Solucions primera entrega avaluable

E.2.

(a)

$X_1, \dots, X_n$ r.v. i.i.d with the following pdf:

$$f_X(x|\alpha) = \frac{\alpha}{x^2} \exp\left(\frac{-\alpha}{x}\right),$$

$x>0,\ \alpha>0.$

The likelihood function of the model is:

$$L(\alpha | \underbrace{x_1, \dots, x_n}_{\text{realizations of } X_1, \dots, X_n}) = \prod_{i=1}^{n} f_{X_i}(x_i|\alpha) = \prod_{i=1}^{n} \frac{\alpha}{x_i^2} \exp\left(\frac{-\alpha}{x_i}\right) = \frac{\alpha^n}{\prod_{i=1}^{n} x_i^2} \exp\left(-\alpha \sum_{i=1}^{n} \frac{1}{x_i}\right).$$

The log-likelihood function can thus be writen as:

$$\ell(\alpha | x_1, \dots, x_n) = \log\left(L(\alpha|x_1, \dots, x_n)\right) = n\log(\alpha) - 2\sum_{i=1}^{n} \log(x_i) - \alpha \sum_{i=1}^{n} \frac{1}{x_i},$$

and the score function is:

$$s(\alpha | x_1, \dots, x_n) = \frac{\delta \ell(\alpha | x_1, \dots, x_n)}{\delta \alpha} =$$

$$= \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{x_i} .$$

$$\Longrightarrow s(\alpha | x_1, \dots, x_n) = 0$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{x_i} = 0$$

$$n - \alpha \sum_{i=1}^{n} \frac{1}{x_i} = 0$$

$$\underbrace{\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} 1/x_i}}_{\substack{\text{point estimate} \\ \text{(based on the} \\ \text{realizations)}}} \; ; \quad \underbrace{\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} 1/X_i}}_{\substack{\text{point estimator} \\ \text{(based on the} \\ \text{random variables)}}}$$

Finally, we will check that $\hat{\alpha}$ is actually the MLE of α; that is, is a maximizer of the likelihood function. Hence,

$$\frac{\delta s(\alpha | x_1, \dots, x_n)}{\delta \alpha} = \underbrace{- \frac{n}{\alpha^2}}_{\substack{\text{This is always} \\ \text{negative given that} \\ \alpha^2 > 0 \text{ and } n \text{ (sample size)} \\ \text{is always positive as well.}}} < 0$$

(b)

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} 1/x_i}$$ (Maximum Likelihood Estimator)

$$\hat{\alpha} \overset{a}{\sim} N\left(\alpha, \frac{\alpha^2}{n}\right)$$

the MLE is always asymptotically unbiased.

The MLE is asympt. efficient.

Note that the asympt. variance of the MLE is the inverse of the expected Fisher information. In our case here, the expected Fisher information is:

$$\underbrace{F(\alpha \mid x_1, \dots, x_n)}_{\text{Fisher information}} = -\frac{\delta s(\alpha \mid x_1, \dots, x_n)}{\delta \alpha} = \frac{n}{\alpha^2}$$

in this case, those are the same

$$F^*(\alpha) = E\left(F(\alpha \mid \underbrace{X_1, \dots, X_n}))\right) = \frac{n}{\alpha^2}$$

Note that here we use random variables as we'll compute the expectation of the Fisher information.

Finally, we can see that the asympt. variane of $\hat{\alpha}$ is α^2/n.

(c)

$\eta = \alpha / \log(2)$

Then, for the property of invariance of the MLE,

$\hat{\eta} = \hat{\alpha} / \log(2) \Longrightarrow$

$$\left[\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} 1/x_i} \right] \qquad \hat{\eta} = \frac{n}{\sum_{i=1}^{n} 1/x_i} \cdot \frac{1}{\log(2)} \, .$$

From part (c).

(d)

$\eta = g(\alpha) = \alpha / \log(2) \quad / \quad \hat{\eta} = \frac{n}{\sum_{i=1}^{n} 1/x_i} \cdot \frac{1}{\log(2)}$

$$g'(\alpha) = \frac{\log(2)}{[\log(2)]^2} = \frac{1}{\log(2)}$$

$$[g'(\alpha)]^2 = \left(\frac{1}{\log(2)}\right)^2$$

$$V(\hat{\eta}) \approx [g'(\alpha)]^2 \underbrace{V(\hat{\alpha})}_{\text{asympt. variance}} = \left(\frac{1}{\log(2)}\right)^2 \frac{\alpha^2}{n}$$

Finally,

$$\hat{\eta} \overset{a}{\sim} N\left(\eta, \left(\frac{1}{\log(2)}\right)^2 \frac{\alpha^2}{n}\right) .$$

E.1.

(a)

$$\hat{\mu}_1 = n \min(X_1, \ldots, X_n)$$
$$\hat{\mu}_2 = \frac{1}{n-1} \sum_{i=1}^{n} X_i$$

To estimators for the unknown parameter μ.

Recall also that:

$X_1, \ldots, X_n \sim \text{Exponential}(\lambda)$ and,

$$\mu = \frac{1}{\lambda}.$$

We already know from exercice sessions that:

If $X_1, \ldots, X_n \sim \text{Exponential}(\lambda)$,

then $Y = \min(X_1, \ldots, X_n) \sim \text{Exponential}(n\lambda)$

In addition, it is easy to see that

$E(X_i) = 1/\lambda$ and $E(Y) = \dfrac{1}{n\lambda}$.

Accordingly,

$$\mathbb{E}(\hat{\mu}_1) = \mathbb{E}[n \min(X_1, \ldots, X_n)] = n\, \mathbb{E}(\min(X_1, \ldots, X_n)) =$$
$$= n \frac{1}{n\lambda} = \frac{1}{\lambda} = \mu.$$

$$\mathbb{E}(\hat{\mu}_2) = \mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^{n} X_i\right) = \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) =$$

$$= \frac{1}{n-1}\underbrace{n\,\mathbb{E}(X_1)}_{\text{because } X_1,\dots,X_n \text{ are i.i.d}} = \frac{n}{n-1}\cdot\frac{1}{\lambda} \rightarrow \mu$$

$$\text{Bias}(\hat{\mu}_1) = \mu - \mu = 0.$$

$$\text{Bias}(\hat{\mu}_2) = \frac{n}{n-1}\mu - \mu = \mu\left(\frac{n}{n-1} - 1\right) =$$

$$= \mu\left(\frac{n-n+1}{n-1}\right) = \frac{\mu}{n-1}.$$

Therefore, $\hat{\mu}_1$ is an unbiased estimator for μ as $\text{Bias}(\hat{\mu}_1) = 0$, but $\hat{\mu}_2$ is a biased estimator for μ because $\text{Bias}(\hat{\mu}_2) = \mu/(n-1)$.

(b)

To compute $MSE(\hat{\mu}_1)$ and $MSE(\hat{\mu}_2)$ we need $\mathbb{V}(\hat{\mu}_1)$ and $\mathbb{V}(\hat{\mu}_2)$. Therefore:

$$\mathbb{V}(\hat{\mu}_1) = \mathbb{V}(n\min(X_1,\dots,X_n)) = n^2\underbrace{\mathbb{V}(\min(X_1,\dots,X_n))}$$

$$= \cancel{n^2}\frac{1}{\cancel{n^2}\lambda^2} = \frac{1}{\lambda^2} = \mu^2$$

$Y \sim Exp(n\lambda)$

$\mathbb{E}(Y) = 1/n\lambda$

$\mathbb{V}(Y) = 1/n^2\lambda^2$

$$MSE(\hat{\mu}_1) = [\text{Bias}(\hat{\mu}_1)]^2 + \mathbb{V}(\hat{\mu}_1) =$$

$$= 0 + \mu^2 = \mu^2.$$

$$\mathbb{V}(\hat{\mu}_2) = \mathbb{V}\left(\frac{1}{n-1}\sum_{i=1}^{n} X_i\right) = \frac{1}{(n-1)^2}\mathbb{V}\left(\sum_{i=1}^{n} X_i\right) =$$

$$\underset{\substack{\downarrow \\ X_1,\dots,X_n \text{ i.i.d}}}{=} \frac{1}{(n-1)^2}\, n\, \underbrace{V(X_1)}_{1/\lambda^2} = \frac{n}{(n-1)^2\lambda^2} \underset{\substack{\downarrow \\ \mu^2 = 1/\lambda^2}}{=} \frac{n}{(n-1)^2}\mu^2.$$

$$MSE(\hat{\mu}_2) = \left[\text{Bias}(\hat{\mu}_2)\right]^2 + \mathbb{V}(\hat{\mu}_2) =$$

$$= \left[\frac{\mu}{n-1}\right]^2 + \frac{n}{(n-1)^2}\mu^2 =$$

$$= \frac{\mu^2}{(n-1)^2} + \frac{n\mu^2}{(n-1)^2} = \mu^2\frac{(1+n)}{(n-1)^2}.$$

(c)

We'll study here consistency in MSE. Hence,

$$MSE(\hat{\mu}_1) = \mu^2 \longrightarrow \underbrace{\lim_{n\to\infty} MSE(\hat{\mu}_{1n}) = \mu^2}$$

$\hat{\mu}_1$ is an unbiased estimator for μ, but is not consistent in MSE.

$$MSE(\hat{\mu}_2) = \frac{n+1}{(n-1)^2}\mu^2 \longrightarrow \lim_{n\to\infty} MSE(\hat{\mu}_{2n}) =$$

$$= \underbrace{\lim_{n\to\infty}\frac{n+1}{(n-1)^2}\mu^2 = 0}$$

$\hat{\mu}_2$ is a biased estimator for μ, but is consistent in MSE.