Fourier Analysis

The Fourier Transform

Recall that the Laplace transform is a powerful analysis tool that allows the conversion of information between two different domains, i.e. the time and frequency domains.

The Fourier transform is another such mathematical transformation, operating on signals of the form $Ae^{j(\omega t+\theta)}$ and combinations thereof.

Although these types of signals represent just a subset of what can be described with the Laplace transform, this subset's importance makes the Fourier transform a ubiquitous analysis method.

While not every mathematical function has a Fourier transform, any physically realizable signal *does*. That's good enough for us!

So what?

There are a number of reasons this is important, although they may not be apparent until you begin applying the transform in practical situations.

Some important mathematical operations are easier to perform in the frequency domain.

- The significance of various components of a signal may be more obvious in the frequency domain.
- The Fourier transform converts a signal into time domain to its equivalent in the frequency domain.

Why are we interested in the frequency domain?

1. If we are interested in the frequency content of a signal, the frequency domain representation is an intuitive way of display this information.
2. Some useful signal processing operations are much easier to perform in the frequency domain. This applies both to intuition and to processing efficiency, through the development of what are known as *fast Fourier transform* (FFT) algorithms.

The Fourier transform and its inverse, while perhaps sounding intimidating, are simply integrals:

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t)\exp(-j\omega t)\,dt \\ g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\exp(j\omega t)\,d\omega \end{aligned} \tag{1}$$

These equations represent the decomposition and reconstruction of a signal into components of the form $Ae^{j(\omega t+\theta)}$, known as the transform's *basis functions*.

What the %*$&# does all this mean?

The forward transform answers a pair of questions:

Is a sinusoidal wave of a particular frequency ω present in a signal?

If so, with what amplitude A and phase θ?

- A particular solution of the forward Fourier transform integral is thus a single complex number for a single frequency.
- This solution is the complex coefficient $Ae^{j\theta}$ of the complex exponential $e^{j\omega t}$.
- To determine the total content of a signal i.e. all of its frequency components, this integral must be repeatedly calculated, at every frequency!
- Fortunately, if the signal can be described in symbolic form (as a general equation), its transform can often be likewise expressed in symbolic form.

How does the integral work?

The **forward** Fourier transform utilizes an important characteristic of signals of the form $e^{j\omega}$ known as *orthogonality*:

$$G(\omega) = \int_{-\infty}^{\infty} \exp(j\omega_1 t) \exp(-j\omega_2 t)\, dt = 2\pi \text{ when } \omega_1 = \omega_2, 0 \text{ otherwise.}$$

- In non-technical terms, the integral over all time of the conjugate product of complex exponentials at two different frequencies is always zero.
- Thus, when computing the Fourier integral at a particular frequency, the component of the signal at that frequency will be "sifted" out and its amplitude and phase identified.
- Analyzing a signal with the Fourier transform is akin to describing a "recipe" for the signal, listing the magnitude and phase of components of the signal as a function of frequency.
- In this light, the *inverse* Fourier transform can be seen as taking that recipe and using it to build up the original signal again.
- A particular solution of the inverse integral provides the value of the signal in question *at a single time*.

Calculating the integral

We can't put it off any longer! It's time to compute the Fourier transform of something.

Let's start at the very beginning, with $\cos(\omega t)$:

$$
\begin{aligned}
\text{Recall}\quad \cos(\omega_c t) &= \frac{1}{2}[\exp(j\omega_c t) + \exp(-j\omega_c t)] \\
G(\omega) &= \int_{-\infty}^{\infty} \cos(\omega_c t)\exp(-j\omega t)\, dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}[\exp(j\omega_c t) + \exp(-j\omega_c t)]\exp(-j\omega t)\, dt \\
&= 2\pi\frac{1}{2}\int_{-\infty}^{\infty} \exp(j\omega_c t)\exp(-j\omega t)\, dt + 2\pi\frac{1}{2}\int_{-\infty}^{\infty} \exp(-j\omega_c t)\exp(-j\omega t)\, dt \\
&= 2\pi\frac{1}{2}[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)] = \pi[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)]
\end{aligned}
\tag{2}
$$

In this solution, we introduced an important function known as the *delta* function:
$\delta(x) = 1$ for $x = 0,\;\; 0$ otherwise.

The orthogonality principle is at work here, in that the Fourier integrals above are zero except at the frequencies $\exp(j\omega_c t)$ and $\exp(-j\omega_c t)$.

But wait! The original signal seems to only have *one* frequency component in it, namely $\cos(\omega t)$, but the Fourier transform has *two*. What's going on?

- The original signal is a solely real signal, while the basis functions of the Fourier transform are complex-valued.
- To construct a solely-real signal, we have to take two complex exponentials of equal frequency and magnitude but opposite phase and add them together such at their imaginary parts *cancel out*.

In light of this, we see Euler's cosine identity with a whole new perspective:

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

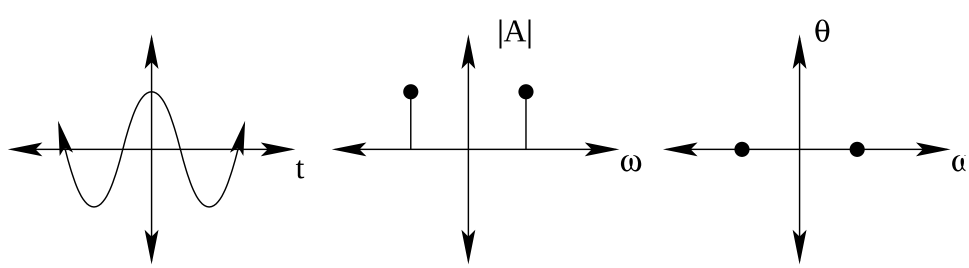

Figure 1: The Fourier transform of $\cos(\omega t)$ in magnitude and phase.

Some observations:

- $e^{j\omega t}$ and $e^{-j\omega t}$ are both phasors, rotating counter-clockwise and clockwise.

- The frequency domain is divided into both signs of the exponent, referred to as the positive and negative frequencies.
- There is one unique frequency that is neither positive nor negative at $\omega = 0$, corresponding to the constant mean value of the signal.
- This is typically referred to as the "DC" component even though the signals involved are often not current signals.

Let's move to the next logical choice of a fundamental function, with $\sin(\omega t)$:

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \sin(\omega_c t) \exp(-j\omega t)\; dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2j}[\exp(j\omega_c t) - \exp(-j\omega_c t)] \exp(-j\omega t)\; dt \\ &= 2\pi \frac{1}{2j} \int_{-\infty}^{\infty} \exp(j\omega_c t) \exp(-j\omega t)\; dt - 2\pi \frac{1}{2j} \int_{-\infty}^{\infty} \exp(-j\omega_c t) \exp(-j\omega t)\; dt \\ &= 2\pi \frac{1}{2j}[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)] = \frac{\pi}{j}[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)] \end{aligned} \tag{3}$$

Once again, we can relate this result to one of Euler's identities, this time for sine:

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

This result is the same as for cosine in magnitude, but different in phase.

This makes intuitive sense, because the only difference between a sine wave and a cosine wave of the same frequency and amplitude is their *phase*.

Let's consider the transform of the *rectangle* function, defined as $x/2$ for $|x| < 1$:

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} \text{rect}(t) \exp(-j\omega t)\; dt \\ &= \frac{1}{2} \int_{-1}^{1} \exp(-j\omega t)\; dt = \frac{\exp(-j\omega t)}{-j\omega} \;|_{-1}^{1} \\ &= \frac{\exp(-j\omega) - \exp(j\omega)}{-j2\omega} \\ &= \frac{\sin \omega}{\omega} \end{aligned} \tag{4}$$

This result is known as the $sinc$ function, where $sinc(\omega) = \frac{\sin \omega}{\omega}$.

As in the case of the Laplace transform, the Fourier transforms of many fundamental functions are listed in reference tables.

Symmetry: evenness and oddness

The cosine and sine functions exhibit a pair of properties that are important to understand, as they are frequently mentioned in discussions of the Fourier transform and its applications.

The function $\cos(\omega t)$ is what is known as an *even* function, which has form $f(t) = -f(t)$. An even function is thus *symmetric* about the origin.

The function $\sin(\omega t)$ is what is known as an *odd* function, which has form $f(t) = -f(-t)$. An even function is thus *anti-symmetric* about the origin.

Fourier series

A function and it Fourier transform solution constitute a Fourier transform *pair.*

Before considering more Fourier transform pairs, lets consider a special case of the transform, which is that for periodic signals of finite power, and which produces a *Fourier series*.

A periodic signal of period T has form $f(t + nT) = f(t)$, n an integer.

- It turns out that a periodic signal of finite power can be decomposed into what is known as a *harmonic series*.
- This is the weighted combination of a orthogonal signals with the same period, beginning with the *fundamental* frequency and integer multiples of this frequency known as the *harmonics*.
- Thus, a repetitive or periodic signal will have a discrete frequency spectrum.
- (Note that sometimes the fundamental is called the *first harmonic*. This is just a convention, like calling the ground floor of a building the first floor.)
- For such signals, the Fourier integral can be calculated over a single period of the signal rather than over all time.

Signals $g(t)$ and $f(t)$, each with period T, are orthogonal if

$$\int_{-T/2}^{T/2} f(t)g(t)dt = 0$$

The Fourier series decomposition of the signal $f(t)$ has the form:

$$f(t) = a_0/2 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

This weighted combination of sines and cosines is governed by the coefficients a_n and b_n, calculated with integrals similar to the Fourier integral:

$$a_n = \frac{2}{T_0}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)dt$$
$$b_n = \frac{2}{T_0}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)dt$$

- $a_0/2 =$ the mean, or "DC," component of signal. $\omega_0 = 0$ at DC.
- Even signals comprise cosine terms only ($b_n = 0$ for all n).
- Odd signals comprise sine terms only ($a_n = 0$ for all n).

If you can determine the coefficients of a Fourier series representation of the signal, the magnitude-squared value of these coefficients indicates the energy associated with each discrete frequency component in the series.

Note that the spectrum of periodic signals is always discrete!

Let's calculate the Fourier series solution for a square wave:

$$\begin{aligned}
f(t+nT) &= -1 \text{ for } T/2 < t \leq 0, 1 \text{ for } 0 < t \leq T/2 \\
a_n &= -\frac{2}{T_0}\int_{-T/2}^{0}\cos(n\omega_0 t)dt + \frac{2}{T_0}\int_{0}^{T/2}\cos(n\omega_0 t)dt \\
b_n &= -\frac{2}{T_0}\int_{-T/2}^{0}\sin(n\omega_0 t)dt + \frac{2}{T_0}\int_{0}^{T/2}\sin(n\omega_0 t)dt \\
a_n &= 0 \text{ by inspection for cosine, an even function,} \\
b_n &= \frac{4}{T_0}\int_{0}^{T/2}\sin(n\omega_0 t)dt \text{ by inspection for sine, an odd function,} \\
b_n &= -\frac{4}{nT_0\omega_0}\cos(n\omega_0 t)\mid_0^{T/2} \\
b_n &= -\frac{4}{nT_0\omega_0}\left(\cos(n\omega_0 T/2) - 1\right) \\
b_n &= 0 \text{ for even integer n,} \\
b_n &= \frac{8}{nT_0\omega_0} \text{ for odd integer n,} \\
f(t) &= \sum_{n=1,3,5\ldots}^{\infty}\frac{8}{nT_0\omega_0} = \sum_{n=1,3,5\ldots}^{\infty}\frac{4}{n\pi} \text{ since } \omega_0 = \frac{2\pi}{T_0}
\end{aligned}$$

One observes that a square wave is the sum of the odd integer sine harmonics.

This results is illustrated in Figure 2.

A certain degree of overshoot is evident in the reconstruction. This is present even for a sum of a large number of harmonics, and is known as *Gibb's phenomenon.*

Let's consider the Fourier series solution of a sequence of delta functions known as the *comb* function:

$$
\begin{aligned}
f(t) &= comb(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \\
a_n &= \frac{2}{T_0}\cos(0) = \frac{2}{T_0} \\
f(t) &= \frac{1}{T_0} + \sum_{n=-\infty}^{\infty} \frac{2}{T_0}\cos n\omega_0 t \\
\omega_0 &= \frac{2\pi}{T_0} \\
\therefore F(\omega) &= \frac{2\pi}{T_0} comb(\omega)
\end{aligned}
$$

Thus, the Fourier transform of a *comb* is a *comb*!

- Multiplication of a signal by the *comb* function is used to represent the sampling operation of analog to digital conversion.
- We will find that the digitally sampled signal will have a periodic spectrum.
- A general observation is that sampling in one of domain produces periodicity in the other domain.

Fourier transform pairs

$$
\begin{aligned}
f(t) = A\cos(\omega_c t) &\Longleftrightarrow F(\omega) = A\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] \\
f(t) = A\sin(\omega_c t) &\Longleftrightarrow F(\omega) = \frac{A\pi}{j}[\delta(\omega - \omega_c) - \delta(\omega + \omega_c)] \\
f(t) = rect(t) &\Longleftrightarrow F(\omega) = \frac{\sin\omega}{\omega} = sinc(\omega) \\
f(t) = A &\Longleftrightarrow F(\omega) = 2\pi A\delta(\omega) \\
f(t) = \delta(t - t_0) &\Longleftrightarrow F(\omega) = \exp(-j\omega t_0), \ = 1 \text{ for } t_0 = 0.
\end{aligned}
$$

Properties of the Fourier transform

Extensive tables of Fourier transform *pairs* exist showing fundamental functions and their transforms.

The Fourier transform exhibits a series of properties that can greatly simplify analysis. These properties also allow *extension* of the fundamental transform pairs by generalization.

Given $f(t)$ and its Fourier transform $F(\omega)$:

$$
\begin{aligned}
\text{Scaling: } f(at) &\iff \frac{1}{|a|}F\left(\frac{\omega}{a}\right)\\
\text{Superposition: } af(t)+bg(t) &\iff aF(\omega)+bG(\omega)\\
\text{Translation / shift: } f(t-t_0) &\iff \exp\big(-j\omega t_0\big)F(\omega)\\
\text{Convolution: } f(t)*g(t) &\iff F(\omega)\;G(\omega)\\
\text{Autocorrelation: } f(t)*f^*(-t) &\iff |F(\omega)^2|\\
\text{Cross-correlation: } f(t)*g^*(-t) &\iff F(\omega)\;G^*(\omega)
\end{aligned}
$$

$$
\text{Power (Parseval's Theorem): } \int_{-\infty}^{\infty} f(t)g^*(t)\;dx \;=\; \int_{-\infty}^{\infty} F(\omega)\;G^*(\omega)\;d\omega
$$

Note that the scaling property has an amplitude term $1/|a|$ out front. *For reasons beyond the scope of this discussion, this factor is only included for functions having finite energy, such as a square pulse.*

This amplitude scaling does not apply to periodic signals. Thus the Fourier transform of $\cos(2\omega_0 t) = \pi[\delta(\omega+2\omega_0)+\delta(\omega-2\omega_0)]$, not $\pi/2[\delta(\omega+2\omega_0)+\delta(\omega-2\omega_0)]$.

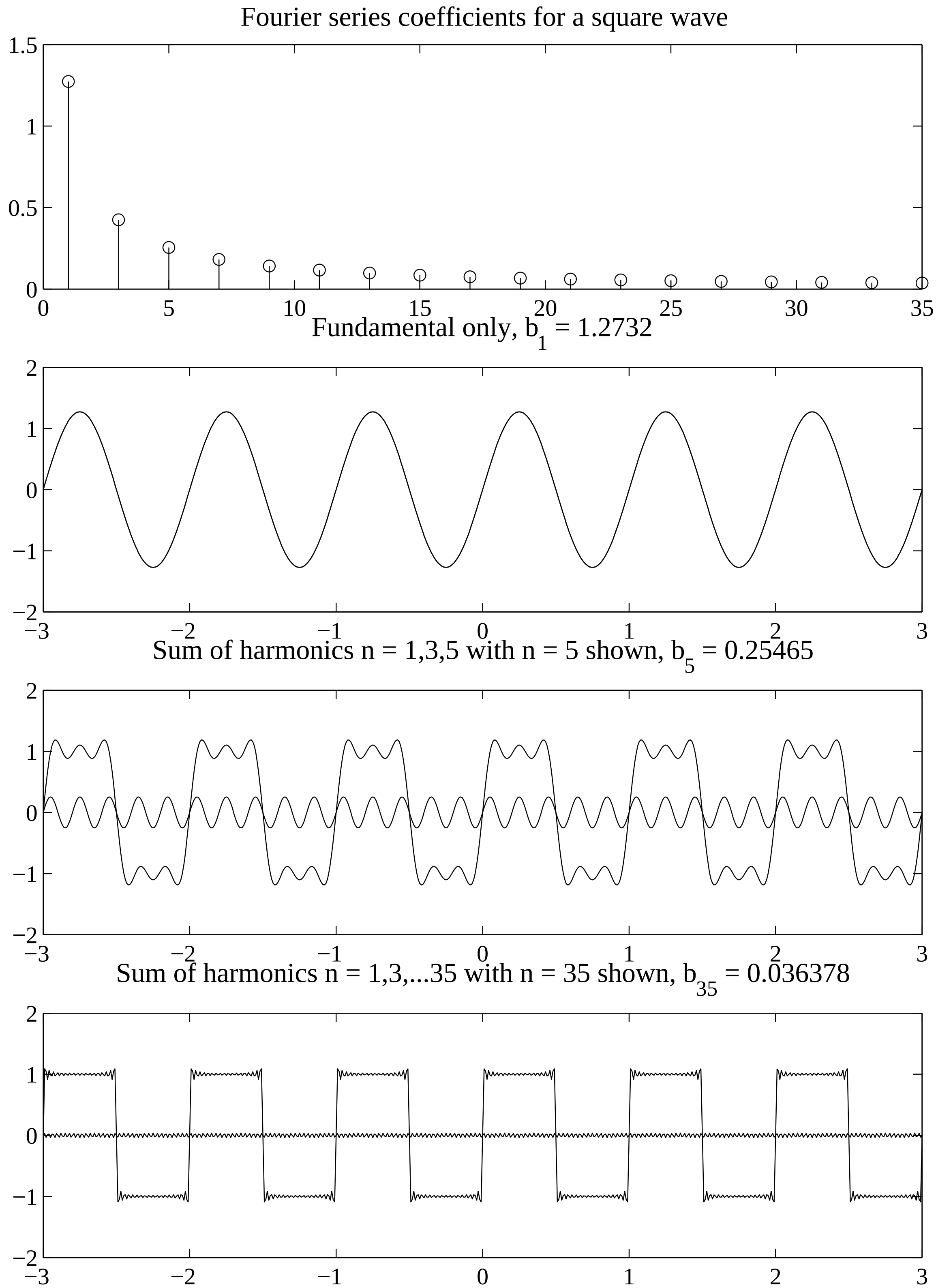

Figure 2: The Fourier series deconstruction of a square wave is illustrated, with sine component amplitudes (the function's *discrete magnitude spectrum*) shown in the top figure and series sums over time shown for $n = 1$, $n = [1 : 5]$, and $n = [1 : 35]$.