de Finetti Lattices and Magog Triangles

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joint work with
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Canadian Discrete and Algorithmic Mathematics Conference
May 2021

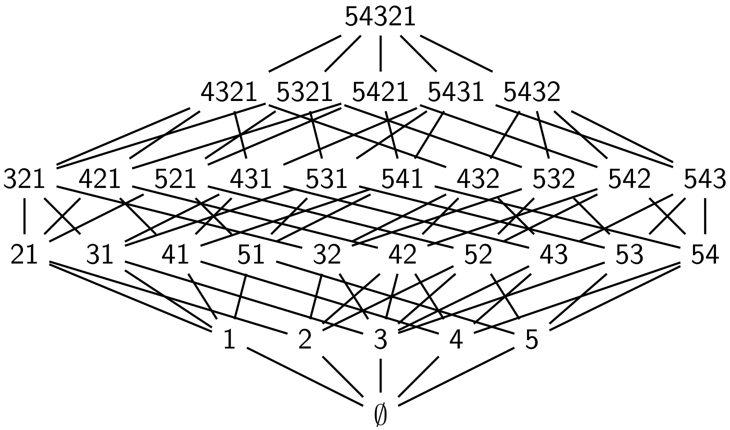
MACALESTER

Introduction

Poset Definitions

- A **partially ordered set** (or *poset* for short) consists of a set P and a binary relation $\preceq$ that is reflexive ($x \preceq x$), antisymmetric (if $x \preceq y$ and $y \preceq x$ then $x = y$) and transitive (if $x \preceq y$ and $y \preceq z$ then $x \preceq z$).
- An **order ideal** is a subset $I \subset P$ that is downward-closed: if $x \in I$ and $y \preceq x$ then $y \in I$.
- A **lattice** is a poset such that every pair of elements have a least upper bound and a greatest lower bound.
- A **refinement** R of poset P adds more relations.
- A **linear extension** L of poset P is a refinement that is a **total order**: every pair of elements are comparable.

Boolean Lattice B_n and Order Ideal $B_{n,2}$



B_5 = subsets of $\{1, 2, 3, 4, 5\}$ ordered by inclusion

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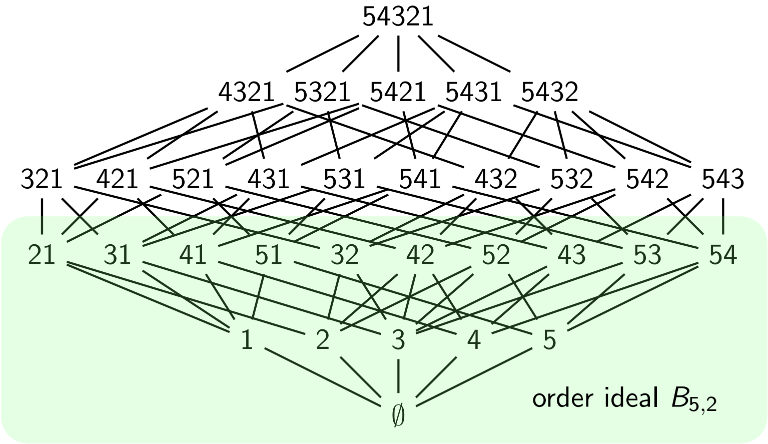

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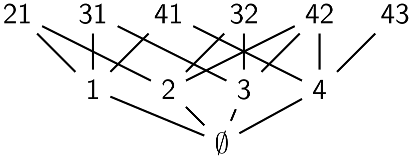
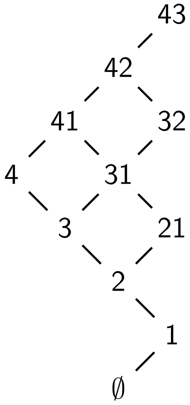
de Finetti Lattice $F_{n,2}$



order ideal $B_{4,2}$
of Boolean Lattice B_4



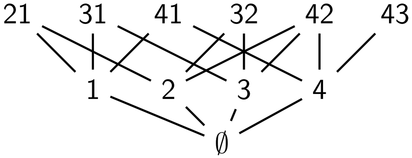
de Finetti Lattice $F_{4,2}$

de Finetti Lattice $F_{n,2}$

$$\{i\} \prec \{i,j\}$$

$$\{i\} \prec \{j\} \longleftrightarrow i < j$$

$$\{i,j\} \preceq \{k,\ell\} \longleftrightarrow i \leq k \text{ and } j \leq \ell$$



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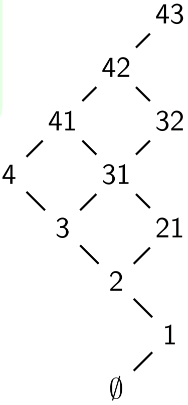

de Finetti Lattice $F_{4,2}$

Motivation for de Finetti Lattice $F_{n,2}$

Definition

A *de Finetti refinement* $(F, \prec_F)$ of the Boolean lattice $(B_n, \prec)$ is a poset refinement that adheres to

(F1) $\emptyset \prec \{1\} \prec \{2\} \prec \ldots \prec \{n\}$, and

(F2) $X \prec Y$ if and only if $X \cup Z \prec Y \cup Z$ for all $Z \subset [n]$ such that $(X \cup Y) \cap Z = \emptyset$.

for all sets $X, Y \subset [n]$ that are comparable in F.

A *de Finetti total order* is a linear extension of B_n that adheres to (F1) and (F2).

Condition (F2) is de Finetti's axiom (1931)

de Finetti Total Orders

de Finetti total orders of B_n appear in various settings:

- Comparable Probability Orders (probability)
- Boolean Term Orders (computational algebra)
- Completely Separable Preferences (social choice theory)

Open Question: General formula for the number of de Finetti total orders (OEIS A005806)

$$1 \quad 1 \quad 2 \quad 14 \quad 546 \quad 169,444 \quad 560,043,206$$

(The first 14 terms are known.)

Our Results for Refinements of $F_{n,2}$

Theorem [B, Calaway, Heysse (2021)]

The collection $\mathcal{F}_{n,2}$ of linear extensions of $F_{n,2}$ are in bijection with Strict Sense Ballots for n candidates.

strict sense ballots $\longleftrightarrow$ shifted standard Young tableaux of shape $(n, n-1, \ldots .1)$

Theorem [B, Calaway, Heysse (2021)]

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with Magog Triangles of size $n-1$.

magog triangles $\longleftrightarrow$ totally symmetric self-complementary plane partitions

Linear Extensions of $F_{n,2}$

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Theorem [B, Calaway, Heysse (2021)]

The collection $\mathcal{F}_{n,2}$ of linear extensions of $F_{n,2}$ are in bijection with Strict Sense Ballots for n candidates.

The strict-sense ballot number sequence (OEIS A003121) begins with

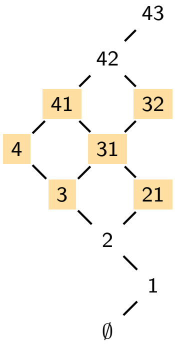
$$1 \quad 1 \quad 2 \quad 12 \quad 286 \quad 33,592 \quad 23,178,480 \quad \ldots$$

and the general formula for the nth strict-sense ballot number is

$$\binom{n+1}{2}! \, \frac{\prod_{k=1}^{n-1} k!}{\prod_{k=1}^{n} (2k-1)!}.$$

Proof Part 1: Linear Extension to Shifted SYT

Linear Extension of $F_{n,2}$ $\iff$ Shifted Standard Young Tableau of shape $(n, n-1, \dots, 1)$.



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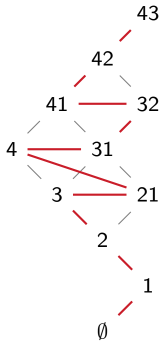

pick a valid total order

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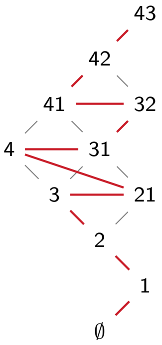

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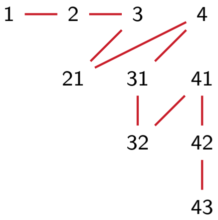

rotate and omit $\emptyset$

1	2	3	5
	4	6	8
		7	9
			10

record order visited

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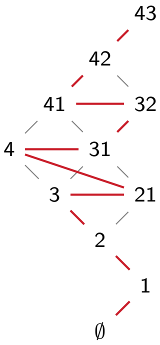

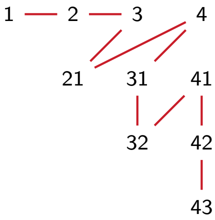

1	2	3	5
	4	6	8
		7	9
			10

pick a valid total order

rotate and omit $\emptyset$

record order visited

de Finetti refinement $\iff$ rows and columns increase

Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with n candidates:

- Candidate k receives exactly $n+1-k$ votes for $1 \leq k \leq n$
- During the vote count, candidate k always strictly ahead of candidate ℓ for all $1 \leq k < \ell \leq n$

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	4	6	8
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Shifted SYT $\iff$ Strict Sense Ballot

1	2	3	5
	4	6	8
		7	9
			10

$\iff$ 1, 1, 1, 2, 1, 2, 3, 2, 3, 4

Row i gives the indices of the votes for candidate i. $\square$

Singleton Refinements of $F_{n,2}$

Singleton Refinements of $F_{4,2}$

Theorem [B, Calaway, Heysse (2021)]

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with Magog Triangles of size $n-1$.

The magog triangle sequence (OEIS A005130) begins with

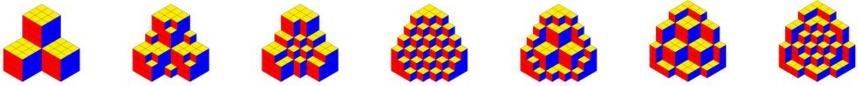
$$1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7,436 \quad 218,348 \quad \ldots$$

and the general formula is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

TSSCPP are in bijection with Magog Triangles

Totally Symmetric Self-Complementary Plane Partition (TSSCPP): A stack of cubes in a $2n \times 2n \times 2n$ box is a $2n \times 2n$ with the maximum possible symmetry.



$$\begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{1} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|cc}\hline 1 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

Magog Triangle: triangular array $M(i,j)$ where $1 \leq j \leq i \leq n$ and $1 \leq M(i,j) \leq j$ with weakly increasing columns and weakly increasing rows.

Image adapted from Fischer and Konvalinka, PNAS (2020)

TSSCPP and ASM and DPP and AST are equinumerous

Four distinct families enumerated by $\displaystyle\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$

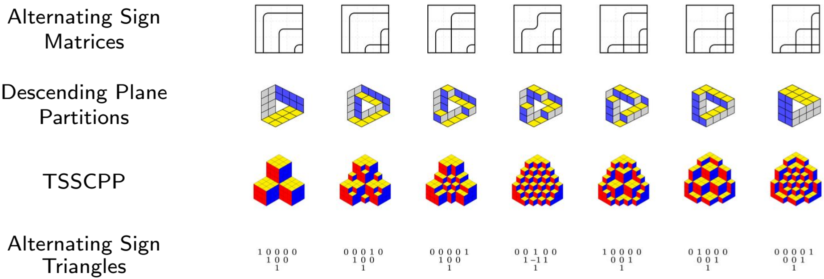
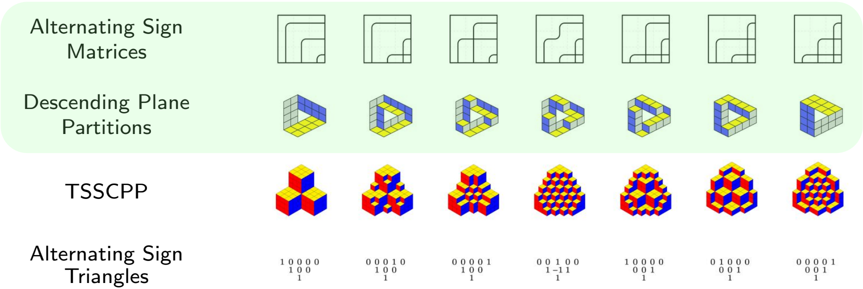


Image adapted from Fischer and Konvalinka, PNAS (2020)

TSSCPP and ASM and DPP and AST are equinumerous

~~Four~~ Three distinct families enumerated by $\displaystyle\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$



Breakthrough: Fischer and Konvalinka (2019) found a bijection between ASMs and DPP!

Image adapted from Fischer and Konvalinka, PNAS (2020)

Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- n rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

1					
1	1				
1	1	1			
1	1	2	2		
1	2	2	2	3	
1	2	2	4	5	5

Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- n rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

omagog triangle $M^{\circ}(i,j)$

- $n-1$ rows
- $0 \leq M^{\circ}(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

1					
1	1				
1	1	1			
1	1	2	2		
1	2	2	2	3	
1	2	2	4	5	5

subtract all-ones triangle

0				
0	0			
0	1	1		
1	1	1	2	
1	1	3	4	4

Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- n rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

omagog triangle $M^{\circ}(i,j)$

- $n-1$ rows
- $0 \leq M^{\circ}(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

NEW!

kagog triangle $K(i,j)$

- $n-1$ rows
- $0 \leq K(i,j) \leq j$
- columns weakly decreasing
- rows can start with multiple zeros, then positive entries strictly increasing

1					
1	1				
1	1	1			
1	1	2	2		
1	2	2	2	3	
1	2	2	4	5	5

subtract all-ones triangle

0				
0	0			
0	1	1		
1	1	1	2	
1	1	3	4	4

two-color cube pyramid

1				
0	1			
0	1	2		
0	0	2	4	
0	0	0	1	3

Magog Triangles are in bijection with Kagog Triangles

Theorem [B, Calaway, Heysse (2021)]

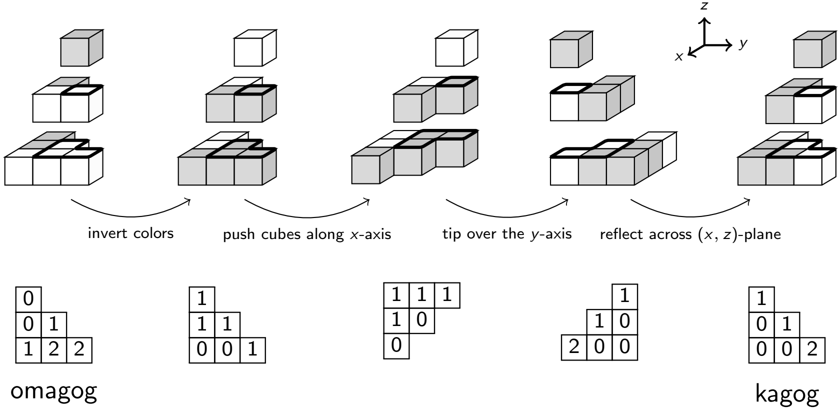
Kagog triangles are in bijection with omagog triangles (and therefore with magog triangles and TSSCPPs).

Proof Sketch:

- Represent an omagog triangle as a **two-color cube pyramid**
  - White cubes are present
  - Grey cubes are missing
- Invert the colors
- Perform a series of elementary transformations
- The result is a kagog pyramid

□

Example of omagog to kagog mapping

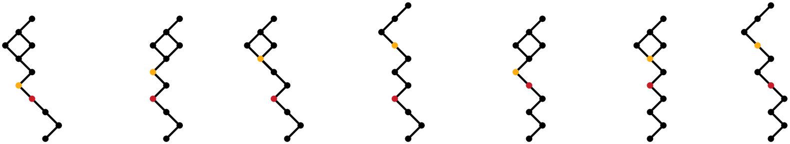


The kagog triangle is made from the "missing cubes" of the omagog triangle.

Back to Singleton Refinements of $F_{4,2}$

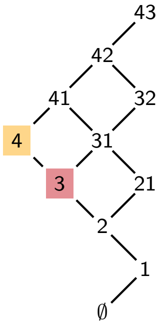
Theorem [B, Calaway, Heysse (2021)]

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with ~~Magog Triangles of size $n-1$~~ Kagog Triangles of size $n-2$.

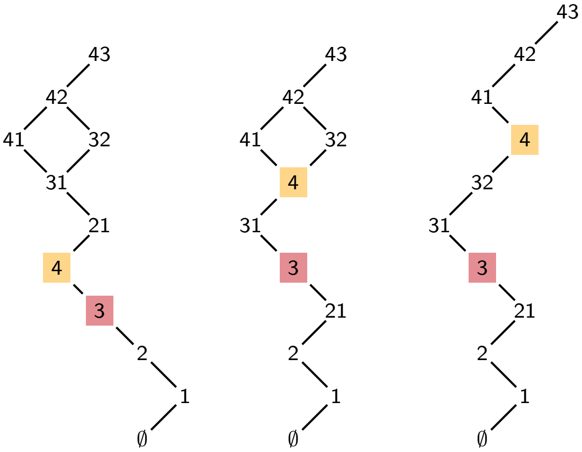


1	1	1	1	0	0	0
1 2	0 2	0 1	0 0	0 2	0 1	0 0

Singleton Refinements of $F_{4,2}$

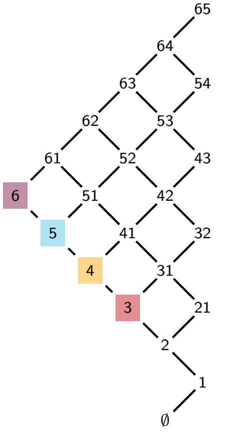


de Finetti Lattice $F_{4,2}$



three different refinements (out of seven)

Proof By Example: Refining $F_{6,2}$ and Creating Kagog



de Finetti Lattice $F_{6,2}$

3

21	

4

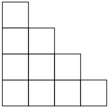
31	32
21	

5

41	42	43
31	32	
21		

6

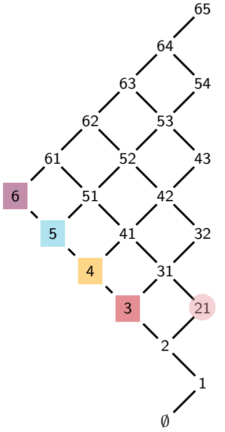
51	52	53	54
41	42	43	
31	32		
21			



kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

Proof By Example: Refining $F_{6,2}$ and Creating Kagog



de Finetti Lattice $F_{6,2}$

3

21

$21 \prec 3$

4

31	32
21	

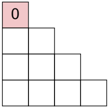
5

41	42	43
31	32	
21		

6

51	52	53	54
41	42	43	
31	32		
21			

count white boxes in each row



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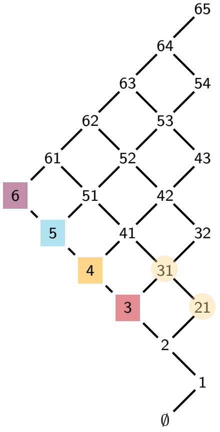


de Finetti Lattice $F_{6,2}$

3

21

$21 \prec 3$

4

31	32
21	

$31 \prec 4 \prec 32$

5

41	42	43
31	32	
21		

6

51	52	53	54
41	42	43	
31	32		
21			

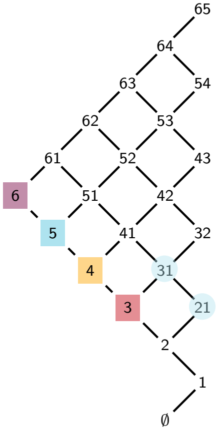
count white boxes in each row

0			
0	1		

kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

Proof By Example: Refining $F_{6,2}$ and Creating Kagog



de Finetti Lattice $F_{6,2}$

3

21

$21 \prec 3$

4

31	32
21	

$31 \prec 4 \prec 32$

5

41	42	43
31	32	
21		

$31 \prec 5 \prec 32$
$5 \prec 41$

6

51	52	53	54
41	42	43	
31	32		
21			

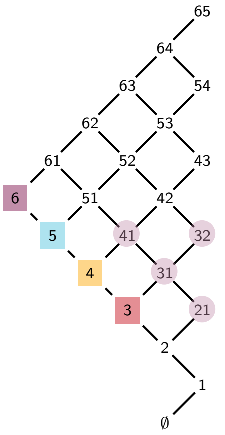
count white boxes in each row

0			
0	1		
0	1	3	

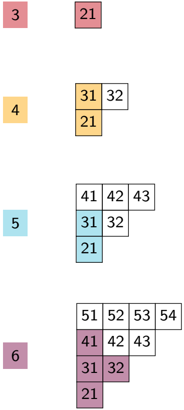
kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

Proof By Example: Refining $F_{6,2}$ and Creating Kagog



de Finetti Lattice $F_{6,2}$



$21 \prec 3$

$31 \prec 4 \prec 32$

$31 \prec 5 \prec 32$
$5 \prec 41$

$41 \prec 6 \prec 42$
$6 \prec 51$

count white boxes in each row

0			
0	1		
0	1	3	
0	0	2	4

kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

The de Finetti constraints correspond to the kagog constraints. □

Summary and Future Work

Refinements and linear extensions of de Finetti Lattice $F_{n,2}$

Our Bijective Results

Linear extensions of $F_{n,2}$	$\longleftrightarrow$	Strict Sense Ballots
Singleton refinements of $F_{n,2}$	$\longleftrightarrow$	Kagog Triangles
Kagog Triangles	$\longleftrightarrow$	Magog Triangles

Future Directions: find enumerative formulas for

- de Finetti refinements of $F_{n,k}$ for $3 \leq k \leq n$
- de Finetti total orders of $F_{n,n}$ = de Finetti total orders of B_n (aka comparative probability orders, OEIS A005806)

Alternating Sign Matrices and Gog Triangles

Alternating Sign Matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Column Sum Matrices

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} \begin{bmatrix}1&0&0\\1&0&1\\1&1&1\end{bmatrix} \begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix} \begin{bmatrix}0&1&0\\1&0&1\\1&1&1\end{bmatrix} \begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix} \begin{bmatrix}0&0&1\\1&0&1\\1&1&1\end{bmatrix} \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$$

Gog Triangles (aka Monotone Triangles)

$$\begin{array}{|c|cc}\hline 1 \\ \hline 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline\end{array} \quad \begin{array}{ccc} 1 \\ 1 & 3 \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 \\ 1 & 2 \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 \\ 1 & 3 \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 \\ 2 & 3 \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 3 \\ 1 & 3 \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 3 \\ 2 & 3 \\ 1 & 2 & 3 \end{array}$$

Ogog Triangles (subtract minimal Gog Triangle)

$$\begin{array}{cc} 0 \\ 0 & 0 \end{array} \quad \begin{array}{cc} 0 \\ 0 & 1 \end{array} \quad \begin{array}{cc} 1 \\ 0 & 0 \end{array} \quad \begin{array}{cc} 1 \\ 0 & 1 \end{array} \quad \begin{array}{cc} 1 \\ 1 & 1 \end{array} \quad \begin{array}{cc} 2 \\ 0 & 1 \end{array} \quad \begin{array}{cc} 2 \\ 1 & 1 \end{array}$$

Two-Color Cube Pyramid Involution for Gog Triangles

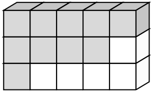
Theorem [B, Calaway, Heysse (2021)]

There is a gog triangle involution ϕ that corresponds to both:

- an affine transformation of two-color cube pyramids, and
- reversing the order of the rows of the corresponding alternating sign matrix.

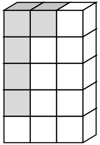
$$\begin{array}{ccccccc} 5 \\ 3 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \\ \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{2} \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}$$

G°



$\triangle G_5^\circ$

ϕ



$\triangle H_3^\circ$

$$\begin{array}{ccccccc} 2 \\ 2 & 4 \\ \boxed{1} & \boxed{4} & \boxed{5} \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

H°