# Math Camp Session I

## Notation, sets and intervals, functions, sequences

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August 13, 2018

# Organization of Math Camp

Camp Schedule

- **Fundamentals** (notation, sets and intervals, functions, sequences)
- **Calculus** (limits, continuity, derivative, intervals)
- **Probability** (set theory, distributions, random variables, hypothesis testing)
- **Lab sessions** with R
- + **Fun time** (mentoring sessions with faculty, graduate students + food + coffee)

# This session

1. Sets and intervals
2. Functions
3. Sequences
4. $\sum, \prod$
5. R prep session & break (10:40)

# Sets

A set is a collection of objects (elements).

$$
\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{a, b, c, d\} \\
C &= \{\text{Natural numbers}\} \\
D &= \{\text{UIUC PSGSA}\}
\end{aligned}
$$

# Sets

## Elements

If A is a set, we say that x is an element of A by writing, x $\in$ A. If x is not an element of A then, we write x $\notin$ A.

- $3 \in \{1, 2, 3\}$
- $3 \notin \{a, b, c, d\}$
- $10 \in$ {Natural numbers}
- Sanghoon $\notin$ {UIUC PSGSA}

### A=B

If A and B are sets, then we say that $A = B$ if, for all $x \in A$ then $x \in B$ and for all $y \in B$ then $y \in A$.

### Subset

If A and B are sets, then we say that $A \subset B$ if, for all $x \in A$, then $x \in B$.

- $A = \{a, b, c\}$, $B = \{a, b, c, d, e\}$
- Then, $A \supset B$, or $A \subset B$ ?
- If two sets are equal (A=B), then $A \subset B$ and $B \subset A$.

# Set Builder Notation

- Some famous sets
  - $N = \{1, 2, 3, \dots\}$
  - $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - $\mathbb{R} =$ real numbers
- Set builder notation to identify subsets
  - $[a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$
  - $(a, b) = \{x : x \in \mathbb{R} \text{ and } a < x < b\}$
  - $[a, b) = \{x : x \in \mathbb{R} \text{ and } a \leq x < b\}$
  - $(a, b] = \{x : x \in \mathbb{R} \text{ and } a < x \leq b\}$
- Practice questions
  - Let A $= \{1, 2, 3, 4, 5\}$
  - $(1, 4) =$
  - $[1, 4) =$
  - $[2, 3] =$
  - $(2, 3) =$

# Set Operations

We can build new sets with set operations.

## Union

Suppose A and B are sets. Define the **Union** of sets A and B as the new set that contain all elements in set A **or** in set B. In notation,

$$\begin{aligned} C &= A \cup B \\ &= \{x : x \in A \text{ or } x \in B\} \end{aligned}$$

- $A = \{a, b, c\}, B = \{c, d, e\}$, then, $A \cup B = \{a, b, c, d, e\}$

## Intersection

Suppose A and B are sets. Define the **Intersection** of sets A and B as the new set that contains all elements in set A **and** in set B. In notation,

$$\begin{aligned} C &= A \cap B \\ &= \{x : x \in A \text{ and } x \in B\} \end{aligned}$$

- $A = \{a, b, c\}, B = \{c, d, e\}$, then, $A \cap B = \{c\}$

# Functions

## Definition

Functions map one variable to another

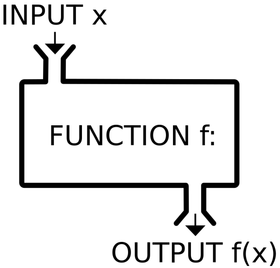
$$F(X) : X \to Y$$

where A is domain and B is codomain.

- Functions can also map multiple variables onto a single variable.
  - $F(x) = x^2$
  - $F(x) = x^2 + 2x + 4$
  - $F(x, y) = x^2 + 2xy + y^2$

# Functions

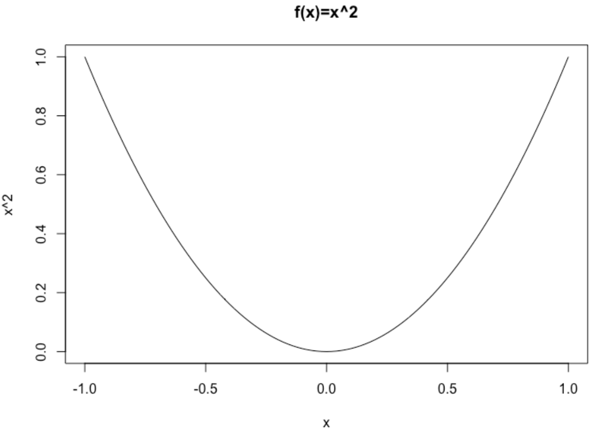
- $F(x, y) = x^2 + 2xy + y^2$
- Variables on the right side of the equation are called *input variables*, and the variable on the left side is the *output*.
- The input variables are also called **Independent variables**, and the output is called a **Dependent variable**.



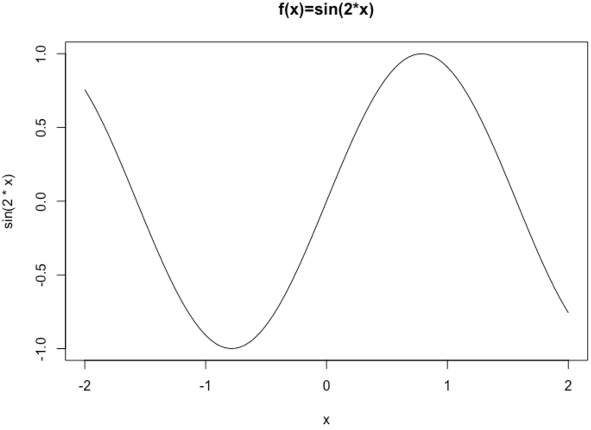
# Functions

- $F(x, y) = x^2 + 2xy + y^2$
- Variables on the right side of the equation are called *input variables*, and the variable on the left side is the *output*.
- The input variables are also called **Independent variables**, and the output is called a **Dependent variable**.
- For example...
  - Acemoglu, Johnson, and Robinson (2001) argued that the quality of institutions leads to economic growth. There is a positive relationship between the quality of institutions and economic growth.
    $\rightarrow$ $Growth_i = f(institution\ quality_i)$
    $\Rightarrow$ $\log y_i = \mu + \alpha R_i + X_i'\gamma + \epsilon_i$

# Plotting functions

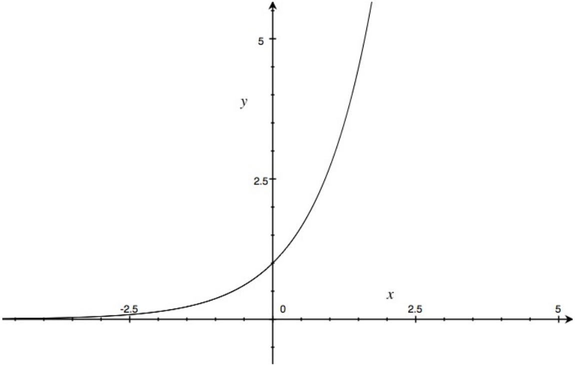


# Plotting functions



# Plotting functions

$y = e^x$



# Plotting functions

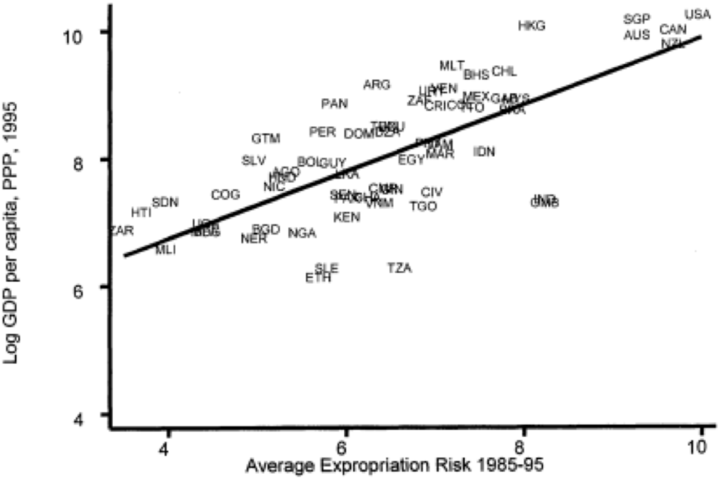


FIGURE 2. OLS RELATIONSHIP BETWEEN EXPROPRIATION RISK AND INCOME

# So far...

1. Sets and intervals
2. **Functions**
   - Definition, plotting functions
   - Exponents
   - Logarithms
   - Inverse functions
   - Functions and estimation
3. Sequences
4. $\sum, \prod$

# Exponental Function

$$f(x) = 2^x$$
$$g(x) = e^x$$

Some rules of exponents

$$a^x \times a^y = a^{x+y}$$
$$(a^x)^y = a^{xy}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$\frac{1}{a^y} = a^{1-y}$$
$$a^x \times b^x = (ab)^x$$
$$a^0 = 1$$
$$a^1 = a$$

# Exponental Function

Before we move on, what is $e$?

## Mathematical constant $e$

The number $e$ is a mathematical constant, approximately equal to 2.718. $e$ rises as the limit of $(1 + \frac{1}{n})^n$ as $n$ approaches infinity. Also, $e$ can be calculated as the sum of the infinite series,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots \approx 2.718$$

The function $f(x) = e^x$ is called the **(natural) exponential function**, and is the unique exponential function equal to its own derivative.

The **natural logarithm**, or logarithm to **base** e, is the inverse function to the natural exponential function ($log_e = log$).

# Composite and Inverse Functions

Log and exponents are inverse functions to each other. We start with the concept of Composite Functions.

## Composite Functions

Suppose $f : A \to B$ and $g : B \to C$. Then, define,

$$g \circ f = g(f(x))$$

- $f(x) = x, g(x) = x^2. Then, g \circ f = g(f(x)) = x^2$
- $f(x) = \sqrt{x}, g(x) = e^x. Then, g \circ f = e^{\sqrt{x}}$
- $f(x) = sin(x), g(x) = |x|. Then, g \circ f = |sin(x)|$

# Composite and Inverse Functions

## Inverse function

Suppose a function f is 1-1. Then, we'll define $f^{-1}$ as its inverse if,

$$f^{-1}(f(x)) = x$$

- $f(x) = x^2. Then, f^{-1}(f(x)) = \sqrt{x}$
- $f(x) = 2x + 3. Then, f^{-1}(f(x)) = \frac{x-3}{2}$
- $f(x) = log(x). Then, f^{-1}(f(x)) = e^x$

# Logarithms

Logarithm is a class of functions and is the **inverse function** to the exponential function.

- This means that $log_e z = x$ solves $e^x = z$
- e.g. $6^x = 216 \rightarrow x = log_6 216 = log_6 6^3 = 3log_6 6 = 3$
- Once again, we call $log_e$ natural logarithm and often we note $log_e = log$.
- $log\,(e) = log_e e = 1(because\ e^1 = e)$
- $log_{10} 1000 = 3$

# Logarithms

Some rules of logarithms

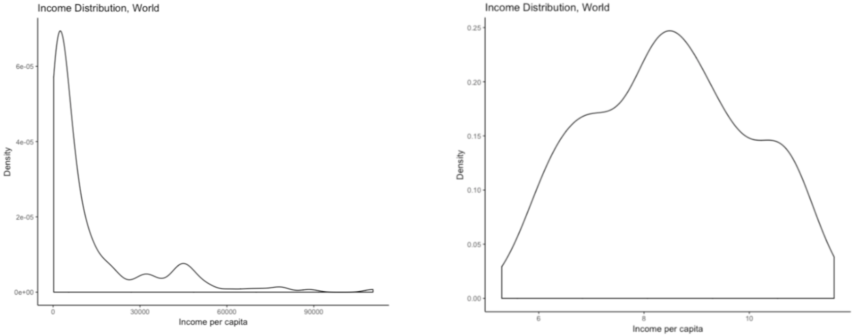
- $\log(a \times b) = \log(\text{a}) + \log(\text{b})$
- $\log(\frac{a}{b}) = \log(\text{a}) - \log(\text{b})$
- $\log(a^b) = b\log(a)$
- $\log(1) = 0$ why?
- $\log(\text{e}) = 1$

Some rules of exponents

$$a^x \times a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$
$$(a^x)^y = a^{xy}$$
$$a^x \times b^x = (ab)^x$$

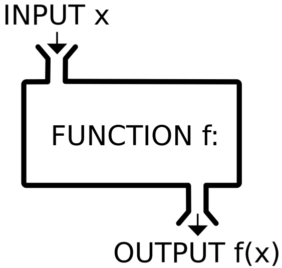
# "We use log GDP per capita as our dependent variable..."

(150, 298, 600, 1,000, 2,000, 6,000, 15,000)
$\rightarrow$
$(log150, log298, log600, log1000, log2000, log6000, log15000)$
$= (5.01, 5.70, 6.40, 6.99, 7.60, 8.70, 9.62)$




Log transformation can make the data to look similar to the nomal distribution. Why is it important to a bell curve?
$\rightarrow$ Stay tuned for the second half of Math Camp!

# Functions and estimation



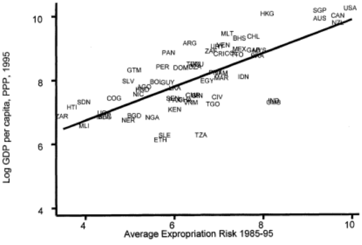




FIGURE 2. OLS RELATIONSHIP BETWEEN EXPROPRIATION RISK AND INCOME

$$F(x) = 2 + 3x + x^2$$

$$\log\ y_i = \mu + \alpha R_i + X_i'\gamma + \epsilon_i$$

- In a mathematical calculation, we know the **coefficients**, and we are interested in the **output**.
- In a statistical estimation, we know the values of variables, and we are interested in *estimating the coefficient* ($\alpha$).

# $\sum, \prod$ argh!

We start with the sequence.

**Sequence**

A sequence is a function whose domain is the set of positive integers.

We write a sequence as,

$$\{a_n\}_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots, a_N, \ldots)$$

Examples:

- $a_n = \{n\} = (1, 2, 3, 4, \ldots, n, \ldots)$
- $b_n = \{\frac{1}{n^2}\} = \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots\right)$
- $c_n = \{\frac{1}{2^n}\} = \left(\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right)$

# $\sum, \prod$

Summations and products

- $\displaystyle\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n$
- $\displaystyle\prod_{i=1}^{n} x_i = x_1 x_2 x_3 \cdots x_n$

# $\sum, \prod$

Summations and products: Examples

- Sequence: $a_n = \{n\} = (1, 2, 3, 4, \ldots, n, \ldots)$
- Summation: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$
- Product: $\prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n = 1 \cdot 2 \cdot 3 \cdot 4 \cdots$

How about...

- $b_n = x^2$
- $\sum_{i=1}^{n} b_i = ?$
- $\prod_{i=1}^{n} b_i = ?$

Practice?

$$log(\prod_{i=1}^{n} x_i) =$$