The Complex Exponential

We engineers like to describe signals as sinusoids, i.e. in terms of sine and cosine functions. Why is this? For a number of reasons, including that:

- these functions efficiently describe the natural behavior of a variety of natural and electrical systems of interest,
- we utilize Fourier methods which allow us to express any physically realizable signal as a weighted combination of sinusoids, and
- we utilize Fourier methods to describe LSI (linear, shift-invariant) systems, and by extension, to describe how such a system responds to any physically realizable signal. The object is to predict the output given some input.

A "generalized" sinusoid of amplitude A and phase θ has the form:

$$f(t) = A\cos(\omega t + \theta) \tag{1}$$

This seems like a perfectly good expression that is familiar to us all.

Why mess with it?

Because trigonometric forms such as that in Equation **??** become quite inconvenient when you start performing mathematical transformations on them! Transformations of interest include advancing or retarding the phase, integrating, differentiating, multiplying by another signal, and of course, the Fourier Transform. Trigonometric identities will take you only so far.

To avoid these difficulties, we adopt notation using the complex exponential, $e^{j\omega t}$. By adopting a more complicated description of the signal, we can actually make the mathematical transformations of interest *much* easier.

We recall Euler's Law, and use it to construct a "generalized" complex exponential:

$$\begin{aligned} e^{j\phi} &= \cos\phi + j\sin\phi \\ Ae^{j(\omega t+\theta)} &= A[\cos(\omega t + \theta) + j\sin(\omega t + \theta)] \end{aligned} \tag{2}$$

The complication we have accepted in adopting this notation is that we have converted a real signal into a complex signal. The imaginary part has been introduced as a mathematical device with which to efficiently *keep track of phase*.

In describing the response of some circuit to a real sinusoidal input signal, we begin by describing the input signal in complex notation, perform operations on this signal to reflect the circuit's response, and then solve for the system output by taking the real part of the result.

The generalized complex exponential itself can be interpreted to represent a complex vector rotating about the origin having length A and instantaneous angle θ with respect to the real, positive axis, as shown in Fig. **??**.

Figure 1: A phasor is a complex number that describes a particular sinusoidal function represented with the complex exponential $e^{j\omega t}$. The phasor has magnitude A and phase θ. The real component of this phasor is $Re[Ae^{j\theta}] = A\cos\theta$, while the imaginary component is $Im[Ae^{j\theta}] = A\sin\theta$. The generalized complex exponential function itself is of the form $Ae^{j\theta}e^{j\omega t}$. At time $t = 0$, the phasor diagram shows the instantaneous value of the complex exponential function.

The following identities follow:

$$
\begin{aligned}
Re[Ae^{j\theta}] &= A\cos\theta \\
Im[Ae^{j\theta}] &= A\sin\theta \\
\theta &= \arctan\frac{Im[Ae^{j\theta}]}{Re[Ae^{j\theta}]} \\
|A| &= \sqrt{Re[Ae^{j\theta}]^2 + Im[Ae^{j\theta}]^2} \\
e^{j0} = 1, &\qquad e^{j\pi/2} = j \\
e^{-j\pi/2} = -j, &\qquad e^{j\pi} = e^{-j\pi} = -1
\end{aligned}
\tag{3}
$$

Mathematical manipulations include:

$$
\begin{aligned}
\text{Phase shift of } \phi \text{ radians:} &\quad e^{j\phi}\cdot Ae^{j(\omega t+\theta)} = Ae^{j(\omega t+\theta+\phi)} \\
\text{Differentiation:} &\quad \frac{d}{dt}[Ae^{j(\omega t+\theta)}] = j\omega Ae^{j[\omega t+\theta]}
\end{aligned}
\tag{4}
$$

Example: If a voltage function of the form $v_C(t) = Re[Ae^{j\omega t}]$ is applied across a capacitor of value C, what is the current through the capacitor?

$$
\begin{aligned}
& i_C(t) = Re\big[C\frac{d}{dt}[Ae^{j\omega t}]\big] \\
\therefore\quad & i_C(t) = Re\big[j\omega CAe^{j\omega t} = e^{j\pi/2}\omega CAe^{j\omega t}\big] \quad \text{(current leads voltage by } 90^\circ\text{)} \\
& \qquad\;\; = Re\left[Ae^{j\omega t}/\frac{1}{j\omega C}\right] \\
\therefore\quad & i_C(t) = Re\left[\frac{Ae^{j\omega t}}{Z_C}\right] \quad \text{(Ohm's Law!)}
\end{aligned}
\tag{5}
$$

You **must** be comfortable intra-converting the sin/cosine, complex exponential, polar, and complex number forms of a signal at some time t.

The complex exponential times a phasor has the forms $Ae^{j\theta}e^{j\omega t}$, $Ae^{j(\omega t+\theta)}$, and $A[\cos(\omega t + \theta) + j\sin(\omega t + \theta)]$.

The complex phasor can also be represented in **polar form** $A\angle\theta$ or **rectangular form** $A(\cos\theta + j\sin\theta)$.

Recall that for complex numbers A and B:

$$\begin{aligned} |AB| = |A||B| \quad |A/B| = |A|/|B| \\ \angle AB = \angle A + \angle B \quad \angle A/B = \angle A - \angle B \end{aligned} \tag{6}$$

These identities are very useful in simplifying the magnitude and phase of a complicated transfer function.

Note that a $\pm 180°$ or $\pm\pi$ phase shift corresponds to a sign inversion.

Note that when using a calculator, the $\arctan$ function only gives correct phases in the first and fourth quadrants, i.e. for a positive real part. If the real part is negative, you must shift the 1st or 4th quadrant answer by $180°$ to get the correct answer.

Note that Ohm's law and other circuit analysis principles can be applied to circuits with capacitors and inductors using the complex exponential form of signals and the complex impedance description of these components.

Common mistakes include forgetting to convert between radians and degrees (related by the factor $180/\pi$), and between frequency measured in radians/s and Hertz (related by the factor $1/(2\pi)$).