2. a. $\dfrac{z_{k+1}-z_k}{z_{k+1}} = \dfrac{\dfrac{3-\mathrm{i}\sqrt{3}}{4}z_k - z_k}{\dfrac{3-\mathrm{i}\sqrt{3}}{4}z_k} = \dfrac{\cancel{z_k}\left(\dfrac{3-\mathrm{i}\sqrt{3}}{4}-1\right)}{\dfrac{3-\mathrm{i}\sqrt{3}}{4}\cancel{z_k}} = \dfrac{\dfrac{3-\mathrm{i}\sqrt{3}}{4}-1}{\dfrac{3-\mathrm{i}\sqrt{3}}{4}} = \dfrac{-1-\mathrm{i}\sqrt{3}}{4} \times \dfrac{4}{3-\mathrm{i}\sqrt{3}} = \dfrac{-1-\mathrm{i}\sqrt{3}}{3-\mathrm{i}\sqrt{3}}$

On multiplie par le conjuguée du dénominateur :

$\dfrac{z_{k+1}-z_k}{z_{k+1}} = \dfrac{(-1-\mathrm{i}\sqrt{3})(3+\mathrm{i}\sqrt{3})}{(3-\mathrm{i}\sqrt{3})(3+\mathrm{i}\sqrt{3})} = \dfrac{-3-\mathrm{i}\sqrt{3}-3\mathrm{i}\sqrt{3}+3}{9+3} = \dfrac{-4\mathrm{i}\sqrt{3}\times\sqrt{3}}{12\times\sqrt{3}} = \dfrac{-12\mathrm{i}}{12\sqrt{3}} = -\dfrac{1}{\sqrt{3}}\mathrm{i}$

On a donc $\left|\dfrac{z_{k+1}-z_k}{z_{k+1}}\right| = \left|-\dfrac{1}{\sqrt{3}}\mathrm{i}\right| \iff \dfrac{|z_{k+1}-z_k|}{|z_{k+1}|} = \dfrac{1}{\sqrt{3}} \iff \dfrac{A_kA_{k+1}}{OA_{k+1}} = \dfrac{1}{\sqrt{3}} \iff A_kA_{k+1} = \dfrac{1}{\sqrt{3}}OA_{k+1}$.

(a) D'après la question précédente, pour tout entier naturel $k$,

$A_kA_{k+1} = \dfrac{1}{\sqrt{3}}OA_{k+1} = \dfrac{1}{\sqrt{3}}|z_{k+1}| = \dfrac{1}{\sqrt{3}} \times 8 \times \left(\dfrac{\sqrt{3}}{2}\right)^{k+1} = \dfrac{8}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{2}\right)^{k+1}$

Donc $\ell_{2022} = \dfrac{8}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{2}\right)^{1} + \dfrac{8}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{2}\right)^{2} + \cdots + \dfrac{8}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{2}\right)^{2022} = \dfrac{8}{\sqrt{3}} \times \dfrac{\sqrt{3}}{2}\left(1 + \left(\dfrac{\sqrt{3}}{2}\right)^{1} + \cdots + \left(\dfrac{\sqrt{3}}{2}\right)^{2021}\right)$

Puis $\ell_{2022} = 4 \times \dfrac{1-\left(\dfrac{\sqrt{3}}{2}\right)^{2022}}{1-\left(\dfrac{\sqrt{3}}{2}\right)} = 4 \times \dfrac{1-\left(\dfrac{\sqrt{3}}{2}\right)^{2022}}{\dfrac{2-\sqrt{3}}{2}} = \dfrac{8}{2-\sqrt{3}} \times \left(1-\left(\dfrac{\sqrt{3}}{2}\right)^{2022}\right)$