Intro to Linear Algebra I

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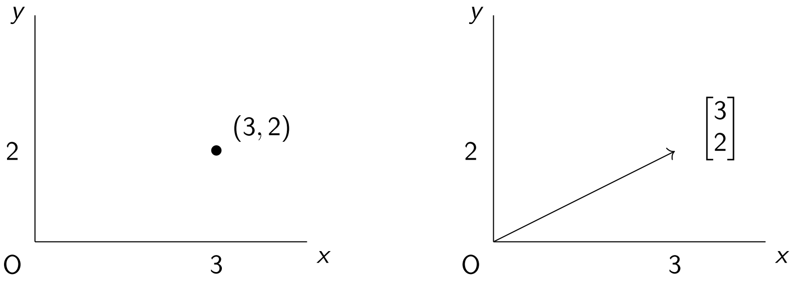
What are vectors?

- Vector: A serial listing of numbers where the order matters (Gill, 83)



Coordinates on a graph $\Leftrightarrow$ A vector

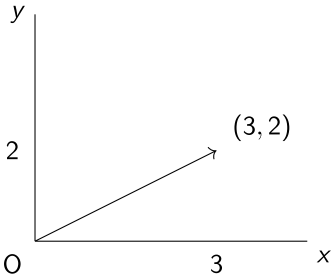
What are vectors?

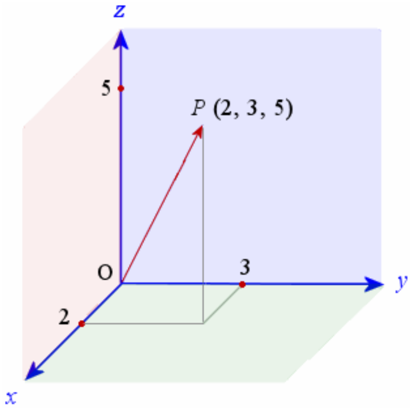
a row vector: $[1, 2, 3, 4]$

a column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

- Vector is written in bold type: $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- Vector in a more general form: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

Vector visualization





Vector calculation

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

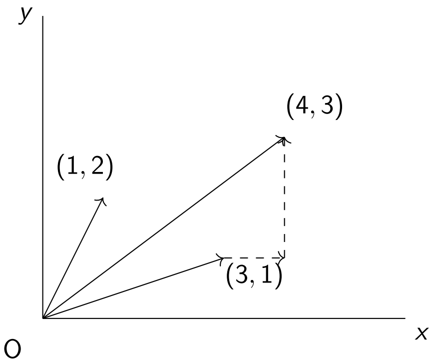
- **u+v**
- **u-v**

Vectors should be **conformable** to conduct calculation.

1. $\underset{1\times 4}{\mathbf{u}} + \underset{1\times 4}{\mathbf{v}}$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 0 \\ 5 \end{bmatrix}$
2. $\underset{1\times 4}{\mathbf{u}} + \underset{1\times 3}{\mathbf{v}}$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \\ -3 \end{bmatrix} = \cancel{\begin{bmatrix} 8 \\ 5 \\ 0 \\ 4 \end{bmatrix}}$

Vector calculation visualization

$$\underset{1\times 2}{\mathbf{u}} + \underset{1\times 2}{\mathbf{v}}: \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Vector multiplication and product

1. Vector scalar multiplication

$$3\mathbf{v} = 3 \cdot \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \\ -9 \\ 3 \end{bmatrix}$$

2. Vector inner product

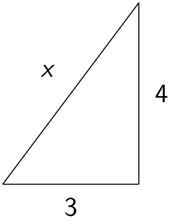
$$\mathbf{u} \cdot \mathbf{v} = [u_1 v_1 + u_2 v_2 + \cdots + u_k v_k] = \sum u_i v_i$$

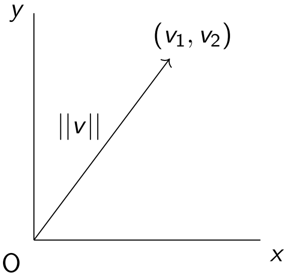
$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 7 + 2 \cdot 3 + 3 \cdot -3 + 4 \cdot 1 = 8$$

Practice!

Vector distance visualization





Distance of vector: vector norm $||\mathbf{v}||$

- For example, with vector $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
- The length of the diagonal line is: $\sqrt{4^2 + 2^2} = \sqrt{20}$

Vector norm

$$||v|| = (v_1^2 + v_2^2 + \dots v_n^2)^{\frac{1}{2}} = (\mathbf{v}'\mathbf{v})^{\frac{1}{2}}$$

- Practice: calculate vector norms

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$$

- Which vector is longer? $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$

Matrix

What are matrices?

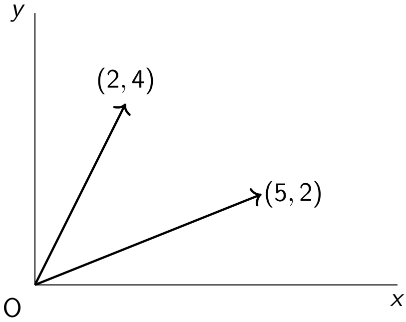
- two dimensions: rows and columns $\underset{2\times 2}{\mathbf{X}} = \begin{bmatrix} 1 & 3 \\ 7 & 3 \end{bmatrix}$
- Examples: dimensions?

$$\begin{bmatrix} 1 & 3 & 4 \\ 7 & 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 \\ 3 & 9 \\ 8 & 10 \\ 9 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 9 & 2 & 7 \\ 8 & 10 & 2 & 1 \end{bmatrix}$$

$\underset{i\times j}{\mathbf{X}}$: i for rows, j for columns.

Matrix visualization

With a matrix of $\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$,



Matrix calculations

1. Addition, subtraction

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 1-2 & 2+2 \\ 3+0 & 4+1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 1+2 & 2-2 \\ 3-0 & 4-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

2. Scalar Multiplication

$$s = 5$$

$$s\mathbf{X} = \begin{bmatrix} 5 \times 1 & 5 \times 2 \\ 5 \times 3 & 5 \times 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

Matrix Multiplication

$$\underset{(k\times n)}{\mathbf{X}}\underset{(n\times p)}{\mathbf{Y}} = \underset{(k\times p)}{\mathbf{XY}}$$

$$\begin{aligned}\mathbf{XY} &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\ &= \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}\end{aligned}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{XY} =$$

Matrix transposition

The transpose of an $i \times j$ matrix X is the $j \times i$ matrix X' (“X prime”). For example,

$$X' = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}' = \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \\ x_{13} & x_{23} \end{bmatrix}$$

$$X' = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}' =$$

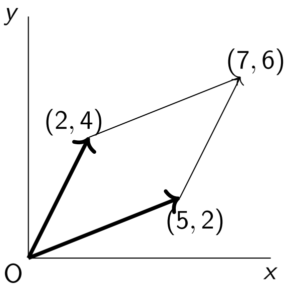
Matrix properties

Matrix inversion $XX^{-1} = X^{-1}X = I$

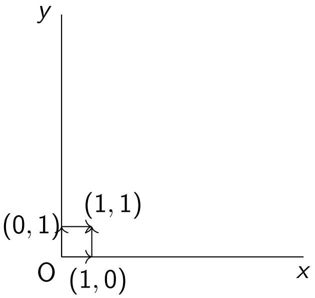
- When $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

  $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

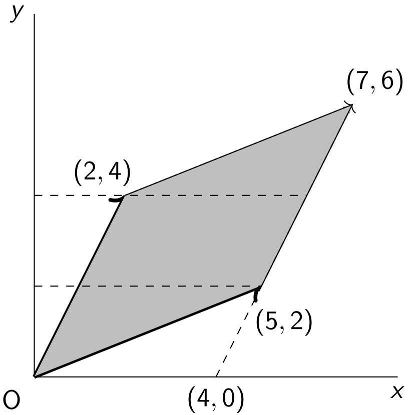
  $|A| = ad - bc$, where $|A|$ is the **determinant** of matrix A.



$\Rightarrow$



Determinant



- Size of the shaded area: $4 \times 4 = 16$
- In a matrix form, we have $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$
- Determinant of the matrix: $ad - bc = 5 \cdot 4 - 2 \cdot 2 = 16$
- $|A| =$ shaded area of the matrix

Inverse

Show that when $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $A \cdot A^{-1} = A^{-1} \cdot A = I$

From equations to matrices

Vectors, Matrices are useful when it comes to dealing with complex algebraic calculations, especially when we delve into linear regressions. For example, in the next session, James will talk more about what it means to solve the following equation:

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

But, for now, it's the best to understand that linear regression is one of different equations that we want to find a solution to. So, as our final step, let's look at how we can solve equations with linear algebra.

Consider the following system of equations,

$$
\begin{aligned}
2x_1 - 3x_2 &= 4 \\
5x_1 + 5x_2 &= 3,
\end{aligned}
$$

we can rewrite them into matrix form,

$$
\begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}
$$

x_1 and x_2 can be derived by multiplying the inverse of the 2×2 matrix on both sides

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 5 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.16 \\ -0.56 \end{bmatrix}
$$

The above system of equations can be simplified as:

$$
\mathbf{Ax} = \mathbf{b}
$$

Linear Systems of Equations

For the following expression on the relationship between political blame and regional political variables, simplify it in matrix algebra form.

$$
\begin{aligned}
Y_i = {} & \beta_0 + \beta_1 CHANGELIV + \beta_2 BLAMECOMM + \beta_3 INCOME \\
& + \beta_4 FARMER + \beta_5 OWNER + \beta_6 BLUESTATE \\
& + \beta_7 WHITESTATE + \beta_8 FORMMCOMM + \beta_9 AGE \\
& + \beta_{10} SQAGE + \beta_{11} SEX + \beta_{12} SIZEPLACE \\
& + \beta_{13} EDUC + \beta_{14} FINHS + \beta_{15} ED * HS \\
& + \beta_{16} RELIG + \beta_{17} NATION + E_i, \text{for } i{=}1 \text{ to } n
\end{aligned}
$$

$\Rightarrow$