$

$$\tilde{I}_x\left(R + \frac{1}{j\omega C} + j\omega L\right) = \left(R + \frac{1}{j\omega C}\right)\tilde{I}_1 - \tilde{V}_1$$

$$\tilde{I}_x = \frac{\left(R + \frac{1}{j\omega C}\right)\tilde{I}_1 - \tilde{V}_1}{R + \frac{1}{j\omega C} + j\omega L} = \frac{\left(R + \frac{1}{j\omega C}\right)\tilde{I}_1 - \tilde{V}_1}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \tag{19.35}$$

This is the final symbolic expression for $\tilde{I}_x$. There is no need for any further manipulation (In fact, making further manipulations increases the likelihood of error). Check the units of each term and the equation as a whole for consistency before numbers are inserted. It is also a good idea to double-check each step of your algebraic manipulations to ensure that a subscript hasn't accidentally changed from a 1 to a 2 or an L accidentally gets replaced with a C.

$$\tilde{I}_x = \frac{\left(150 + \frac{1}{j(2\pi \cdot 10^3)10^{-6}}\right)0.25 - 40}{150 + j\left[(2\pi \cdot 10^3)(25 \cdot 10^{-3}) - \frac{1}{(2\pi \cdot 10^3)(10^{-6})}\right]}$$

Once numbers are inserted, double-check to make sure that you have transcribed the correct number for each symbol, before you use your calculator to make the final calculation.

$$\tilde{I}_x = 0.27\angle - 93° \tag{19.36}$$

Transforming back to the time domain:

$$\boxed{i_x(t) = 0.27\cos(2\pi \cdot 10^3 t - 93°)\ A}$$

Chapter 20

Dependent Sources

Dependent sources are a fundamental set of circuit elements that include transistors. Transistors are the primary building block of all integrated circuits, and hence it is very important to understand the way dependent sources behave from both a terminal perspective as well as how they are implemented using transistors. Transistors will be analyzed in chapter 22. In the current chapter, analysis of circuits containing dependent sources is presented using their terminal characteristics.

Types of Dependent Sources

There are four types of dependent sources. They are characterized by whether they are producing a voltage or a current and whether they are controlled by a voltage or a current. There are four possible combinations.

The symbol for a dependent source is a diamond with either a set of polarities indicating a voltage source or an arrow indicating a current source inside the diamond. The control variable is scaled to produce the value of the dependent source.

Introducing the dependent sources in no particular order is presented in figures 20.1-4.

voltage controlled voltage source (VCVS)

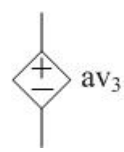


Figure 20.1 Voltage controlled voltage source (VCVS).

"a" is a dimensionless constant that multiplies the control variable v_3, a voltage somewhere else in the circuit, to produce the local voltage av_3. A good question to ask is, "How can a voltage somewhere else in a circuit control the voltage at another pair of terminals?" This question will be answered in chapter 22. In the meantime, we will work with the terminal characteristics of the VCVS. v_3 may be the unknown quantity being sought or some other unknown (for which you must produce another equation called a "control equation.").

current controlled voltage source (CCVS)

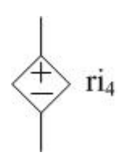


Figure 20.2 Current controlled voltage source (CCVS).

i_4 is the control current somewhere else in the circuit, and r is the transfer resistance. The dependent source is a voltage source indicated by the polarities inside the diamond. The value of that source voltage is a scaled version of the current, i_4, located somewhere else in the circuit. The coefficient that scales i_4 must have units of Ohms and is called the transfer resistance. It is not a physical resistor, but it is a resistance and is therefore designated by a lower case “r.” There remain two dependent sources, both of which are current sources.

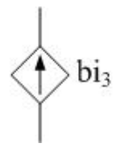


Figure 20.3 Current controlled current source (CCCS).

The arrow inside the diamond indicates that the element is a current source. The value of the current produced by the source is bi_3, where i_3 is a current somewhere else in the circuit. “b” is a dimensionless constant. The only dependent source remaining is a voltage controlled current source.

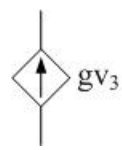
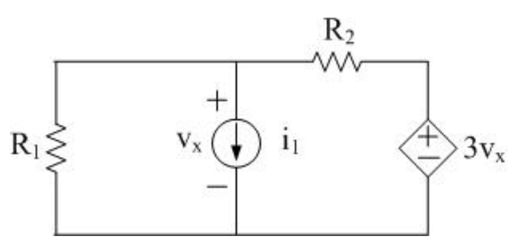


Figure 20.4 Voltage controlled current source (VCCS).

The control variable is the voltage v_3 and is located somewhere else in the circuit, controlling the value of the dependent current source gv_3. The coefficient of v_3 is the *transconductance* g. The value of a constant that multiplies a voltage to produce a current must have units of mhos since it is an inverse resistance. The transconductance is not a physical element. It represents the gain the voltage v_3 is multiplied by to produce the current gv_3.

The best way to illustrate how to solve circuits with dependent sources is through examples.

Example 20.1



Given the element values, calculate the symbolic expression for the unknown voltage v_x.

Using the NVM with the bottom node as ground, KCL at the upper left node produces

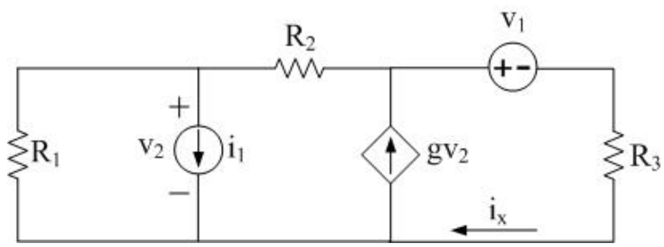
$$\frac{v_x}{R_1} + i_1 + \frac{v_x - 3v_x}{R_2} = 0 \tag{20.1}$$

Solving for v_x:

$$v_x\left(\frac{1}{R_1}-\frac{2}{R_2}\right)=-i_1 \qquad \boxed{v_x=\frac{-i_1}{\frac{1}{R_1}-\frac{2}{R_2}}} \qquad (20.2)$$

There is an important observation to be made about the resulting symbolic solution for v_x. The observation is that there is only a single unknown, v_x. The presence of the VCVS did not add any complexity to the solution. That will always be true if the control variable is the same as the unknown being sought. The next problem does not possess this quality.

Example 20.2



Given v_1, i_1, R_1, R_2, R_3 and g, calculate the symbolic solution for i_x.

Solution: A direct approach to this problem is the branch current method. The currents through each of the resistors must be labeled, and KVL must be applied around loop(s) with no current source in them as part of the BCM. That leaves only the outside loop, the only loop with no current source in it. Therefore, according to the BCM properties, there should be only one unknown current. As will be seen, there will be a second unknown in the KVL besides i_x. The other unknown will be v_2, which is **not given**. This is the case in which the control variable (v_2) is different from the unknown being sought (i_x).

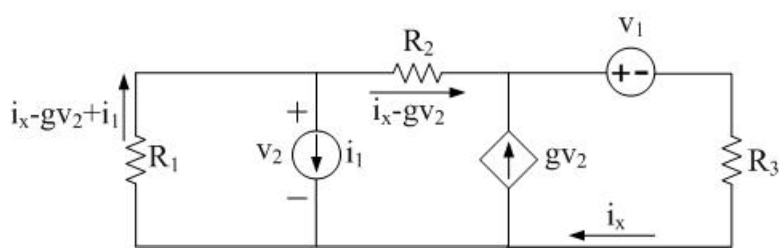


Figure 20.5 Circuit of example 20.2 labeled for applying KVL to the outer loop as part of the BCM.

KVL outside loop: $R_1(i_x-gv_2+i_1)+R_2(i_x-gv_2)+v_1+R_3i_x=0$ (20.3)

There are two unknowns in this equation: i_x and v_2. Therefore, another equation is needed. This other equation is called the **control equation**. The control equation is just another equation using a basic law (Ohm's law, KVL, KCL) that relates the control variable to other quantities in the circuit. For this particular circuit, the control equation is Ohm's law applied to resistance R_1.

Control Equation: $$v_2 = R_1(-i_x + gv_2 - i_1) \tag{20.4}$$

Factoring (20.3) and (20.4):

$$i_x(R_1 + R_2 + R_3) + v_2(-gR_1 - gR_2) = -R_1 i_1 - v_1 \tag{20.5}$$

$$i_x(-R_1) + v_2(gR_1 - 1) = i_1 R_1 \tag{20.6}$$

Cramer's Rule
$$i_x = \frac{\begin{vmatrix} -R_1 i_1 - v_1 & -gR_1 - gR_2 \\ i_1 R_1 & gR_1 - 1 \end{vmatrix}}{\begin{vmatrix} R_1 + R_2 + R_3 & -gR_1 - gR_2 \\ -R_1 & gR_1 - 1 \end{vmatrix}} \tag{20.7}$$

Check units before proceeding. If any terms are incompatible with the other terms, you must trace your steps backwards and find out where the error occurred, correct the error and proceed forward again. All of the terms in (20.7) are consistent, with the result being a current in amps. Therefore it is reasonable to proceed.

$$\boxed{i_x = \frac{(-R_1 i_1 - v_1)(gR_1 - 1) - (i_1 R_1)(-gR_1 - gR_2)}{(R_1 + R_2 + R_3)(gR_1 - 1) - (-R_1)(-gR_1 - gR_2)}} \tag{20.8}$$

Suppose you want to be certain that your solution for i_x is correct. If you solve the problem using the NVM and get the same result, the likelihood of getting the exact same wrong answer is extremely small. The amount of work required to solve the problem using the BCM required about one page. Let's compare the amount of work required by the NVM as well as check our answer for i_x. The circuit redrawn and labeled for using NVM is shown in figure 20.6.

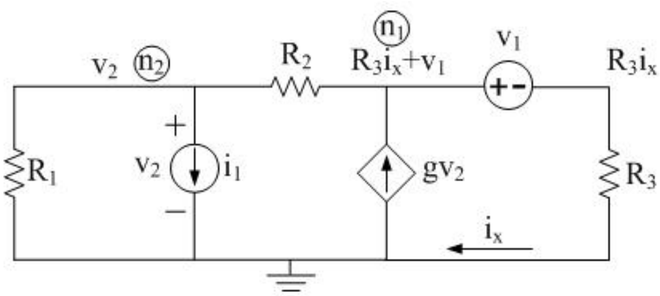


Figure 20.6 Circuit of example 20.2 labeled for solution using NVM.

With the circuit labeled for solution using the NVM, the control variable v_2 becomes an unknown node voltage. Therefore the control equation will be a KCL at node n_2.

KCL n_1: $$\frac{R_3 i_x + v_1 - v_2}{R_2} - gv_2 + i_x = 0 \tag{20.9}$$

KCL n_2:

$$\frac{v_2}{R_1} + i_1 + \frac{v_2 - (R_3 i_x + v_1)}{R_2} = 0 \tag{20.10}$$

KCL n_1 Factored:

$$i_x\left(1 + \frac{R_3}{R_2}\right) + v_2\left(\frac{-1}{R_2} - g\right) = \frac{-v_1}{R_2} \tag{20.11}$$

KCL n_2 Factored:

$$i_x\left(\frac{-R_3}{R_2}\right) + v_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = -i_1 + \frac{v_1}{R_2} \tag{20.12}$$

Cramer's Rule

$$i_x = \frac{\begin{vmatrix} \frac{-v_1}{R_2} & \frac{-1}{R_2} - g \\ -i_1 + \frac{v_1}{R_2} & \frac{1}{R_1} + \frac{1}{R_2} \end{vmatrix}}{\begin{vmatrix} 1 + \frac{R_3}{R_2} & \frac{-1}{R_2} - g \\ \frac{-R_3}{R_2} & \frac{1}{R_1} + \frac{1}{R_2} \end{vmatrix}} \tag{20.13}$$

$$\boxed{i_x = \frac{\left(\frac{-v_1}{R_2}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \left(-i_1 + \frac{v_1}{R_2}\right)\left(\frac{-1}{R_2} - g\right)}{\left(1 + \frac{R_3}{R_2}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \left(\frac{-R_3}{R_2}\right)\left(\frac{-1}{R_2} - g\right)}} \tag{20.14}$$

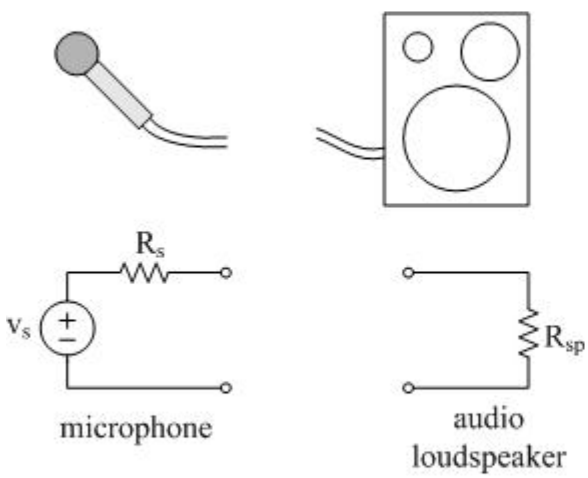
It is left to the reader to perform the algebraic manipulation of (20.14) that proves that NVM gives the exact same solution as BCM (20.8). It could be argued the NVM actually takes less work than the BCM. The solution effort is virtually the same with either technique for this problem.

Chapter 21

General Voltage Amplification and Operational Amplifiers (Op-Amps)

General Voltage Amplification

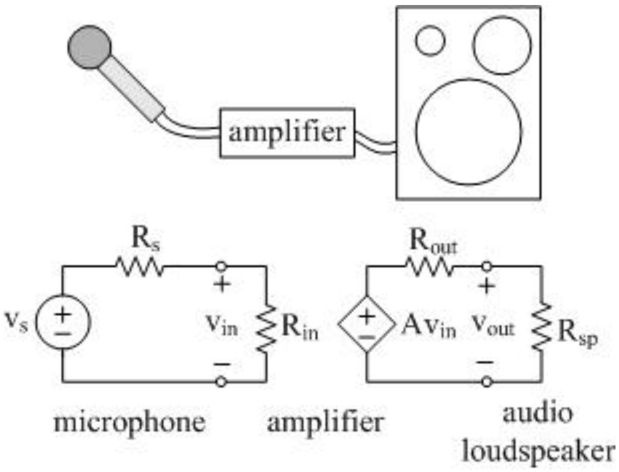
Consider a microphone and an audio loudspeaker. A typical unpowered microphone can produce on the order of 10-20 mV. With an internal resistance of 150-300 Ω, the maximum current that the microphone can produce is on the order of tens of μA. Audio loudspeakers are designed to have input resistances of 4, 8, 16, or 32 Ω with 8 Ω being the most typical. A pictorial diagram along with the schematic of the equivalent circuit is shown below in figure 21.1.



The audio loudspeaker is shown with three drivers, a woofer, a midrange, and a tweeter. These drivers each respond to a different set of frequencies in the audio spectrum (20 Hz – 20 kHz for humans). Inside the speaker there are circuit elements (inductors and capacitors) that provide the frequency filtering needed for each driver, and to provide a constant 8 Ω of impedance through the entire audio frequency range as seen at the speaker's input terminals. The drivers and the electronic crossover network (filter and impedance matcher just described) can be accurately represented by a single 8 Ω resistor.

Figure 21.1 Pictorial diagram and schematic of an audio microphone and audio loudspeaker.

If the microphone is connected to the audio loudspeaker, do you think that if you talk into the microphone anything will be heard from the loudspeaker? You should be able to explain why or why not. The most direct approach to answering the question would be to connect the microphone in the schematic to the audio loudspeaker (8 Ω resistor) and solve for the amount of power delivered to the speaker. Then question whether or not that is enough power to produce sound by the speaker. The answer is that you would hear nothing. If you solve for the power delivered to the speaker it is on the order of tens of nW, which is much too small to power the speaker. What is needed? Hopefully you concluded that an amplifier is needed. If we represent an amplifier by its Thevenin equivalent circuit at its input and its Thevenin equivalent circuit at its output and connect the amp to the microphone and the speakers, the diagram and circuit that results is shown in Figure 21.2. There are three parameters of the amplifier's equivalent circuit; the input resistance to the amplifier R_{in}, the output resistance of the amp R_{out}, and A, the voltage gain of the amp. The objective of the amplifier is to produce a usable output voltage at the audio loudspeaker's input that is an exact replica of the electrical signal produced by the microphone.



When a person speaks into the microphone, the acoustical wave vibrates a magnet relative to a coil, inducing a voltage across the microphone output terminals in accordance with the acoustic signal. The amplifier takes this weak signal and amplifies it to a value that is large enough to drive the audio loudspeaker. Every speaker has a power rating, which, if exceeded, could damage the speaker. In general, it is a good idea to match the system components such as the amplifier and the speakers.

Figure 21.2 Pictorial diagram and schematic of audio public address system (PA system).

If the power rating of the speakers is 100 W, then it would make sense to utilize a 100 W amplifier. This is not necessary, however. A 40 W amp can be used to drive the speaker. It just will not produce an output sound from the speaker as loud as the 100 W amp.

By applying voltage division twice it is possible to get an expression for the output voltage v_{out} in terms of the circuit elements and amp parameters. The result is

$$v_{out} = v_s \, \mathrm{A}\left[\frac{R_{in}}{R_{in}+R_s}\right]\left[\frac{R_{sp}}{R_{sp}+R_{out}}\right]$$

It should be apparent from both the circuit diagram and the analytical expression for v_{out} that we desire:

R_{in} largest reasonable value
R_{out} smallest reasonable value
A as large as possible

You should be able to explain why these characteristics of an amplifier are desirable.

Operational Amplifiers (Op-Amps)

An op-amp is an integrated circuit voltage amplifier first developed in vacuum tube technology in 1941, refined in 1947 and formally defined in 1947. In 1963 the first monolithic (all parts on a single chip) was invented by Bob Widlar at Fairchild electronics. A “chip” or “microchip” is an electronic circuit manufactured by integrating several electronic devices into a single semiconductor substrate. The single transistor (solid state replacement of the vacuum tube) was invented in 1947. The number of transistors per unit area of an integrated circuit has grown exponentially following Moore’s law, which states that integrated circuit transistor density doubles every year, has been very accurate since the Intel 4004 (2,300 transistors in 1971) to 5.5

million transistors on the Pentium pro in 1975. It is believed that Moore's law is nearing an end due to the miniaturization of the transistor to the atomic level. Gordon E. Moore was the Cofounder of Intel Electronics in 1965.

If you were to take the cover off virtually any consumer electronic system (e.g. DVD players, amplifiers, TVs, radios to name a few) you would find integrated circuits. An example of an integrated circuit, consider the LM386 audio amplifier shown in Figure 21.3. This integrated circuit or IC has eight pins, which are identified by counting counter clockwise beginning from the notch at one end of the IC. The LM386 shown is in a dual in-line package (DIP), and will fit very easily into a breadboard or printed circuit board. The most common type of op-amp is the type 741 and has eight pins and looks exactly like the DIP in Figure 21.3.

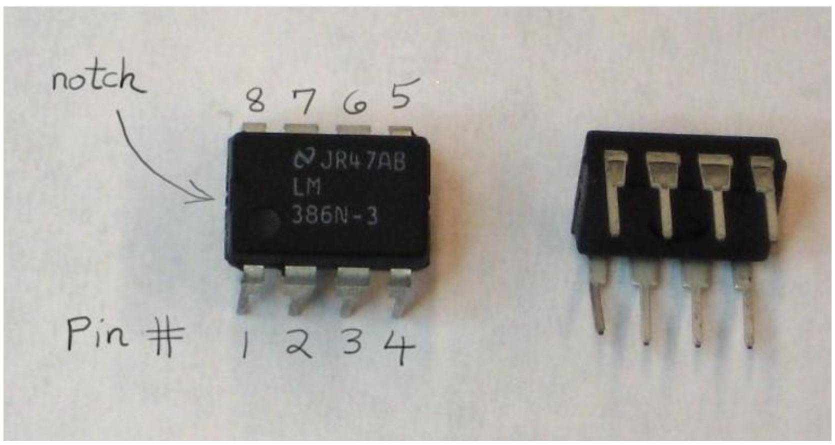
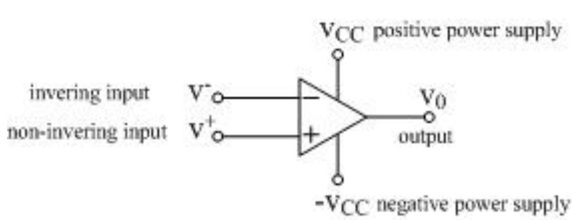


Figure 21.3 An example of a dual in-line package integrated circuit, the LM386 audio amplifier.

Op-Amp Symbol

The symbol for an op-amp is a triangle. There are three signal terminals whose node voltages are (v^+,v^-,v_o) and two power supply terminals with node voltages (v_{CC} and $-v_{CC}$). All of these voltages are node voltages. Therefore there must be a ground node in all op-amp circuits.



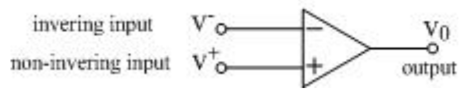


Figure 21.4 Op-Amp symbol with and without the external DC power supply connections shown.

When analyzing the signal path in an op-amp circuit, the symbol is typically shown without the DC power supply connections. A common range of values of v_{CC} is 5-15 V although there are op-amps that are designed for high power applications that operate at much higher voltages. For example, the LM675 op-amp operates with $16\ V < v_{CC} < 60\ V$. It is important to recognize that it is physically impossible for the output voltage to be driven higher than v_{CC} by the op-amp.

Op-Amp Model

An op-amp is modeled by its Thevenin equivalent circuit at its input and its Thevenin equivalent circuit at its output.

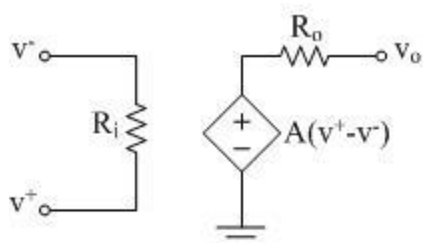
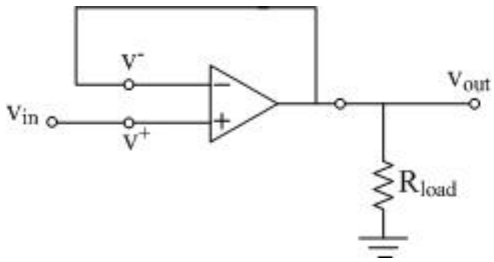


Figure 21.5 Model of an operational amplifier based on Thevenin equivalent circuits.

This model is very similar to the model for a general voltage amplifier. However, notice the following differences; the input resistance to the op-amp is R_i not R_{in}, the output resistance is R_o instead of R_{out} and there is a ground node on the bottom of the dependent voltage source. There are three parameters in the model. The input resistance to the op-amp is R_i, the output resistance of the op-amp is R_o and the voltage amplification factor is A. As previously mentioned, the most commonly used op-amp is the type 741. If you google "741 op-amp" under images, you can see exactly what is inside the op-amp. The type 741 op-amp has the following parameter values;
$A = 2\cdot 10^5$, $R_i = 10\ M\Omega$, $R_o = 75\ \Omega$, $v_{CC} \leq 18\ V$, $R_{load} \geq 2\ k\Omega$. As an example of using the op-amp model to make calculations, consider the voltage follower circuit shown below.

Example 21.1 Voltage Follower Circuit

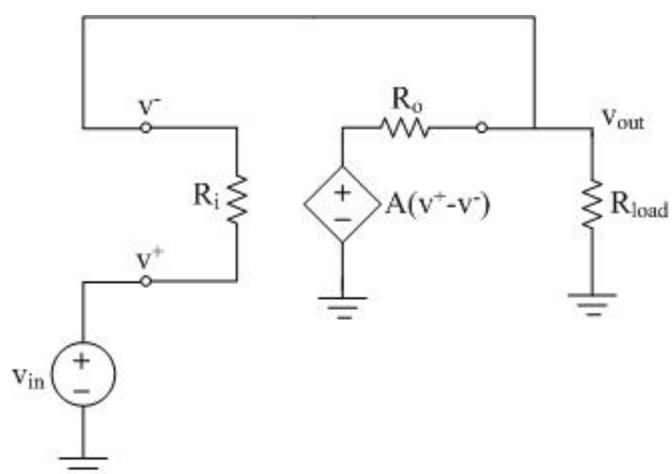


Given R_{load} = 2 kΩ and a 741 op-amp, calculate a) the voltage gain and b) the input resistance.

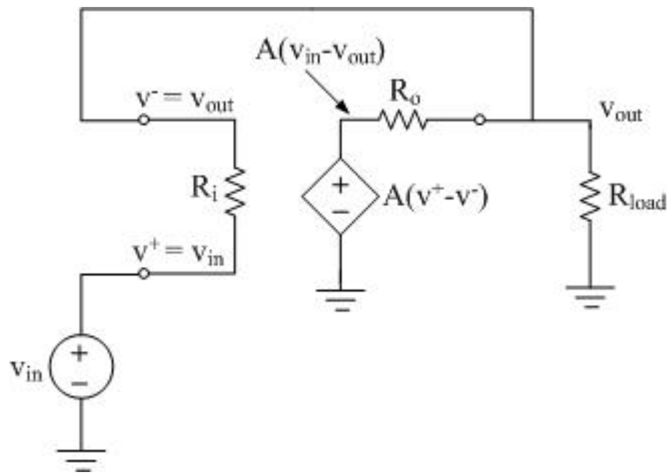
Solution: The first thing to notice is that the node voltage at the input is v_{in}, and it is implied that a source whose value is v_{in} is connected to that node. This is the stimulus to the circuit. Even though v_{out} is shown in the very same manner as v_{in}, there is no source present with value v_{out}. v_{out} is the response of the circuit. You must also understand what the problem is asking for. Part a) is asking for the voltage gain, and part b) is asking for the input resistance. These quantities are defined as follows.

$$\text{Voltage gain} \triangleq \frac{v_{out}}{v_{in}} \qquad R_{in} \triangleq \frac{v_{in}}{i_{in}}$$

In order to solve this problem, we must first replace the op-amp symbol with its model.



When solving for the voltage gain of a circuit, treat v_{in} as if it were known and v_{out} as the unknown. Solve for v_{out} in terms of v_{in}, and then factor out v_{in} from the right hand side of the equation. Once you have done this, forming the ratio of v_{out} over v_{in} is very easy. In the voltage follower circuit it can be seen that $v^+ = v_{in}$, $v^- = v_{out}$ and therefore the node voltage above the dependent voltage source is $A(v_{in} - v_{out})$. With this information inserted into the circuit diagram, we get the following circuit.



Applying KCL at the output node yields the equation:

$$\frac{v_{out} - v_{in}}{R_i} + \frac{v_{out} - A(v_{in} - v_{out})}{R_o} + \frac{v_{out}}{R_{load}} = 0$$

Factoring v_{out} on the left side of the equation and v_{in} on the right side yields:

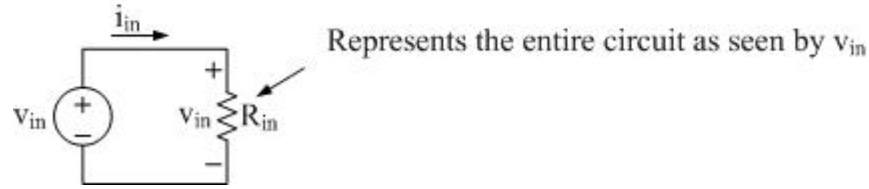
$$v_{out}\left(\frac{1}{R_i} + \frac{1+A}{R_o} + \frac{1}{R_{load}}\right) = v_{in}\left(\frac{1}{R_i} + \frac{A}{R_o}\right)$$

The solving for the voltage gain results in the following:

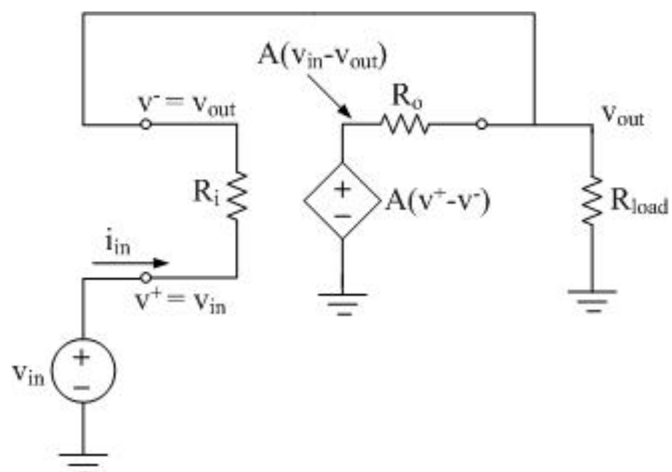
$$\frac{v_{out}}{v_{in}} = \frac{\frac{1}{R_i} + \frac{A}{R_o}}{\frac{1}{R_i} + \frac{1+A}{R_o} + \frac{1}{R_{load}}} = \frac{\frac{1}{2\cdot 10^6} + \frac{2\cdot 10^5}{75}}{\frac{1}{2\cdot 10^6} + \frac{1+2\cdot 10^5}{75} + \frac{1}{2\cdot 10^3}} = 0.999995$$

Voltage gain is a dimensionless quantity and hence has no units. Once again, it is important to check the units of the final symbolic result to ensure that all the units are consistent and the resulting expression is dimensionless. Only when that has been established does it make sense to substitute numerical values.

To obtain the input resistance, the input current must be identified. Imagine the entire circuit as seen by the input voltage source v_{in} as being equivalent to a single resistance whose value is R_{in}. Such a circuit would be a simple series (and parallel) combination of v_{in} and R_{in}. Using Ohm's law, the input current must be in the direction of the voltage drop for the resistant R_{in} and consequently in the direction of voltage rise for the source v_{in}.



The correct location and direction of the input current i_{in} is now inserted in the proper location of the circuit diagram.



The process of finding the input resistance is to obtain an expression for the input current i_n in terms of v_{in}. Once again we will treat v_{in} as a known quantity and solve for i_{in} as the unknown, factor out i_{in} on the left side of the equation and v_{in} on the right side and finally solve for the ratio of v_{in} to i_{in} to obtain R_{in}. With this approach applied to the voltage follower circuit, the following results.

$$i_{in} = \frac{v_{in} - v_{out}}{R_i} = v_{in}\left(\frac{1 - \frac{v_{out}}{v_{in}}}{R_i}\right)$$

Solving for the input resistance yields:

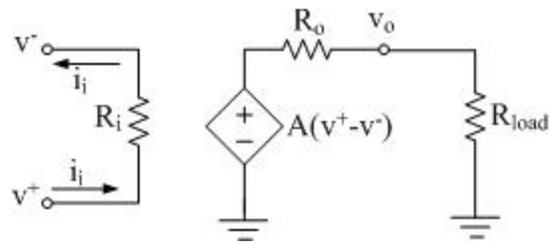
$$R_{in} = \frac{v_{in}}{i_{in}} = \frac{R_i}{1 - \frac{v_{out}}{v_{in}}} = \frac{2 \cdot 10^6}{1 - 0.999995} = 400\ G\,\Omega$$

which is a very large value. Any source connected to a voltage follower would be required to produce a very small current. This is the key to the purpose of a voltage follower circuit. It provides the same voltage as its input voltage (hence the name voltage follower) and provides a huge input resistance thereby drawing a miniscule current from the source to the voltage follower. It is very important to recognize the difference between R_{in} and R_i. R_i.is the input resistance to the op-amp, that is, the resistance looking into the input terminals of the op-amp. R_{in}

is the input resistance to the whole circuit. That is, R_{in} is the resistance seen by the input voltage source v_{in} looking into the circuit as a whole. R_{in} is the resistance looking into the circuit between the node above v_{in} and ground. R_{in} is a very different quantity than the op-amp input resistance R_i.

The Ideal Op-Amp

Consider the op-amp model driving a load resistance R_{load}.



A **single assumption** is made to derive the ideal op-amp. Namely, that the voltage amplification factor A → ∞. This is clearly a false assumption. In engineering there are two reasons for making an assumption. First, the assumption should make computations easier. Second, the assumption should make the results easier to interpret. There is a caveat of course, that the easier computation must result in accurate results. It would be useless to make an assumption if the resulting computation is inaccurate. Therefore, as we solve ideal op-amp circuits, there should be a lingering question regarding the accuracy of the result. If the ideal op-amp results in an inaccurate result, the op-amp model must be utilized. Examining the output side of the op-amp circuit above allows us to use voltage division to obtain an expression for v_o.

$$v_o = A(v^+ - v^-)(\frac{R_{load}}{R_{load} + R_o})$$

Solving for the difference between the op-amp input voltages produces

$$v^+ - v^- = v_o \frac{(R_{load} + R_o)}{A(R_{load})}$$

V_o is constrained by the DC power supply voltage such that $-v_{CC} < v_o < v_{CC}$. R_{load} is finite and non-zero. R_o is finite and non-zero. Therefore, we can take the limit of the difference between v^+ and v^- as A → ∞.

$$\lim_{A\to\infty} v^+ - v^- = \lim_{A\to\infty} v_o \frac{(R_{load} + R_o)}{A(R_{load})} = 0$$

This leads to the first ideal op-amp condition $v^- = v^+$. If we define the input current to the op-amp as the current that enters the non-inverting input and exits the inverting input as i_i and take the limit of this quantity as A → ∞

$$\lim_{A\to\infty} i_i = \lim_{A\to\infty} \frac{v^+ - v^-}{R_i} = \lim_{A\to\infty} v_o \frac{(R_{load} + R_o)}{A(R_{load})R_i} = 0$$

Which leads us to the second ideal op-amp condition $i_i = 0$.

To summarize the derivation of the ideal op-amp, a **single assumption** is made: A → ∞. This single assumption results in two ideal op-amp conditions, which are consequences of the single assumption. It is incorrect to call these ideal op-amp conditions assumptions.

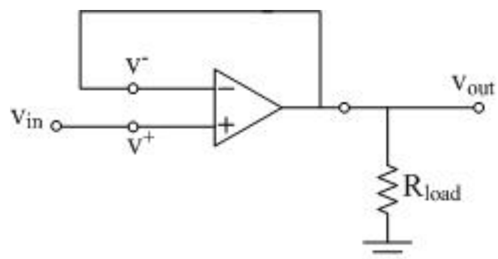
Ideal op-amp conditions: 1) $v^- = v^+$ and 2) $i_i = 0$

If you consider the ideal op-amp conditions together, they are contradictory. The first condition states that there is a short circuit at the input and the second condition states that there is an open circuit at the input. Therefore, the ideal op-amp leads to erroneous results. But the question remains, is the accuracy acceptable? Is analysis made easier? These questions should be at the forefront of your consciousness as you solve ideal op-amp circuits.

Ideal Op-Amp Circuits

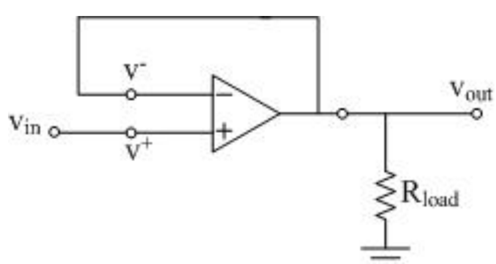
As the first example of solving an op-amp circuit using the ideal op-amp, let us reconsider the voltage follower circuit.

Example 1 Voltage Follower Circuit



Given the op-amp is ideal, find a) the voltage gain and b) the input resistance.

Solution: The way to begin all ideal op-amp circuit analyses is to write down the two ideal op-amp conditions and show them on the circuit diagram. After labeling the voltage follower circuit with the ideal op-amp conditions, the following circuit diagram results.



With $v^- = v^+$ and $i_i = 0$ clearly labeled on the circuit diagram, it is immediately evident that $v^+ = v_{in}$ and $v^- = v_{out}$. Therefore, $v_{out} = v_{in}$ and the voltage gain is 1. To obtain the input current i_{in}, it is recognized that $i_{in} = i_i = 0$. Therefore the input resistance is calculated to be ∞. Do not automatically assume that the input resistance of an ideal op-amp circuit is ∞. For many ideal op-amp circuits, $R_{in} \neq \infty$, but is a finite. Are the values for the voltage gain and the input resistance accurate? This question can be answered analytically by calculating percent error. The percent error for the voltage gain is

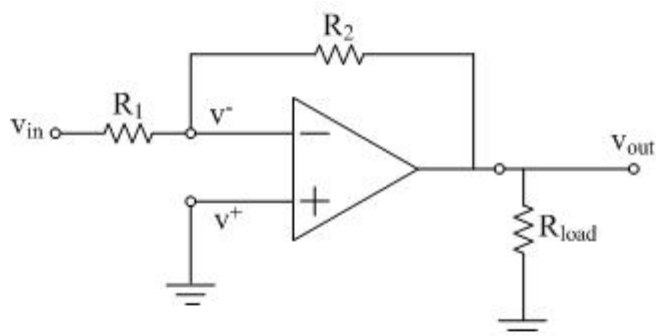
$$\%\,error = \frac{1 - 0.999995}{0.999995} \times 100 = 5 \cdot 10^{-4}\ \%$$

which indicates that the model is not needed to solve for voltage gain. If the work required using the ideal op-amp is compared to the work required using the op-amp model, the difference is significant. Use of the ideal op-amp to calculate the input resistance is a different story. Consider the percent difference in the calculation of the input resistance

$$\%\,error = \frac{\infty - 400 \cdot 10^9}{400 \cdot 10^9} \times 100 = \infty$$

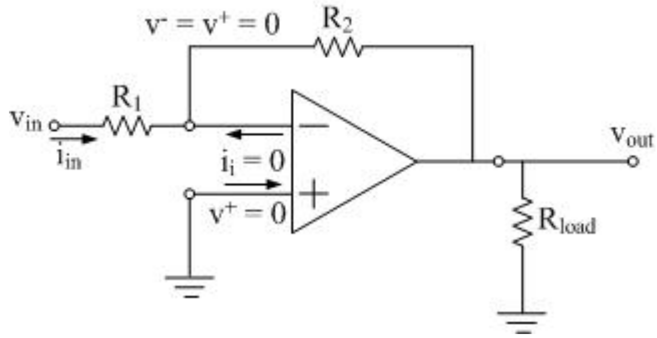
which indicates that use of the ideal op-amp to accurately calculate the input resistance of the voltage follower circuit, the model must be used.

Example 2. Inverting Amplifier



Given that the op-amp is ideal and R_1, R_2 and R_{load}, calculate a) the voltage gain and b) the input resistance.

Solution: The first thing to do in any ideal op-amp problem is to write down the two ideal op-amp conditions ($v^- = v^+$ and $i_i = 0$) on the circuit diagram. Since v^+ is connected directly to ground, its node voltage is zero making $v^- = 0$.



Applying KCL at the inverting node gives

$$\frac{0 - v_{in}}{R_1} - \ i_i + \frac{0 - v_{out}}{R_2} = 0$$

But we know that $i_i = 0$. Solving for the ratio of v_{out} to v_{in} we get

$$\frac{v_{out}}{v_{in}} = -\frac{R_2}{R_1}$$

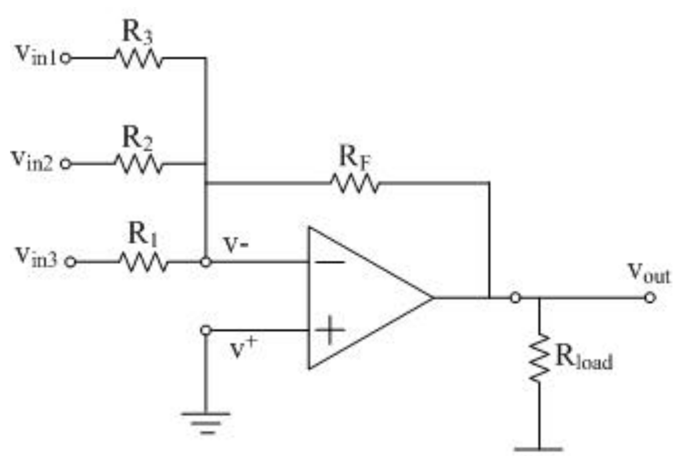
The minus sign in the result for the voltage gain is the reason why this amplifier is called an inverting amplifier. The input resistance is a ratio of v_{in} to i_{in} with i_{in} defined as shown. In this circuit

$$i_{in} = \frac{v_{in} - 0}{R_1}$$

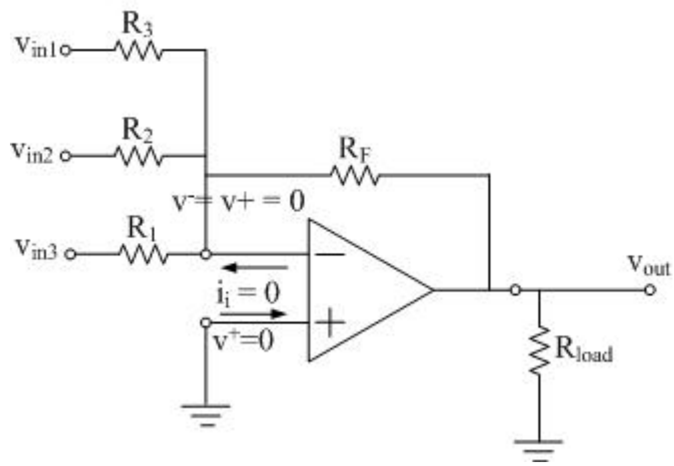
which leads to the result that the input resistance is $R_{in} = \frac{v_{in}}{i_n} = R_1$. This is an important example that illustrates that the input resistance to an ideal op-amp circuit is not ∞ in every case. Only those cases in which $i_{in} = i_i$ is the input resistance ∞, making use of the op-amp model necessary to obtain an accurate result.

Another commonly used circuit is essentially an expansion of the inverting amplifier and is called a **summing amplifier**.

Example 3 Summing Amplifier



Given the op-amp is ideal, calculate the output voltage v_{out} as a function of the input voltages v_{in1}, v_{in2}, and v_{in3}. As in all ideal op-amp problems, the first step is to label, on the circuit diagram, the two ideal op-amp conditions.



Applying KCL at the inverting node yields

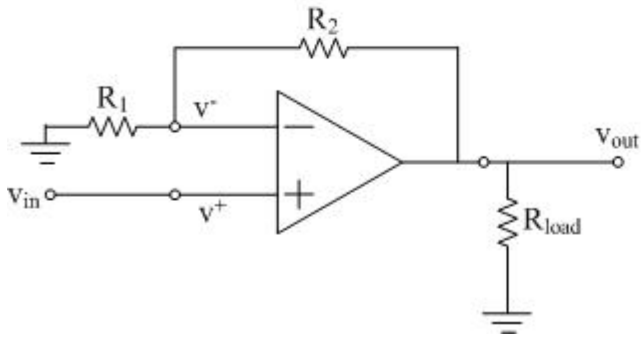
$$\frac{0 - v_{in1}}{R_1} + \frac{0 - v_{in2}}{R_2} + \frac{0 - v_{in3}}{R_3} - i_i + \frac{0 - v_{out}}{R_F} = 0$$

where RF refers to the feedback resistance. Algebraic manipulation of the equation above leads to the relationship between v_{out} and the three input resistances.

$$v_{out} = -R_F\left(\frac{v_{in1}}{R_1} + \frac{v_{in2}}{R_2} + \frac{v_{in3}}{R_3}\right)$$

If the resistors were replaced with variable resistors (potentiometers used with the wiper terminal and one of the other terminals), the summing amplifier becomes a very useful circuit for audio mixing. The feedback resistance R_F acts as a volume control affecting the gain of all input channels equally. Each of the input resistances R_1, R_2, and R_3 become balance controls for the mixer. A large value of R_1 fades channel 1 out of the output. A small value of R_1, compared to R_2 and R_3, would make channel 1 dominant.

Example 4 Non-Inverting Amplifier

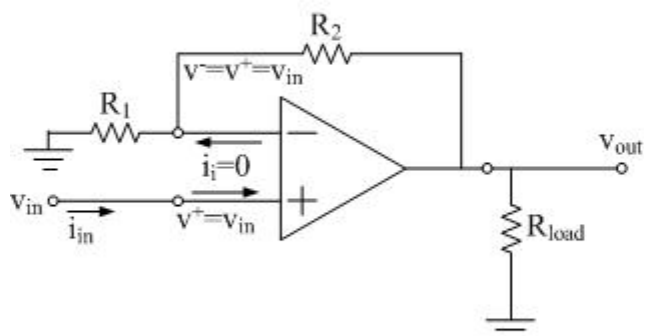


$R_1 = 1\ k\Omega$
$R_2 = 9\ k\Omega$
$R_{load} = 2\ k\Omega$

Figure 21.4 A non-inverting amplifier circuit

Given the op-amp is ideal, calculate both the voltage gain and input resistance.

Solution: As with all ideal op-amp circuits, the first step is to label the ideal op-amp conditions directly on the diagram. Since v^+ is connect directly to v_{in}, then v^- is also equal to v_{in}.



Applying KCL at the inverting node yields

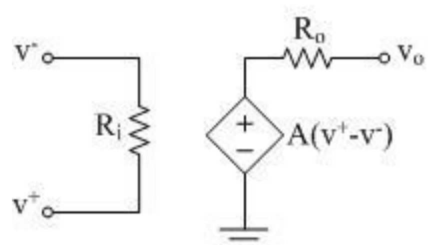
$$\frac{v_{in} - 0}{R_1} - i_i + \frac{v_{in} - v_{out}}{R_2} = 0$$

Since the op-amp is ideal, $i_i = 0$. Algebraic manipulation results in the voltage gain

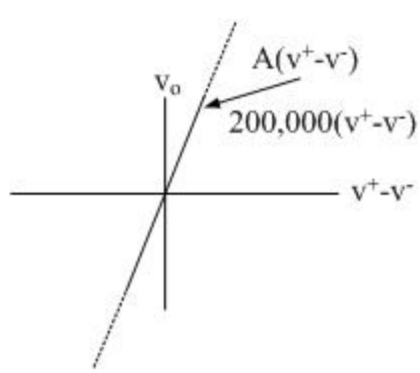
$$\frac{v_{out}}{v_{in}} = 1 + \frac{R_2}{R_1} = 1 + \frac{9 \cdot 10^3}{1 \cdot 10^3} = 10$$

The input resistance cannot be calculated accurately because $i_{in} = i_i$ and a result of $R_{in} = \infty$ occurs. Therefore, the model for the op-amp must be used if an accurate value for R_{in} is required. Many students apply KCL at the output node of the op-amp. This process requires great care because one must include the current flowing into the op-amp output. Forgetting this term is a serious error. If KCL were applied to the output node of the op-amp yields the same result as the KCL at the inverting node. As an exercise, you are to perform the KCL at the output node of the non-inverting amplifier. The key to obtaining the current that flows into the op-amp output is to realize that this current flows through R_o and the dependent voltage source to ground. Therefore this current is the sum of all the currents flowing out of all the other grounds.

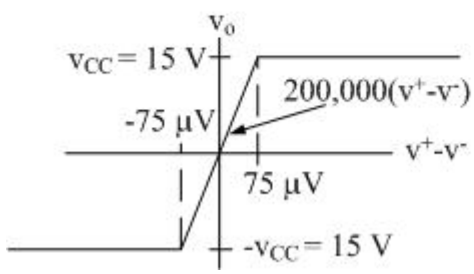
In each of the examples of op-amp circuits so far, the op-amp has been operating in its **linear** region. **In the linear region of operation, the op-amp produces whatever current is necessary to make $v^- = v^+$.** In order to understand what it means for the op-amp to operate in its linear region, we will examine the **voltage transfer characteristic** of an open loop, open circuit output 741 op-amp. The voltage transfer characteristic is the functional relationship between the input and the output. Consider a type 741 op-amp with the output connected to nothing external to the op-amp and consider the relationship between the output voltage v_o and the input voltage v^+-v^-. To do this, we will need to know v_{CC}. We will use v_{CC} = 15 V.



If we were to graph v_o as a function of v^+-v^-, what would the function look like? Many students come to the conclusion that it looks like a line as in

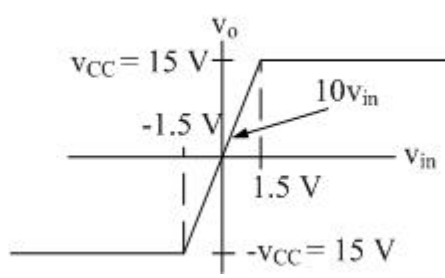


ince the voltage drop across Ro is zero (no current through it). When asked what the output voltage is when $v^+-v^- = 1$ V, students eagerly answer 200,000 V! How can an op-amp with a DC power supply of 15 V produce 200,000 V? With a little thought, students come to the conclusion that the voltage transfer function is only linear until the output voltage v_0 reaches $\pm v_{CC} = \pm 15$ V. The correct voltage transfer function expressed graphically is shown below.

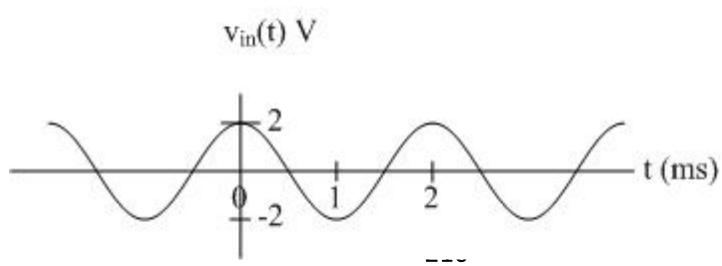


When $|v^+-v^-|$ exceeds 75 μV, the op-amp **saturates** at the power supply rails. As an exercise, graph the voltage transfer function of the non-inverting amplifiers in Figure X using the values given and $\pm v_{CC} = \pm 15$ V

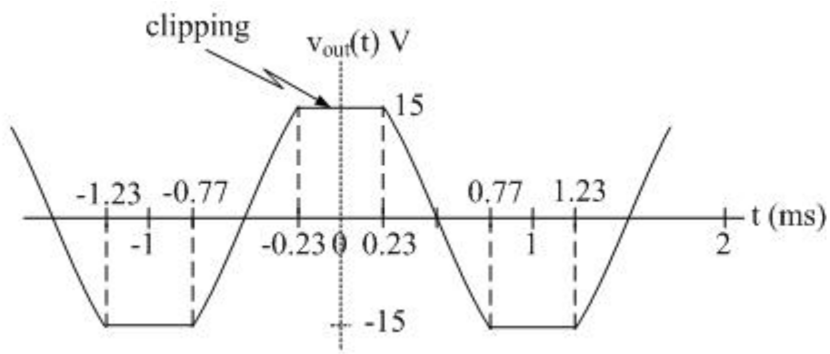
Solution: The voltage transfer function is the output voltage as a function of the input voltage. We know from the analysis of the non-inverting amplifier in Figure X that the voltage gain is 10. However, once the input voltage reaches |1.5 V| or greater, the output of the op-amp will saturate. Taking this into account, the voltage transfer function is shown graphically in Figure X.



Exercise: Given the non-inverting amplifier in Figure X, $\pm v_{CC} = \pm 15$ V, and the input sinusoidal waveform as shown below, accurately graph $v_{out}(t)$.

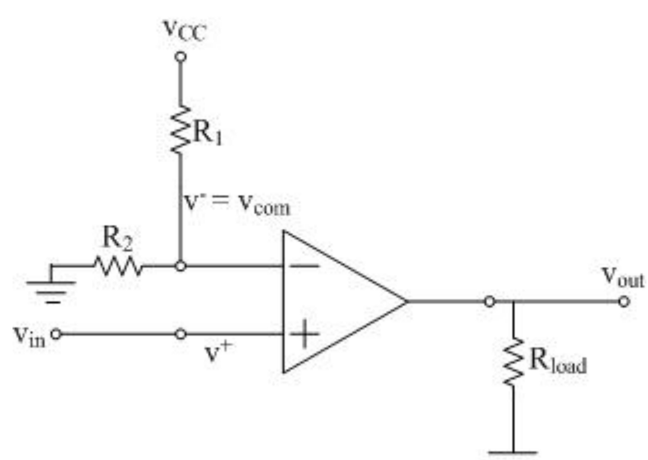


Solution: The non-inverting amplifier in Figure X has a voltage gain of 10. Therefore, when the input voltage magnitude exceeds 1.5 V, the output will be clipped at ± 15 V depending on the sign of the input voltage. We need to find the times at which the input voltage is equal to ±1.5 V. To find these times, we set $2\cos(\pi \cdot 10^3 t) = \pm 1.5$ and solve for time t. Note that this is a 500 Hz sinusoid, and remember that $\omega = 2\pi f$. Solving for the times when $2\cos(\pi \cdot 10^3 t) = \pm 1.5$, we get t = ±230 μs centered on each *ms* of time. The resulting output voltage graph looks like



Comparators

Op-amps can also be used in their non-linear region as a comparator. Consider an op-amp with one input tied to a DC voltage v_{com}. This voltage could be produced easily with a voltage divider between v_{CC} and ground or $-v_{CC}$ and ground.



Resistors R_1 and R_2 can be chosen such that $V^- = v_{com}$ is any voltage between v_{CC} and ground in this example. Notice that the feedback loop is open. This means that the op-amp has no means by which it can force $v^- = v^+$. The output voltage will be either v_{CC} or $-v_{CC}$ depending on the comparison of v_{in} with v_{com}. If $v_{in} > v_{com}$ then $v_{out} = v_{CC}$. If $v_{in} < v_{com}$ then $v_{out} = -v_{CC}$. There exists no feedback loop between the output of the op-amp and the inverting input that is required in order for the op-amp to behave linearly.

To illustrate the behavior of a comparator consider the case in which $\pm v_{CC} = \pm 12$ V, $v_{com} = 4$ V and the input voltage is the waveform shown in Figure 21.5.

Example 24.5 Comparator

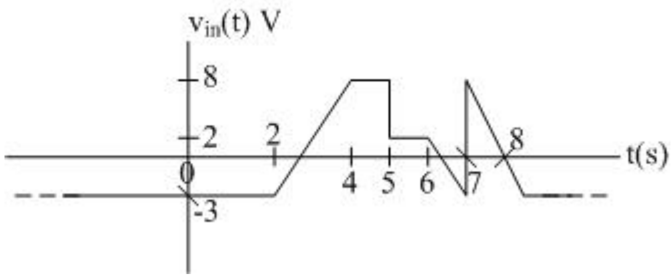


Figure 21.5 Input voltage to a comparator circuit with $v_{com} = 4$ V.

Given the input voltage waveform, obtain and graph the output voltage waveform for all time.
Solution: The output voltage will be equal to $-v_{CC} = -15$ V when $v_{in} < v_{com}$ and equal to $+v_{CC}$ when $v_{in} > v_{com}$. Compare the input voltage to 4 V. You will need equations for the line segments between 2 and 4 seconds and during the interval 7-8 seconds, and set those equations equal to 4 V to determine the output voltage transition times. The first line segment (with non-zero slope) is $v_{in}(t) = -14 + 5.5t$ V. The second line segment (with non-zero slope) is $v_{in}(t) = 64 - 8t$ V. The resulting output waveform is shown in Figure 21.

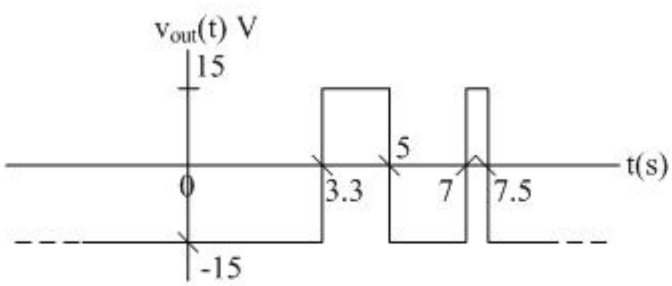


Figure 21.6 Output voltage waveform for the comparator circuit with $v_{com} = 4$ V

Chapter 22

Semiconductors, Diodes, and MOSFETS

Semiconductors

The electrical conductivity of a material is its ability to pass electric current. Examples of good electrical conductors include silver, copper, gold, and aluminum. Poor electrical conductors are called electrical insulators. Examples of good electrical insulators are diamond, glass, plastic, paper, rubber, air, and wood. Electrical semiconductors are materials exhibiting electrical conductivity that can vary with temperature over a wide range of conductivity values (insulators at low temperature and conductors at room temperature and above. Semiconductors used in electronic devices are typically silicon crystals with some of the silicon lattice atoms replaced with an impurity called a dopant.

Consider the periodic tables of the elements in the vicinity of silicon as shown in Figure 22.1.

III	IV	V
B	C	N
Al	Si	P
Ga	Ge	As

Figure 22.1 Periodic table of the elements surrounding silicon.

The column number indicates the number of electrons in the valence shell of the given element. Pure silicon at absolute zero temperature is a crystal with the repeated shape of a tetrahedron. In two dimensions, this shape can be expressed by the diagram in Figure 22.2 of a subsection of the silicon crystal.

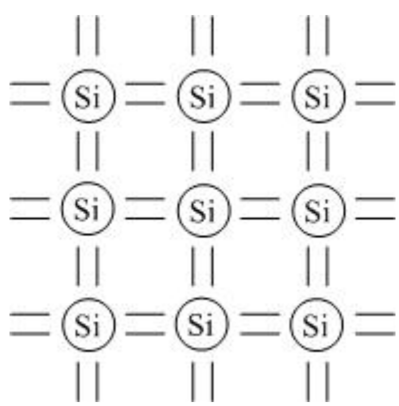


Figure 22.2 Silicon lattice in two dimensions at a temperature of absolute zero.

The lines represent electrons being shared with neighboring atoms in covalent bonds. At absolute zero temperature, there are no electrons available for conduction and hence the conductivity is zero. At room temperature (75 °F) some of the electrons in covalent bonds become free, forming electron-hole pairs. A hole is a vacant covalent bond, and for our purposes, we can consider it a positive charge carrier with the same charge magnitude as an electron. The structure of the crystal lattice at room temperature is depicted in Figure 22.3.

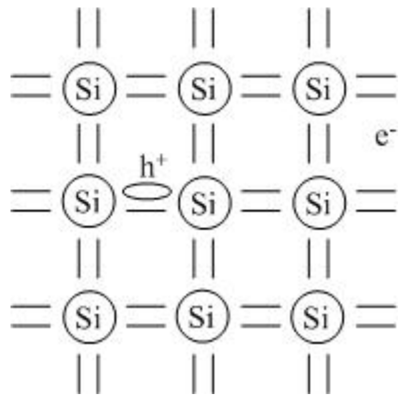


Figure 22.3 Silicon lattice shown in two dimensions at room temperature emphasizing the presence of electron-hole pairs due to room temperature.

The electron is free to move interstitially, or within the crystal (part of the lattice). The hole can move too, but its movement is restricted to swapping locations with a nearby covalently bonded electron. The hole is constricted to moving only within the lattice. At room temperature (75 °F) the density of electron-hole pairs (available for conduction) in silicon is approximately $1 \cdot 10^{10}$ per cm^3. For comparison, a good conductor has approximately 10^{22} electrons per cm^3 available for conduction at room temperature. Therefore, **pure (also called *intrinsic*) silicon is a poor conductor at room temperature**. To analytically characterize the resistance of a sample of silicon, we need to examine **resistivity**.

The resistance in Ω of a material is given by $R = \frac{\rho l}{A}$ where ρ is the resistivity, *l* is the length of the material in the direction of current flow (in *m*) and *A* is the cross-sectional area of the material perpendicular to the current flow (in m^2). The resistivity of copper at room temperature is $\rho_{copper} = 1.7 \cdot 10^{-8}\ \Omega m$. Calculation of the resistance of a copper wire with length 1 *cm* and diameter 5 *mm* produces a resistance of 8.7 μΩ. The resistivity of intrinsic silicon at room temperature is 2.5 kΩm. Calculation of the resistance of a cylinder of intrinsic silicon with a length of 1 *cm* and diameter of 5 *mm* at room temperature produces a resistance of 1.3 MΩ. By comparing the resistance of the copper wire (8.7 μΩ) with a cylinder of intrinsic silicon at room temperature with the same dimensions (1.3 MΩ) once again shows that **intrinsic silicon is a very poor conductor**.

It is possible, however, to increase the conductivity of the silicon material by introducing impurities, or dopants, into the crystal lattice. For example, boron, from column III of the periodic table, can replace up to 10^{21} silicon atoms per cm^3 in the lattice. The common range of doping concentrations is 10^{15} to 10^{21} dopant atoms per cm^3. P-type semiconductor is created in this manner. The boron doped semiconductor looks like the lattice shown in Figure 22.4. Since boron only has three valence electrons, it cannot complete the fourth covalent bond with its own valence electrons. At room temperature, the boron atom is ionized by accepting an electron, making it negative. The electron used to complete the fourth covalent bond with the boron atom in the silicon lattice comes from nearby covalent bonded electrons (leaving behind a hole) or from free electrons moving interstitially.

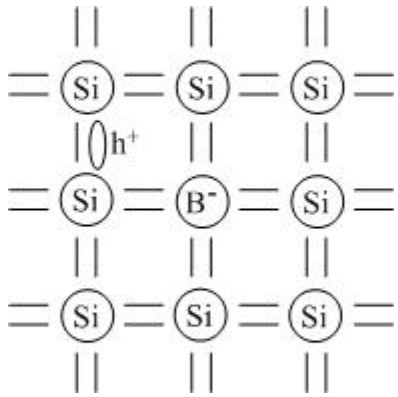


Figure 22.4 Boron doped silicon creating p-type semiconductor.

The "p" in "p-type semiconductor" stands for "positive." This is because the majority of the charge carriers in p-type semiconductor are holes, which we treat as positively charged particles. It is important to note that p-type semiconductor, which is obtained by doping silicon with an element from column III of the periodic table, is electrically neutral. There are local regions, such as the boron ions, where there is net charge, but the bulk semiconductor is neutral. Elements from column III of the periodic table are called "acceptors" because they "accept" an electron to complete the four neighboring covalent bonds. Every acceptor atom contributes a hole to the silicon crystal.

If silicon is doped with 10^{21} boron or phosphorus atoms per cm^3 (max. doping concentration), the resistivity becomes $1 \cdot 10^{-4}$ Ωcm, and a cylinder with a length of 1 *cm* and diameter of 5 *mm* at room temperature produces a resistance of 5.1 Ω. Hence, the range of resistance of this cylinder for intrinsic (pure) silicon to the most heavily doped silicon is 1.3 MΩ to 5.1 Ω at room temperature. This example illustrates the extremely wide range of resistance achievable in a semiconductor region depending on the dopant concentration.

If silicon is doped with an element from column V of the periodic table such as phosphorus, only four of the five valence electrons are needed to complete the four neighboring covalent bonds. At room temperature, the phosphorus atom is ionized because its fifth valence electron is donated to the crystal and drifts interstitially within the silicon lattice. The diagram in Figure 22.5 illustrates phosphorus doped silicon. Silicon doped with an element from column V of the periodic table is termed "n-type semiconductor" where the "n" stands for negative. This is because the majority of charge carriers in n-type semiconductor are electrons, which are negatively charged. Elements from column V are called "donors" because they donate an electron to the semiconductor as a mobile charge carrier. Every donor atom contributes an electron to the silicon crystal.

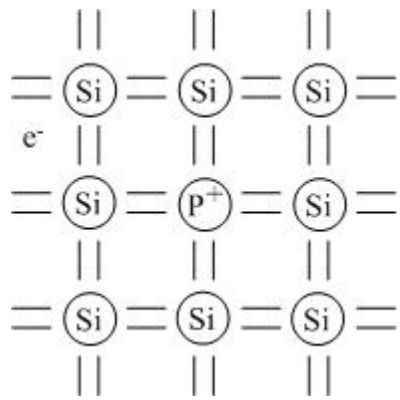


Figure 22.5 Phosphorus doped silicon creating n-type semiconductor.

When a small slab of p-type semiconductor is linked to a small slab of n-type semiconductor, a pn junction is formed. With metal contacts on the ends and terminals attached, the entire system forms a semiconductor **diode** as shown in Figure 22.6. With no applied voltage, the diode is unbiased, but there exists a region on both sides of the pn junction depleted of charge carriers. Electrons from the n-type semiconductor have diffused into the p-type semiconductor recombined with the holes in the p-doped region thereby preventing them from acting as mobile charge carriers. There is a thermal equilibrium that determines the thickness of the unbiased **depletion region**. As the diode voltage increases beyond zero (called **forward bias**), the **depletion region narrows** and current can flow. When the diode voltage is negative (called **reverse bias**), the **depletion region widens** and current is limited to its saturation current. The current-voltage relationship for a diode is given by

$$i_D = i_s(e^{\frac{v_D}{nv_T}} - 1)$$

where i_s is the saturation current determined by the doping concentration of each side of the junction, the cross sectional area of the pn junction and temperature, n is the nonideality factor (typically between 1 and 2) and v_T is the thermal voltage given by $\frac{kT}{q}$ where k is Boltzmann's, and q is the absolute value of the charge of an electron constant (v_T = 25 mV at room temperature).

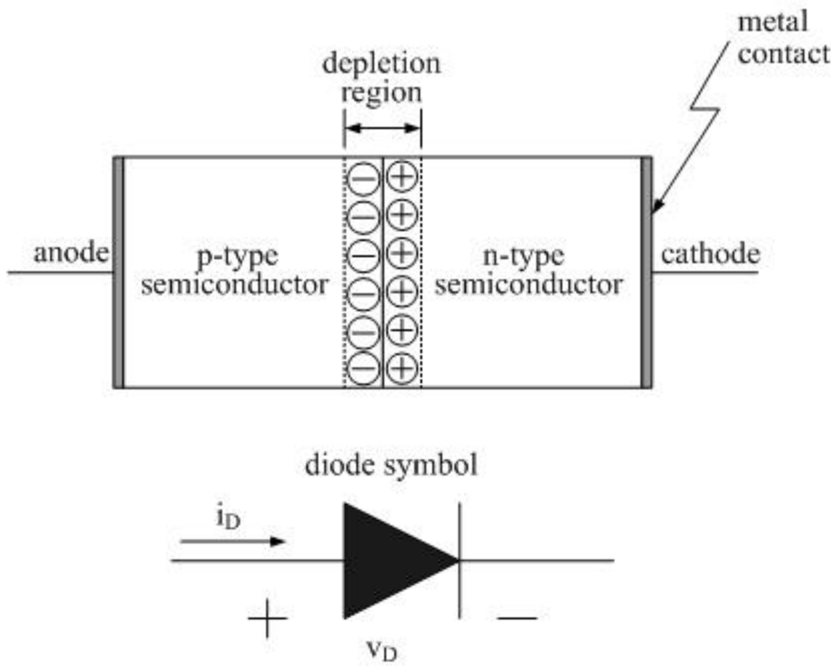


Figure 22.6 pn junction with no applied voltage and the diode symbol.

When the diode voltage is negative (reverse bias), the current is essentially $i_D = -i_s$, and the diode is typically considered "off" and not conducting. When the diode voltage is positive (forward bias), current grows exponentially with voltage. Significant current can flow through a silicon diode at a diode voltage $v_D = 0.8$ V. The most important point here is that a diode behaves as a one way valve for current. Current flows easily from p to n, but not from n to p. Graphically, the current-voltage relationship is shown in figure 22.7

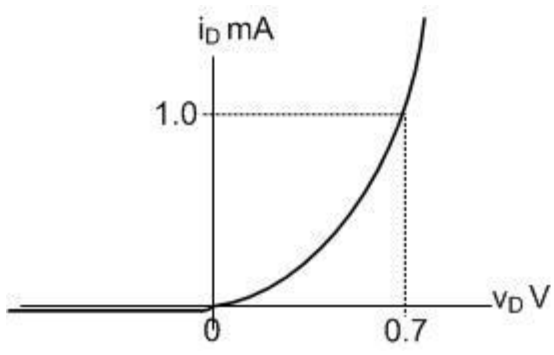


Figure 22.7 Current-voltage relationship for a silicon diode. The specific numerical values for i_D and v_D are for illustrative purposes only.

A picture of several light emitting diodes and one signal diode are shown in Figure 22.8.

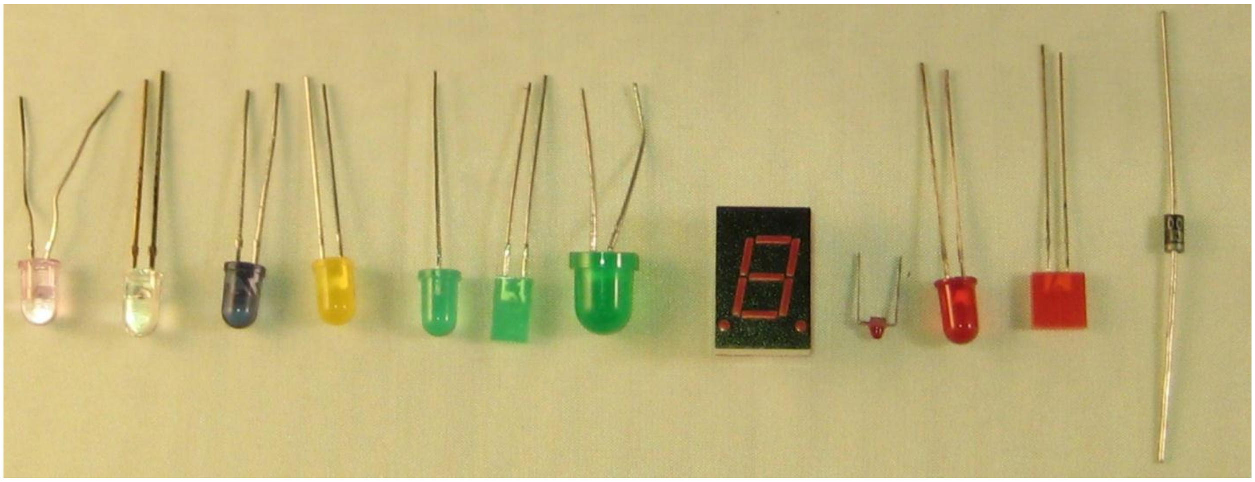


Figure 22.8 Several light emitting diodes (LEDs) and one signal diode. LEDs include infrared, white, blue, yellow, green with different shapes and power output, red seven segment LED display, and red LEDs of different size and shape. The signal diode is on the far right.

Metal Oxide Semiconductor Field Effect Transistors (MOSFETs)

A metal oxide semiconductor field effect transistor (MOSFET) is a four-terminal device that has a physical structure similar to the simplified diagram in Figure 22.9.

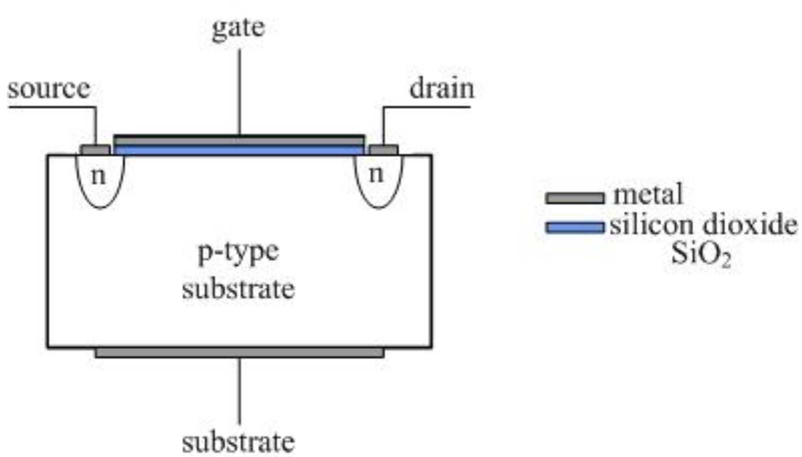


Figure 22.9 Basic structure of an n-channel enhancement mode MOSFET.

If an external voltage source $v_{DS} > 0$ is placed from the drain to the source as shown in figure 22.10, the drain current i_D will be zero.

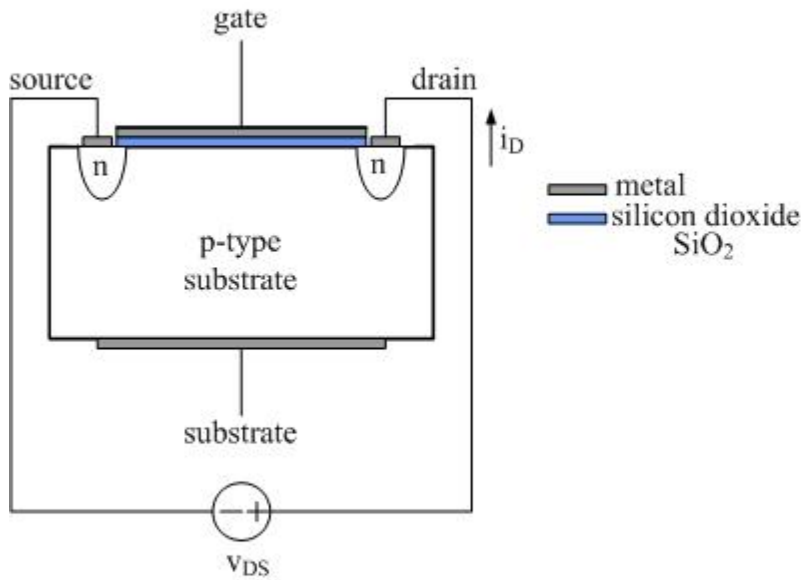


Figure 22.10 n-channel MOSFET with drain-to-source voltage present.

The drain current i_D, defined to be the current flowing from the drain to the source within the MOSFET, is zero because of the reverse biased pn junction between the substrate and the drain. Remember, current can flow easily from p to n but not from n to p. If a voltage source is placed between the gate and the substrate with the substrate connected to the source as shown in Figure 22.11, and $v_{GS} > 0$ but less than the threshold voltage v_T of the MOSFET, the MOSFET is off and the drain current i_D is zero. However, due to the positive voltage between the metal gate and the p-type substrate, electrons begin to form an inversion layer or channel between the source and the drain directly below the silicon dioxide layer (a very good insulator). When $v_{GS} > v_T$, where v_T is the threshold voltage of the MOSFET, an electron channel is formed between the drain and the source, the device is on and $i_D > 0$ (See Figure 22.12).

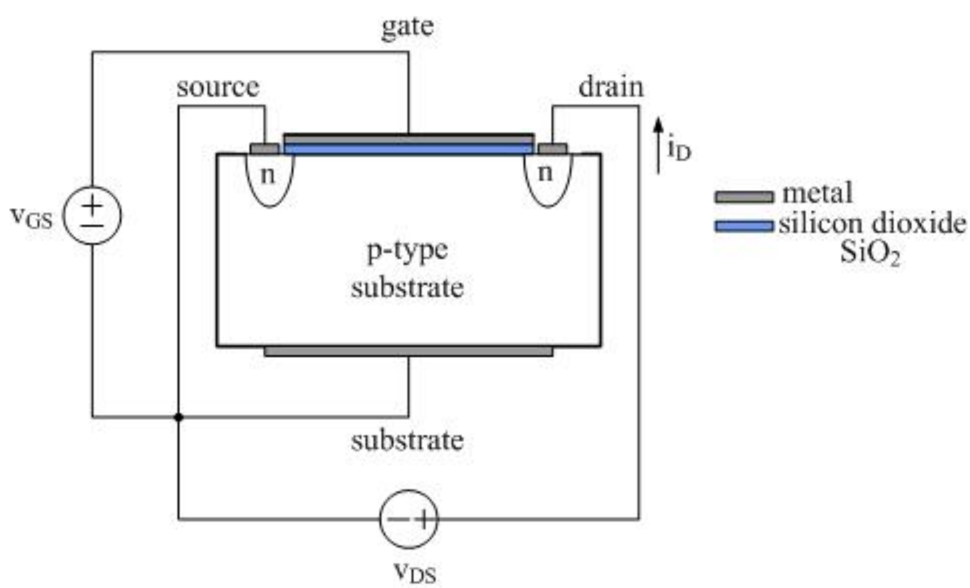


Figure 22.11 MOSFET with external voltage sources v_{DS} and v_{GS} present. Note that that source is connected to the substrate.

The device is turned on and off by v_{GS}. If $v_{GS} < v_T$, then the MOSFET is off and the drain current $i_D = 0$. If $v_{GS} > v_T$, the MOSFET is on, and if $v_{DS} > 0$, drain current i_D will flow. You can think of the MOSFET as a voltage controlled resistance where v_{GS} is the controlling voltage and the resistance of the channel is the resistance being controlled. Once v_{GS} is increased to the threshold voltage v_T, the electron channel (hence n-channel) is just barely formed and the resistance of the channel is still very high. Once v_{GS} is increased above v_T, the electron channel thickens and the resistance of the channel is reduced, and for a given v_{GS} and v_{DS} a specific drain current i_D will flow. Another way of thinking about a MOSFET is as an electronic voltage controlled valve – the valve being between the source and the drain.

A MOSFET may be operated in either an analog or a digital mode. If the device is operated in the two extreme cases where the transistor is either off or on hard (v_{GS} is significantly greater than v_T), then the MOSFET is operating in a digital mode. The symbol for an n-channel MOSFET is shown in Figure 22.13 where it is assumed that the substrate and source have been connected to produce a three terminal device.

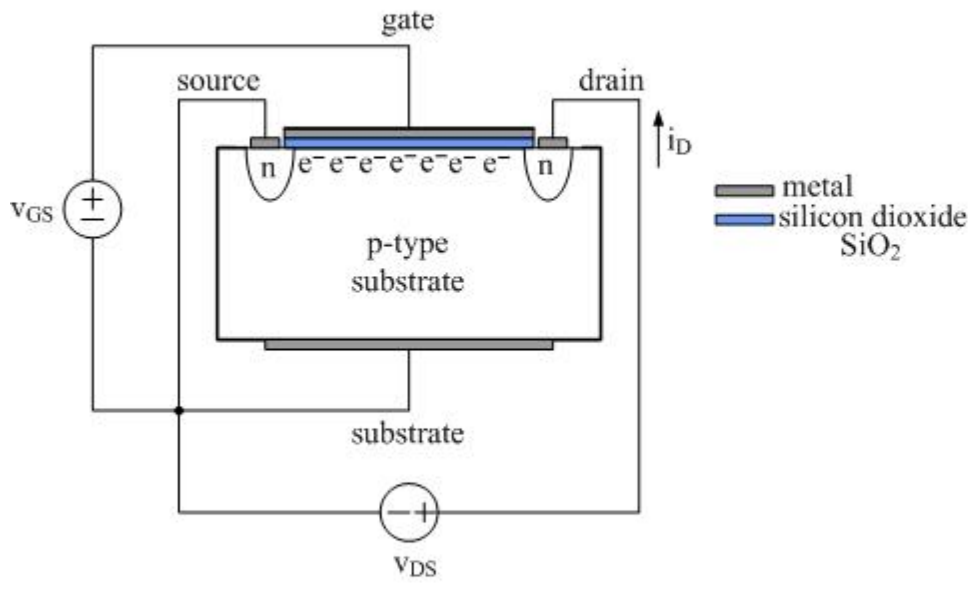


Figure 22.12 MOSFET with $v_{DS} > 0$ and $v_{GS} > V_T$ turning the MOSFET on, allowing the drain current i_D to flow through the transistor.

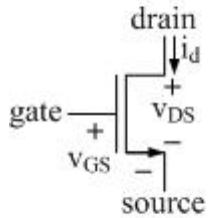


Figure 22.13 n-channel MOSFET symbol

Example 22.1 NMOS Inverter

n-channel MOSFETs are the fundamental building blocks of NMOS digital circuits. Assuming that the node voltage being applied to the gate is either 0 or significantly greater than the threshold voltage v_T, and the source and substrate nodes are at ground potential, the most fundamental NMOS gate, the inverter or NOT gate, is produced with a drain resistor connected to the DC power supply v_{DD} and is shown in Figure 22.14.

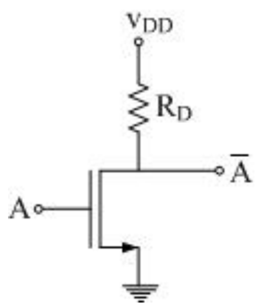


Figure 22.14 NMOS inverter.

When the input is 0, the MOSFET is off and the drain current is zero. The drain current is the same as the current flowing down through R_D. Since this current is zero, the voltage drop across R_D is zero and the output is V_{DD} (“high” or digital “1”). If the input is “high,” then the MOSFET is on “hard,” and the drain-to-source voltage is small as the result of the drain current flowing through R_D and creating a voltage drop very close to v_{DD} across R_D. The result is a “low” output. Combining these two extreme behaviors, the circuit operates as a NOT gate or inverter.

Example 22.2 NMOS NOR Gate

If multiple n-channel MOSFETs are connected in parallel and operated in their digital mode, the result is an NMOS NOR gate. A three input NMOS NOR gate is shown in Figure 22.15. If any one of the MOSFETs is on, the output is “pulled” low. The output is high (v_{DD}) only in the case that all three MOSFETs are off.

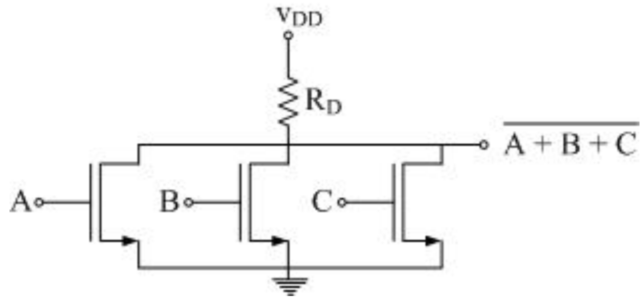


Figure 22.15 Three input NMOS NOR gate.

Example 22.3 NMOS NAND Gate

If multiple n-channel MOSFETs are connected in series, this results in an NMOS NAND gate. An example an NMOS NAND gate is shown in Figure 22.16.

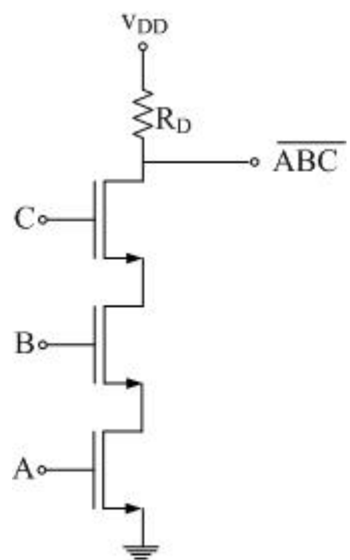


Figure 22.16 NMOS NAND gate.

Current flows through the resistor if and only if all inputs are high. In this case the output will be “pulled” to ground. If any of the MOSFETs are off, no current will flow and the voltage drop across the drain resistance R_D will be zero. Therefore the output will be high (v_{DD}). When combined, these behaviors constitute the functionality of a NAND gate.

The three “natural” gates that arise in NMOS are the NOT, NOR and NAND gates. To obtain the OR and AND functions, the output of NOR and NAND gates must be inverted.

NAND gates and NOR gates are universal gates, meaning that any logic expression can be implemented using only NAND gates or NOR gates. The NOT function is implemented by connecting the inputs of either a NAND or a NOR gate together.

Example 22.4

Given the digital circuit below, draw its implementation using NMOS gates and calculate its complexity.

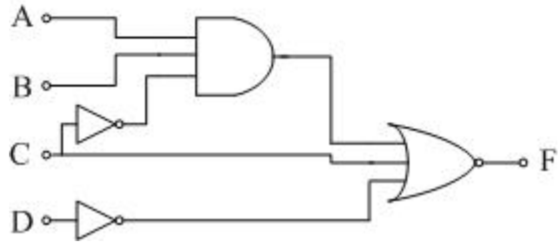


Figure 22.17 Digital circuit to be implemented in NMOS technology.

Solution: The output function F is given by

$$F = \overline{AB\bar{C} + C + \bar{D}}$$

The NMOS realization of this function is shown below.

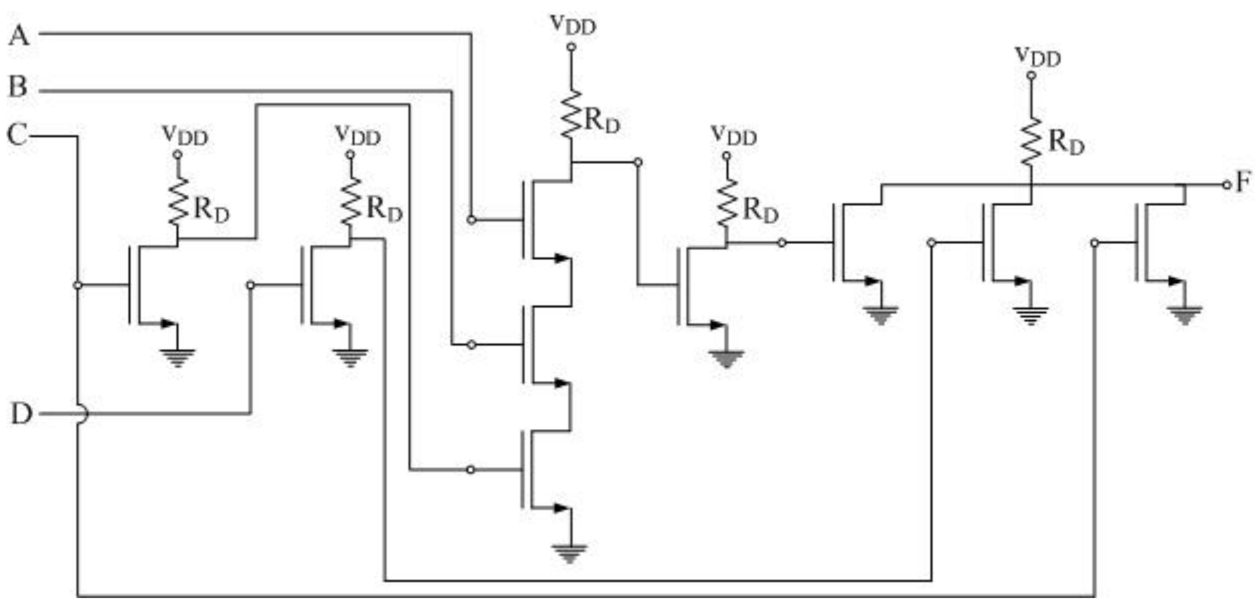


Figure 22.18 NMOS implementation of the digital circuit in Figure 22.17

It is left to the student to label the signal at each node to verify that the function F is in fact realized by this circuit. The complexity is 14 (5 gates + 9 inputs). As an extension to this exercise, is it possible to manipulate F using Boolean algebra such that the realization in NMOS is reduced in complexity?

Appendix 1

Superposition and Linearity

A circuit is linear if each element is linear. An element is linear if it passes the linearity test. The linearity test is to apply two sources, one at a time, and then add the two responses. Then, find the output (response) with both sources acting together. If the sum of the individual responses is the same as the response when both sources are acting together, superposition holds, and the element is linear. Any circuit composed of any number of those elements is also linear. The concept of linearity is critically important in 1) equivalent circuits and 2) response to a periodic input.

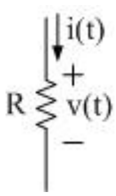
In the case of equivalent circuits, in order for one circuit to be equivalent to another circuit, an element attached to the terminals of equivalency in both circuits has to produce the same voltage and the same current, and this must hold for any element. The only way the two circuits can be equivalent is if they are composed of linear circuit elements.

In the case of response to a periodic input, a periodic input is represented by its Fourier series, which is an infinite sum of harmonically related sinusoids. To find the output of a circuit, the response of the circuit to each of the harmonically related sinusoids is summed. The only way the response could be correct is for the circuit to be linear.

Linearity Test

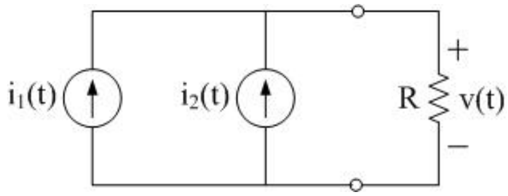
Example 1: Element Under Test – <u>Resistor</u>

The current-voltage relationship for a resistor is called <u>Ohm's law</u>.



Ohm's Law $v(t) = Ri(t)$

Set-up for linearity test



Output for source $i_1(t)$ acting alone: $v_1(t) = R(i_1(t))$

Output for source $i_2(t)$ acting alone: $v_2(t) = R(i_2(t))$

Output obtained by adding result from $i_1(t)$ and $i_2(t)$ acting together

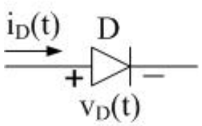
$$v_1(t) + v_2(t) = \mathrm{R}(\mathrm{i}_1(\mathrm{t})) + \mathrm{R}(\mathrm{i}_2(\mathrm{t}))$$

Output obtained by both sources acting simultaneously

$$v(t) = R(i_1(t) + i_2(t))$$

Since the output due to $i_1(t)$ and $i_2(t)$ acting together is the same as the sum of the output for source $i_1(t)$ acting alone and source $i_2(t)$ acting alone, a resistor obeys superposition. Therefore **all resistive circuits are linear**.

Example 2: Element Under Test – <u>Diode</u>

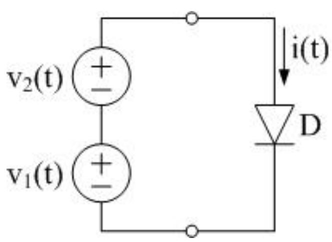


The current-voltage relationship for a diode is given by

$$i_D = i_s\left(e^{\frac{v_D}{v_T}} - 1\right) \tag{A.1}$$

where i_D is the diode current, i_s the saturation current of the diode, v_D is the diode voltage and v_T is the thermal voltage defined by $v_T = \frac{KT}{q}$. K is Boltzmann's constant, T is temperature and q is the magnitude of the charge of an electron. At room temperature $v_T \cong 26$ mV. There is no single individual who can be identified as the discoverer of this equation.

<u>Set-up for linearity test</u>



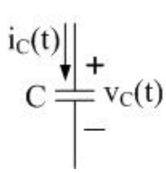
With $v_1(t)$ acting alone: $i_{D1} = i_s\left(e^{\frac{v_1}{v_T}} - 1\right)$

With $v_2(t)$ acting alone: $i_{D2} = i_s\left(e^{\frac{v_2}{v_T}} - 1\right)$

With $v_1(t)$ and $v_2(t)$ acting together: $i_D = i_s\left(e^{\frac{v_1+v_2}{v_T}} - 1\right) \neq i_{D1} + i_{D2}$

Therefore a **diode is a non-linear element**. Therefore **circuits with diodes in them are non-linear**.

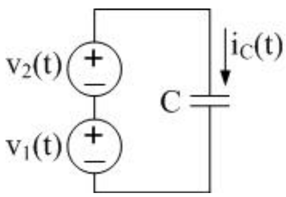
Example 3: Element Under Test – <u>Capacitor</u>



Maxwell's Law

$$i(t) = C\,\frac{dv(t)}{dt}$$

Set-up for linearity test:



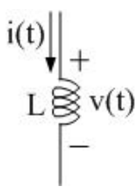
With $v_1(t)$ acting alone, $i_1(t)$ is C $\frac{dv_1(t)}{dt}$.

With $v_2(t)$ acting alone, $i_2(t)$ is C $\frac{dv_2(t)}{dt}$.

With both voltage sources acting together i(t) = C $\frac{d(v_1(t)+v_2(t))}{dt} = \text{C}\ \frac{dv_1(t)}{dt} + \text{C}\ \frac{dv_2(t)}{dt}$

Therefore a **capacitor is a linear circuit element**. Therefore a circuit that is composed of any number of capacitors and other linear circuit elements, such as resistors, is linear.

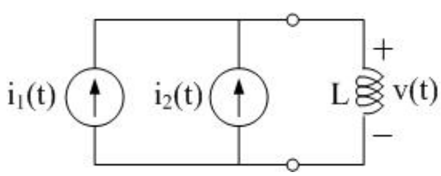
Example 4: Element Under Test – <u>Inductor</u>



Faraday's Law

$v(t) = L\frac{di(t)}{dt}$

Set-up for linearity test



With $i_1(t)$ acting alone, $v_1(t) = L\frac{di_1(t)}{dt}$

With $i_2(t)$ acting alone, $v_2(t) = L\frac{di_2(t)}{dt}$

With both sources acting together, $v(t) = L\frac{d\left(i_1(t)+i_2(t)\right)}{dt} = L\frac{di_1(t)}{dt} + L\frac{di_2(t)}{dt}$

Therefore inductors are linear. Any circuit containing any number of inductors and other linear circuit elements are linear.

Combining the results of these four examples, we have determined that circuits containing any number of resistors, capacitors and inductors are linear circuits.