Mathcamp Day 2: Integrals

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Outline

Section 1: Introduction

Introduction: Some ideas

As a political science graduate student...

- Why do we care about math?

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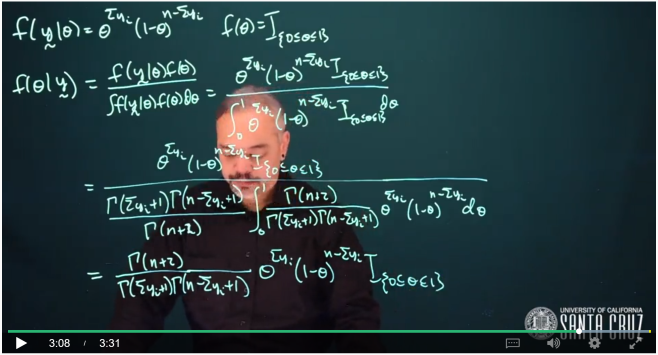


Section 1: Introduction

- Why do we care about learning integral?
- What is the learning objective in today's math camp?
- How do you participate in the morning sessions?

Introduction: Link to Day 1

A brief overview of functions and derivatives

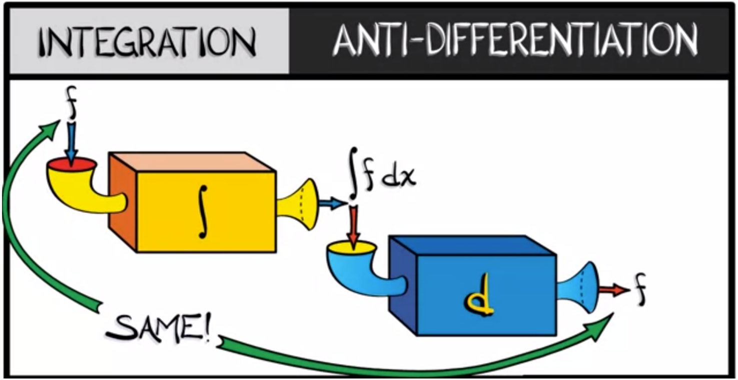


Figure 1: Integration and differentiation

For further references see Calculus: Single Variable Part 3 - Integration

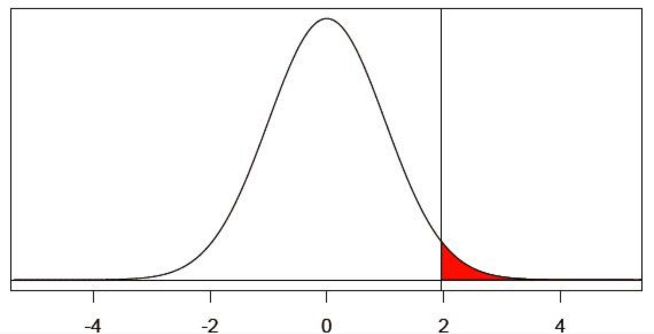
Section 2: Concept of Integral

Area under curve:

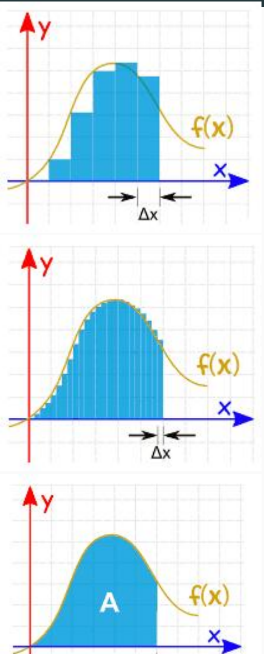
Let's take the standard normal curve as an example.

What if we want to know the area below the curve when x $\geq$ 1.96?

We'll come back to this example later.



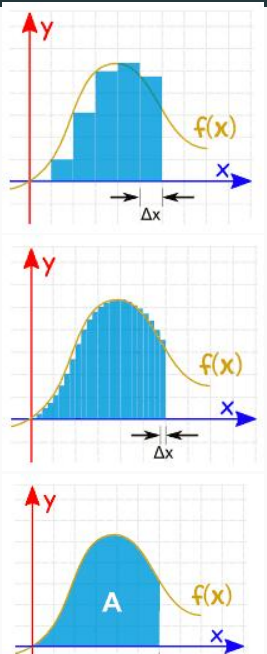
Concept of Integrals



i. We could calculate the function at a few points and add up slices of width Δx.

[a]Reference: Math is fun: Introduction to Integration

Concept of Integrals



i. We could calculate the function at a few points and add up slices of width Δx.
ii. We can make Δx a lot smaller and add up many small slices.

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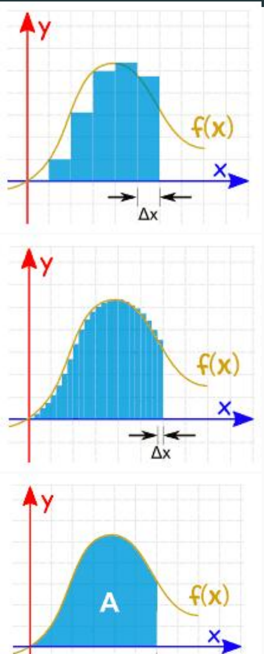
Concept of Integrals



i. We could calculate the function at a few points and add up slices of width Δx.
ii. We can make Δx a lot smaller and add up many small slices.
iii. As the slices approach zero in width, the answer approaches the true answer. [a]

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Concept of Integral

The way to find the area **under a curve** is to take the integral.

What is an integral?

- The area under the curve f(x) for some range of x = (a, b) is defined as the definite integral for f from a to b.

$$\int_a^b f(x)dx$$

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. [a, b].

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. [a, b].
- Useful illustration in Coursera (7 minutes)

Concept of Integral

The integral is the inverse function of the derivative:

$$F(x) = \int_a^b f(x)dx$$

$$f(x) = \frac{dF(x)}{dx}$$

Then, for any x,

$$F(x) = \int_a^b f(x)dx = \int_a^b \frac{dF(x)}{dx}dx = F(x)$$

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^b f(x)dx = f(x)$$

In other words, F(x) is the anti-derivative of f(x).

Concept of Integral

Too abstract?

Some examples...

What's an integral of 2x?

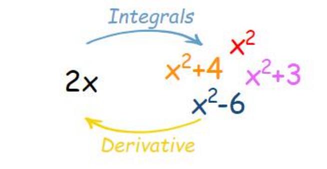
- We know that the derivative of x^2 is ......
- An integral of 2x is ......
- We will write $\int 2x dx = x^2 + C$

Power rule:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Why plus C?

It is the "Constant of Integration". It is there because of all the functions whose derivative is 2x:



- The derivative of $x^2 + 4$ is 2x, and the derivative of $x^2 + 99$ is also 2x, and so on! Because the derivative of a constant is **zero**.
- So when we reverse the operation (to find the integral) we only know 2x, but there could have been a constant of any value.

Concept of (Indefinite) Integrals

The anti-derivative, also called the indefinite integral, is not unique.

- There are multiple anti-derivatives for each function (for each C).
- This shifts the curve up or down the y-axis
- With information, such as a point the function passes through, you can solve for c.

Concept of (Indefinite) Integrals: Some Classic Anti-derivative Formulas

antiderivative = indefinite integral

$$\begin{aligned}
\int 1dx &= x + c \\
\int kdx &= kx + c \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + c \\
\int \frac{1}{x} dx &= \log x + c \\
\int e^x dx &= e^x + c \\
\int a^x dx &= \frac{a^x}{\log a} + c
\end{aligned}$$

Concept of (Indefinite) Integrals

Pause......Questions?

Recall Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Practices:

i $\int 8x dx = ...$

ii $\int 6x^2 dx = ...$

iii $\int x^3 dx = ...$

iv $\int e^x dx = ...$

Section 3: Fundamental Theorem of Calculus and Definite Integral

Fundamental Theorem of Calculus and Definite Integral

A **deep** connection between derivatives and integrals makes integration much easier.

The definite integral, again, is the defined by a specific range:

Fundamental Theorem of Calculus

Suppose F is differentiable on $[a, b]$ and that its derivative, f, is integrable. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

- This formula helps you to find area under the curve f (x) for some range of x from a to b.
- Simply find an antiderivative F, evaluate at b, evaluate at a, and take $F(b) - F(a)$.

Fundamental Theorem of Calculus and Definite Integral: Uniform Distribution

Example I:

Uniform Distribution Suppose $f : \Re \rightarrow \Re$, with

$$\begin{aligned} f(x) &= 1 \text{ if } x \in [0, 1] \\ f(x) &= 0 \text{ otherwise} \end{aligned}$$

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We will call $f(x) = 1$ the uniform distribution.

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Example II:

Area Under a Line

Suppose $f : \Re \rightarrow \Re$, with

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$$\int_2^t f(x)dx = \int_2^t xdx$$

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$$\begin{aligned} \int_2^t f(x)dx &= \int_2^t xdx \\ &= \frac{x^2}{2}|_2^t \end{aligned}$$

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$$\begin{aligned} \int_2^t f(x)dx &= \int_2^t xdx \\ &= \frac{x^2}{2}|_2^t \\ &= \frac{t^2}{2} - \frac{2^2}{2} \end{aligned}$$

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Fundamental Theorem of Calculus and Definite Integral: Integration Facts

If $f_1, f_2 : [a, b] \to \Re$ and f_1, f_2 are integrable on [a,b], then

- Consider the interval $[a, b]$ and $c \in [a, b]$. Then,

$$
\begin{aligned}
\int_c^c f_1^{'}(x)dx &= f_1(c) - f_1(c) = 0 \\
\int_a^b f_1^{'}(x)dx &= \int_a^c f_1^{'}(x)dx + \int_c^b f_1^{'}(x)dx \\
&= (f_1(c) - f_1(a) + (f_1(b) - f_1(c)) \\
&= f_1(b) - f_1(a)
\end{aligned}
$$

Section 4: Integrals and Distributions

Integrals and Distributions

- What is distribution, and why should we care?

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- What is distribution, and why should we care?
- Examples of discrete variables and continuous variables
- What is probability mass function (PMF)? What is probability density function (PDF)? What is the difference between the two?
  1. Intuitively, PDF is the probability that a continuous X drawn in a region that is indefinitely small at the limit of a shrinking process.
  2. Formally, $Pr(X \in [a, b]) = \int_a^b f(x)dx$. The probability of being in some interval [a, b] is the integral of the PDF over that region.

Integrals and Distributions

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    1. Intuitively, if we pick a number, and we want to know the probability that a random draw from a population that produces a value less than that specified number, we sum the individual probabilities of each value below the specified value.

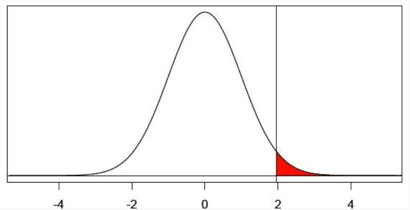
Integrals and Distributions

- What is cumulative distribution function (CDF) for both discrete and continuous variables?
    1. Intuitively, if we pick a number, and we want to know the probability that a random draw from a population that produces a value less than that specified number, we sum the individual probabilities of each value below the specified value.
    2. For discrete variable is $\mathrm{Pr(Y \leq y) = \sum_{i \leq y} p(i)}$.
    3. For continuous variable, we replace the sum with an integral to get the equation for the CDF: $Pr(X \leq \mathrm{x}) = \mathrm{F(x)} = \int_{-\infty}^{x} f(t)dt$.
    4. $0 \leq \mathrm{CDF} \leq 1$.

Integrals and Distributions

Back to the normal distribution that started this off:

What if we want to know the area below the curve when x $\geq$ 1.96?



To find the area, we need to integrate the function:

$$f(x) = \frac{1}{2\pi} \exp^{-\frac{x^2}{2}}$$

Integrals and Distributions

To find the area, we need to integrate the PDF function of the normal distribution:

$$f(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

Pause...

What's the next step to calculate the area under the curve when x ≥ 1.96?

Integrals and Distributions

To find the area, we need to integrate the PDF function of the normal distribution:

$$f(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}}$$

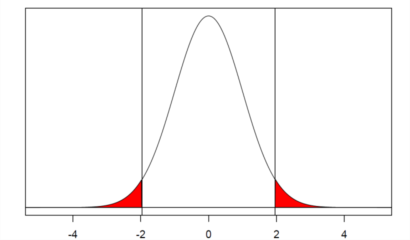
Pause...

What's the next step to calculate the area under the curve when x $\geq$ 1.96?

$$\int_{1.96}^{\infty} f(x) = \int_{1.96}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2}} = 0.025$$

Integrals and Distributions

The area under the curve when x $\geq$ 1.96 is 0.025. The function is symmetric, so the area when x $\leq$ 1.96 is also 0.025. Together it is 0.05.



P-values are integrals under the normal curve. This is where the famous P-value coming from.

One of the most important **assumption** is the continuous variable X follows a normal distribution.

Section 5: References

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Alicia Uribe-McGuire. Math Camp Day 1. University of Illinois at Urbana-Champaign. August 22, 2017.

Justin Grimmer. Math Camp Day 4. University of Chicago. August 31, 2017.

Moore, Will H. A Mathematics Course for Political and Social Research. Princeton, NJ: Princeton University Press, 2013. Ch. 3, 10, 11.

Math is Fun: Introduction to Integration

Calculus: Single Variable Part 3 - Integration