# Approval Ballot Triangles

Andrew Beveridge[1]

joint work with

Ian Calaway[2]

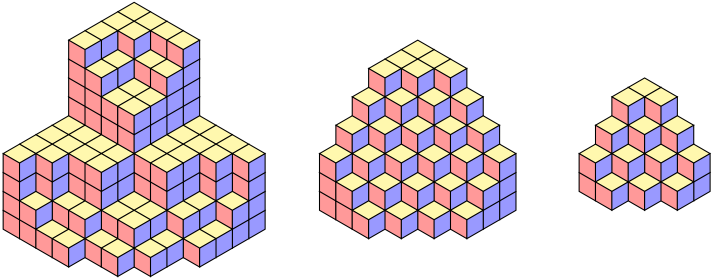
Macalester College[1] and Stanford University[2]

Joint Mathematics Meetings

April 2022

MACALESTER

# Cold Open



$$= (1,1,1,2,2,3,1,2,3,1,2,4,3,4,5)$$

# Introduction

# Bertrand's Ballot Problem (1887*)

In a two candidate election,

- Candidate $A$ receives $a$ votes.
- Candidate $B$ receives $b < a$ votes.

There are

$$\frac{a-b}{a+b}\binom{a+b}{a}$$

orderings of the ballots so that $A$ is always ahead of $B$ during the vote count.

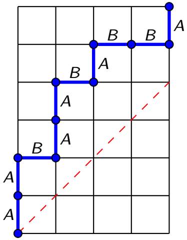
Joseph Bertrand

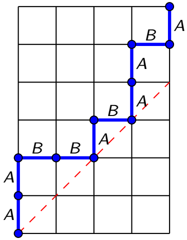

William Allen Whitworth

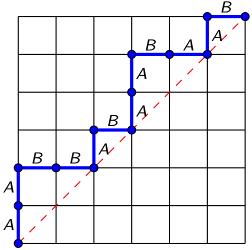
*Fun Fact: Bertrand actually rediscovered William Allen Whitworth's 1878 result.

# Ballot Problems as Lattice Paths

$a > b$, no ties allowed $\quad$ $a > b$, ties allowed $\quad$ $a = b$, ties allowed




$$\frac{a-b}{a+b}\binom{a+b}{a} \qquad \frac{a+1-b}{a+b}\binom{a+b}{a} \qquad C_a = \frac{1}{a+1}\binom{2a}{a}$$

# Ballot Sequences: $\ell$ never trails $\ell + 1$

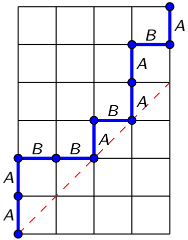
**Definition**

The sequence $b_1, \ldots, b_n$ where $1 \leq b_k \leq k$ is a **ballot sequence** when every partial sequence $b_1, \ldots, b_k$ contains at least as many $\ell$'s as $(\ell + 1)$'s for all $1 \leq \ell < k$,

| examples | non-examples | |
|---|---|---|
| $1, 1, 1$ | $1, 2, 2$ | too many 2's |
| $1, 1, 2$ | $1, 1, 3$ | no 2 before the 3 |
| $1, 2, 1$ | $2, 1, 3$ | must start with 1 |
| $1, 2, 3$ | $1, 3, 2$ | no 2 before the 3 |

In a ballot sequence, the final tally for $\ell$ is greater than or equal to the final tally for $\ell + 1$.

# Ballot Sequences generalize Ballot Problems




$A, \quad A, \quad B, \quad B, \quad A, \quad B, \quad A, \quad A, \quad B, \quad A$

$1, \quad 1, \quad 2, \quad 2, \quad 1, \quad 2, \quad 1, \quad 1, \quad 2, \quad 1$

Bertrand's Ballot Problem is a ballot sequence $b_1, b_2, \ldots, b_n$ where $b_k \in \{1, 2\}$ for $1 \leq k \leq n$.

# A Voting Procedure that Creates a Ballot Sequence

Here is a voting procedure that creates a sequence $b_1, b_2, \ldots, b_n$ such that $b_k \in [k]$.

- People enter a room, one at at item.
- Person $k$ casts a ballot for any of the $k$ people **currently in the room**.

The ballots $b_1, b_2, \ldots, b_n$ are a **ballot sequence** provided that "person $\ell$ never trails person $\ell + 1$" as the votes are cast.

# Ballot Sequences are Counted by the Involution Numbers

Ballot sequences of length $n$ are in bijection with standard Young tableaux (SYT) of size $n$.

- $b_k$ records the SYT **row** that contains element $k$

$$(1,1,2,1,3,2,1,2) \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 & 8 & \\ \cline{1-3} 5 & & & \\ \cline{1-1}\end{array} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 6 & 2 & 4 & 8 & 7 \end{pmatrix}$$

SYT of size $n$ are in bijection with involutions of $[n]$ via the Robinson-Schensted correspondence. So ballot sequences are counted by the **involution numbers** (OEIS A000085)

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, \ldots$$

with recurrence

$$t_0 = 1, \qquad t_1 = 1, \qquad \text{and} \qquad t_n = t_{n-1} + (n-1)t_{n-2} \quad \text{for} \quad n \geq 2.$$

# Lazy Ballot Sequences: Voters Can Abstain

**Definition**

The sequence $b_1, \ldots, b_n$ where $0 \leq b_k \leq k$ is a **lazy ballot sequence** when every partial sequence $b_1, \ldots, b_k$ contains at least as many $\ell$'s as $(\ell + 1)$'s for all $1 \leq \ell < k$.

- We allow $b_k = 0$ which corresponds to an **abstention**.
- We do not care about the (relative) number of abstentions.

| examples | | | | |
|---|---|---|---|---|
| $0,0,0$ | $0,0,1$ | $0,1,1$ | $0,1,2$ | $1,1,1$ |
| | $0,1,0$ | $1,0,1$ | $1,0,2$ | $1,1,2$ |
| | $1,0,0$ | $1,1,0$ | $1,2,0$ | $1,2,1$ |
| | | | $1,2,3$ | |

# Lazy Ballot Sequences Counted by Switchboard Numbers

The number of lazy ballot sequences is

$$s_n = \sum_{k=0}^{n} \binom{n}{k} t_k$$

where $t_k$ is the number of (regular) ballot sequences of length $k$.

These are the **switchboard numbers** (OEIS A005425).

$$1, 2, 5, 14, 43, 142, 499, 1850, 7193, 29186, 123109, \ldots$$

which obey the recurrence

$$s_0 = 1, \qquad s_1 = 2, \qquad \text{and} \qquad s_n = 2s_{n-1} + (n-1)s_{n-2} \quad \text{for} \quad n \geq 2.$$

# Approval Voting

## Approval Voting

- Each voter specifies their **subset** of approved candidates.
- Each approved candidate receives one vote in their favor.
- The winner is the candidate with the most approval votes.

Example with candidates $A$, $B$ and $C$.

| | |
|---|---|
| Voter 1: | $\{A, B\}$ |
| Voter 2: | $\{B, C\}$ |
| Voter 3: | $\{B\}$ |
| Voter 4: | $\{A, C\}$ |
| Voter 5: | $\emptyset$ |
| Voter 6: | $\{B, C\}$ |

| Final Tally |
|---|
| A: 2 |
| B: 4 |
| C: 3 |

$B$ is the winner.

# Approval Ballot Sequence

## Definition

The sequence $B_1, B_2, \ldots, B_n$ of (possibly empty) sets $B_k \subset [k]$ is an **approval ballot sequence** when for every $1 \leq \ell < k \leq n$, the partial set sequence $B_1, B_2, \ldots, B_k$ contains at least as many $\ell$'s as $(\ell + 1)$'s.

Example

| Ballots | | Partial Tallies |
|---|---|---|
| $B_1 =$ | $\{1\}$ | $(1, 0, 0)$ |
| $B_2 =$ | $\emptyset$ | $(1, 0, 0)$ |
| $B_3 =$ | $\{1, 2\}$ | $(2, 1, 0)$ |
| $B_4 =$ | $\{3\}$ | $(2, 1, 1)$ |
| $B_5 =$ | $\{1, 2\}$ | $(3, 2, 1)$ |
| $B_6 =$ | $\{2, 3\}$ | $(3, 3, 2)$ |

partial tallies are weakly decreasing

# Approval Ballot Sequences for $n = 2$

There are 7 approval ballot sequences of length 2

$$\emptyset, \quad \emptyset$$
$$\emptyset, \quad \{1\}$$
$$\emptyset, \quad \{1,2\}$$

$$\{1\}, \quad \emptyset$$
$$\{1\}, \quad \{1\}$$
$$\{1\}, \quad \{2\}$$
$$\{1\}, \quad \{1,2\}$$

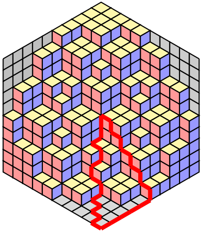
The number of approval ballot sequences of length $n$ is

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots$$

# Approval Ballot Sequences are TSSCPPs

**Proposition (B, Calaway, 2022+)**

Approval ballot sequences $B_1, \ldots, B_{n-1}$ are in bijection with totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box.



TSSCPP

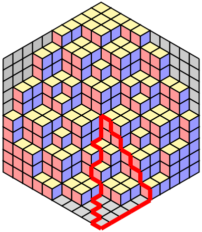
$$\{1\}, \quad \{1\}, \quad \{2,3\}, \quad \{2\}$$

approval ballot sequence

# Approval Ballot Sequences are TSSCPPs

**Proposition** ~~(B, Calaway, 2022+)~~ (Doran 1993, Striker 2018)

Approval ballot sequences $B_1, \ldots, B_{n-1}$ are in bijection with totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box.

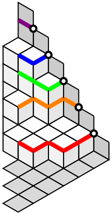


TSSCPP

$$\{1\}, \quad \{1\}, \quad \{2,3\}, \quad \{2\}$$

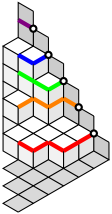
approval ballot sequence

# TSSCPP to Approval Ballot Sequence

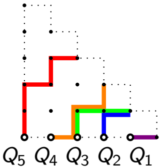


TSSCPP
fundamental domain

# TSSCPP to Approval Ballot Sequence

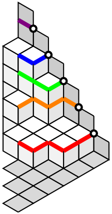


TSSCPP
fundamental domain

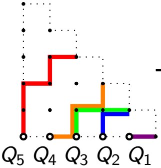



noncrossing
lattice paths

# TSSCPP to Approval Ballot Sequence



TSSCPP
fundamental domain

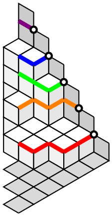



noncrossing
lattice paths

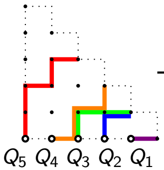
| | | | | | |
|---|---|---|---|---|---|
| 0 | $Q_1$ | | | | |
| 0 | 1 | $Q_2$ | | | |
| 0 | 0 | 1 | $Q_3$ | | |
| 1 | 0 | 1 | 0 | $Q_4$ | |
| 0 | 1 | 0 | 1 | 1 | $Q_5$ |

approval ballot
triangle

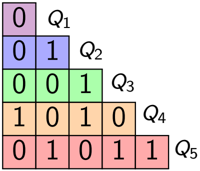
# TSSCPP to Approval Ballot Sequence



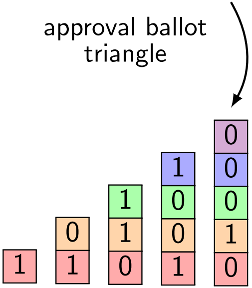
TSSCPP
fundamental domain



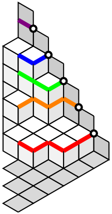

noncrossing
lattice paths



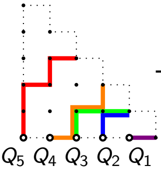

approval ballot
triangle



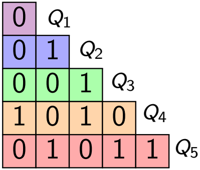
# TSSCPP to Approval Ballot Sequence



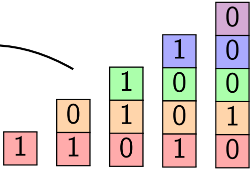
TSSCPP
fundamental domain




noncrossing
lattice paths




approval ballot
triangle



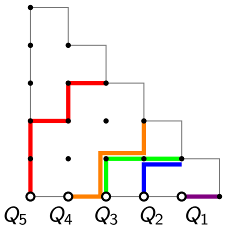

$$\{1\}, \quad \{1\}, \quad \{2,3\}, \quad \{1,4\}, \quad \{2\}$$

approval ballot sequence

# Non-Crossing Lattice Paths

## Definiton

A *nest of noncrossing lattice paths* (NCLP) of order $n$ is a sequence of noncrossing paths $Q_1, \ldots, Q_{n-1}$ where path $Q_i$ starts at $(n-i, 1)$ and ends at the diagonal $D = \{(n+1-j, j) : 1 \leq j \leq n\}$, taking only east $(1,0)$ steps and north $(0,1)$ steps.




NCLP of order 6

# Approval Ballot Triangles

## Definiton

An *approval ballot triangle* (ABT) of order $n$ is a binary triangular array $A(i,j)$ for $1 \leq j \leq i \leq n-1$ satisfying the row compatibility condition

$$\sum_{k=j}^{i} A(i,k) \leq \sum_{k=j}^{i+1} A(i+1,k) \quad \text{for} \quad 1 \leq j \leq i \leq n-2.$$

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| 0 | 1 | | | |
| 0 | 0 | 1 | | |
| 1 | 0 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 1 |

ABT of order 6