First assessment

Víctor Ballester Ribó
NIU: 1570866

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Universitat Autònoma de Barcelona
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Exercise 1. *Let* $X_1, \dots X_n \overset{i.i.d.}{\sim}$ *Exp*(λ), $\lambda > 0$ *(rate) and* $n \in \mathbb{N}$ *with* $n \geq 2$. *Consider the following estimators for the parameter* $\mu = \frac{1}{\lambda}$*:*

$$\hat{\mu}_1 = n \min(X_1, \dots, X_n) \qquad \hat{\mu}_2 = \frac{1}{n-1}\sum_{i=1}^n X_i$$

1. *Compute the expectation of* $\hat{\mu}_1$ *and* $\hat{\mu}_2$. *Are the estimators unbiased for* μ*? Justify your answer. [1.5 points]*

   Resolution. Let's determine first the distribution of $\hat{\mu}_1$. Given $t \leq 0$, clearly we have $\mathbb{P}(\hat{\mu}_1 \leq t) = 0$. Given $t > 0$, we have:

$$\begin{aligned}
\mathbb{P}(\hat{\mu}_1 \leq t) &= \mathbb{P}\left(\min(X_1, \dots, X_n) \leq \frac{t}{n}\right) \\
&= 1 - \mathbb{P}\left(\min(X_1, \dots, X_n) > \frac{t}{n}\right) \\
&= 1 - \mathbb{P}\left(X_1 > \frac{t}{n}\right) \cdots \mathbb{P}\left(X_n > \frac{t}{n}\right) \\
&= 1 - \mathbb{P}\left(X_1 > \frac{t}{n}\right)^n \\
&= 1 - \left(\int_{\frac{t}{n}}^{\infty} \lambda \mathrm{e}^{-\lambda x}\,\mathrm{d}x\right)^n \\
&= 1 - \left(-\mathrm{e}^{-\lambda x}\Big|_{\frac{t}{n}}^{\infty}\right)^n \\
&= 1 - \left(\mathrm{e}^{-\frac{\lambda t}{n}}\right)^n \\
&= 1 - \mathrm{e}^{-\lambda t}
\end{aligned}$$

   where in the forth equality we have used that the random variables are i.i.d. Note that the result obtained is precisely the cdf of a $\mathrm{Exp}(\lambda)$. Hence, $\hat{\mu}_1 \sim \mathrm{Exp}(\lambda)$ and therefore $\mathbb{E}(\hat{\mu}_1) = \frac{1}{\lambda} = \mu$. Regarding the estimator $\hat{\mu}_2$, we have:

$$\mathbb{E}(\hat{\mu}_2) = \mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^n X_i\right) = \frac{1}{n-1}\sum_{i=1}^n \mathbb{E}(X_i) = \frac{1}{n-1}\sum_{i=1}^n \frac{1}{\lambda} = \frac{n}{n-1}\cdot\frac{1}{\lambda} = \frac{n}{n-1}\mu$$

   Since $\mathrm{bias}(\hat{\mu}_1) = \mathbb{E}(\hat{\mu}_1) - \mu = \mu - \mu = 0$, $\hat{\mu}_1$ is unbiased. On the other hand:

$$\mathrm{bias}(\hat{\mu}_2) = \mathbb{E}(\hat{\mu}_2) - \mu = \frac{n}{n-1}\mu - \mu = \frac{1}{n-1}\mu \neq 0$$

   Thus, $\hat{\mu}_2$ is biased.

2. *Show that* $\mathrm{MSE}(\hat{\mu}_1) = \mu^2$ *and* $\mathrm{MSE}(\hat{\mu}_2) = \frac{n+1}{(n-1)^2}\mu^2$. *[2 points]*

   Resolution. We know that given an estimator $\hat{\theta}$ of a parameter θ, the $\mathrm{MSE}(\hat{\theta})$ is given by:

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{Var}(\hat{\theta}) + \left(\mathrm{bias}(\hat{\theta})\right)^2$$

Let's compute the variance of $\hat{\mu}_1$ and $\hat{\mu}_2$. For the first case, we already know that $\text{Var}(\hat{\mu}_1) = \frac{1}{\lambda^2} = \mu^2$ because $\hat{\mu}_1 \sim \text{Exp}(\lambda)$. For the second case, using the fact that $X_1, \ldots, X_n$ are i.i.d., we have:

$$\text{Var}(\hat{\mu}_2) = \text{Var}\left(\frac{1}{n-1}\sum_{i=1}^{n} X_i\right) = \frac{1}{(n-1)^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{(n-1)^2}\sum_{i=1}^{n}\frac{1}{\lambda^2} = \frac{n}{(n-1)^2}\cdot\frac{1}{\lambda^2} = \frac{n}{(n-1)^2}\mu^2$$

From here the desired result follows immediately:

$$\text{MSE}(\hat{\mu}_1) = \text{Var}(\hat{\mu}_1) + (\text{bias}(\hat{\mu}_1))^2 = \mu^2 + 0 = \mu^2$$
$$\text{MSE}(\hat{\mu}_2) = \text{Var}(\hat{\mu}_2) + (\text{bias}(\hat{\mu}_2))^2 = \frac{n}{(n-1)^2}\mu^2 + \left(\frac{1}{n-1}\mu\right)^2 = \frac{n+1}{(n-1)^2}\mu^2$$

3. *Are $\hat{\mu}_1$ and $\hat{\mu}_2$ consistent? Justify your answer. [0.5 points]*

   Resolution. Recall that an estimator estimator $\hat{\theta}_n$ of a parameter θ is consistent if $\lim\limits_{n\to\infty}\text{MSE}(\hat{\theta}_n) = 0$. So in our case, we have:

$$\lim_{n\to\infty}\text{MSE}(\hat{\mu}_1) = \lim_{n\to\infty}\mu^2 = \mu^2 \neq 0$$
$$\lim_{n\to\infty}\text{MSE}(\hat{\mu}_2) = \lim_{n\to\infty}\frac{n+1}{(n-1)^2}\mu^2 = 0$$

   Thus, $\hat{\mu}_2$ is consistent but $\hat{\mu}_1$ isn't.

Exercise 2. *For $n \in \mathbb{N}$ let $X_1, \ldots, X_n$ be i.i.d. random variables with density function:*

$$f_X(x;\alpha) = \frac{\alpha}{x^2}\mathrm{e}^{-\frac{\alpha}{x}} \qquad x > 0, \alpha > 0$$

In the following, $x_1, \ldots, x_n$ denotes a sample from these random variables.

1. *Find the maximum likelihood estimator (*MLE*), $\hat{\alpha}$, for α. [2.5 points]*

   Resolution. Let $\mathbf{X} := (X_1, \ldots, X_n)$ and $\mathbf{x} := (x_1, \ldots, x_n)$. We know that (in the continuous case) the likelihood function $L(\alpha;\mathbf{x})$ is joint pdf of $X_1, \ldots, X_n$ evaluated at $\mathbf{x}$. Since $X_1, \ldots, X_n$ are i.i.d. random variables we will have:

$$L(\alpha;\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x};\alpha) = \prod_{i=1}^{n} f_{X_i}(x_i;\alpha) = \prod_{i=1}^{n}\frac{\alpha}{{x_i}^2}\mathrm{e}^{-\frac{\alpha}{x_i}} = \frac{\alpha^n}{(x_1\cdots x_n)^2}\mathrm{e}^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}$$

   Thus, the log-likelihood function is:

$$\ell(\alpha;\mathbf{x}) = \log L(\alpha;\mathbf{x}) = n\log\alpha - 2\log(x_1\cdots x_n) - \alpha\sum_{i=1}^{n}\frac{1}{x_i}$$

   And the score function is:

$$S(\alpha;\mathbf{x}) = \frac{\partial\ell}{\partial\alpha}(\alpha;\mathbf{x}) = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{1}{x_i} \tag{1}$$

   In order to find $\hat{\alpha}$, we have to find the zero (if any) of $S(\alpha;\mathbf{x})$. So we obtain an estimation for α isolating α from equation (1) equated to 0:

$$S(\alpha;\mathbf{x}) = 0 \iff \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{1}{x_i} = 0 \iff \alpha = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$$

   Since $\frac{\partial S}{\partial\alpha}(\alpha;\mathbf{x}) = -\frac{n}{\alpha^2} < 0\ \forall\alpha > 0$ we have that the value found above is indeed a maximum. Hence, the MLE $\hat{\alpha}$ is:

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\frac{1}{X_i}}$$

2. *Compute the asymptotic distribution of $\hat{\alpha}$. [2 points]*

Resolution. Let's compute first the observed information J:

$$J(\alpha; \mathbf{X}) = -\frac{\partial^2 \ell}{\partial \alpha^2}(\alpha; \mathbf{X}) = \frac{n}{\alpha^2}$$

Thus, the Fisher information is:

$$I(\alpha) = \mathbb{E}(J(\alpha; \mathbf{X})) = \mathbb{E}\left(\frac{n}{\alpha^2}\right) = \frac{n}{\alpha^2}$$

But we know that the asymptotic distribution of $\hat{\alpha}$ is normal with expectation α and variance $I(\alpha)^{-1}$. So finally we get:

$$\hat{\alpha} \overset{\mathrm{a}}{\sim} N\left(\alpha, \frac{\alpha^2}{n}\right)$$

3. *The median η of the distribution specified above is given as $\eta = \alpha/(\log 2)$. Obtain the* MLE, *$\hat{\eta}$, for η. [0.5 points]*

   Resolution. Let $g(x) = \frac{x}{\log 2}$. We know that $\eta = g(\alpha)$, so because of the invariance of the MLE under transformations, we have:

$$\hat{\eta} = g(\hat{\alpha}) = \frac{\hat{\alpha}}{\log 2} = \frac{1}{\log 2} \cdot \frac{n}{\sum_{i=1}^{n} \frac{1}{X_i}}$$

4. *Use the univariate Delta method to obtain the asymptotic distribution of the estimator $\hat{\eta}$. [1 points]*

   Resolution. Delta method tell us that the asymptotic expectation of $\hat{\eta}$ is given by $\eta = g(\alpha)$ and that the asymptotic variance of $\hat{\eta}$ is given by $\mathrm{Var}(\hat{\eta}) = \frac{g'(\alpha)^2}{I(\alpha)}$. We have that $g'(x) = \frac{1}{\log 2}$, and so at the limit we will have:

$$\hat{\eta} \overset{\mathrm{a}}{\sim} N\left(g(\alpha), \frac{g'(\alpha)^2}{I(\alpha)}\right) = N\left(\frac{\alpha}{\log 2}, \frac{\alpha^2}{n(\log 2)^2}\right)$$