Exercise 1. Let $X_1, \cdots, X_n \overset{\text{i.i.d}}{\sim} \text{Exp}(\lambda)$ for $\lambda > 0$ (rate) and $n \in \mathbb{N}$ with $n \geq 2$. Consider the following estimators for the parameter $\mu = \frac{1}{\lambda}$:

$$\hat{\mu}_1 = n \min\left(X_1, \cdots, X_n\right), \qquad \hat{\mu}_2 = \frac{1}{n-1} \sum_{i=1}^{n} X_i.$$

(a) Compute the expectation of $\hat{\mu}_1$ and $\hat{\mu}_2$. Are the estimators unbiased for μ? Justify your answer. [1.5 points]

(b) Show that $\text{MSE}(\hat{\mu}_1) = \mu^2$ and $\text{MSE}(\hat{\mu}_2) = \frac{n+1}{(n-1)^2}\mu^2$. [2 points]

(c) Are $\hat{\mu}_1$ and $\hat{\mu}_2$ consistent? Justify your answer. [0.5 points]

Exercise 2. For $n \in \mathbb{N}$ let $X_1, \cdots, X_n$ be i.i.d random variables with density function:

$$f_X(x|\alpha) = \frac{\alpha}{x^2} \exp\left(-\frac{\alpha}{x}\right), \quad x > 0,\ \alpha > 0.$$

In the following, $x_1, \cdots, x_n$ denotes a sample from these random variables.

(a) Find the maximum likelihood estimator (MLE), $\hat{\alpha}$, for α. [2.5 points]

(b) Compute the asymptotic distribution of $\hat{\alpha}$. [2 points]

(c) The median η of the distribution specified above is given as $\eta = \alpha/(\log 2)$. Obtain the MLE, $\hat{\eta}$, for η. [0.5 points]

(d) Use the univariate Delta method to obtain the asymptotic distribution of the estimator $\hat{\eta}$. [1 point]