Math Camp: Limits, Continuity, & Derivatives

Calc I

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Why does math matter?

- You can read articles and understand them better.
- Certain topics—like game theory and formal theory—rely heavily on math.
- Your future research and dissertation could (and most likely will) involve methods you learn on your own.

Why does math matter?

- Political questions, on the surface, do not require math:
  - Why do people vote for certain candidates?
  - Why do countries go to war?

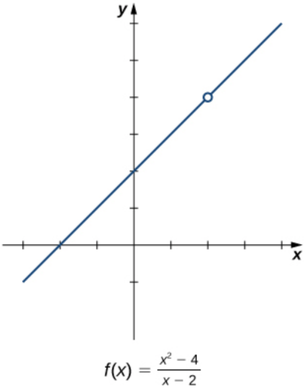
Disclaimer

- I am not doing proofs.
- Most undergrads spend a semester on these concepts.
- Don't get discouraged if it's your first time or you're a little rusty.

Intuition about Limits

- Suppose we have a function $f(x) = \frac{(x^2-4)}{(x-2)}$
- How does the function behave around x=2?

Figure 1



Intuition about Limits

- As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4.
- In more mathy terms, we say that the limit of $f(x)$ as x approaches 2 is 4.
- Notationally, we express the limit in the following form: $\lim_{x \to 2} f(x) = 4$

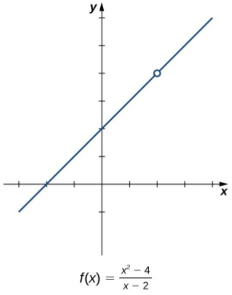
Formal Definitions of Limits

- Definition of Limit: L is the limit of $f(x)$ as x approaches c when the value of $f(x)$ nears L as x nears c
  - $\lim_{x \to c} f(x) = L$
- Right-hand Limits: The value of $f(x)$ when approaching c from the right-hand side of the graph.
  - $\lim_{x \to c^+} f(x) = L$
- Left-hand Limits: The value of $f(x)$ when approaching c from the left-hand side of the graph.
  - $\lim_{x \to c^-} f(x) = L$
- The value of $f(x)$ as the function approaches infinity.
  - $\lim_{x \to \infty} f(x) = L$

Example of Right-handed & Left-Handed Limits

- What is the $\lim\limits_{x \to 2^+} \frac{(x^2-4)}{(x-2)}$?
- What is the $\lim\limits_{x \to 2^-} \frac{(x^2-4)}{(x-2)}$?

Figure 1



(a)

Properties of Limits

- Let $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$
  - If $f = g$, then $L_1 = L_2$
  - $\lim_{x\to c} k = k$
  - $\lim_{x\to c} x = c$
  - $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L_1 + L_2$
  - $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) = kA$
  - $\lim_{x\to c} f(x)g(x) = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)] = AB$
  - $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{A}{B}$ for all $B \neq 0$
  - If $\lim_{x\to L_2} f(x) = L_3$ and $\lim_{x\to c} g(x) = L_2$, then $\lim_{x\to c} f(g(x)) = L_3$

Definition of Continuity at a Point

A function $f(x)$ is **continuous at a point** a if and only if all three conditions are satisfied:

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

A function $f(x)$ is **discontinuous at a point** a if it fails to be continuous at a.

Formal Definitions of Discontinuity at a Point

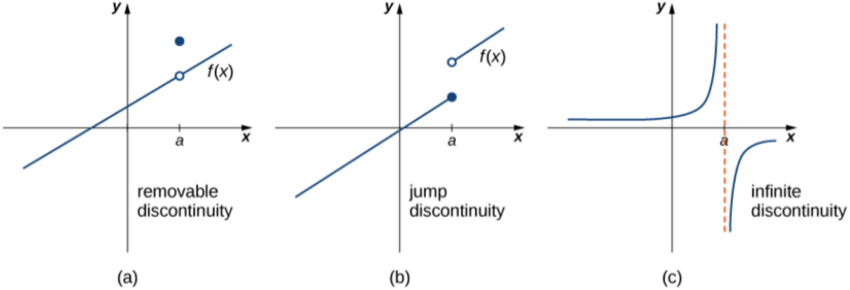
Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Graphic Representation of Discontinuity at a Point

Figure 1



Continuity over an Interval

- A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.
- A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$.

Continuity over an Interval

- A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.
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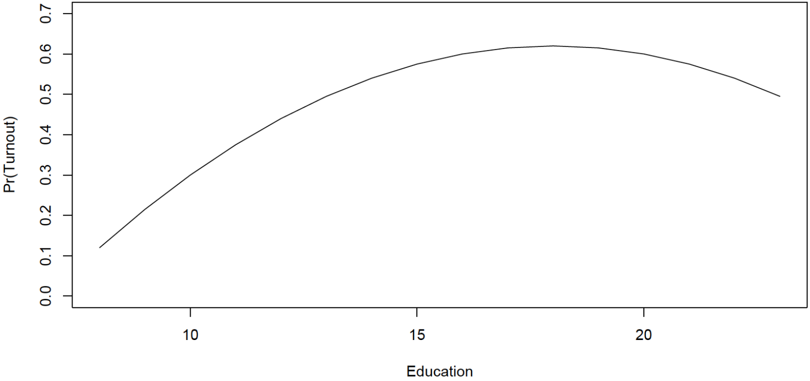
Intuition: Generally, if we can use a pencil to trace a function between any two points in the interval, then the function is continuous.

Derivatives

- The type of limit we compute to find the slope of the line tangent to a function occurs in many settings.
- This limit occurs so frequently that we call it **the derivative**.
- The process of finding a derivative is called **differentiation**.

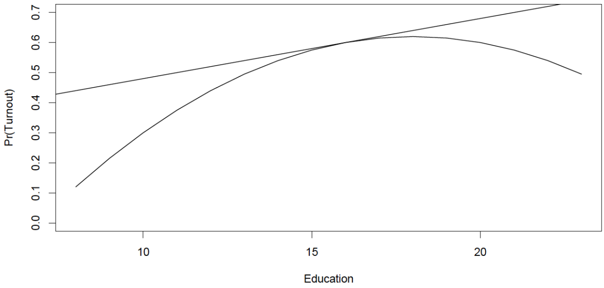
Derivative Example

- Suppose the probability of turning out to vote has a quadratic relationship with education.
- Let's assume the following relationship: $Pr(Turnout) = -0.05Ed^2 + 0.18Ed - 1$ (where Ed=years of education).



Derivative Example

- Suppose I want to know how the likelihood of turnout is changing at 16 years of college education. I theorize those with bachelor's degrees should vote at a higher rate of change.
- I need to know the slope of the function at this point. How do I do this?
- I can determine the slope of the tangent line touching the function at that point (aka the derivative).



Derivative Notation

Notationally, derivatives are written in many ways.

- $f(x)' = \frac{dy}{dx} = \frac{d}{dx}y = \mathrm{D}f(x) = \mathrm{D}f$
- You can differentiate a derivative. (You can take a derivative of a derivative.)
- A second derivative is the derivative of the first derivative. A third derivative is the derivative of the second derivative. Etc.

Why take Derivatives?

- Can tell us whether the function is increasing or decreasing at a particular point.
- Can help us find critical points: minima, maxima, and inflection points (changes in concavity) of the function.

The First & Second Derivative

- The first derivative tells us whether the formula is increasing or decreasing.
- The second derivative tells us whether the derivative is increasing or decreasing.
- This also tells us the concavity of the function.
- This helps us determine the maxima and minima of the function.

Critical Points

- To find critical points, the slopes of tangents at maxima or minima are 0.
- The derivative tells us the slope of a tangent line.

Inflection Points

- When a function changes concavity, this is called an inflection point.
- Inflection points occur where the $f''(x) = 0$
- When $f''(x) = 0$, there is no information on the concavity.

Differentiation Rules

- $f(x) = e^x$; $f'(x) = e^x$
- $f(x) = c^x$; $f'(x) = ln(c)c^x$
- $f(x) = ln(x)$; $f'(x) = \frac{1}{x}$
- $f(x) = log_n(x)$; $f'(x) = \frac{1}{xln(n)}$
- $f(x) = sin(x)$; $f'(x) = cos(x)$
- $f(x) = cos(x)$; $f'(x) = -sin(x)$
- $f(x) = tan(x)$; $f'(x) = sec^2(x)$

Derivative Basics

- Not all functions have derivatives for all values.
  - The value of a derivative does not exist wherever the function is non-continuous.
- The derivative of a scalar is 0.
- For derivatives of polynomials with degree 1, the derivative is a scalar.
- For polynomials of degree 2 or higher, the derivative will depend on the value of x.

Helpful Derivatives

- Power Rule
  - If $f(x) = \sum_{k=0}^{n} a_k x^k$, then
  - $f'(x) = \sum_{k=0}^{n} k a_k x^{k-1}$
- Constant Rule: $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$
- Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
- Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} g(x)$
- Quotient Rule:
  $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{\text{Low}d\text{High} - \text{High}d\text{Low}}{\text{Denominator}^2}$
- Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

References & Resources

Alicia Uribe-McGuire. Math Camp Day 1. August 22, 2017.
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Strang, G. (2016). *Calculus*. Houston, TX: OpenStax College.