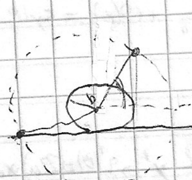
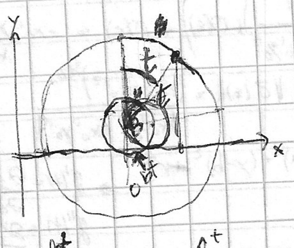
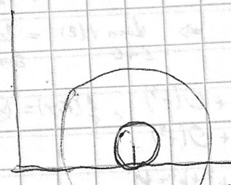
List 1

(1.1)







$x(t) = a\sin t + bt,\ y(t) = a\cos t + b$

$\alpha(t) = (a\sin t + at,\ a\cos t + a)$, $a = b$

$\alpha'(t) = (a\cos t + a,\ -a\sin t)$

$\|\alpha'(t)\| = a\sqrt{\cos^2 t + 1 + 2\cos t + \sin^2 t} = \sqrt{2}a\sqrt{1+\cos t}$

(1.1) $s(t) = \int_0^t \sqrt{2}a\sqrt{1+\cos t}\,dt = \sqrt{2}a\int_0^t \sqrt{1+\cos t}\,dt = \sqrt{2}a\int_0^t \sqrt{2\cos^2\frac{t}{2}}\,dt = 2a\int_0^t \left|\cos\frac{t}{2}\right| \underset{t<\pi}{=} 2a\sin\frac{t}{2}\Big|_0^t = 2a\sin\frac{t}{2}$

(1.2) i) cicloide $\rightarrow$ $\tilde{\alpha}(t) = (a\sin t + at,\ a\cos t + a)$ $\Rightarrow$ cicloide invertida: $\gamma(\lambda) = (a\sin\lambda + a\lambda,\ -a\cos\lambda + a)$, $\lambda \in [0, \lambda]$

ii) $E_1 = E_2 \iff mgh_1 = \frac{1}{2}m v(\theta)^2 + mgh_2 \iff g\cdot a(1+\cos\alpha) = \frac{1}{2}|\dot\gamma(\theta)|^2 + ga(1+\cos\theta) \Rightarrow$

$\Rightarrow |v(\theta)| = \sqrt{2ga(\cos\alpha - \cos\theta)}$

iii) $s = \int_\alpha^\theta \|(a\cos\lambda + a,\ a\sin\lambda)\|\,d\lambda = \int_\alpha^\theta a\sqrt{\cos^2\lambda + 2\cos\lambda + 1 + \sin^2\lambda} = \sqrt{2}a\int_\alpha^\theta \sqrt{1-\cos\lambda}\,d\lambda \underset{\lambda\in[0,\pi]}{=} \sqrt{2}a\int_\alpha^\theta \sqrt{2}\sin\frac{\lambda}{2}\,d\lambda = -2a\cos\frac{t}{2}\Big|_\alpha^\theta = 4a\left[\cos\frac{\alpha}{2} - \cos\frac{\theta}{2}\right]$

iv) ~~$t(\theta) = \frac{s(\theta)}{v(\theta)} = \frac{2a\left[\sin\frac{\theta}{2} - \sin\frac{\alpha}{2}\right]}{\sqrt{2ga}\sqrt{1-\cos\alpha}} \Rightarrow t(\pi) = \frac{2a\sqrt{1-\sin\frac{\alpha}{2}}}{\sqrt{2ga}\sqrt{1-\cos\alpha}}$~~

~~$t(\theta) = \int_\alpha^\theta \frac{s(\theta)}{|v(\theta)|}d\theta = \int_\alpha^\theta \frac{4a\left(\cos\frac{\alpha}{2} - 2\cos\frac{\theta}{2}\right)}{\sqrt{2ga}\sqrt{\cos\alpha - \cos\theta}}d\theta = \dots$~~

~~$\Rightarrow t(\pi) = \int_\alpha^\pi \frac{2a\left(\cos\frac{\alpha}{2} - \cos\frac{\theta}{2}\right)}{\sqrt{2ga}\sqrt{\cos\alpha - \cos\theta}}d\theta \Rightarrow \frac{dt(\pi)}{d\alpha} = \dots$~~

$v = \frac{ds}{dt} = \frac{2a\sin\frac{\theta}{2}\,d\theta}{dt} \Rightarrow t(\alpha) = \int_\alpha^\pi \frac{2a\sin\frac{\theta}{2}\,d\theta}{\sqrt{2ga}\sqrt{\cos\alpha - \cos\theta}} = \int_{\cos\frac{\alpha}{2}}^0 \frac{2a(-2\,du)}{\sqrt{2ga}\sqrt{\cos\alpha + 1 - 2u^2}} = \frac{4a}{\sqrt{2ga}}\int_0^{\cos\frac{\alpha}{2}} \frac{1}{\sqrt{\cos\alpha + 1 - 2u^2}}\,du =$

$u = \cos\frac{\theta}{2} \Rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \Rightarrow \cos\theta = 2u^2 - 1$

$\frac{\sqrt{2}\cos\frac{\alpha}{2}}{\sqrt{\cos\alpha + 1}} = \frac{\sqrt{2}\cos\frac{\alpha}{2}}{\sqrt{2}\cos\frac{\alpha}{2}} = 1$

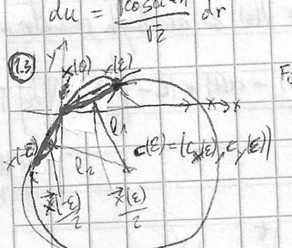
$= \frac{4a}{\sqrt{2ga}}\int_0^1 \frac{1/\sqrt{2}\,dr}{\sqrt{1-r^2}} = \frac{4a}{2\sqrt{ga}}\arcsin(r)\Big|_0^1 = 2\sqrt{\frac{a}{g}}\frac{\pi}{2} = \pi\sqrt{\frac{a}{g}}$, which does not depend on α.

$du = -\frac{1}{2}\sin\frac{\theta}{2}d\theta$

$\frac{\sqrt{2}u}{\sqrt{\cos\alpha + 1}} = r$

$du = \frac{\sqrt{\cos\alpha + 1}}{\sqrt{2}}dr$

(1.3)



For a small enough ε, we have that: $x(\pm\varepsilon) = \pm x_0'\varepsilon + x_0''\frac{\varepsilon^2}{2} + O(\pm\varepsilon^3) \Rightarrow x(\pm\varepsilon)^2 = x_0'^2\varepsilon^2 + O(\varepsilon^3)$

$y(\pm\varepsilon) = \pm y_0'\varepsilon + y_0''\frac{\varepsilon^2}{2} + O(\pm\varepsilon^3) \Rightarrow$ ~~$\frac{x(\pm\varepsilon)}{y(\pm\varepsilon)}$~~

Equation for l_1: slope $= -\frac{1}{m}$, where $m = \frac{y(\varepsilon)}{x(\varepsilon)}$ $\Rightarrow l_1(s) = -\frac{x(\varepsilon)}{y(\varepsilon)}\left(s - \frac{x(\varepsilon)}{2}\right) + \frac{y(\varepsilon)}{2}$

Equation for l_2: $l_2(s) = -\frac{x(-\varepsilon)}{y(-\varepsilon)}\left(s - \frac{x(-\varepsilon)}{2}\right) + \frac{y(-\varepsilon)}{2}$

$\Rightarrow l_1(s) = l_2(s) \iff -\frac{x(\varepsilon)}{y(\varepsilon)}\left(s - \frac{x(\varepsilon)}{2}\right) + \frac{y(\varepsilon)}{2} = -\frac{x(-\varepsilon)}{y(-\varepsilon)}\left(s - \frac{x(-\varepsilon)}{2}\right) + \frac{y(-\varepsilon)}{2} \iff \frac{y(\varepsilon) - y(-\varepsilon)}{2} + \frac{x(\varepsilon)^2}{2y(\varepsilon)} - \frac{x(-\varepsilon)^2}{2y(-\varepsilon)} =$

Let's find the center.

$= s\left(\frac{x(\varepsilon)}{y(\varepsilon)} - \frac{x(-\varepsilon)}{y(-\varepsilon)}\right) \Rightarrow s = \dfrac{\frac{y(\varepsilon) - y(-\varepsilon)}{2} + \frac{x(\varepsilon)^2 y(-\varepsilon) - x(-\varepsilon)^2 y(\varepsilon)}{2y(\varepsilon)y(-\varepsilon)}}{\frac{x(\varepsilon)y(-\varepsilon) - x(-\varepsilon)y(\varepsilon)}{\ }} = \dfrac{\frac{y(\varepsilon)y(-\varepsilon)(y(\varepsilon) - y(-\varepsilon))}{2} + \frac{x(\varepsilon)^2 y(-\varepsilon) - x(-\varepsilon)^2 y(\varepsilon)}{2y(\varepsilon)y(-\varepsilon)}}{x(\varepsilon)y(-\varepsilon) - x(-\varepsilon)y(\varepsilon)} =$

$= \dfrac{\left(y_0'\varepsilon + y_0''\frac{\varepsilon^2}{2} + O(\varepsilon^3)\right)\left(-y_0'\varepsilon + y_0''\frac{\varepsilon^2}{2} + O(\varepsilon^3)\right)\left(y_0' + O(\varepsilon^3)\right) + \frac{y(\varepsilon)y(-\varepsilon)}{x_0}}{\left(x_0'\varepsilon + x_0''\frac{\varepsilon^2}{2} + O(\varepsilon^3)\right)\left(-y_0'\varepsilon + y_0''\frac{\varepsilon^2}{2} + O(\varepsilon^3)\right) - \left(-x_0'\varepsilon + x_0''\right.}$

Other approach: ~~$O(\varepsilon^3)$~~ Recall that $\frac{1}{1 \pm x + O(x^2)} = 1 \mp x + O(x^2) \Rightarrow \frac{x(\pm\varepsilon)}{y(\pm\varepsilon)} = \frac{x(\pm\varepsilon)}{\pm\varepsilon y_0'\left(1 \pm \frac{y_0''}{y_0'}\frac{\varepsilon}{2} + O(\varepsilon^2)\right)} = \frac{\pm x_0'\varepsilon + x_0''\frac{\varepsilon^2}{2} + O(\varepsilon^3)}{\pm\varepsilon y_0'}\left(1 \mp \frac{y_0''}{y_0'}\frac{\varepsilon}{2} + O(\varepsilon^2)\right)$

$= \left(\frac{x_0'}{y_0'} \pm \frac{x_0''}{y_0'}\frac{\varepsilon}{2} + O(\varepsilon^2)\right)\left(1 \mp \frac{y_0''}{y_0'}\frac{\varepsilon}{2} + O(\varepsilon^2)\right) = \frac{x_0'}{y_0'} \pm \frac{\varepsilon}{2}\left(\frac{x_0''}{y_0'} - \frac{y_0'' x_0'}{y_0'^2}\right) + O(\varepsilon^2)$

$\Rightarrow \varepsilon y_0' + O(\varepsilon^3) + \frac{1}{2}x(\varepsilon)\left(\frac{x_0'}{y_0'} + \frac{\varepsilon}{2}\left(\frac{x_0''y_0' - y_0''x_0'}{y_0'^2}\right) + O(\varepsilon^2)\right) - \frac{1}{2}x(-\varepsilon)\left(\frac{x_0'}{y_0'} - \frac{\varepsilon}{2}\left(\frac{x_0''y_0' - y_0''x_0'}{y_0'^2}\right) + O(\varepsilon^2)\right) = s\left(\varepsilon\frac{x_0''y_0' - y_0''x_0'}{y_0'^2} + O(\varepsilon^2)\right) \iff$

$\iff \varepsilon y_0' + O(\varepsilon^3) + \frac{1}{2}\left(\frac{x_0'^2}{y_0'}\varepsilon + \dots + O(\varepsilon^2)\right) - \frac{1}{2}\left(-\frac{x_0'^2\varepsilon}{y_0'} + O(\varepsilon^2)\right) = s\left(\varepsilon\frac{x_0''y_0' - y_0''x_0'}{y_0'^2} + O(\varepsilon^2)\right) \iff$

$$\Leftrightarrow \frac{y_0'^2 + \frac{x_0'^2}{y_0'}\varepsilon + O(\varepsilon^2)}{\varepsilon\frac{x_0''y_0' + x_0'y_0''}{y_0'^2} + O(\varepsilon^2)} = \frac{\frac{y_0'^2+x_0'^2}{y_0'}\varepsilon + O(\varepsilon^2)}{\varepsilon\left(\frac{x_0''y_0'+x_0'y_0''}{y_0'^2}\right)(1+O(\varepsilon))} = \frac{\left(\frac{y_0'^2+x_0'^2}{y_0'} + O(\varepsilon)\right)(1+O(\varepsilon))}{\frac{x_0''y_0'+x_0'y_0''}{y_0'^2}} = y_0'\frac{y_0'^2+x_0'^2}{x_0''y_0'+x_0'y_0''} + O(\varepsilon) = c_x(\varepsilon)$$

$$l_1(c_x(\varepsilon)) = -\frac{x(\varepsilon)}{y(\varepsilon)}\left(c_x(\varepsilon) - \frac{x(\varepsilon)}{2}\right) + \frac{y(\varepsilon)}{2} = \left(-\frac{x_0'}{y_0'} + O(\varepsilon)\right)\left(y_0'\frac{y_0'^2+x_0'^2}{x_0''y_0'+x_0'y_0''} + O(\varepsilon)\right) + O(\varepsilon) = -x_0'\frac{y_0'^2+x_0'^2}{x_0''y_0'+x_0'y_0''} + O(\varepsilon) = c_y(\varepsilon)$$

$$\Rightarrow \lim_{\varepsilon\to 0}\vec{c}(\varepsilon) = \frac{y_0'^2+x_0'^2}{x_0''y_0'+x_0'y_0''}(y_0', -x_0') \quad\Rightarrow\quad \lim_{\varepsilon\to0} r(\varepsilon) = \lim_{\varepsilon\to 0}\|\vec{c}(\varepsilon)\| = \frac{(y_0'^2+x_0'^2)^{3/2}}{|x_0''y_0'+x_0'y_0''|}$$

1.4 $g(t) = f(\vec{s}(t))$, where $x(t) = x_0't + x_0''\frac{t^2}{2} + O(t^3)$, $y(t) = y_0't + y_0''\frac{t^2}{2} + O(t^3)$, $f(x,y) = (x-a)^2 + (y-b)^2 - r^2$

$$\begin{cases} g(0)=0 \\ g'(0)=0 \\ g''(0)=0\end{cases} \Leftrightarrow \begin{cases} a^2+b^2-r^2=0 \\ -2ax_0'-2by_0'=0 \\ 2x_0'^2+2y_0'^2-2ax_0''-2by_0''=0\end{cases} \Leftrightarrow \begin{cases} a^2+b^2=r^2 \\ ax_0'+by_0'=0 \\ x_0'^2+y_0'^2 = ax_0''+by_0''\end{cases} \to b = -\frac{ax_0'}{y_0'}$$

$$g'(t) = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' \Rightarrow g'(0) = (-2a)\cdot x_0' + (-2b)\cdot y_0'$$

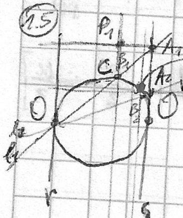
$$g''(t) = \frac{\partial^2 f}{\partial x^2}x'^2 + \frac{\partial f}{\partial x}x'' + \frac{\partial^2 f}{\partial y^2}y'^2 + \frac{\partial^2 f}{\partial y\partial x}y'x' + \frac{\partial f}{\partial y}y'' + \frac{\partial^2 f}{\partial x\partial y}y'x' \underset{\text{Schwarz}}{=} \frac{\partial^2 f}{\partial x^2}x'^2 + \frac{\partial^2 f}{\partial y^2}y'^2 + 2\frac{\partial^2 f}{\partial x\partial y}x'y' + \frac{\partial f}{\partial x}x'' + \frac{\partial f}{\partial y}y''$$

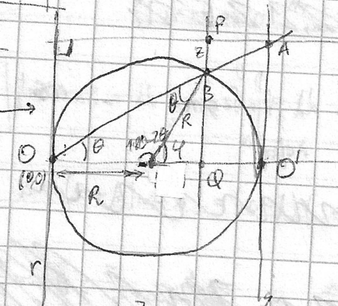
$$g''(0) = 2x_0'^2 + 2y_0'^2 + 0 + (-2a)\cdot x_0'' + (-2b)y_0''$$

$$\Rightarrow x_0'^2 + y_0'^2 = ax_0'' - a\frac{x_0'y_0''}{y_0'} \Leftrightarrow \boxed{a = \frac{x_0'^2+y_0'^2}{x_0''y_0'-x_0'y_0''}\cdot y_0'} \Rightarrow \boxed{b = \frac{x_0'^2+y_0'^2}{x_0''y_0'-x_0'y_0''}(-x_0')}$$

$$\Rightarrow \boxed{r = \frac{(x_0'^2+y_0'^2)^{3/2}}{|x_0''y_0'-x_0'y_0''|}}$$

1.5





$B := (b_x, b_y)$ $\quad P := (x,y)$. Obs: $x = b_x$, $y = a_y$, $a_x = 2R$

$A := (a_x, a_y)$

Obs: $\varphi = 2\theta \Rightarrow B = (R\cos\varphi + R, R\sin\varphi) = (R\cos 2\theta + R, R\sin 2\theta)$

By the similarity of the triangles OBQ and BPA $\Rightarrow \frac{z}{a_x - b_x} = \frac{b_y}{b_x} \Leftrightarrow$

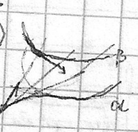
$$\Leftrightarrow z = \frac{b_y}{b_x}\cdot(a_x - b_x) = \frac{\sin 2\theta}{1+\cos 2\theta}(2R - R - R\cos 2\theta) \Rightarrow y = b_y + z = R\sin 2\theta + \frac{R\sin 2\theta}{1+\cos 2\theta}(1-\cos 2\theta) =$$

$$= \frac{R\sin 2\theta}{1+\cos 2\theta}\left[1+\cos 2\theta + 1 - \cos 2\theta\right] = \frac{2R\sin 2\theta}{1+\cos 2\theta} \Rightarrow P = \left(R\cos 2\theta + R, \frac{2R\sin 2\theta}{1+\cos 2\theta}\right) = \boxed{(2R\cos^2\theta, 2R\tan\theta)}$$

$$\frac{2R\,2\sin\theta\cos\theta}{2\cos^2\theta}$$

List 2

2.1



a) Let $\alpha(s) = (x(s), y(s))$ be an arc-length parametrization of α and $\beta(s)$ be an arc-length parametrization (from α) of β.

By hypothesis $\langle \beta'(s), \alpha'(s)\rangle = 0$ $\quad (a(s), b(s))$

β differentiable, $\beta(s) = \alpha(s) + \lambda(s)\alpha'(s)$, for some $\lambda(s)$ $\Rightarrow 0 = \langle\beta'(s), \alpha'(s)\rangle = \langle \alpha'(s) + \lambda'(s)\cdot\alpha'(s) + \lambda(s)\cdot\alpha''(s), \alpha'(s)\rangle$

it is in the tangent line

Obs: $\lambda(s)$ differentiable because $\lambda(s) = \langle\beta(s) - \alpha(s), \alpha'(s)\rangle \in C^\infty$

$$= \langle\alpha'(s), \alpha'(s)\rangle + \lambda'(s)\langle\alpha'(s), \alpha'(s)\rangle + \lambda(s)\underbrace{\langle\alpha''(s), \alpha'(s)\rangle}_{0} = (1+\lambda'(s))\underbrace{\|\alpha'(s)\|^2}_{1} = 1 + \lambda'(s) \Rightarrow \lambda'(s) = -1 \Rightarrow \lambda(s) = -s + C \Rightarrow$$

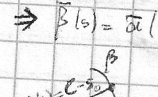
$$\Rightarrow \beta(s) = \alpha(s) + (-s + C)\alpha'(s)$$

b) Sigui $\alpha(t)$ una corba qualsevol, $I \to \mathbb{R}^2$, i $s: I \to J$ la parametrització per l'arc. $\Rightarrow h(t) = s(t) \Rightarrow \bar{\alpha} = (\alpha\circ h^{-1})(s) \leftarrow$ arc-length parametrized.

$$\Rightarrow \bar\beta(s) = \bar\alpha(s) + (-s+C)\bar\alpha'(s) = \alpha(h^{-1}(s)) + \left(-\int_{t_0}^t\|\alpha'(u)\|du + C\right)\alpha'(h^{-1}(s))\cdot h'(h^{-1}(s))^{-1}$$

$$\bar\alpha'(s) = \alpha'(h^{-1}(s))(h^{-1})'(s) = \alpha'(h^{-1}(s))\cdot(h'(h^{-1}(s)))^{-1}$$

Or easily: $\beta(t) = \alpha(t) + \left(C - \int_{t_0}^t\|\alpha'(u)\|du\right)T_\alpha(t)$, $\quad T_\alpha(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$



c) We fix a rope and measure from s_0 to s_1 and then, we start to rotate and we'll get the involute.

d) $\alpha(t) = (t, \cosh t) \Rightarrow \alpha'(t) = (1, \sinh t) \Rightarrow \|\alpha'(t)\| = \sqrt{1+\sinh^2 t} = \cosh t \Rightarrow s(t) = \int_0^t \cosh u\, du = \sinh u\Big|_0^t = \sinh t - 1 \Rightarrow$

$$\Rightarrow \beta(t) = (t, \cosh t) + (C - \sinh t + 1)\left(\frac{1}{\cosh t}, \tanh t\right) = \text{At } t=0, \text{ we have } \beta(t) = (C+1, 1) = (0,1) \Rightarrow C = -1 \Rightarrow$$

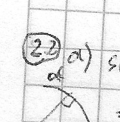
$$\Rightarrow \beta(t) = \left(t - \tanh t, \cosh t - \frac{\sinh^2 t}{\cosh t}\right) = \left(t - \tanh t, \frac{1}{\cosh t}\right) \leftarrow \text{parametrization of a tractrix.}$$

e) Circle: $\alpha(t) = (R\cos t, R\sin t) \Rightarrow \alpha'(t) = R(-\sin t, \cos t) \Rightarrow \|\alpha'(t)\| = R \Rightarrow s(t) = Rt \Rightarrow \beta(t) = (R\cos t, R\sin t) + (C - Rt)\frac{(-R\sin t, R\cos t)}{R} =$

$$= (R\cos t - (C-Rt)\sin t, R\sin t + (C-Rt)\cos t)$$

Cycloid: $\alpha(t) = (Rt + R\sin t, R + R\cos t) \Rightarrow \alpha'(t) = R(1+\cos t, -\sin t) \Rightarrow \|\alpha'(t)\| = R\sqrt{2+2\cos t} = 2R|\cos\frac{t}{2}|$

2.2



a) Similarly we must have $\langle\beta'(t), \alpha'(t)\rangle = 0$. Moreover $\beta(t) \in$ {normal curve of $\alpha(t)$} $= \alpha(t) + N_\alpha(t)\lambda(t)$. $\Rightarrow \beta'(s) = \alpha'(t) + N_\alpha'(s)\lambda(s) + N_\alpha(s)\lambda'(s)$

$$\Rightarrow 0 = \langle\beta'(s), \alpha'(s)\rangle = \langle v(t)(1 - \lambda(t)K_\alpha(t))T_\alpha(t) + N_\alpha(t)\lambda'(t), v(t)T(t)\rangle = v(t)^2(1-\lambda(t)K_\alpha(t))T_\alpha \Rightarrow \lambda(t) = \frac{1}{K_\alpha(t)} \Rightarrow$$

$$T_\alpha(t)\|\alpha'(t)\| \quad -K_\alpha(s)T_\alpha(s)\|\alpha'(t)\|$$

$$\Rightarrow \beta(t) = \alpha(t) + \frac{1}{K_\alpha(t)}\cdot N_\alpha(t), \quad \frac{1}{K_\alpha(t)} = \rho_\alpha(t)$$

$K_\alpha(t) \neq 0$.

curvatura de totes els cercles osculadors de $\alpha(t)$.

b) $\beta(t)$ és la corba formada per tots els centres de curvatura de tots els cercles osculadors de $\alpha(t)$.

c) $L_\beta(s_0,s_1) = \int_{s_0}^{s_1} \|\beta'(s)\|\, ds \underset{\uparrow}{=} \int_{s_0}^{s_1} \rho'(s)\, ds = \rho(s_1)-\rho(s_0)$

$\beta'(s) = \underset{T(s)}{\alpha'(s)} + \dfrac{1}{K_\alpha(s)} N_\alpha(s) + \rho(s) \underset{-K(s)T(s)}{N_\alpha'(s)} \Rightarrow \|\beta'(s)\| = |\rho'(s)|\cdot\|N(s)\| = |\rho'(s)| \underset{\uparrow}{=} \rho'(s)$

sps ρ' has constant positive sign.

d) Proved in c): $\beta'(s) = \rho'(s)\cdot N_\alpha(s)$.

e) ~~$\beta'(s_0)=\rho'(s_0)\cdot N_\alpha(s_0)$, $\beta'(s_1)=\rho'(s_1) N_\alpha(s_1)$~~ $\Rightarrow$ ~~$\beta'(s_0+\varepsilon)=\rho'(s_0+\varepsilon)N_\alpha(s_0+\varepsilon)$, $s_1=s_0+\varepsilon$~~

Normal line of α at s_1: $\ell_1(s)=\alpha(s_1)+\lambda(s)N(s_1)$; $\ell_2(s)=\alpha(s_2)+\mu(s)N(s_2)$ sps they coincide at

$\Rightarrow \alpha(s_1)+\lambda(s)N(s_1)=\alpha(s_2)+\mu(s)N(s_2) \Rightarrow \langle \alpha(s_1)-\alpha(s_2)+\lambda(s)N(s_1), T(s_2)\rangle = 0 \Leftrightarrow \langle \alpha_1(s_1)-\alpha(s_2), T(s_2)\rangle = -\lambda(s)\langle N(s_1), T(s_2)\rangle$

$\Leftrightarrow \left\langle \dfrac{\alpha_1(s_1)-\alpha(s_2)}{s_1-s_2}, T(s_2)\right\rangle = -\lambda(s)\left\langle \dfrac{N(s_1)-N(s_2)}{s_1-s_2}, T(s_2)\right\rangle - \dfrac{\lambda(s)}{s_1-s_2}\underset{0}{\langle N(s_2),T(s_2)\rangle} \xrightarrow[s_2\to s_1]{} \langle \underset{T(s_1)}{\alpha_1'(s_1)}, T(s_1)\rangle = -\lambda(s)\langle \underset{-K(s_1)T(s_1)}{N'(s_1)}, T(s_1)\rangle \Leftrightarrow$

$\Leftrightarrow 1 = \lambda(s)K(s_1) \Rightarrow \lambda(s) = \dfrac{1}{K(s_1)} \Rightarrow \ell_1(s) = \alpha(s_1) + \dfrac{1}{K(s_1)}N(s_1) = \beta(s_1)$. ✓

f) Coordinate equations for the evolute of $\alpha(t)=(x,y)$:
$$\begin{cases} X(t) = x - y'\dfrac{(x'^2+y'^2)}{x'y''-x''y'} \\ Y(t) = y + x'\dfrac{x'^2+y'^2}{x'y''-x''y'} \end{cases}$$

Cycloid: $x(t)=a(t+\sin t)$, $y(t)=a(1+\cos t)$ $\to x'=a(1+\cos t) \to x''=-a\sin t$; $y'=-a\sin t \to y''=-a\cos t$

$\Rightarrow x'^2+y'^2 = a^2\left((1+\cos t)^2+(\sin t)^2\right) = a^2(1+(\cos t)^2+2\cos t+(\sin t)^2) = 2a^2(1+\cos t)$; $x'y''-x''y' = a(1+\cos t)(-a\cos t)+a\sin t(-a\sin t) = a^2\left[\cos t+(\cos t)^2+(\sin t)^2\right] =$

$= -a^2(1+\cos t) \Rightarrow \dfrac{x'^2+y'^2}{x'y''-x''y'} = -2 \Rightarrow X(t) = a(t+\sin t) - 2a\sin t = a(t-\sin t)$ $\to$ cycloid

$Y(t) = a(1+\cos t) - 2a(1+\cos t) = -a(1+\cos t)$

g) $K(t) = \dfrac{\|\alpha'\times\alpha''\|}{\|\alpha'\|^3}$; $\alpha'(t)=(1,\sinh t)$, $\alpha''(t)=(0,\cosh t)$ $\Rightarrow \alpha'\times\alpha'' = \begin{vmatrix} i & j & k \\ 1 & \sinh t & 0 \\ 0 & \cosh t & 0\end{vmatrix} = (0,0,\cosh t)$; $\|\alpha'\|^3 = (1+(\sinh t)^2)^{3/2} = (\cosh t)^3 \Rightarrow K(t) = \dfrac{\cosh t}{(\cosh t)^3} = \dfrac{1}{(\cosh t)^2}$

$B = \dfrac{(0,0,\cosh t)}{\cosh t} = (0,0,1)$, $T = \dfrac{(1,\sinh t)}{\cosh t}$; $N = B\times T = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{\cosh t} & \tanh t & 0\end{vmatrix} = \left(-\tanh t, \dfrac{1}{\cosh t}, 0\right) \Rightarrow \beta(t) = \alpha(t) + \dfrac{1}{K(t)}N_\alpha(t) = (t,\cosh t)+(\cosh t)^2\left(-\tanh t, \dfrac{1}{\cosh t}\right) =$
$= (t-\sinh t\cosh t,\ 2\cosh t)$.

③ a) $s(t) = \int_0^t \|(1,\sinh t)\|\,dt = \int_0^t \cosh t\, dt = \sinh t \Rightarrow t(s) = \operatorname{arcsinh} s \Rightarrow \alpha(t(s)) = \bar\alpha(s) = (\operatorname{arcsinh}(s), \sqrt{1+s^2}) \Rightarrow$

$\Rightarrow \bar\alpha'(s) = \left(\dfrac{1}{\sqrt{1+s^2}}, \dfrac{s}{\sqrt{1+s^2}}\right) \Rightarrow \bar\alpha''(s) = \dfrac{1}{(1+s^2)^{3/2}}(-s,1) \Rightarrow \|\bar\alpha''(s)\| = \dfrac{\sqrt{1+s^2}}{(\sqrt{1+s^2})^3} = \dfrac{1}{1+s^2}$

b) Tractrix (β) is the involute of the catenary (α) $\Rightarrow \beta(s) = \alpha(s) + (\underset{\text{const.}}{s_0}-s)\alpha'(s)$ (s is arc-length parameter of α) $= (\operatorname{arcsinh}(s), \sqrt{1+s^2}) + \dfrac{s_0-s}{\sqrt{1+s^2}}(1,s) = \left(\operatorname{arcsinh}(s)+\dfrac{s_0-s}{\sqrt{1+s^2}},\right.$

$\left.\dfrac{(\sqrt{1+s^2})^2+s_0s-s^2}{\sqrt{1+s^2}}\right) \underset{\text{take } s_0=0}{=} \left(\operatorname{arcsinh}(s) - \dfrac{s}{\sqrt{1+s^2}}, \dfrac{1}{\sqrt{1+s^2}}\right) \Rightarrow \beta'(s) = \left(\dfrac{1}{\sqrt{1+s^2}} - \dfrac{1}{(\sqrt{1+s^2})^3}, \dfrac{s}{(\sqrt{1+s^2})^3}\right) = \dfrac{1}{(1+s^2)^{3/2}}(s^2, s) \to \|\beta'(s)\|^2 = \dfrac{s(1+s^2)^{1/2}}{(1+s^2)^{3/2}} = \dfrac{s}{1+s^2}$

$\to \beta''(s) = \dfrac{1}{(1+s^2)^{5/2}}(2s-s^3, 1-2s^2) \Rightarrow K(t) = \dfrac{\det(\beta',\beta'')}{\|\beta'(t)\|^3} \Rightarrow \det(\beta',\beta'') = \dfrac{1}{(1+s^2)^{3/2}}\dfrac{1}{(1+s^2)^{5/2}}\begin{vmatrix} s^2 & s \\ 2s-s^3 & 1-2s^2\end{vmatrix} = \dfrac{s^2-2s^4-2s^2+s^4}{(1+s^2)^4} = \dfrac{-s^2(1+s^2)}{(1+s^2)^4} = \dfrac{-s^2}{(1+s^2)^3}$

$\Downarrow$

$K(s) = \dfrac{\frac{-s^2}{(1+s^2)^3}}{\frac{s^3}{(1+s^2)^3}} = \boxed{-\dfrac{1}{s}}$

c) We know that $K_\beta(s) = \dfrac{\|\beta'(s)\times\beta''(s)\|}{\|\beta'(s)\|^3}$ (arc-length parameter of α). Moreover from ② $\Rightarrow \beta(s) = \alpha(s)+(s_0-s)\alpha'(s)$.

$\Rightarrow \beta'(s) = \alpha'(s) + (s_0-s)\alpha''(s) - \alpha'(s) \Rightarrow \beta''(s) = \alpha'''(s)(s_0-s) - \alpha''(s) = (x'''(s_0-s)-x'', y'''(s_0-s)-y'')$

$(s_0-s)(x'',y'')$

$\Rightarrow \beta'\times\beta'' = \begin{vmatrix} i & j & k \\ x'' & y'' & 0 \\ x'''(s_0-s)-x'' & y'''(s_0-s)-y'' & 0\end{vmatrix}(s_0-s) = (s_0-s)\left(x''y'''(s_0-s) - y''x'''(s_0-s)\right)(0,0) \Rightarrow \underset{\|\beta'\times\beta''\|}{(s_0-s)^2\left[x''y'''-y''x'''\right]} \Rightarrow K(s) = \dfrac{(s_0-s)^2}{|s_0-s|^3} = \dfrac{1}{|s_0-s|}$

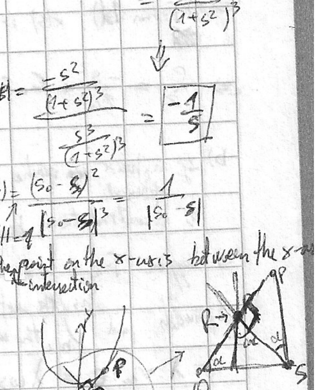
$\|\alpha''\| = 1$

④ Catenary: curve s.t. the slope at each point is $s(t)$ (arclength parameter).
$(t,\cosh t) \Rightarrow$ slope $\sinh t = s(t)$ ✓

Tractrix: distance from a point on the curve and the point on the x-axis between the x-axis interaction and the tangent line, is $=1$

Let P be a point of the catenary and R be a point on the tangent line at a distance $\sinh t$ from P. Construct Q and S as shown in the figure. We shall prove that $RS=1$, and that would imply that the points R form the tractrix.

By def. of the catenary $\tan\alpha = \sinh t \Rightarrow \tan\alpha = \dfrac{RP}{RS} = \dfrac{\sinh t}{RS} \Rightarrow RS = 1$ //



List 3

③.1 a) $f(s) = \|x(s)-c(s_0)\|^2 - r(s_0)^2 \to f(s_0) = 0 \Rightarrow \|x(s)-c(s_0)\|^2 = r(s_0)^2$

Let $T=T(s_0)$, $N=N(s_0)$ and $B=B(s_0)$

$f'(s) = \langle x'(s), x(s)-c(s_0)\rangle\cdot 2 \Rightarrow f'(s_0) = 2\langle T, x(s_0)-c(s_0)\rangle = 0 \Rightarrow$

$f''(s) = 2\langle x''(s), x(s)-c(s_0)\rangle + 2\langle x'(s), x'(s)\rangle \to f''(s_0) = 2\langle T', x(s_0)-c(s_0)\rangle + 2 = 0 \to \langle KN, x(s_0)-c(s_0)\rangle = -1$

$f'''(s) = 2\langle x'''(s), x(s)-c(s_0)\rangle + 2\langle x''(s), x'(s)\rangle + 4\langle x''(s), x'(s)\rangle \to f'''(s_0) = 2\langle T'', x(s_0)-c(s_0)\rangle = 0$

$\langle T', T\rangle = 0$ $\langle T', T\rangle = 0$

$f'''(s_0) \rightsquigarrow$ Let $x(s_0)-c(s_0)=\lambda T+\mu N+\nu B \Rightarrow$

$$\begin{cases} \lambda^2+\mu^2+\nu^2 = r(s_0)^2 \\ \langle T, \lambda T+\mu N+\nu B\rangle = 0 \to \lambda = 0 \\ \langle KN, \lambda T+\mu N+\nu B\rangle = -1 \to \mu = -1/K \\ \langle K'N-K^2T-\tau KB, \lambda T+\mu N+\nu B\rangle = 0 \Leftrightarrow K'\mu - K^2\lambda - \tau K\nu = 0 \end{cases}$$

$T'' = K'N+KN' = K'N - K^2T - \tau KB$

$\nu = \frac{K'}{\tau K}\mu = -\frac{K'}{\tau K^2}$

$\Rightarrow x(s_0)-c(s_0) = -\frac{1}{K}N - \frac{K'}{\tau K^2}B \to c(s_0) = x(s_0)+\rho(s_0)N+\frac{K'}{\tau K^2}B = x(s_0)+\rho(s_0)N+\frac{\rho'}{\tau}B.$

$\rho = \frac{1}{K} \to \rho' = \frac{-K'}{K^2}$

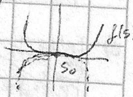
$\Rightarrow r(s_0)^2 = \mu^2+\nu^2 = \frac{1}{K^2}+\left(\frac{K'^2}{\tau K^4}\right) = \rho(s_0)^2 + \frac{\rho'(s_0)^2}{\tau(s_0)^2}$

b) $f^{(4)}$ ~~$= 2\langle T''', x(s)\rangle + 2\langle T'', x(s_0)-c(s_0)\rangle$~~

Per taylor: $f(s) = \frac{f^{(4)}(s_0)}{4!}(s-s_0)^4 + \mathcal{O}((s-s_0)^5)$

si $f^{(4)}(s_0)<0 \to$ the curve is inside the sphere

$f^{(4)}(s_0)>0 \to$ the curve is outside the sphere.



c) $c'(s) = x'(s)+\rho'(s)N(s)+\rho(s)N'(s) - \left(\frac{\rho'}{\tau}\right)'B - \frac{\rho'}{\tau}B' = T+\rho'N-\rho KT-\rho\tau B - \left(\frac{\rho'}{\tau}\right)'B - \rho'N = -\lambda B$

$N' = -KT-\tau B$, $B' = \tau N$, $\lambda = \rho\tau + \left(\frac{\rho'}{\tau}\right)'$

$2rr' = 2\rho\rho' + 2\frac{\rho'}{\tau}\left(\frac{\rho'}{\tau}\right)' \Rightarrow r' = \frac{1}{r}\rho'\left[\rho+\frac{1}{\tau}\left(\frac{\rho'}{\tau}\right)'\right] = \frac{\rho'}{r\tau}\left(\rho\tau+\frac{1}{\tau}\left(\frac{\rho'}{\tau}\right)'\right) = \frac{\rho'\lambda}{r\tau}$

d) $\boxed{\Rightarrow}$ the sphere has to be the osculating sphere $\Rightarrow$ the $c(s)$ is constant (and r' is constant too) $\Rightarrow c'(s) = \lambda B \Rightarrow \lambda = 0$ ✓.

$\boxed{\Leftarrow}$ sps $\lambda = 0 \Rightarrow c'(s) = 0$; $r' = 0. \to r = r_0 = \text{cont.} \to f(s) = \|x(s)-c_0\|^2 - r_0^2 \to f(s) = 0 \quad \forall s$ and with the same center and radius

$\to c = c_0 = \text{cont}$

$\to x(s)$ is containing in a sphere.

e) Consider $g(s_0,s) = \frac{\partial^3}{\partial s^3}(\|c(s_0)-x(s)\|^2 - r(s_0)^2) \Rightarrow g(s_0,s_0) = 0 \Rightarrow \partial_s g(s_0,s_0) + \partial_{s_0}g(s_0,s_0) = 0$

because the osculating sphere has contact ≥ 3; $\partial_s g(s_0,s_0) = f^{(4)}(s_0)$

We shall prove $f^{(4)}(s_0) \Leftrightarrow \lambda(s_0) = 0$. But note from the above observation $f^{(4)}(s_0) = 0 \Leftrightarrow \partial_{s_0}g(s_0,s_0) = 0 \to \partial_{s_0}g(s_0,s_0) \overset{\text{schwartz}}{=} \frac{\partial^3}{\partial s^3}(2\langle c'(s_0), c(s_0)-x(s)\rangle - 2r(s_0)r'(s_0))\Big|_{s=s_0} \overset{c)}{=}$

$\frac{\partial^3}{\partial s^3}(-2\lambda(s_0)\langle B(s_0), c(s_0)-x(s)\rangle - 2r(s_0)r'(s_0))\Big|_{s=s_0} = -2\lambda(s_0)\frac{\partial^2}{\partial s^2}(\langle B(s_0), -x'(s)\rangle)\Big|_{s=s_0} = 2\lambda(s_0)\frac{\partial}{\partial s}(\langle B(s_0), K(s)N(s)\rangle) = 2\lambda(s_0)\langle B(s_0), -K(s)^2T(s) - K(s)\tau(s)B(s)\rangle\Big|_{s=s_0}$

$= 2\lambda(s_0)K(s_0)\tau(s_0) = 0 \Leftrightarrow \lambda(s_0) = 0 \quad K,\tau \neq 0.$

f) $\partial_s F_s(p) = \langle c'(s), c(s)-p\rangle \cdot 2 - 2r(s)r'(s) \overset{c)}{=} \langle -\lambda(s)B(s), c(s)-p\rangle 2 - 2r\frac{\lambda\rho'}{\tau r} = 2\lambda\left[\langle -B(s), c(s)-p\rangle - \frac{\rho'}{\tau}\right] = 0 \to$

$\to$ equation of a plane perpendicular to B that passes through $x(s)$ $\equiv$ osculating plane

(intuitively $((x-c_x(s))^2A + (y-c_y(s))^2B + (z-c_z(s))^2C + D = 0)$)

g) $\lim_{\varepsilon\to 0} \Sigma(s_0)\cap\Sigma(s_0+\varepsilon) \overset{?}{=} C(s_0) \leftarrow$ osculating circle of $x(s)$ at $x(s_0)$

The equation of $\Sigma(s_0)$ is $F_{s_0}(p) = 0$ and the equation of $\Sigma(s_0+\varepsilon)$ is $F_{s_0+\varepsilon}(p) = 0 \Rightarrow \Sigma(s_0)\cap\Sigma(s_0+\varepsilon) = \{p : F_{s_0}(p) - F_{s_0+\varepsilon}(p) = 0\}\cap\Sigma(s_0)$

$F_{s_0}(p) - F_{s_0+\varepsilon}(p) = \|c(s_0)-p\|^2 - r(s_0)^2 - \|c(s_0+\varepsilon)-p\|^2 + r(s_0+\varepsilon)^2 = \|c(s_0)\|^2 + \|p\|^2 - 2\langle c(s_0), p\rangle - \|c(s_0+\varepsilon)\|^2 - \|p\|^2 + 2\langle c(s_0+\varepsilon), p\rangle +$

$r(s_0+\varepsilon)^2 - r(s_0)^2 = 0 \Leftrightarrow \|c(s_0+\varepsilon)\|^2 - \|c(s_0)\|^2 + r(s_0)^2 - r(s_0+\varepsilon)^2 = 2\langle c(s_0+\varepsilon) - c(s_0), p\rangle \Leftrightarrow \frac{\|c(s_0+\varepsilon)\|^2 - \|c(s_0)\|^2}{\varepsilon} + \frac{r(s_0)^2 - r(s_0+\varepsilon)^2}{\varepsilon} =$

$2\left\langle \frac{c(s_0+\varepsilon)-c(s_0)}{\varepsilon}, p\right\rangle \xrightarrow[\varepsilon\to 0]{} \underbrace{\|c'(s_0)\|^2 - r'(s_0)^2}_{\text{const}} = 2\langle c'(s_0), p\rangle \overset{c)}{=} -2\lambda(s_0)\langle B(s_0), p\rangle \leftarrow$ equation of a plane $\perp B(s_0) \Rightarrow$ it is the osculating plane

$\Rightarrow \Sigma(s_0)\cap\Sigma(s_0+\varepsilon) \xrightarrow[\varepsilon\to 0]{} \{\text{osculating plane}\}\cap\Sigma(s_0) = \text{osculating circle } C(s_0)$ because $C(s_0)\subset\{\text{osculating plane}\}$ and $C(s_0)\subset\Sigma(s_0)$.

h) Envolvent de la familia d'esferes osculatrius $\{F_s(p) = 0\}$ (variants) per equacions: $\{p : \exists s, F_s(p) = \partial_s F(p) = 0\} = \bigcup_s(\Sigma(s)\cap \text{pla osculador de centre } s) \overset{g)}{=} \bigcup_s C(s)$

def de envolvents; pla tangent a l'envolvent trobades; the osculating circle.

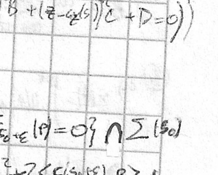
② a) From 1d) $x(s)$ is spherical $\Leftrightarrow \lambda = \tau\rho + \left(\frac{\rho'}{\tau}\right)' = 0 \overset{\tau\rho = \frac{1}{a}}{\Leftrightarrow} \frac{1}{a} + (\rho'\rho a)' = 0 \Leftrightarrow (\rho'\rho)' + \frac{1}{a^2} = 0 \Leftrightarrow \rho'\rho = -\frac{s}{a^2} + C_0 \Leftrightarrow$

$\Leftrightarrow \frac{\rho^2}{2} = \frac{-s^2}{2a^2} + C_0 s + C_1 \overset{K=\frac{1}{\rho}}{\Leftrightarrow} K^2 = \frac{1}{2}\frac{1}{\frac{-s^2}{2a^2}+C_0 s + C_1} = \frac{1}{\frac{-s^2}{4a^2}+Bs+C} \Rightarrow A = \frac{1}{a} = \cot\alpha$

b) If a spherical helix reaches the north pole, at $\alpha = \frac{\pi}{2}$ (because it reaches the north pole orthogonally to the axis). But then $\frac{K}{\tau} = \tan\frac{\pi}{2} = \infty \Rightarrow \tau = 0$, which is a contradiction, because the curve is not contained in a plane. it can only

c) Suppose $T\cdot e = \cos\alpha$ (axis of the earth). At the point $x(s)$ we'll have $x'(s) = \lambda v_\alpha + \mu v_n$, with $\lambda^2+\mu^2 = 1$ and where v_n is the unit vector in the direction of parallels and v_θ, the unit vector in the direction of the meridians. Then, if θ is the altitude from the equator $\to T\cdot e =$

$= \lambda v_\alpha\cdot e = \lambda\cos\theta = \cos\alpha \Rightarrow \theta$ can only be at most α and that happens if $\lambda = 1 \Rightarrow \mu = 0 \Rightarrow x' \perp$ a parallel



d) From 2a) ~~we know that $r^2 = \rho^2 + \frac{\rho'^2}{\tau^2}$ and $K = \tau\tan\alpha \Rightarrow \tau = \frac{1}{\rho\tan\alpha}$, $r^2 = \rho^2(1+\tan^2\alpha\,\rho'^2)$~~ we have that $\rho^2 = -A^2s^2 + Bs + C = -\left(As - \frac{B}{2A}\right)^2 + C + \frac{B^2}{4A^2} \Rightarrow \rho^2 + \left(As - \frac{B}{2A}\right)^2 = C + \frac{B^2}{4A^2}$, which is the equation of an epicycloid.

Now let $\alpha_1 = \pi_e\alpha$ the projection on the plane perpendicular to $e \Rightarrow \alpha_1(s) = \alpha(s) - \langle\alpha(s), e\rangle e \Rightarrow \alpha_1'(s) = \alpha'(s) - \langle T, e\rangle e = T - \cos\alpha\, e$

$\Rightarrow s_1(s) = \int_0^s \|\alpha'(u)\|du = \int_0^s \sqrt{1+\cos^2\alpha}\ldots$

List 4

4.1 Let $T(s) = x'(s) := (u(s), v(s))$. Since x is arc-length parametrized $\Rightarrow \|T\| = 1 \Rightarrow u^2+v^2=1$. Consider $\int_0^s [u(t)v'(t) - v(t)u'(t)]\,dt$

However $T' = KN = K(-v,u)$ ($T'=(u',v')$, K signed curvature) $\Rightarrow u' = -Kv$, $v' = Ku$ $\Rightarrow \int_0^s K[u^2+v^2]\,dt = \int_0^s K(t)\,dt$. (Now consider $\theta(s) = \theta_0 + \int_0^s K(t)\,dt$. The function will be)

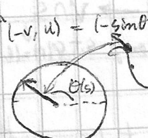
which is a differentiable function.

Sps $T = (\cos\theta(s), \sin\theta(s))$ (u,v) $\Rightarrow T' = (-\theta'\sin\theta(s), \theta'\cos\theta(s)) = \theta'(-\sin\theta(s), \cos\theta(s)) \Rightarrow$

KN, $(-v,u) = (-\sin\theta(s), \cos\theta(s))$

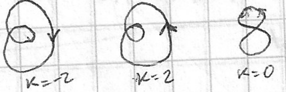
$\theta' = K(s) \Rightarrow \theta(s) = \theta_0 + \int_0^s K(t)\,dt$

Geometric interpretation:



Obs: $x(s) = x_0 + \int_0^s \cos(\theta(t))\,dt$, $y(s) = y_0 + \int_0^s \sin(\theta(t))\,dt$

4.2 Example



Consider $\alpha(s) = (\cos(s), \sin(s))$ defined on $[0, 2K\pi] \Rightarrow \alpha' = (-\sin s, \cos s) = (\cos(s+\frac{\pi}{2}), \sin(s+\frac{\pi}{2})) \Rightarrow \theta(s) = s + \frac{\pi}{2} \Rightarrow \theta(2K\pi) - \theta(0) = 2K\pi$

Other option: let $r(t) = 1 + \frac{1}{2}\cos(\frac{t}{K})$ and consider $\alpha(t) = r(t)(\cos t, \sin t)$

4.3 $y(s) = x(s) + \frac{1}{K(s)}N(s) \Rightarrow y'(s) = x'(s) - \frac{K'(s)}{K(s)^2}N(s) + \frac{1}{K(s)}(-KT) = 0$.

($x' = T$; $K'(s) = 0$; curvature plane $\Rightarrow \tau = 0$)

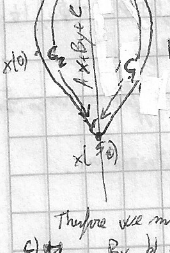
4.4 a) $\int_0^\ell (Ax(s) + By(s) + C)K'(s)\,ds = A\int_0^\ell x(s)K'(s)\,ds + B\int_0^\ell y(s)K'(s)\,ds + C\int_0^\ell K'(s)\,ds = A(1) + B(2) + C(3)$. Recall that $x'(s) = \cos\theta(s), \sin\theta(s)$ and with $\theta'(s) = K$

$(1) = A\int_0^\ell x(s)K'(s)\,ds = K(s)x(s)\Big|_0^\ell - \int_0^\ell K(s)x'(s)\,ds = -\int_0^\ell K(s)x'(s)\,ds = +A\int_0^\ell \theta'\sin\theta\,ds = A\int_0^\ell (\sin\theta)'\,ds = A[\sin\theta(\ell) - \sin\theta(0)] = 0$

$K'(s)ds = dv \to K(s) = v$, $u = x(s) \to du = x'(s)\,ds$; $K(\ell) = K(0)$ and $x(\ell) = x(0)$: the curve is closed and regular. $\theta(\ell) = \theta(0) = 2\pi K$ for some $K \in \mathbb{Z}$

Similarly $(2) = 0$ and $(3) = \int K(\ell) - K(0) = 0$ (curve closed and regular).

b)



$\Rightarrow K'(s_1) = 0$, $K'(s_0) = 0$. Consider the line $Ax + By + C$ passing through s_0 and s_1.

From a), we know that $I = \int (Ax(s) + By(s) + C)K'(s)\,ds = 0 = \int_{C_1} (Ax(s)+By(s)+C)K'(s)\,ds + \int_{C_2} (Ax(s)+By(s)+C)K'(s)\,ds = (1) + (2)$.

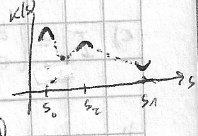
We can suppose $\underbrace{Ax(s)+By(s)+C}_{L(s)} \leq 0$ in C_2 (WLOG) and $Ax(s)+By(s)+C \geq 0$ in C_1 (the figure). If $K' > 0$ in $C_2 \Rightarrow L(s)\cdot K'(s) < 0$ in C_2.

In that case if $K' < 0$ in $C_1 \Rightarrow L(s)K'(s) < 0$ in C_1 and this leads to a contradiction, because the whole integral $I = 0$. If $K' > 0$ in $C_1 \Rightarrow$ (unless $K'(s) = 0\ \forall s \Rightarrow \infty$ vertices ✓)

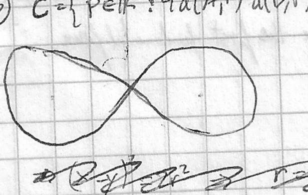
$\Rightarrow C\int_0^\ell K'(s)\,ds \Leftrightarrow \int_{C_1} K'(s)\,ds + \int_{C_2} K'(s)\,ds = 0 \Rightarrow K'(s) = 0\ \forall s \Rightarrow \infty$ vertices. The case $K' < 0$ in C_2 is similar.

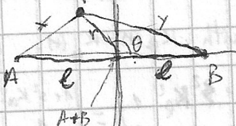
Therefore we must have at some side of $Ax + By + C = 0$ curvature >0 and <0.

c) By b) we know that K has a local extremum at some point s_2 between s_0 and s_1. But this implies that at least there is another extremum (see figure) $\Rightarrow$ we get 4 vertices.



4.5 $C = \{P \in \mathbb{R}^2 : 4d(A,P)\,d(B,P) = d(A,B)^2\}$





$P = r(\cos\theta, \sin\theta)$

$4x\cdot y = (2\ell)^2 \Rightarrow \ell^2 = xy$

$\Rightarrow y^2 = r^2 + \ell^2 - 2r\ell\cos\theta$

(cosine theorem) $x^2 = r^2 + \ell^2 - 2r\ell\cos(\pi - \theta) = r^2 + \ell^2 + 2r\ell\cos\theta$

$\ell^4 = (xy)^2 = x^2y^2 = (r^2+\ell^2 - 2r\ell\cos\theta)(r^2+\ell^2+2r\ell\cos\theta) = (r^2+\ell^2)^2 - 4r^2\ell^2\cos^2\theta = r^4 + \ell^4 + 2r^2\ell^2 - 4r^2\ell^2\cos^2\theta \Rightarrow$

$\Rightarrow r^2 = 4\ell^2\cos^2\theta - 2\ell^2 = 2\ell^2(2\cos^2\theta - 1) = 2\ell^2\cos 2\theta \Rightarrow \boxed{r^2 = 2\ell^2\cos 2\theta}$, $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$

4.6 a) $\lim_{\varepsilon \to 0^+} \Phi(s, s+\varepsilon) = \lim_{\varepsilon\to 0} \frac{x(s+\varepsilon) - x(s)}{\|x(s+\varepsilon) - x(s)\|} = \lim_{\varepsilon\to 0} \frac{x(s+\varepsilon) - x(s)}{\varepsilon} \cdot \frac{1}{\frac{\|x(s+\varepsilon) - x(s)\|}{\varepsilon}} = \lim_{\varepsilon\to 0} \frac{x(s+\varepsilon)-x(s)}{\varepsilon} \left\| \lim_{\varepsilon\to 0} \frac{x(s+\varepsilon) - x(s)}{\varepsilon} \right\|^{-1} = \frac{x'(s)}{\|x'(s)\|}$ ($s \neq 0$; $\|\cdot\|$ is cont.)

$= x'(s)$. (s is the arc-length parameter.) $\rightarrow$ esencia entregable.

~~$\lim_{s\to 0, t\to\ell} \Phi(s,t) = \lim \frac{x(t) - x(s)}{\|x(t) - x(s)\|}$~~

List 5

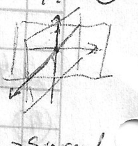
5.1 a) $\Phi = xyz - 1 \rightarrow D\Phi = (yz \;\; xz \;\; xy) = 0 \Leftrightarrow$ WLOG $x = z = 0$. But in that case the point $(0, y, 0) \notin S \Rightarrow S$ is regular

b) $\Phi = z^2 + x^2y^2 - 2xyz \rightarrow D\Phi = (2xy^2 - 2yz \;\; 2x^2y - 2xz \;\; 2z - 2xy) = 0 \Leftrightarrow z = xy$. In that case we have $x^2y^2 + x^2y^2 - 2x^2y^2 = 0$ (we can't deduce anything) $\Leftarrow (x, y, xy) \in S$

Obs: $\Phi = (z - xy)^2 = 0 \Leftrightarrow \underbrace{z - xy}_{\phi} = 0 \rightarrow D\phi = (-y \;\; -x \;\; 1) \neq 0 \rightarrow$ regular surface

c) $\Phi = x^2+y^2-z^2-\lambda \to D\Phi = (2x \;\; 2y \;\; -2z) = 0 \Leftrightarrow x=y=z=0 \to$ If $\lambda=0$, $(0,0,0)\in S \Rightarrow S$ is not regular surface. Why? $\to$ ⊛ (for $\lambda\neq 0 \Rightarrow S$ is regular)

d) $\Phi = xy \to D\Phi = (y \;\; x \;\; 0) = 0 \Leftrightarrow x=y=0$, and $(0,0)\in S \Rightarrow S$ is not a regular surface. Why? ⊕



e) $\Phi = x^2+y^2-\cosh^2 z \to D\Phi = (2x \;\; 2y \;\; \underbrace{-2\cosh z \sinh z}_{\sinh(2z)}) = 0 \Leftrightarrow x=y=z=0$. But $(0,0,0)\notin S \Rightarrow S$ is regular

f) $\Phi = y\cos z - x\sin z \to D\Phi = (-\sin z \;\; \cos z \;\; -y\sin z - x\cos z) \neq 0$ because $\sin z$ and $\cos z$ do not vanish simultaneously $\Rightarrow S$ is regular.

(7.1) ⊛ Because there are 3 tangent vectors and thus, they are not coplanar.  in $(0,0,0)$

Other approach: Suppose there is a homeo $\varphi: U \longrightarrow V\cap S$ s.t. $\varphi(p)=(0,0,0)$

$\Rightarrow \tilde{\varphi}: \underset{\text{connected}}{U\setminus\{p\}} \cong \underset{\text{disconnected}}{V\cap S\setminus\{(0,0,0)\}} \Rightarrow$ Contradiction!

Revolution de teoremes

Teorem $S\subseteq\mathbb{R}^3$ superfície regular $\Leftrightarrow \forall p\in S\ \exists U\subseteq\mathbb{R}^2, \exists V\subseteq\mathbb{R}^3$ oberts: $\exists\ \varphi_i: U \to V\cap S \subseteq \mathbb{R}^3$ parametrització regular ($\Leftrightarrow \varphi$ diferenciable, $\varphi_u\times\varphi_v\neq 0$), homeomorfisme

Prop: $F: \underset{\text{obert}}{\Omega}\subseteq\mathbb{R}^3 \to \mathbb{R}$ diferenciable, $\nabla F(p)\neq 0\ \forall p\in F^{-1}(0)$ $\overset{\text{T.F.I}}{\Rightarrow} S=F^{-1}(0)$ superfície regular $\Downarrow$ S té pla tangent en cada punt (vectors tangents a corbes diferenciables contingudes en S)

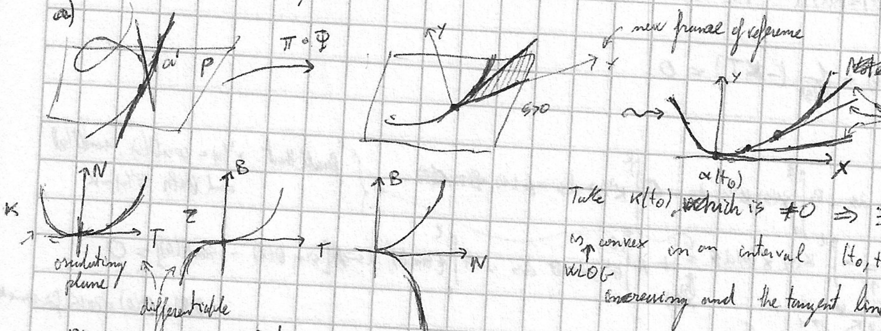
⊕ Because there are 3 tangent vectors $\Rightarrow$ they are not planar $\Rightarrow$ the tangent plane does not exist

(7.2) $\Phi(t,s) = \alpha(t) + s\,\alpha'(t)$

Obs: Φ is regular parametrization because it is differentiable and $\Phi_t\times\Phi_s = (\alpha'(t)+s\alpha''(t))\times(\alpha'(t)) = s(\underset{T'}{\alpha''}\times\underset{T}{\alpha'}) = sK(N\times T) = -sKB \neq 0 \Leftrightarrow \underset{K\neq 0}{}$

$s=0 \Rightarrow \Phi$ is regular $\Leftrightarrow s\neq 0$

a)



Note that $\pi\circ\Phi$ injective $\Leftrightarrow$ the rays do not intersect between them.

Take $K(t_0)$, which is $\neq 0 \Rightarrow \exists\varepsilon>0$ s.t. $K(t)\neq 0\ \forall t\in(t_0,t_0+\varepsilon) \Rightarrow$ the function is convex in an interval $(t_0,t_0+\varepsilon)$ WLOG $\Rightarrow$ Since the derivative of a convex function is increasing and the tangent line lies always below the graph $\Rightarrow \pi\circ\Phi$ is injective

Obs: $\pi\circ\Phi$ injective $\Rightarrow \Phi$ injective, because Φ not injective $\Rightarrow \pi\circ\Phi$ not injective.

b) Let's see that Φ is homeomorphism, we need to prove that the inverse Φ^{-1} is continuous. Note that $\Phi_t = T+sKN$; $\Phi_s = T \Rightarrow D(\pi\circ\Phi) = D\pi\, D\Phi = \pi D\Phi \Rightarrow \det D(\pi\circ\Phi) = \det_P(\pi\Phi_t, \pi\Phi_s)(t,s)\big|_{t=t_0} =$ (we want to use the inverse function theorem; π is linear; determinant in the plane P). By doing so, let's prove that $(\pi\circ\Phi)^{-1}$ is

$\det_P(T(t_0)+sKN(t_0), T(t_0)) = -sK(t_0)\det_P(T(t_0),N(t_0)) \neq 0\ \forall s \Rightarrow \exists\varepsilon$ s.t. $D(\pi\circ\Phi)\neq 0 \Rightarrow \pi\circ\Phi$ is locally a diffeomorphism. In particular $(\pi\circ\Phi)^{-1}$ is locally continuous $\Rightarrow$ But now $\Phi^{-1} = \underset{\text{continuous}}{(\pi\circ\Phi)^{-1}}\circ\underset{\text{continuous}}{\pi}$ = continuous $\Rightarrow \Phi$ is a local parametrization. (I.F.Theorem)

c) $I_p = \begin{pmatrix} E & F \\ F & G\end{pmatrix} = \begin{pmatrix} \langle\Phi_t,\Phi_t\rangle & \langle\Phi_t,\Phi_s\rangle \\ \langle\Phi_s,\Phi_t\rangle & \langle\Phi_s,\Phi_s\rangle\end{pmatrix} = \begin{pmatrix} 1+s^2K^2 & 1 \\ 1 & 1\end{pmatrix}$ which does not depend on Z.

d) Let $\beta(t)$ be a plane curve with $K_\beta(t) = K_\alpha(t)$. Define $\Phi_p^\beta(t,s) = \beta(t)+s\,\beta'(t)$. Obs: they are defined in the same open set.

Moreover $I_p^\beta = \begin{pmatrix} E_\beta & F_\beta \\ F_\beta & G_\beta\end{pmatrix} \underset{\text{same as once}}{=} \begin{pmatrix} 1+s^2K_\beta^2 & 1 \\ 1 & 1\end{pmatrix} \underset{K_\beta=K_\alpha}{=} \begin{pmatrix} 1+s^2K_\alpha^2 & 1 \\ 1 & 1\end{pmatrix} = I_p^\alpha \Rightarrow$ They are isometric because they have the first fundamental form.

e) $\lim_{h\to 0}\dfrac{\Phi(t,h)-\Phi(t,0)}{|\Phi(t,h)-\Phi(t,0)|} = \lim_{h\to 0}\dfrac{\alpha(t)+h\alpha'(t)-\alpha(t)}{|\alpha(t)+h\alpha'(t)-\alpha(t)|} = \lim_{h\to 0}\dfrac{h\,\alpha'(t)}{|h|\,|\alpha'(t)|}$

$\lim_{h\to 0}\dfrac{\Phi(t-h,h)-\Phi(t,0)}{|\Phi(t-h,h)-\Phi(t,0)|} = \lim_{h\to 0}$

$\Phi(t-h,h)-\Phi(t,0) = \alpha(t-h)-h\alpha'(t-h)-\alpha(t)$

$\lim_{h\to 0}\dfrac{\Phi(t+h,-h)-\Phi(t-h,h)}{|\Phi(t+h,-h)-\Phi(t-h,h)|} = \lim_{h\to 0}\dfrac{\alpha(t+h) - h\alpha'(t+h) - \alpha(t-h) - h\alpha'(t-h)}{}$

f)

(5.3) S_1, S_2 are locally isometric if they have the same fundamental 1st form.

Surface A $\to \varphi^A(u,v) = (u,v,0) \to \varphi^A_u = (1,0,0),\ \varphi^A_v = (0,1,0) \longrightarrow I_A = \begin{pmatrix} E_A & F_A \\ F_A & G_A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

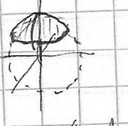
Surface B $\to \varphi^B(u,v) = (\cos u, \sin u, v) \to \varphi^B_u = (-\sin u, \cos u, 0),\ \varphi^B_v = (0,0,1) \to I_B = \begin{pmatrix} E_B & F_B \\ F_B & G_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ } local isometric. everywhere

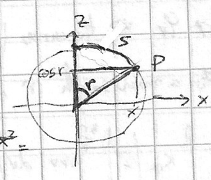
Surface C $\to \varphi^C(u,v) = (u, v, \sin v) \to \varphi^C_u = (1,0,0),\ \varphi^C_v = (0,1,\cos v) \longrightarrow I_C = \begin{pmatrix} E_C & F_C \\ F_C & G_C \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\cos^2 v \end{pmatrix} \to$ local isometry with the previous ones $\forall v = \frac{\pi}{2}$.

(5.4) a) ~~thats~~ For D_r is clear since the ~~fastest~~ shortest way ~~to go~~ in distance between two point is a straight line. $\Rightarrow$ the points $\notin D_r$ are at a distance $> r$ from the origin whereas the points $\in D_r$ are at a distance $\leq r$, so ~~[illegible]~~ in the latter case, there exists a curve (the line segment) joining $(0,0)$ with the point.

For $\overline{D_r}$ the furthest point satisfy  , which are (in the image) the points like P

$z = \cos r$





We have that, for being $P \in S^2$, ~~$\cos^2 r + x^2 = 1$~~ $\Rightarrow$ ~~$(1-\cos r)^2 + x^2 = 1 + 2\cos r + \cos^2 r + x^2 =$~~

we have that the radius of S^2 is 1 ~~we've projected on y=0~~ ~~$= 2(1-\cos r) =$~~

and therefore $s = r$ ✓.

b) We already know that area $D_r = \pi r^2$. A parametrization of the sphere is: $\varphi(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

$\overline{D_r} = \{(u,v) \in (0,2\pi)\times(0,\pi) : v \leq r\} \leftrightarrow \varphi_u = (-\sin u \sin v, \cos u \sin v, 0)$, $\varphi_v = (\cos u \cos v, \sin u \cos v, -\sin v) \to E = (\sin v)^2,\ F = 0,\ G = 1 \to$ area $\overline{D_r} = \int_0^{2\pi} du \int_0^r dv (\sin v) = -2\pi \cos v \Big|_0^r =$

$\sqrt{EG-F^2} = \sin v$

$= 2\pi(1-\cos r)$.

c) Suppose there exists an isometry φ between the ~~[illegible]~~ open sets Int D_r and Int $\overline{D_r}$. By continuity, which we can by a) ~~[illegible]~~ $\partial D_r \xrightarrow{\varphi} \partial \overline{D_r}$. But by b), these two sets has different area and an isometry preserves area. Contradiction.

(5.5) $\varphi(z,w) = (w\cos z, w \sin z, z) \rightsquigarrow I_\varphi = \begin{pmatrix} (\cosh v)^2 & 0 \\ 0 & (\cosh v)^2 \end{pmatrix}$; $\psi(u,v) = (\cosh v \cos u, \cosh v \sin u, v) \rightsquigarrow I_\psi = \begin{pmatrix} (\cosh v)^2 & 0 \\ 0 & (\cosh v)^2 \end{pmatrix}$ Since $I_\varphi = I_\psi$

$w = \sinh v$, $z = u$

~~If $w=0 \Rightarrow$ [illegible] $= I_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$~~ $\Rightarrow$ They are locally isometric. and clearly this isometry cannot be extended globally because ~~[illegible] $= (\cosh v)^2 = I_\psi$~~. In C $\to u\to 0,\ u \to 2\pi \Rightarrow$ the curve is $(\cosh v, 0, v)$, $v \in \mathbb{R}$ ↗ which aren't topologically equivalent

In H $\to z\to 0,\ z \to 2\pi \Rightarrow$ the curve is $(w,0,0)$, $w \in \mathbb{R}$

$\varphi(F_1(u,v), F_2(u,v))$

Furthermore a local isometry between H and C is given by F^{-1}.

List 7

(7.1) $A(S) = \int_U \|\varphi_u \times \varphi_v\| \,du\,dv$; $\psi = \varphi \circ F \Rightarrow \psi(V) = \varphi(F(V)) = \varphi(U) = S$ $\quad \psi_u = \frac{\partial}{\partial u}(\varphi(F(u,v))) = \varphi_u(F) F_{1u} + \varphi_v(F) F_{2u}$

$\psi_v = \varphi_v(F) F_{1v} + \varphi_v(F) \frac{F_2}{\partial v}$

$\Rightarrow \int_V \|\psi_u \times \psi_v\| \,du\,dv =$ ~~[illegible]~~ $\int_V \|\varphi_u(F(u,v)) \times \varphi_u(F(u,v))\| |\det DF(u,v)| \,du\,dv \underset{\uparrow}{=} \int_U \|\varphi_u \times \varphi_v\| |\det DF| |(\det DF)|^{-1} du\,dv =$

$\psi_u \times \psi_v = \varphi_u \times \varphi_v \cdot F_{1u} F_{1v} + \varphi_v \times \varphi_u \cdot F_{2u} F_{1u} = \varphi_u \times \varphi_v \cdot \det DF \qquad u,v \mapsto F^{-1}(u,v)$; $F^{-1}: U \to V$. $\det DF^{-1} = \frac{1}{\det DF}$

$= \int_U \|\varphi_u \times \varphi_v\| \,du\,dv = A(S)$

(7.2) a) Let's parametrize the circle C_u of center $\alpha(u)$ and radius $r(u)$ $\quad C_u = \alpha(u) + r(u)(\cos\theta\, m(u) + \sin\theta\, b(u)) \quad \theta \in [0, 2\pi]$.

Obs: $r(u) \neq 0$ is differentiable

$\Rightarrow \varphi: I \times \mathbb{R} \longmapsto \mathbb{R}^3$

$(u,v) \longmapsto \{\alpha(u) + r(u)(\cos v\, m(u) + \sin v\, b(u))$.

b) $\varphi_u = \overset{t(u)}{\alpha'(u)} + r'(u)(\cos v\, m(u) + \sin v\, b(u)) + r(u)\Big[\cos v\, \overbrace{(-K(u)t(u) - \tau(u) b(u))}^{m'(u)} + \sin v(\tau(u) \cdot m(u))\Big] = (1 - rK\cos v)\vec{t} + (r'\cos v + r\tau \sin v)\vec{m} + (r'\sin v - r\tau\cos v)\vec{b}$

$\varphi_v = r(u)(-\sin v\, m(u) + \cos v\, b(u))$

$\Rightarrow \varphi_u \times \varphi_v = -r(u)\sin v\, t\times m + \cos v\, t \times b + r r'(u)\cos^2 v\, m \times b - r r' \sin^2 b \times m + r^2 \Big[+K \sin v \cdot t \times m - \cos^2 v\, K\, t \times b + \cos v \sin v\, \tau\, b \times m$

$+ \sin v \cos v\, \tau\, m \times b\Big] = -r\sin v\, b(u) - r\cos v \cdot n(u) + r r' t(u) + r^2 \cdot K \sin v \cos v \cdot b(u) + r^2 \cos^2 v \cdot K \cdot n(u) =$

$= r r' t(u) + r\cos v(r\cos v \cdot K - 1) n(u) + r\sin v(rK\cos v - 1) b(u)$ $\quad$ Obs: $r > 0 \Rightarrow r\cos v(r\cos v K - 1) = 0 \Leftrightarrow$ compute

$\Rightarrow \begin{cases} \cos v = 0 \Rightarrow \sin v \neq 0 \text{ and } r\sin v(r\cos v - 1) \neq 0 \\ r\cos v K = 1 \to r\cos v \cdot K < \frac{1}{K} \cdot \cos v \cdot K = \cos v \leq 1 = r\cos v \cdot K \text{ ! impossible} \end{cases} \Big\}\ \varphi_u \times \varphi_v \neq 0$.

Obs: $\varphi: I \times [v_0 - \pi, v_0 + \pi]$ compact $\downarrow$ S Hausdorff $\Downarrow$ φ homeo

Since $\varphi_u \times \varphi_v \neq 0 \Rightarrow \|\varphi_u \times \varphi_v\| \neq 0 \Rightarrow \det I \neq 0 \to$ the surface is regular. Because is injective in U ($\Rightarrow$ homeomorphism).

c) $E = \langle\varphi_u, \varphi_u\rangle = (1 - rK\cos v)^2 + (r'\cos v + r\tau\sin v)^2 + (r'\sin v - r\tau\cos v)^2 = 1 - 2rK\cos v + r^2K^2\cos^2 v + r'^2 + r^2\tau^2$

$F = \langle\varphi_u, \varphi_v\rangle = r r' \sin v \cos v + r^2 \tau \sin^2 v + r r' \sin v \cos v - r^2 \tau \cos^2 v = -r^2\tau$

$G = \langle\varphi_v, \varphi_v\rangle = r^2$

Another approach: $\Phi(u,v,w) = \alpha(u) + (r(u) + w)(\cos v\, m + \sin v\, b(u)) = \varphi(u,v) + w(\cos v\, m + \sin v\, b(u))$

$\det(D\Phi(u_0, v_0, 0)) \neq 0 \underset{\text{T.F. Inversa}}{\Rightarrow} \Phi^{-1}$ diffe local

d) $A(S) = \iint_U \sqrt{EG - F^2}\,du\,dv$; $EG - F^2 = r^2(1 - rK\cos v)^2 + r'^2 r^2 + r^4\tau^2 - r^2\tau^2 = r^2(1 - rK\cos v)^2 + r'^2 r^2$

which does not depend on the parametrization

e) If $r = r_0 \Rightarrow r' = 0 \Rightarrow \sqrt{EG - F^2} = r_0(1 - r_0K\cos v) \Rightarrow A(S) = \int_a^b \int_{v_0 - \pi}^{v_0 + \pi} r_0 - r_0^2 K \cos v = \int_a^b 2 r_0 \pi$, which does not depend on K.

$1 > rK\cos v \qquad u \in (a,b)$

e) If $r = r_0 \Rightarrow \varphi_u = (1 - r_0K\cos v) t(u) + r\tau(\sin v\, m(u) - \cos v\, b(u))$

$\varphi_v = r_0(-\sin v\, m(u) + \cos v\, b(u))$

$\Rightarrow N = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|} = \frac{r_0\cos v(r_0K\cos v - 1)m(u) + r_0\sin v(r_0K\cos v - 1)b(u)}{r_0(1 - r_0K\cos v)} =$

$= -\cos v\, m(u) - \sin v\, b(u)$

$N_u = -\cos v(-\kappa t - \tau b) - \sin v(\tau n)$

$N_v = \sin(v)\, n(u) - \cos v\, b(u)$ $\Rightarrow$ $e = -\langle \varphi_u, N_u\rangle = -\kappa\cos v(1 - r_0\kappa\cos v) + \sin^2 v\, r\tau^2 + r_0\tau^2\cos^2 v = -\kappa\cos v(1 - r_0\kappa\cos v) + r_0\tau^2$

$f = -\langle \varphi_u, N_v\rangle = -r_0\tau\sin^2 v - r_0\tau\cos^2 v = -r_0\tau$

$g = -\langle \varphi_v, N_v\rangle = r_0\sin^2 v + r_0\cos^2 v = r_0$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-r_0\kappa\cos v(1 - r_0\kappa\cos v)}{r_0^2(1 - r_0\kappa\cos v)^2} = \boxed{\frac{-\kappa\cos v}{r_0(1 - r_0\kappa\cos v)}}$$

f) $\underset{\tau=0}{\mathrm{II}} = \begin{pmatrix} -\kappa\cos v(1 - r_0\kappa\cos v) & 0 \\ 0 & r_0 \end{pmatrix}$ ~~let $\gamma(t) = (x, y)$ be a line of curvature~~ ~~s.t. it is arc-length parametrized~~ ~~$\Rightarrow \kappa_n(\gamma') = \mathrm{II}(\gamma') = 0 \Leftrightarrow$~~

α is planar

$\Rightarrow W_p = \mathrm{I}^{-1}\mathrm{II} = \begin{pmatrix} (1 - r\kappa\cos v)^2 & 0 \\ 0 & r^2 \end{pmatrix}^{-1} \begin{pmatrix} -\kappa\cos v(1 - r\kappa\cos v) & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} \frac{-\kappa\cos v}{1 - r\kappa\cos v} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$ $\Rightarrow$ the lines of curvature are the coordinate lines

$\varphi_u \neq \varphi_v \Rightarrow$ the curves $\{u = \text{const}\}$ and $\{v = \text{const}\}$ are lines of curvature.

(7.3) a) For a small enough r we have that the Gauß curvature is positive for $v \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ $\Rightarrow \int_R K\,dS = \int_{S_1}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} K\,dv = \int_{S_1}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{-\kappa_\alpha\cos v}{r(1 - r\kappa_\alpha\cos v)} \overbrace{r(1 - r\kappa_\alpha\cos v)}^{\|\varphi_u \times \varphi_v\|}\, dv\, dt =$

$= \int_{S_1} \kappa_\alpha \left(-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos v\, dv\right) = 2\int_{S_1} \kappa_\alpha$

b)

List 9

(9.1) a) $x(u,v) = \alpha(u) + v\gamma(u)$ $\Rightarrow$ $x_u = \alpha' + v\gamma'$, $x_v = \gamma(u)$ $\Rightarrow$ $x_{uu} = \alpha'' + v\gamma''$, $x_{uv} = \gamma'$, $x_{vv} = 0$. $x_u \times x_v = \alpha' \times \gamma + v\gamma' \times \gamma$. The points where $x_u \times x_v = 0$ are isolated or is a curve. In both cases this set of points is a closed set so $\exists U \underset{\text{open}}{\subseteq} \mathbb{R}^2$ such that $x|_U$ is regular.

Hem de suposar que x_u, x_v siguin l.i. $\Rightarrow$ parametrització regular.

$E = \langle x_u, x_u\rangle = \langle \alpha', \alpha'\rangle + v^2\langle \gamma', \gamma'\rangle + 2v\langle \alpha', \gamma'\rangle$

$\Rightarrow F = \langle \alpha' + v\gamma', \gamma\rangle = \langle \alpha', \gamma\rangle$, $\langle \gamma, \gamma'\rangle = 0$ because $|\gamma| = 1 = \text{const}$.

$G = \langle \gamma, \gamma\rangle = 1$

$e = \langle x_{uu}, N\rangle = \left\langle \alpha'' + v\gamma'', \frac{\alpha' \times \gamma + v\gamma' \times \gamma}{\|x_u \times x_v\|}\right\rangle$

$f = \langle x_{uv}, N\rangle = \left\langle \gamma', \frac{\alpha' \times \gamma + v\gamma' \times \gamma}{\|x_u \times x_v\|}\right\rangle = \frac{\langle \gamma', \alpha' \times \gamma\rangle}{\|x_u \times x_v\|}$

$g = \langle x_{vv}, N\rangle = 0$

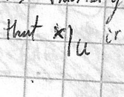
$\circledast = \dots$ $\quad K = \frac{eg - f^2}{EG - F^2} = \frac{-f^2}{\underbrace{EG - F^2}_{\|x_u \times x_v\| > 0}} \leq 0$. Obs: $K = 0 \Leftrightarrow f = 0$. ~~...~~ Pero tenemos que $g = 0 \Rightarrow \langle N_v, x_v\rangle = 0$

$\Rightarrow f = 0 \Leftrightarrow \langle N_v, x_u\rangle = 0 \Leftrightarrow N_v = 0$ $\quad u_0 \longmapsto N(u_0, v)$ is constant.

N_v ortogonal a una base

b) $K = 0 \Leftrightarrow f = 0 \Leftrightarrow \langle \gamma', \alpha' \times \gamma\rangle = 0 \Leftrightarrow \det(\gamma', \alpha', \gamma) = 0 \Leftrightarrow \alpha' = \lambda\gamma + \mu\gamma'$ $\Rightarrow$

suposem $\gamma' \neq 0 \Rightarrow$ si $\gamma' = 0 \Leftrightarrow \gamma = \text{const} \to$ cilindre generalitzat



$\vec{0} = x_u \times x_v = (\lambda\gamma + \mu\gamma') \times \gamma + v\gamma' \times \gamma = \mu\gamma' \times \gamma + v\gamma' \times \gamma = (\mu + v)\gamma' \times \gamma \Rightarrow \boxed{v = -\mu}$ $\quad \gamma'' \perp \gamma$

The lines $u = \text{const}$ are in the direction of x_v $\Rightarrow$ $\beta'(u) = \alpha' - \mu'\gamma - \mu\gamma'$. The curve is $\beta(u) = x(u, -\mu)$, where $\alpha' = \lambda\gamma + \mu\gamma'$, $\mu = \frac{\langle \alpha', \gamma'\rangle}{\langle \gamma', \gamma'\rangle}$

$\Rightarrow \beta' \parallel x_v$ ✓. Otherway: $\beta' = \alpha' - \mu'\gamma - \mu\gamma' = \lambda\gamma + \mu\gamma' - \mu'\gamma - \mu\gamma' = (\lambda - \mu')\gamma$ ✓ and $\beta' \times x_v = (\alpha' - \mu'\gamma - \mu\gamma') \times \gamma = \alpha' \times \gamma - \mu\gamma' \times \gamma = 0$ $\Rightarrow$ we are in the regression axis

(9.2) a) $\Phi(t,s) = \alpha(t) + s\alpha'(t)$. $\Phi_t = \alpha' + s\alpha''$, $\Phi_s = \alpha'$ $\Rightarrow$ $\Phi_t \times \Phi_s = (\alpha' + s\alpha'') \times \alpha' = s\alpha'' \times \alpha' \neq 0$

b) Clearly it is reglated with $\gamma = \alpha'$. Moreover $K = 0 \Leftrightarrow f = 0 \Leftrightarrow \langle \alpha'', \underbrace{\alpha' \times \alpha'}_{0}\rangle = 0 \Rightarrow K = 0$ ✓. $\alpha' \perp \alpha''$ because $\|\alpha'\| = 1 = \text{const}$.

c) By (9.1) $G = 1$, $F = \langle \alpha', \alpha'\rangle = 1$ and $E = \langle \alpha', \alpha'\rangle + v^2\langle \alpha'', \alpha''\rangle + 2v\underbrace{\langle \alpha', \alpha''\rangle}_{0} = 1 + v^2\kappa^2$ $\quad \gamma = \alpha'$

d) This is due to the non-dependence of τ in the 1st fundamental form. $\|\alpha''\|^2 = \kappa^2$

(9.3) $G = 1$, $F = \langle \underset{T}{\alpha'}, N\rangle = 0$, $E = \langle \alpha', \alpha'\rangle + v^2\langle N', N'\rangle + 2v\langle \alpha', N'\rangle = 1 + v^2\langle -\kappa T - \tau B, -\kappa T - \tau B\rangle + 2v\langle T, -\kappa T - \tau B\rangle =$

Formules of (9.1) using $\gamma = N$.

$= 1 + v^2(\kappa^2 + \tau^2) - 2v\kappa = (1 - v\kappa)^2 + v^2\tau^2$

$g = 0$, $x_s \times x_\lambda = \alpha' \times N + vN' \times N = B - v\kappa B + \tau vT = (1 - v\kappa)B + \tau vT$ $\Rightarrow$ $\quad T \times N = B$, $N' = -\kappa T - \tau B$

$\Rightarrow N = \frac{(1 - v\kappa)B + \tau vT}{\sqrt{(1 - v\kappa)^2 + \tau^2 v^2}}$ $\Rightarrow f = \frac{\langle N', \alpha' \times N\rangle}{\sqrt{(1 - v\kappa)^2 + \tau^2 v^2}} = \frac{\langle -\kappa T - \tau B, B\rangle}{\sqrt{(1 - v\kappa)^2 + \tau^2 v^2}} = \frac{-\tau}{\sqrt{(1 - v\kappa)^2 + \tau^2 v^2}} \Rightarrow K = \frac{-f^2}{\underset{G=1}{EG}} = \frac{-\frac{\tau^2}{(1 - v\kappa)^2 + \tau^2 v^2}}{(1 - v\kappa)^2 + v^2\tau^2} =$

$= \boxed{\frac{-\tau^2}{[(1 - v\kappa)^2 + v^2\tau^2]^2}}$

(9.4) $G = 1$, $F = \langle \alpha', B\rangle = 0$, $E = \langle T, T\rangle + v^2\langle B', B'\rangle + 2v\langle T, \underset{\tau N}{B'}\rangle = 1 + v^2\tau^2$.

$g = 0$, $f = \frac{\langle B', T \times B\rangle}{\|x_s \times x_\lambda\|} = \frac{\langle \tau N, N\rangle}{\sqrt{1 + v^2\tau^2}} = \frac{\tau}{\sqrt{1 + v^2\tau^2}} \Rightarrow K = \frac{-f^2}{\underset{G=1}{EG}} = \frac{\frac{-\tau^2}{1 + v^2\tau^2}}{1 + v^2\tau^2} = \boxed{\frac{-\tau^2}{(1 + v^2\tau^2)^2}}$

$x_s \times x_\lambda = T \times B + vB' \times B = N + v\tau T \Rightarrow \|x_s \times x_\lambda\| = \sqrt{1 + v^2\tau^2}$ $\quad B' = \tau N$

(9.5) → ~~Entregable exercise~~

~~Bertrand... $T\!\!\!\!T(t)\alpha(t)+\lambda N(t)+\mu B(t)=0$ ... $=T+\lambda N'+\mu B'=0=T-\lambda\kappa T+\lambda\tau B$ ... $\mu\tau N=$~~

~~$=(1-\lambda\kappa)T+(\lambda'+\mu\tau)N+(\mu'-\lambda\tau)B=0 \Rightarrow 1-\lambda\kappa=0 \Rightarrow \lambda=\frac{1}{\kappa}=\rho$ and $\rho'+\mu\tau=0 \Leftrightarrow \rho'+\mu\tau=0 \Leftrightarrow \mu=-\frac{\rho'}{\tau}$~~

~~Absurd~~ planes:

List 10.

(9.1) $x(u,v)$ and $\alpha(t)=x(u(t),v(t))$ $\Rightarrow \|\alpha'\|^2=\langle\alpha',\alpha'\rangle=(u',v')\begin{pmatrix}E&F\\F&G\end{pmatrix}\begin{pmatrix}u'\\v'\end{pmatrix}=u'^2E+2Fu'v'+v'^2G$

(9.2) We have $L=T=\frac{1}{2}m\|\alpha'\|^2 \underset{m=1}{=} \frac{1}{2}\|\alpha'\|^2=\frac{1}{2}(u'^2E+2u'v'F+v'^2G)$.

Euler-Lagrange eq. tell us that $\begin{cases}\frac{\partial L}{\partial u}-\frac{d}{dt}\frac{\partial L}{\partial u'}=0 \to \left[\frac{1}{2}\left(u'^2E_u+2u'v'F_u+v'^2G_u\right)\right]-\frac{d}{dt}\left(\frac{1}{2}(2u'E+2v'F)\right)=\\ \frac{\partial L}{\partial v}-\frac{d}{dt}\frac{\partial L}{\partial v'}=0 \to\end{cases}$

$=\frac{1}{2}(u'^2E_u+2u'v'F_u+v'^2G_u)-\left(u''E+u'^2E_u+u'v'E_v+v''F+v'u'F_u+v'^2F_v\right)$ $(*)$ But also we can also see that $\frac{d}{dt}\frac{\partial L}{\partial u'}=\frac{d}{dt}(u'E+v'F)=$

$=(u',v')\begin{pmatrix}E&F\\F&G\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\frac{d}{dt}\langle\alpha',\varphi_u\rangle=\langle\alpha'',\varphi_u\rangle+\langle\alpha',\varphi_{uu}u'+\varphi_{uv}v'\rangle=\langle\alpha'',\varphi_u\rangle+\langle\varphi_u,\varphi_{uu}\rangle u'^2+\langle\varphi_v,\varphi_{uu}\rangle u'v'+\langle\varphi_u,\varphi_{uv}\rangle u'v'+$ ($\alpha'=u'\varphi_u+v'\varphi_v$)

$+\langle\varphi_v,\varphi_{uv}\rangle v'^2=\langle\alpha'',\varphi_u\rangle+\frac{1}{2}\partial_u\langle\varphi_u,\varphi_u\rangle u'^2+\partial_u\langle\varphi_v,\varphi_u\rangle u'v'+\frac{1}{2}\partial_u\langle\varphi_v,\varphi_v\rangle v'^2=\langle\alpha'',\varphi_u\rangle+\frac{1}{2}E_uu'^2+F_uu'v'+\frac{1}{2}G_uv'^2 \Rightarrow$

$\Rightarrow \frac{\partial L}{\partial u}-\frac{d}{dt}\frac{\partial L}{\partial u'}=\langle\alpha'',\varphi_u\rangle \overset{\text{we impose it}}{=} 0$, Analogously the other equation gives us $\langle\alpha'',\varphi_v\rangle=0 \Rightarrow \alpha''\parallel\nu$.

(9.3) Following $(*)$ $=$ ~~$\frac{1}{2}\,2u''E\ldots-uE_u+u'v'E_v+\frac{1}{2}u'^2E_u+\ldots\frac{v'^2}{2}G_u=0 \Rightarrow u''\ldots$~~ Recall the geodesics eqs. $\begin{cases}u''+\Gamma_{11}^1u'^2+2\Gamma_{12}^1u'v'+\Gamma_{22}^1v'^2=0\\ v''+\Gamma_{11}^2u'^2+2\Gamma_{12}^2u'v'+\Gamma_{22}^2v'^2=0\end{cases}$

we impose $F=0$ $\quad E=1, F=0, G=a^2$ $\quad$ ~~$E=1,\ E_u=E_v=0,\ G_u=2aa_u$~~

$L(u,v,u',v')=\frac{1}{2}u'^2E+\frac{2Fu'v'}{2}+\frac{v'^2}{2}G \overset{\downarrow}{=} \frac{u'^2}{2}+\frac{v'^2a^2}{2}$ $\Rightarrow \begin{cases}\frac{\partial L}{\partial u}-\frac{d}{dt}\frac{\partial L}{\partial u'}=aa_uv'^2-\frac{d}{dt}(u')=aa_uv'^2-u'' \Rightarrow\\ \frac{\partial L}{\partial v}-\frac{d}{dt}\frac{\partial L}{\partial v'}=0-\frac{d}{dt}(v'a^2)=-v''a^2-v'2aa_uu' \Rightarrow\end{cases}$

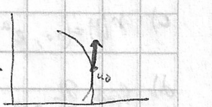
$\Rightarrow \Gamma_{11}^1=\Gamma_{12}^1=0,\ \Gamma_{22}^1=-aa_u$

$\Rightarrow \Gamma_{11}^2=\Gamma_{22}^2=0,\ \Gamma_{12}^2=\frac{a_u}{a}$

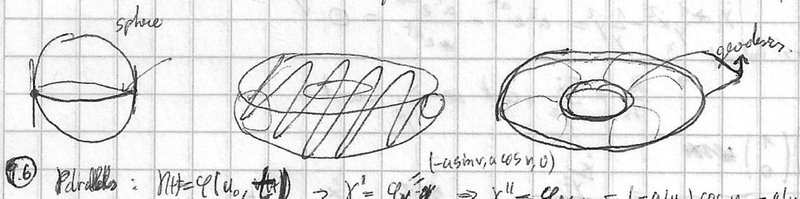
(9.4) If $v=v_0$ (meridian) $\Rightarrow v'=v''=0 \underset{\text{E-L eqs}}{\Rightarrow} \begin{cases}-u''=0\\0=0\end{cases} \Rightarrow u=\alpha t+\beta$ ✓.

(9.5) $\boxed{\Rightarrow}$ If $u=u_0$ (parallel) $\Rightarrow \begin{cases}aa_uv'^2=0\\-v''a^2=0\end{cases} \Rightarrow \begin{cases}a(u_0)a'(u_0)v'^2=0\\-v''=0\end{cases}$ we can't have $v=\text{const}$ because then we won't have a curve $\overset{a\neq0}{\Rightarrow} a'(u_0)=0 \Rightarrow$

$\Rightarrow a(u)$ is the distance to the center. If $a'=0 \to$ local minimum or maximum $\Rightarrow$ the the tangent line must be parallel to the axis. to the meridian



sphere



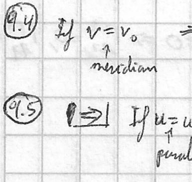
(9.6) Parallels: $\gamma(t)=\varphi(u_0,t)$ $\to \gamma'=\varphi_v \overset{(-a\sin v,\, a\cos v,\, 0)}{=}$ $\Rightarrow \gamma''=\varphi_{vv}=(-a(u_0)\cos v,-a(u_0)\sin v, b(u_0))$ Obs $|\gamma'|=1$ become $a^2+b^2=1$ (by hypothesis).

We know that $\underbrace{\kappa N}_{\gamma''}=\kappa_n\nu+\kappa_g(\nu\times\gamma')$ $\Rightarrow \begin{cases}\kappa_n=\langle\gamma'',\nu\rangle\\ \kappa_g=\langle\gamma'',\nu\times\gamma'\rangle\end{cases}$ $\quad \varphi_u=(a'\cos v, a'\sin v, b')$, $\varphi_v=(-a\sin v, a\cos v, 0)$ $\to \varphi_u\times\varphi_v=(-b'a\cos v,-b'a\sin v, aa')$

$\Rightarrow \nu=\frac{\varphi_u\times\varphi_v}{\|\varphi_u\times\varphi_v\|}=(-b'\cos v,-b'\sin v,a')$ $\Rightarrow \kappa_n=\langle\gamma'',\nu\rangle=+b'(u_0)a(u_0)\cos^2v+b'(u_0)a(u_0)\sin^2v+a'(u_0)b=\boxed{b'(u_0)a(u_0)+a'(u_0)b(u_0)}$

$\nu\times\gamma'=(a'a\cos v,-a'a\sin v,-b'a)$ $\Rightarrow \kappa_g=\langle\gamma'',\nu\times\gamma'\rangle=$ ($\varphi(u(t),v(t))$) $=\boxed{a^2(u_0)a'(u_0)-a(u_0)b(u_0)b'(u_0).}$

(9.7)



T. Clairaut: $a(u(t))\cdot\cos\theta(t)=\text{const}$. Suppose γ is arc-length parametrized. $\Rightarrow \cos\theta(t)=\frac{\langle\gamma'(t),\varphi_v\rangle}{\|\varphi_v\|}=\frac{\langle\gamma',\varphi_v\rangle}{a} \Rightarrow$ ($\|\varphi_v\|=\sqrt{G}=a$)

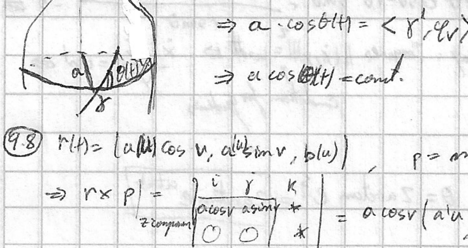
$\Rightarrow a\cdot\cos\theta(t)=\langle\gamma',\varphi_v\rangle=\langle u'\varphi_u+v'\varphi_v,\varphi_v\rangle=v'\langle\varphi_v,\varphi_v\rangle=a^2v'$. $\Rightarrow (a\cos(\theta(t)))'=(a^2v')'=v''a^2+v'2aa'u'=0$ (γ is geodesic)

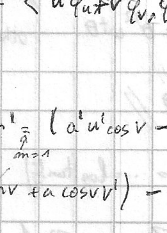
$\Rightarrow a\cos\theta(t)=\text{const}$.

(9.8) $r(t)=(a(u)\cos v, a(u)\sin v, b(u))$, $\quad p=mr' \underset{m=1}{=} (a'u'\cos v-a\sin v\, v', a'u'\sin v+a\cos v\, v', *)$

$\Rightarrow r\times p|_{z\text{ component}}=\begin{vmatrix}i&j&k\\a\cos v&a\sin v&*\\ \circ&\circ&*\end{vmatrix}=a\cos v(a'u'\sin v+a\cos v\, v')-a\sin v(a'u'\cos v-a\sin v\, v')=a^2v'=\text{const}.$

(9.9)



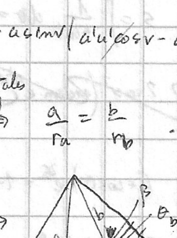


$\overset{\text{Tales}}{\Rightarrow} \frac{a}{r_a}=\frac{b}{r_b}$. Clairaut's theorem says that $r_a\cos\theta_a=\text{const}=r_b\cos\theta_b$

But $\cos\theta_a=\sin\alpha$ and $\cos\theta_b=\sin\beta \Rightarrow$

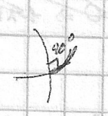
$\Rightarrow a\sin\alpha=b\sin\beta \Rightarrow \frac{\sin\alpha}{b}=\frac{\sin\beta}{a}=\frac{\sin(\pi-\beta)}{a}$

$\Rightarrow$ Sine theorem.



9.10

9.11 Obs $d(\gamma(t))\cos\theta(t) = c_\gamma = 0 \Leftrightarrow \cos\theta(t) = \frac{\pi}{2} \Leftrightarrow \gamma$ is a meridian
$d(\gamma(t)) > 0$



list 11

11.1 Take two curves γ_1, γ_2 whose tangent vectors are $w_1 = \varphi(u_1, v_1)$, $w_2 = \varphi(u_2, v_2)$ $\Rightarrow \cos_h(\widehat{w_1, w_2}) = \frac{\langle w_1, w_2\rangle}{\|w_1\|\|w_2\|} = \frac{(u_1\ v_1)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}}{\sqrt{(u_1\ v_1)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}}\sqrt{(u_2\ v_2)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}}}$ (h: hyperbolic)

$$= \frac{(u_1u_2 + v_1v_2)/y^2}{\sqrt{\frac{u_1^2+v_1^2}{y^2}}\sqrt{\frac{u_2^2+v_2^2}{y^2}}} = \frac{u_1u_2+v_1v_2}{\sqrt{u_1^2+v_1^2}\sqrt{u_2^2+v_2^2}} = \frac{(u_1\ v_1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}}{\sqrt{(u_1\ v_1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_1\\ v_1\end{pmatrix}}\sqrt{(u_2\ v_2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_2\\ v_2\end{pmatrix}}} = \cos(\widehat{w_1, w_2})$$
on the Euclidean plane

11.2
$$A(R_1) = \iint_{R_1} \overbrace{\sqrt{EG-F^2}\,dx\,dy}^{dA} = \int_0^1 dx \int_1^\infty dy \frac{1}{y^2} = \int_1^\infty \frac{1}{y^2}dy = -\frac{1}{y}\Big|_1^\infty = \boxed{1}$$
$EG-F^2 = \frac{1}{y^4}$
$$A(R_2) = \iint_{R_2} \frac{1}{y^2}dx\,dy = \int_0^1 dx \int_0^1 \frac{1}{y^2}dy = \int_0^1 \frac{1}{y^2}dy = \boxed{+\infty}$$

11.3 The lines of curvature cannot be determined because ~~~~ they can be found from the eigenvectors of W ← determined from 1st and 2nd fundamental form.
And the asymptotic lines ~~~~ neither because are the lines where tangent vector vanish II ← 2nd fundamental form. (we cannot determine the normal vector only from 1st fundamental form)

11.4 a) Recall that Euler-Lagrange is:
$$\begin{cases}\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot x} = 0\\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot y} = 0\end{cases}, \text{ where } L(x,y,\dot x,\dot y) = \underbrace{E(x,y)}_{1/y^2}\dot x^2 + 2\underbrace{F(x,y)}_{0}\dot x\dot y + \underbrace{G(x,y)}_{1/y^2}\dot y^2 \Rightarrow \frac{\dot x^2}{y^2} + \frac{\dot y^2}{y^2}$$

$$\to \frac{\partial L}{\partial x} = 0,\ \frac{\partial L}{\partial \dot x} = \frac{2\dot x}{y^2},\ \frac{\partial L}{\partial y} = -\frac{2\dot x^2}{y^3} - \frac{2\dot y^2}{y^3},\ \frac{\partial L}{\partial \dot y} = \frac{2\dot y}{y^2} \Rightarrow \frac{d}{dt}\left(\frac{2\dot x}{y^2}\right) = \frac{2\ddot x y^2 - 4\dot x y\dot y}{y^4} = 2\frac{\ddot x y - 2\dot x\dot y}{y^3}$$

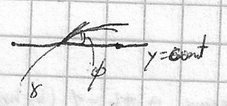
$$\underbrace{\frac{d}{dt}\left(\frac{2\dot y}{y^2}\right)}_{\frac{\partial L}{\partial \dot y}} = 2\frac{\ddot y y^2 - 2y\dot y^2}{y^4} = 2\frac{\ddot y y - 2\dot y^2}{y^3}$$

$$\Rightarrow \begin{cases}\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot x} = -2\frac{\ddot x y - 2\dot x\dot y}{y^3} = 0 \Leftrightarrow \ddot x - \frac{2\dot x\dot y}{y} = 0 \Rightarrow \Gamma_{11}^1 = \Gamma_{22}^1 = 0,\ \Gamma_{12}^1 = -\frac{1}{y}\\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot y} = \frac{-2\dot x^2}{y^3} - \frac{2\dot y^2}{y^3} - 2\frac{\ddot y y - 2\dot y^2}{y^3} = 0 \Leftrightarrow \ddot y + \frac{\dot x^2}{y} - \frac{\dot y^2}{y} = 0 \Rightarrow \Gamma_{12}^2 = 0,\ \Gamma_{11}^2 = \frac{1}{y},\ \Gamma_{22}^2 = -\frac{1}{y}\end{cases}$$

b) $K \underset{\text{formulas}}{=} \frac{1}{E}\left(\underbrace{\Gamma_{11}^1\Gamma_{12}^2}_{0} + \Gamma_{11}^2\Gamma_{22}^2 - \Gamma_{12}^1\Gamma_{11}^2 - \underbrace{\Gamma_{12}^2\Gamma_{12}^2}_{0} + (\Gamma_{11}^2)_y - \underbrace{(\Gamma_{12}^2)_x}_{0}\right) = \frac{1}{1/y^2}\left(-\frac{1}{y}\frac{1}{y} - \left(-\frac{1}{y}\right)\cdot\frac{1}{y} - \frac{1}{y^2}\right) = \boxed{-1}$

c) $\gamma(t) = (x_0, e^{at})$, $\dot x = \ddot x = 0$, $\dot y = ae^{at}$, $\ddot y = a^2e^{at}$
$$\Rightarrow \begin{cases}\ddot x - \frac{2}{y}\dot x\dot y = 0 \checkmark\\ \ddot y + \frac{1}{y}\dot x^2 - \frac{1}{y}\dot y^2 = a^2e^{at} - \frac{a^2e^{2at}}{e^{at}} = 0 \checkmark\end{cases}$$

d) Since $\frac{\partial L}{\partial x} = 0 \underset{\text{Euler-Lagrange}}{\Rightarrow} -\frac{d}{dt}\frac{\partial L}{\partial \dot x} = 0 \Rightarrow \frac{\partial L}{\partial \dot x}$ is a first integral
$$\Rightarrow \frac{\partial L}{\partial \dot x} = \frac{2\dot x}{y^2} = \text{cont} \Leftrightarrow \frac{\dot x}{y^2} = \text{cont}.$$



$$\frac{\cos\phi}{y} = \frac{(\dot x, \dot y)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}}{y\sqrt{(\dot x\ \dot y)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}}\sqrt{(1\ 0)\begin{pmatrix}1/y^2 & 0\\ 0 & 1/y^2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}}} = \frac{\dot x/y^2}{y\sqrt{\underbrace{\frac{\dot x^2+\dot y^2}{y^2}}_{1}\cdot\frac{1}{y^2}}} = \frac{\dot x/y^2}{y\cdot\frac{1}{y}} = \frac{\dot x}{y^2} = \text{cont}$$
we suppose is arc-length parametrized

e) $\gamma(t) = (x_0 + r\cos(\theta), r\sin(\theta))$
$\dot x = -r\theta'\sin(\theta)$, $\dot y = r\theta'\cos(\theta)$
$\ddot x = -r\theta''\sin\theta - r\theta'^2\cos\theta$, $\ddot y = r\theta''\cos\theta - r\theta'^2\sin\theta$
$$\Rightarrow \begin{cases}\ddot x - \frac{2}{y}\dot x\dot y = -r\theta''\sin\theta - r\theta'^2\cos\theta - \frac{2}{r\sin\theta}(-r\theta'\sin\theta)\cdot(r\theta'\cos\theta) = 0 \Leftrightarrow\\ \ddot y + \frac{\dot x^2 - \dot y^2}{y} = r\theta''\cos\theta - r\theta'^2\sin\theta + \frac{r^2\theta'^2\sin^2\theta - r^2\theta'^2\cos^2\theta}{r\sin\theta} = 0 \Leftrightarrow\end{cases}$$

$$\begin{cases}\Leftrightarrow -r\theta''\sin\theta + r\theta'^2\cos\theta = 0 \Leftrightarrow \theta''\sin\theta - \theta'^2\cos\theta = 0\\ \Leftrightarrow r^2\theta''\cos\theta\sin\theta - r^2\theta'^2\cos^2\theta = 0 \Leftrightarrow \theta''\sin\theta - \theta'^2\cos\theta = 0\end{cases} \Leftrightarrow \theta'' = \theta'^2\cot\theta.$$

Other way: Imposes $\|(\dot x,\dot y)\|^2 = \text{cont}$ (condition for geodesics) $\Leftrightarrow \frac{\dot x^2 + \dot y^2}{y^2} = \text{cont} \Leftrightarrow$
$$\Leftrightarrow \frac{r^2\dot\theta^2}{r^2\sin^2\theta} = \text{cont} \Leftrightarrow 2\dot\theta\ddot\theta\sin^2\theta - 2\dot\theta^2\dot\theta\cos\theta\sin\theta = 0 \Leftrightarrow \ddot\theta = \dot\theta^2\cot\theta$$

f) From e) $1 \underset{\text{we impose}}{=} \|(\dot x,\dot y)\| = \frac{\dot\theta}{\sin\theta} = \frac{d\theta/ds}{\sin\theta} \to s = \int\frac{1}{\sin\theta}d\theta = \dots = \log\left(\tan\frac{\theta}{2}\right) \Rightarrow \theta = 2\arctan e^s = \boxed{2\arctan e^{at+b}}$

g) ~~$\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ ⇒ $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\cos^2(\arctan e^{at+b}) - 1 = 2\frac{1}{\tan(\arctan e^{at+b})+1} - 1$~~

$$e^s = \tan\frac{\theta}{2} \Rightarrow \underbrace{\tan^2\frac{\theta}{2}}_{e^{2s}} = \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{1-\cos\theta}{1+\cos\theta} \Rightarrow (1+\cos\theta)e^{2s} = 1-\cos\theta \Rightarrow \cos\theta = \frac{1-e^{2s}}{1+e^{2s}} \Rightarrow \sin\theta = \sqrt{1 - \frac{(1-e^{2s})^2}{(1+e^{2s})^2}} = \frac{2e^s}{1+e^{2s}} \Rightarrow$$

$\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$, $\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta$ ⇒ $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$\Rightarrow x = x_0 + r\frac{1-e^{2s}}{1+e^{2s}},\ y = r\frac{2e^s}{1+e^{2s}}$$

(11.5) $\psi_1(x,y) = \frac{1}{x^2+y^2}(x,y) \to \psi^1_{1x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\psi^2_{1y} = \frac{x^2-y^2}{(x^2+y^2)^2}$, $\Rightarrow d\psi_1 = \frac{1}{(x^2+y^2)^2}\begin{pmatrix} y^2-x^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}$

$\psi^2_{1x} = \frac{-2xy}{(x^2+y^2)^2}$, $\psi^1_{1y} = \frac{-2xy}{(x^2+y^2)^2}$

We have to check that $I_{\psi_1}\left(\frac{\partial\psi_1}{\partial x}, \frac{\partial\psi_1}{\partial x}\right) \overset{?}{=} E$, $I_{\psi_2}\left(\frac{\partial\psi_1}{\partial x}, \frac{\partial\psi_1}{\partial y}\right) \overset{?}{=} F$, $I_{\psi_3}\left(\frac{\partial\psi_1}{\partial y}, \frac{\partial\psi_1}{\partial y}\right) \overset{?}{=} G$.

$\psi_2(x,y) = K(x,y) \to d\psi = K\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\quad I|_{(x,y)=\psi(x,y)} = \frac{1}{(K_y)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \Rightarrow I|_{1x}$

(11.6) $\psi(x,y) = \frac{1+c^2}{(x-c)^2+y^2}(x-c,y) + (c,0)$, $\overset{\gamma(t)}{\psi(0,e^{-t})} = \frac{1+c^2}{c^2+e^{-2t}}(-c, e^{-t}) + (c,0) = \left(c - c\frac{1+c^2}{c^2+e^{-2t}}, e^{-t}\frac{1+c^2}{c^2+e^{-2t}}\right) = (x(t), y(t))$

In particular $\gamma(0) = \left(c - c\frac{1+c^2}{1+c^2}, \frac{1+c^2}{1+c^2}\right) = (0,1)$

$\gamma'(t) = \left(c\frac{1+c^2}{(c^2+e^{-2t})^2}(-2e^{-2t}), -e^{-t}\frac{1+c^2}{c^2+e^{-2t}} + \frac{(1+c^2)(-2e^{-2t})}{(c^2+e^{-2t})^2}\right)$

List 12

(12.1) a) $\omega\wedge\omega = (-1)^{(2m+1)(2m+1)}\omega\wedge\omega = -\omega\wedge\omega \Rightarrow \omega\wedge\omega = 0$. Take $\omega = dx\wedge dy + dz\wedge dt \Rightarrow \omega\wedge\omega = dx\wedge dy\wedge dz\wedge dt - dz\wedge dt\wedge dx\wedge dy =$
$= 2dx\wedge dy\wedge dz\wedge dt \neq 0$. is a basis

b) α is closed $\Rightarrow d\alpha = 0$, β is closed $\Rightarrow d\beta = 0$ $\Big\} \Rightarrow d(\alpha\wedge\beta) = \underset{0}{d\alpha}\wedge\beta + (-1)^{\overset{\text{deg of }\alpha}{p}}\alpha\wedge\underset{0}{d\beta} = 0 + 0 = 0$.

c) α is closed $\Rightarrow d\alpha = 0$; β is exact $\Rightarrow \exists\eta$ s.t. $d\eta = \beta$

Obs: $\alpha\wedge\beta = \alpha\wedge d\eta$. Consider $\omega = (-1)^{\overset{\text{deg of }\alpha}{p}}\alpha\wedge\eta \Rightarrow d\omega = (-1)^p\left[\underset{0}{d\alpha}\wedge\eta + (-1)^p\alpha\wedge d\eta\right] =$
$= \alpha\wedge d\eta = \alpha\wedge\beta \Rightarrow \alpha\wedge\beta$ is exact

d) Sps. $f:\mathbb{R}^n \to \mathbb{R}$ s.t. $df = 0 \Rightarrow \frac{\partial f}{\partial x^i} = 0\ \forall i \Rightarrow f$ is constant on each connected component of the domain.

(12.2) $\omega = x\,dy - dz$, $\eta = 2z^2dx$, $\mu = dx - yz\,dy$

a) i) $x\omega + \eta = (x^2dy - xdz) + 2z^2dx = \boxed{2z^2dx + x^2dy - xdz}$ ii) $z(\eta-\mu) = \boxed{z(2z^2-1)dx + yz^2dy}$

iii) $\omega\wedge\mu = (xdy - dz)\wedge(dx - yz\,dy) = \boxed{x\,dy\wedge dx - dz\wedge dx + yz\,dz\wedge dy}$ iv) $(2\omega - y\mu)\wedge\eta = (-y + 2x)dx + y^2z\,dy - 2dz)\wedge(2z^2dx) =$
$= \boxed{2y^2z^3dy\wedge dz - 4z^2dz\wedge dx}$

v) $\omega\wedge\eta\wedge\mu = (xdy - dz)\wedge[2z^2dx\wedge(dx - yzdy)] = (xdy - dz)\wedge(-2z^3y\,dx\wedge dy) = 2z^3y\,dz\wedge dx\wedge dy = \boxed{2z^3y\,dx\wedge dy\wedge dz}$

b) $X = z^2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$, $Y = y\frac{\partial}{\partial x} + e^x\frac{\partial}{\partial z} \Rightarrow X - Y = (z^2-y)\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - e^x\frac{\partial}{\partial z}$

i) $\omega(X) = x\,dy(X) - dz(X) = -x - 0 = \boxed{-x}$ ii) $\omega\wedge\mu(X, X-Y) = \begin{vmatrix}\omega(X) & \omega(X-Y) \\ \mu(X) & \mu(X-Y)\end{vmatrix} = \begin{vmatrix} -x & -x+e^x \\ z^2+yz & z^2-y+yz\end{vmatrix}$

$= -x(z^2-y+yz) - (z^2+yz)(-x+e^x) = \boxed{xy - e^xz^2 - e^xyz}$ Easily: $(\omega\wedge\mu)(X,X) - (\omega\wedge\mu)(X,Y) = -\begin{vmatrix}\omega(x) & \mu(Y) \\ \mu(X) & \mu(Y)\end{vmatrix} = -\begin{vmatrix} -x & -e^x \\ z^2+yz & y\end{vmatrix} = xy - e^x(z^2+yz)$.

(12.3) a) $T^*\omega = 1ds + s\,d(e^s) = \boxed{ds + se^sds}$ b) $T^*\omega = T^*(zdx)\wedge T^*dz = (st\,dt)\wedge(d(st)) = (st\,dt)\wedge(t\,ds + s\,dt) = \boxed{st^2dt\wedge ds}$

c) $T^*\omega = T^*(x_4^2dx_1)\wedge T^*dx_2\wedge T^*dx_3 = [(s+tu)^2d((s+tu)^2)]\wedge d(tu)\wedge d(us) = [2(s+tu)^3(ds+dt+du)]\wedge(dt + tdu)\wedge(uds + sdu)$
$= [2(s+tu)^3(ds+dt+du)]\wedge(u^2dt\wedge ds + tu\,du\wedge ds + us\,dt\wedge du) = 2(s+tu)^3[u^2du\wedge dt\wedge ds + tu\,dt\wedge du\wedge ds + us\,ds\wedge dt\wedge du] =$
$= \boxed{2(s+tu)^3(us + tu - u^2)ds\wedge dt\wedge du}$

Other way: $T^*\omega = T^*(z)\,T^*(dx)\wedge T^*(dz) = T^*(z)\,d(T^*(x))\wedge d(T^*(z)) = (z\circ T)\,d(x\circ T)\wedge d(z\circ T) =$
$= st\,dt\wedge d(st)$

(12.4) a) $\omega = x\,dy + y\,dx$
$d\omega = dx\wedge dy + dy\wedge dx = \boxed{0}$

b) $\omega = \underset{\alpha}{(dy - xdz)}\wedge\underset{\beta}{(xydx + 3dy + zdz)} \Rightarrow d\omega = (-dx\wedge dz)\wedge\beta \underset{(-1)^1 = -1}{-} \alpha\wedge(xdy\wedge dx) = -3dx\wedge dz\wedge dy + x^2dz\wedge dy\wedge dx = \boxed{(3-x^2)dx\wedge dy\wedge dz}$

$d\alpha = -dx\wedge dz$, $d\beta = x\,dy\wedge dx$

c) $\omega = f(x,y)dx\wedge dy \Rightarrow d\omega = f_x dx\wedge dy\wedge dz + f_y dy\wedge dx\wedge dy = \boxed{0}$

d) $\omega = f(x)dy \Rightarrow d\omega = \boxed{f'(x)dx\wedge dy}$

e) $\omega = \cos(xy^2)dx\wedge dz \Rightarrow d\omega = -\sin(xy^2)\cdot 2yx\,dy\wedge dx\wedge dz = \boxed{2xy\sin(xy^2)dx\wedge dy\wedge dz}$

f) $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy \Rightarrow d\omega = dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy = \boxed{3dx\wedge dy\wedge dz}$

(12.5) a) Clearly $\langle iX, Y\rangle$ is bilinear. Let's see that it is alternating: That is $\forall p\in\mathbb{R}^{2m}$, $\omega_p(u,v) = \langle iu, v\rangle = \omega_p(u,v)$, $\forall u,v\in\mathbb{C}^{2m}$
$\Rightarrow \omega_p(v,u) = \langle iv, u\rangle = \langle -v, iu\rangle = -\langle iu, v\rangle = -\omega_p(u,v)$ and moreover ω_p is constant on $p \Rightarrow \omega_p$ differentiable.

Recall that in $\mathbb{C}^n \to \langle u,v\rangle = \langle\bar{u}, v\rangle$ scalar product on $\mathbb{R}^{2m}$; $\langle a,b\rangle = \langle ia, ib\rangle$ in $\mathbb{C}$ because $\langle a,b\rangle = \mathrm{Re}(\bar{a}_1b_1 + \dots + \bar{a}_mb_m)$

Let $z = (z_1, \dots, z_m) \in \mathbb{C}$, $z_k = x_k + i y_k \in \mathbb{C}$ $\Rightarrow z = (x_1, y_1, \dots, x_m, y_m) \in \mathbb{R}^{2m}$

b) Sps $\omega = \sum_{j<k} \alpha_{jk}\, dx_j \wedge dx_k + \sum_{j,k} \mu_{jk}\, dx_j \wedge dy_k + \sum_{j<k} \nu_{jk}\, dy_j \wedge dy_k$.

Obs: $\alpha_{jk} = \omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = \left\langle i\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle = \left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_k}\right\rangle = 0$, $\nu_{jk} = \omega\left(\frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right) \underset{\text{similarly}}{=} \dots = 0$.

$\mu_{jk} = \omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) = \left\langle i\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right\rangle = \left\langle \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k}\right\rangle = \delta_{jk} \Rightarrow \omega = \sum_{j=1}^{m} dx_j \wedge dy_j = dx_1 \wedge dy_1 + \dots + dx_m \wedge dy_m$.

c) The coefficients of ω are constant $\Rightarrow d\omega = 0$.

d) $\omega \wedge \overset{(m)}{\dots} \wedge \omega$ is a $2m$-form $\Rightarrow$ it has to be of the form $\lambda\, dx_1 \wedge dy_1 \wedge \dots \wedge dx_m \wedge dy_m$ $\Rightarrow \lambda = \omega \wedge \overset{(m)}{\dots} \wedge \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial y_m}\right) = \dots$

Other approach: $\omega \wedge \overset{(m)}{\dots} \wedge \omega = \left(\sum_{j=1}^{m} dx_j \wedge dy_j\right) \wedge \overset{(m)}{\dots} \wedge \left(\sum_{j=1}^{m} dx_j \wedge dy_j\right) = dx_1 \wedge dy_1 \wedge (dx_2 \wedge dy_2 \wedge \dots \wedge dx_m \wedge dy_m) +$

$+ (dx_2 \wedge dy_2) \wedge (dx_1 \wedge dy_1) \wedge (dx_3 \wedge dy_3) \wedge \dots \wedge (dx_m \wedge dy_m) + dx_m \wedge dy_m \wedge (dx_1 \wedge dy_1) \wedge \dots \wedge (dx_{m-1} \wedge dy_{m-1}) =$

$= \boxed{m\, dx_1 \wedge \dots \wedge dy_m}$

e)

(12.6) $\alpha = y\,dx + x\,dy + z\,dz$, $f(u,v) = (\cos u, \sin u, v)$ $\Rightarrow f^*\alpha = \sin u\, d(\cos u) + \cos u\, d(\sin u) + v\,dv = \boxed{v\,dv}$, $d(f^*\alpha) = d(v\,dv) = \boxed{0}$

(12.7) $\omega_X^1 = X^1 dx + X^2 dy + X^3 dz$, $\omega_X^2 = X^1 dy \wedge dz + X^2 dz \wedge dx + X^3 dx \wedge dy$, $\omega_X^3 = f\, dx \wedge dy \wedge dz$

a) i) $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \omega^1_{\operatorname{grad} f}$ ii) $d(\omega_X^1) = \left(\frac{\partial X_1}{\partial y} dy \wedge dx + \frac{\partial X_1}{\partial z} dz \wedge dx\right) + \left(\frac{\partial X_2}{\partial x} dx \wedge dy + \frac{\partial X_2}{\partial z} dz \wedge dy\right) +$

$+ \left(\frac{\partial X_3}{\partial x} dx \wedge dz + \frac{\partial X_3}{\partial y} dy \wedge dz\right) = \left(\frac{\partial X_3}{\partial y} - \frac{\partial X_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial X_1}{\partial z} - \frac{\partial X_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y}\right) dx \wedge dy = \omega^2_{\operatorname{rot} X}$.

iii) $d(\omega^2_X) = \frac{\partial X^1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial X^2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial X^3}{\partial z} dz \wedge dx \wedge dy = \left(\frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z}\right) dx \wedge dy \wedge dz = \omega^3_{\operatorname{div} X}$

b) $0 = d^2 f = d(\omega^1_{\operatorname{grad} f}) = \omega^2_{\operatorname{rot}\operatorname{grad} f} \Rightarrow \underset{\text{all the components are } 0}{\operatorname{rot}(\operatorname{grad} f) = 0}$; $0 = d^2(\omega_X^1) = d(\omega^2_{\operatorname{rot} X}) = \omega^3_{\operatorname{div}\operatorname{rot} X} \underset{\text{all the components are } 0}{\Rightarrow} \operatorname{div}(\operatorname{rot} X) = 0$ //

(12.8) $\rightarrow$ Done it in problems list 13, exercise 13.4

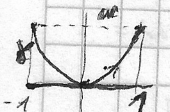
List 13:

(13.1) $\omega = \sin y\, dx + \sin x\, dy$, $\gamma(t) = (0, \pi) + t\underbrace{[(\pi, 0) - (0, \pi)]}_{(\pi, -\pi)} = (0 + t\pi, \pi - \pi t) \Rightarrow \int_\gamma \overbrace{\sin y\, dx + \sin x\, dy}^{\omega} = \int \gamma^* \omega = \int_0^1 \sin(\pi - \pi t)\pi\, dt + \sin(\pi t)(-\pi)\, dt =$

$= \pi \int_0^1 \sin(\pi - \pi t)\, dt - \pi \int_0^1 \sin(\pi t)\, dt = \cos(\pi - \pi t)\Big|_0^1 + \cos(\pi t)\Big|_0^1 = +1 + 1 = 1 - 1 = \boxed{0}$

(13.2) $\omega = \frac{-y\,dx}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+y^2}} dy$ sobre $\gamma(t) = (r\cos t, r\sin t)$. $\Rightarrow \int_\gamma \omega = \int_\gamma \gamma^*\omega = \int_0^{2\pi} \frac{-r\sin t \cdot r(-\sin t)\, dt}{r} + \frac{r\cos t \cdot r\cos t\, dt}{r} = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$

(13.3) $\omega = (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy$ sobre $C = \{(x,y) : |x| \le 1, y = x^2\}$ Obs: $\gamma(t) = (t, t^2)$, $t \in [-1, 1]$.



$\Rightarrow \int_\gamma \omega = \int_{-1}^{1} \gamma^*\omega\, dt = \int_{-1}^{1} (t^2 - 2t^3)\, dt + (t^4 - 2t^3)\, 2t\, dt = 2\int_0^1 (t^2 - 4t^4)\, dt = 2\left(\frac{1}{3} - \frac{4}{5}\right) = \boxed{\frac{-14}{15}}$

(13.4) $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$. Obs: $d\omega = \frac{-(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} dy \wedge dx + \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} dx \wedge dy = \frac{y^2 - x^2}{(x^2+y^2)^2} dy \wedge dx + \frac{-x^2 + y^2}{(x^2+y^2)^2} dx \wedge dy = 0$

Consider $C = \{x^2 + y^2 = 1\}$ If it was exact: $\int_D d(d\alpha) = \int_{\partial D} \underset{d\alpha}{\overset{\omega}{\omega}} = \int_0^{2\pi} \gamma^*\omega\, dt = \int_0^{2\pi} \frac{-\sin t(-\sin t)\, dt}{1} + \frac{\cos t \cos t\, dt}{1} = \int_0^{2\pi} dt = 2\pi \ne 0$.

$\gamma(t) = (\cos t, \sin t)$, $d\alpha = \omega$, $\partial D = C$

$\int_C \omega = \int_C d\alpha = \int_{\partial C = \emptyset} \alpha = 0$ Obs: $\int_D d\omega$ doesn't make sense because ω is not defined at the origin.

Let $C(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ differentiable, $r(t) > 0$ $\Rightarrow \int_C \omega = \int C^*\omega = \int_0^T \frac{-r\sin\theta\,(\dot r\cos\theta\, dt - r\sin\theta\, \dot\theta\, dt)}{r^2} + \frac{r\cos\theta}{r^2} \cdot (\dot r\sin\theta\, dt + r\cos\theta\, \dot\theta\, dt) =$

$= \int_0^T \dot\theta\, dt = \theta(T) - \theta(0) \in 2\pi\mathbb{Z}$ //

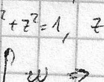
(13.5) $\omega = xy\, dx \wedge dy + 2x\, dy \wedge dz + 2y\, dx \wedge dz$. $\int_A \omega = \int \varphi^*\omega = \int \cos u \sin u \sin^2 v\,(-\sin u \sin v\, du + \cos u \cos v\, dv) = \dots$

$A = \{x^2 + y^2 + z^2 = 1, z \ge 0\}$. $\varphi(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

Stokes: $\int_V d\omega = \int_{\partial V} \omega$, $0 = \int_{\partial V} \omega = \int_A \omega + \int_D \omega \Rightarrow \int_A \omega = -\int_{\underset{z=0}{D}} \omega = -\int_D \varphi^*\omega = -\int_0^1\int_0^{2\pi} r\cos\theta \sin\theta\, d(r\cos\theta) \wedge d(r\sin\theta) = \dots = \boxed{0}$

$d\omega = 2\,dx \wedge dy \wedge dz + 2\,dy \wedge dx \wedge dz = 0$

Volume inside



exposed normal vector

$\varphi(r,\theta) = (r\sin\theta, r\cos\theta, 0)$ Orientation compatible.

$\varphi_r = (\sin\theta, \cos\theta, 0)$ $\rightarrow \varphi_r \times \varphi_\theta = (0, 0, -r)$ ✓ $\rightarrow \hat{N} = (0, 0, -1)$

$\varphi_\theta = (r\cos\theta, -r\sin\theta, 0)$

Otherwise $= -\iint_{x^2+y^2 \le 1} xy\, dx \wedge dy = \iint_{x^2+y^2 \le 1} xy\, dx\, dy = \boxed{0}$

$z = (x, y, 0)$

(13.6) $S = \{x^2 + y^2 - z^2 = 1, a \le z \le b\} \subset \mathbb{R}^3$. $\nu|_{(1,0,0)} = (1, 0, 0)$ $\rightarrow \varphi(u,v) = (\cosh v \cos u, \cosh v \sin u, \sinh v)$, $u \in [0, 2\pi]$, $v \in [\operatorname{arcsinh} a, \operatorname{arcsinh} b]$

a) $\omega = y^2 dx + x\,dy \Rightarrow d\omega = -2y\, dx \wedge dy + dx \wedge dy = (1 - 2y)\, dx \wedge dy \Rightarrow \int_S d\omega = \int \varphi^*(d\omega) = \int_R (1 - 2\cosh v \sin v)\, d(\cosh v \cos u) \wedge d(\cosh v \sin u) =$

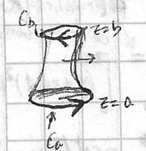
$R = [0, 2\pi] \times [\operatorname{arcsinh} a, \operatorname{arcsinh} b]$

$= \int_R (1-\cancel{2}\cosh v \sin u)\left(-\sin u \cosh v\, du + \sinh v \cos u\, dv\right) \wedge \left(\cos u \cosh v\, du + \sinh v \sin u\, dv\right) \overset{*}{=} \int_R (1-2\cosh \sin u) \sinh v \cosh v\, dv \wedge du =$

$= -2\pi \int_{\operatorname{arcsinh} a}^{\operatorname{arcsinh} b} \sinh v \cosh v \, dv = -2\pi \left.\frac{\sinh^2 v}{2}\right|_{\operatorname{arcsinh} a}^{\operatorname{arcsinh} b} = -\pi(b^2-a^2)$

On the other hand: $\int_{\partial S} \omega = \int_{C_a} \omega + \int_{C_b} \omega = \int_0^{2\pi} (C_a^* \omega + C_b^* \omega) = \int_0^{2\pi} (-r_a^3 \sin^3 t\, dt + r_a^2 \cos^2 t\, dt) + (r_b^3 \cos^3 t\, dt - r_b^2 \sin^2 t\, dt) = r_a^2 \pi - r_b^2 \pi = \pi(1+a^2-1-b^2)$

$= -\pi(b^2-a^2)$ //



$C_a = (r_a \cos t, r_a \sin t, a)$, $r_a^2 = 1+a^2$

$C_b = (r_b \sin t, r_b \cos t, b)$, $r_b^2 = 1+b^2$

b) $\omega = f(x,y,z)dz \rightarrow d\omega = f_x\, dx \wedge dz + f_y\, dy \wedge dz$. $\int_S d\omega = \int_S \varphi^*(d\omega) = \int_R f_x(\varphi) \cdot d(\cosh v \cos u) \wedge d(\sinh v) + f_y(\varphi)\, d(\cosh v \sin u) \wedge d(\sinh v) =$

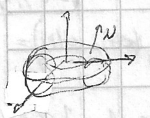
$= \int_R f_x(\varphi)(-\sin u \cosh v\, du + \sinh v \cos u\, dv) \wedge \cosh v\, dv + f_y(\varphi)(\cosh v \cos u\, du + \sinh v \sin u\, dv) \wedge \cosh v\, dv = \int_R f_x(\varphi) \cosh^2 v (\cos u - \sin u)\, du\, dv = 0$.

$\int_0^{2\pi} \cos u - \sin u = 0$.

On the other hand: $\int_{\partial S} \omega = \int_{C_a} C_a^* \omega + \int_{C_b} C_b^* \omega = \int_0^{2\pi} f(C_a) \cdot 0 + f(C_b) \cdot 0 = 0$ //

$d(a) = d(b) = 0$

(13.7) $\omega = z\, dx \wedge dy$, $V = \{(x,y,z) : (\sqrt{x^2+y^2}-R)^2 + z^2 \leq r^2\}$. $\rightarrow d\omega = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz = \eta$.



$\int_V \eta = vol(V) = \iiint_V dx\,dy\,dz = 2\pi \int_{R-r}^{R+r} \rho\, d\rho \int_{-\sqrt{r^2-(\rho-R)^2}}^{\sqrt{r^2-(\rho-R)^2}} dz = 4\pi \int_{R-r}^{R+r} \rho \sqrt{r^2-(\rho-R)^2}\, d\rho = 4\pi \int_{-1}^{1} (ru+R) r \sqrt{1-u^2} \cdot r\, du = (*)$

$u = \frac{\rho - R}{r} \Rightarrow du = \frac{d\rho}{r}$

$\varphi(u,v) = (\cos u (R + r\cos v), \sin u (R + r\cos v), r \sin v)$, $u,v \in [0, 2\pi]$. Obs: cylindrical coordinates: $x = \rho \cos\theta$, $y = \rho \sin\theta$, $z = z$ $\Rightarrow \theta \in [0,2\pi]$, $\rho \in [R-r, R+r]$ $\rightarrow (\rho-R)^2 + z^2 \leq r^2$ $\Rightarrow z^2 \leq r^2 - (\rho-R)^2$

On the other hand:

$\int_{\partial V} \omega = \int_{\partial V} \varphi^* \omega = \int_{[0,2\pi]^2} r \sin v \cdot d(\cos u (R + r\cos v)) \wedge d(\sin u (R + r \cos v)) = \int_{[0,2\pi]^2} r \sin v (-\sin u (R + r\cos v)\, du - r \cos u \sin v\, dv) \wedge (\cos u (R + r\cos v)\, du - r \sin u \sin v\, dv) =$

Torus

$= \int_{[0,2\pi]^2} r \sin v (R + r\cos v) \cdot (r \sin v \underbrace{(\sin^2 u + \cos^2 u)}_{1})\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} r^2 \sin^2 v (R + r\cos v) = 2\pi \int_0^{2\pi} R r^2 \sin^2 v\, dv + 2\pi \int_0^{2\pi} r^3 \sin^2 v \cos v\, dv = \boxed{2\pi^2 R r^2}$

volume of inside the torus.

$(*) = 4\pi r^2 R \int_{-1}^{1} \sqrt{1-u^2}\, du = 4\pi r^2 R \int_{-\pi/2}^{\pi/2} \underbrace{|\cos\theta| \cdot (\cos\theta)}_{\cos^2\theta} d\theta = 4\pi r^2 R \frac{\pi}{2} = \boxed{2\pi^2 r^2 R}$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

(13.8) a) $\int_C \omega_X^1 = \int_C \gamma^* \omega_X^1 = \int_C X_1(\gamma)\, d(\gamma_1) + X_2(\gamma)\, d(\gamma_2) + X_3(\gamma)\, d(\gamma_3) = \int_C X_1(\gamma) \cdot \gamma_1'\, dt + X_2(\gamma) \gamma_2'\, dt + X_3(\gamma) \gamma_3'\, dt = \int_C \langle X(\gamma), \gamma' \rangle\, dt = \int_C X \cdot dL$ //

$C \rightarrow \gamma(t) = (\gamma_1, \gamma_2, \gamma_3)$

b) $\int_S \omega_X^2 = \int_S \varphi^* \omega_X^2 = \int X_1(\varphi)\, d\varphi_2 \wedge d\varphi_3 + X_2(\varphi)\, d\varphi_3 \wedge d\varphi_1 + X_3(\varphi)\, d\varphi_1 \wedge d\varphi_2 = \int X_1(\varphi) \left(\frac{\partial \varphi_2}{\partial u} \frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_2}{\partial v} \frac{\partial \varphi_3}{\partial v}\right) du \wedge dv + X_2(\varphi) \left(\frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_1}{\partial v} - \frac{\partial \varphi_3}{\partial v} \frac{\partial \varphi_1}{\partial u}\right)$

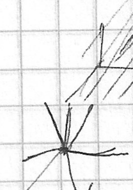
$S \rightarrow \varphi = (\varphi_1, \varphi_2, \varphi_3)$

$\cdot du \wedge dv + \left(\frac{\partial \varphi_1}{\partial u} \frac{\partial \varphi_2}{\partial v} - \frac{\partial \varphi_1}{\partial v} \frac{\partial \varphi_2}{\partial u}\right) du \wedge dv = \int \langle X(\varphi), \varphi_u \times \varphi_v \rangle\, du\, dv = \int_S X \cdot dS$ //

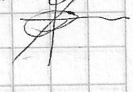
List 14

(14.1) a) $X_0 = (1,0,0) \rightarrow \operatorname{div} X_0 = 0$, $\operatorname{rot} X_0 = 0$. $\begin{cases} x' = 1 \\ y' = 0 \\ z' = 0 \end{cases} \Leftrightarrow \begin{cases} x(t) = t + C_1 \\ y(t) = C_2 \\ z(t) = C_3 \end{cases}$ parallel lines.

b) $X_1 = (x,y,z) \rightarrow \operatorname{div} X_1 = 3$, $\operatorname{rot} X_0 = 0$ $\begin{cases} x' = x \\ y' = y \\ z' = z \end{cases} \Leftrightarrow \begin{cases} x(t) = C_1 e^t \\ y(t) = C_2 e^t \\ z(t) = C_3 e^t \end{cases}$ rays starting at the origin



c) $X_2 = (-y, x, 0) \rightarrow \operatorname{div} X_2 = 0$, $\operatorname{rot} X_0 = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0,0,2)$ $\Leftrightarrow \begin{cases} x' = -y \\ y' = x \\ z' = 0 \end{cases} \Leftrightarrow \begin{cases} x(t) = C_1 \cos t + C_2 \sin t \\ y(t) = C_1 \sin t - C_2 \cos t \\ z(t) = C_3 \end{cases}$ circles centered at the origin



(14.2) a) $\operatorname{div}(fF) = \sum_{i=1}^{3} \frac{\partial (fF_i)}{\partial x_i} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} F_i + f \frac{\partial F_i}{\partial x_i}\right) = \operatorname{grad} f \cdot F + f \operatorname{div} F$

b) $\operatorname{rot}(fF) \rightarrow$ One component is $\frac{\partial (fF_3)}{\partial y} - \frac{\partial (fF_2)}{\partial z} = \frac{\partial f}{\partial y} F_3 + f \frac{\partial F_3}{\partial y} - \frac{\partial f}{\partial z} F_2 - f \frac{\partial F_2}{\partial z} = \frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2 + f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \Rightarrow$

$\Rightarrow \operatorname{rot} fF = \operatorname{grad} f \times F + f \operatorname{rot} F$.

(14.3) - $Y_0 = (y,0,0) = y X_0 \rightarrow \operatorname{div} Y_0 = \boxed{0}$ $\operatorname{rot} Y_0 = \boxed{(0,0,-1)}$

- $Y_1 = (x^2+y^2+z^2)^\varepsilon X_1 = (x^2+y^2+z^2)^\varepsilon (x,y,z)$. $\frac{\partial (x(x^2+y^2+z^2)^\varepsilon)}{\partial x} = (x^2+y^2+z^2)^\varepsilon + \varepsilon x (x^2+y^2+z^2)^{\varepsilon-1} = (x^2+y^2+z^2)^{\varepsilon-1}[\varepsilon x + x^2+y^2+z^2] \Rightarrow$

$\Rightarrow \operatorname{div} Y_1 = \boxed{(x^2+y^2+z^2)^{\varepsilon-1}(\varepsilon x + x^2+y^2+z^2, \varepsilon y + x^2+y^2+z^2, \varepsilon z + x^2+y^2+z^2)}$

$\Rightarrow \operatorname{rot} Y_1 \overset{(14.2)}{=} \operatorname{grad} f \times F + f \underset{0}{\operatorname{rot} F} = (2x\varepsilon(x^2+y^2+z^2)^{\varepsilon-1}, 2y\varepsilon(x^2+y^2+z^2)^{\varepsilon-1}, 2z\varepsilon(x^2+y^2+z^2)^{\varepsilon-1}) \times (x,y,z) \overset{\text{linearly dependent}}{=} \boxed{0}$

- $Y_2 = (x^2+y^2)^\varepsilon X_2$. $\operatorname{div} fF = \operatorname{grad} f \cdot F + f \underset{0}{\operatorname{div} F} = (2x\varepsilon(x^2+y^2)^{\varepsilon-1}, 2y\varepsilon(x^2+y^2)^{\varepsilon-1}, 0) \cdot (-y, x, 0) = \boxed{0}$

$\operatorname{rot} Y_2 = \operatorname{grad} f \times F + f \operatorname{rot} F = 2\varepsilon(x^2+y^2)^{\varepsilon-1}(x,y,0) \times (-y,x,0) + (x^2+y^2)^\varepsilon (0,0,2) = 2\varepsilon(x^2+y^2)^{\varepsilon-1}[x^2+y^2] + 2(x^2+y^2)^\varepsilon = \boxed{2(x^2+y^2)^\varepsilon(1+\varepsilon)}$.

$$\begin{cases} \dot{x} = \dfrac{x}{(x^2+y^2)^\varepsilon} \\ \dot{y} = \dfrac{y}{(x^2+y^2)^\varepsilon} \end{cases} \xrightarrow{\text{polars}} r\dot{r} = x\cdot\dot{x} + y\cdot\dot{y} = \frac{x^2}{(x^2+y^2)^\varepsilon} + \frac{y^2}{(x^2+y^2)^\varepsilon} = \frac{1}{(r^2)^{\varepsilon-1}} \Rightarrow \dot{r} = r^{1-2\varepsilon} \Rightarrow \int r^{2\varepsilon-1}\,dr = t + C \Leftrightarrow$$

$$\Leftrightarrow \frac{r^{2\varepsilon}}{2\varepsilon} = t + C \Leftrightarrow r(t) = (r_0^{2\varepsilon} + 2\varepsilon t)^{\frac{1}{2\varepsilon}} \qquad C = \frac{r_0^{2\varepsilon}}{2\varepsilon}$$

Similarly $\dot\theta = 0 \Rightarrow \theta(t) = \theta_0$

the solution that pass throught (x,y) can be parametrized by $t \longmapsto \phi_\varepsilon^t(x,y) = (r\cos\theta_0, r\sin\theta_0) = \left((r_0^{2\varepsilon} + 2\varepsilon t)^{\frac{1}{2\varepsilon}}\cos\theta_0, (r_0^{2\varepsilon}+2\varepsilon t)^{\frac{1}{2\varepsilon}}\sin\theta_0\right) =$

$$= \left(\frac{x}{r_0^{\frac{2\varepsilon}{2\varepsilon}}}(r_0^{2\varepsilon} + 2\varepsilon t)^{\frac{1}{2\varepsilon}}, \frac{y}{r_0^{\frac{2\varepsilon}{2\varepsilon}}}(r_0^{2\varepsilon}+2\varepsilon t)^{\frac{1}{2\varepsilon}}\right) = \left(x\left(1 + \frac{2\varepsilon t}{(x^2+y^2)^\varepsilon}\right)^{\frac{1}{2\varepsilon}}, y\left(1+\frac{2\varepsilon t}{(x^2+y^2)^\varepsilon}\right)^{\frac{1}{2\varepsilon}}\right)$$

$t=0 \to (r_0\cos\theta_0, r_0\sin\theta_0) = (x,y) \Rightarrow \cos\theta_0 = \frac{x}{r_0}$, $\sin\theta_0 = \frac{y}{r_0} \to r_0^2 = x^2+y^2$

14.4 $F \doteq (y^2, 0)$. ~~By Stokes theorem~~ $\cancel{C_r(x_0,y_0) = \int_{C(x_0,y_0,r)} F\cdot ds = \int_{D(x_0,y_0)} \operatorname{rot} F \cdot dS}$ = Let's calculate $C_r(x_0,y_0) = \int_{C((x_0,y_0),r)} F\cdot ds = \int_0^{2\pi} (y_0 + r\sin t)^2(-r\sin t)\,dt =$

Obs $\operatorname{rot} F = (0,0,-2y)$.

$\alpha(t) = C((x_0,y_0),r) = (x_0 + r\cos t, y_0 + r\sin t, 0)$

$\alpha' = r(-\sin t, \cos t, 0)$

$$= \int_0^{2\pi}(-ry_0^2\sin t - y_0 2r^2\sin^2 t - r^3\sin^3 t)\,dt = \int_0^{2\pi} -2r^2\sin^2 t = -2r^2\pi y_0 \Rightarrow \lim_{r\to 0}\frac{C_r(x_0,y_0)}{\pi r^2} = \lim_{r\to 0}\frac{-2r^2\pi y_0}{\pi r^2} = -2y_0 \Rightarrow \operatorname{rot} F(x_0,y_0) = (0,0,-2y_0)$$

14.5 $L = (\sigma(y-x), x(\rho - z) - y, xy - \beta z)$ $\quad L = 0 \Leftrightarrow \begin{cases} \sigma(y-x) = 0 \\ x(\rho - z) - y = 0 \\ xy - \beta z = 0 \end{cases}$

$\sigma(y-x)=0 \to x=y$

$x(\rho-z)-y=0 \to x(\rho - z - 1) = 0 \to x=0$ or $z = \rho - 1$

$xy - \beta z = 0 \to x^2 - \beta z = 0 \Rightarrow$ $y = x = 0 \Rightarrow z = 0$; $\;z = \rho - 1 \Rightarrow x = \pm\sqrt{\beta(\rho-1)}$

$\to$ Singular points: $(0,0,0)$, $P_\pm = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$

$$\operatorname{rot} L = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \sigma(y-x) & x(\rho-z) & xy-\beta z \end{vmatrix} = (x + x, -y, \rho - z - \sigma) = \boxed{(2x, -y, \rho - z - \sigma)}$$

At the critical points $\operatorname{rot} L(P_\pm) = (\pm 2\sqrt{\beta(\rho-1)}, \mp\sqrt{\beta(\rho-1)}, 1-\sigma)$

14.6 $H = (y + xz, -x + yz, \frac{1}{2}(1 + z^2 - (x^2+y^2)))$. $\operatorname{div} H = z + z + \frac{2z}{2} = \boxed{3z}$, $\operatorname{rot} H = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y+xz & -x+yz & \frac{1}{2}(1+z^2-(x^2+y^2)) \end{vmatrix} = (-y-y, x+x, -1-1) =$

$= \boxed{2(-y, x, -1)}$

14.7 We know that $c'(t) = H(c(t))$ and $K = \frac{\|c' \times c''\|}{\|c'\|^3}$. $\xrightarrow{\text{chain rule}} c'' = DH(c(t))c'(t) = \left(\frac{\partial H}{\partial x}(c(t))\cdot c'(t), \frac{\partial H}{\partial y}(c(t))c'(t), \frac{\partial H}{\partial z}(c(t))\cdot c'(t)\right) =$

$= (DH\cdot H)\big|_{c(t)}$ $\Rightarrow K(x,y,z) = \left.\frac{\|H \times (DH\cdot H)\|}{\|H\|}\right|_{(x,y,z)}$

$$DH = \begin{pmatrix} z & 1 & x \\ -1 & z & y \\ -x & -y & z \end{pmatrix} \Rightarrow DH\cdot H = \left(\tfrac{3}{2}xz^2 + 2yz - \tfrac{1}{2}x - \tfrac{1}{2}x^3 - \tfrac{1}{2}xy^2,\; -2xz - \tfrac{1}{2}y + \tfrac{3}{2}yz^2 - \tfrac{1}{2}x^2y - \tfrac{1}{2}y^3, \right.$$

$\|H\|^2 = y^2 + 2xyz + x^2z^2 + x^2 - 2xyz + y^2z^2 + \dots = (x^2+y^2 \dots$ $\to \|DH\cdot H\| = \frac{(x^2+y^2)(x^2+y^2+z^2+1)^4}{16} - \frac{z(3x^2+3y^2-z^2-1)}{2})$

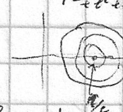
Doing calculations $K(x,y,z) = \frac{2\sqrt{x^2+y^2}}{1+x^2+y^2+z^2}$

14.8 $K(x,y,z) = c = \text{const}$ We want to see that K is constant along the integral curves. $\Leftrightarrow K$ is first integral $\Leftrightarrow \nabla K\cdot H = 0$

$\nabla K \cdot H = \dots = 0 \Rightarrow H \perp \nabla K \Rightarrow H$ is tangent to $K(x,y,z) = \text{const}$.

Let's see $K(x,y,z) = \text{const}$ is a torus. In cylindrical coordinates we have $\frac{2r}{r^2+z^2+1} = c \Leftrightarrow 2r = c(r^2+z^2+1) \Leftrightarrow$

$\Leftrightarrow r^2 + z^2 + 1 - \frac{2r}{c} = 0 \Leftrightarrow (r - \frac{1}{c})^2 + z^2 = \frac{1}{c^2} - 1 \to$ circles



$\xRightarrow{\text{rotating}}$ we get a torus.

14.9 $\tau(t) = \frac{\det(c', c'', c''')}{\|c'\times c''\|^2} = \dots = 0 \Rightarrow$ the curves are contained in a plane and have constant curvature along the integral curve $\Rightarrow$ they are circles.

14.10 $\alpha(t) = ((a + b\cos t)\cos t, (a + b\cos t)\sin t, b\sin t)$ $\quad \alpha' = (-b\sin t\cos t - (a+b\cos t)\sin t, -b\sin^2 t - (a + b\cos t)\sin t, b\cos t) \Rightarrow$