Review - Second Order Systems

$$a_2\frac{d^2v(t)}{dt^2} + a_1\frac{dv(t)}{dt} + a_0v(t) = b_0f(t) \tag{1}$$

$$\frac{1}{\omega_n^2}\frac{d^2v(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dv(t)}{dt} + v(t) = K_sf(t) \tag{2}$$

$$\begin{aligned}
\omega_n &= \sqrt{\frac{a_0}{a_2}} && \longrightarrow \text{ natural frequency}\\
\zeta &= \left(\frac{a_1}{2}\right)\sqrt{\frac{1}{a_0a_2}} && \longrightarrow \text{ damping coefficient}\\
K_s &= \frac{b_0}{a_0} && \longrightarrow \text{ DC gain (steady-state response; not ''t''-damped)}
\end{aligned}$$

Assume solution of the form:

$$v(t) = Ae^{st} \tag{3}$$

Plug into Equation 2:

$$\frac{1}{\omega_n^2}s^2\cancel{Ae^{st}} + \frac{2\zeta}{\omega_n}s\cancel{Ae^{st}} + \cancel{Ae^{st}} \tag{4}$$

Characteristic equation:

$$\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1 = 0 \tag{5}$$

Solve:

$$s^2 + 2\zeta\omega_n + \omega_n^2 = 0 \longrightarrow 2 \text{ roots } (s_1 + s_2) \tag{6}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{7}$$

$\zeta > 1$	2 real + distinct roots	overdamped
$\zeta = 1$	real, repeated root $s_{1,2} = -\zeta\omega_n$	critically-damped
$\zeta < 1$	2 complex conjugate roots $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$	under-damped

Underdamped System Response - pressure response of 'step excitation' of a balloon popping

Sum of 2 sinusoids at ω_d with exponential decay:

$$v(t) = e^{-\zeta\omega_n t}\left[Acos\left(\omega_n t\sqrt{1-\zeta^2}\right) + Bsin\left(\omega_n t\sqrt{1-\zeta^2}\right)\right] \tag{8}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{9}$$

Envelope at:

$$e^{-t/\tau} \tag{10}$$

$$\tau = \frac{1}{\zeta\omega_n} \tag{11}$$

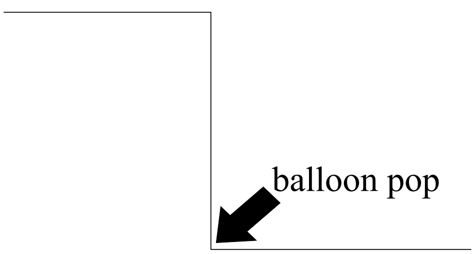

Figure 1: Balloon Pop - step excitation

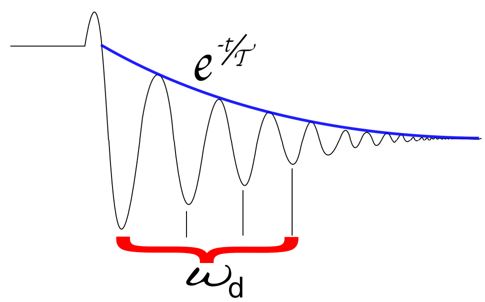

Figure 2: Time Domain - underdamped repsonse

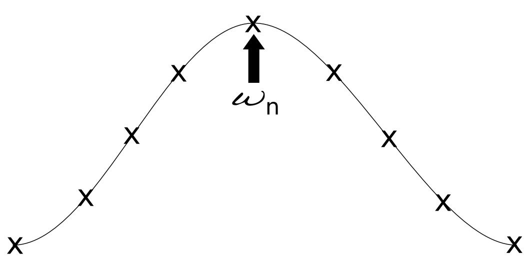

Figure 3: Frequency Domain - underdamped response

From the frequency domain, the half-power bandwidth can be calculated from the -3 dB points ($\omega_2 - \omega_1$).

$$\frac{\omega_2 - \omega_1}{\omega_n} = 2\zeta = \frac{1}{Q} \tag{12}$$

Consider: Step is broadband. What about steady-state sinusoidal inputs?