Limits and Derivatives

Tolgahan Dilgin

University of Illinois at Urbana-Champaign

dilgin2@illinois.edu

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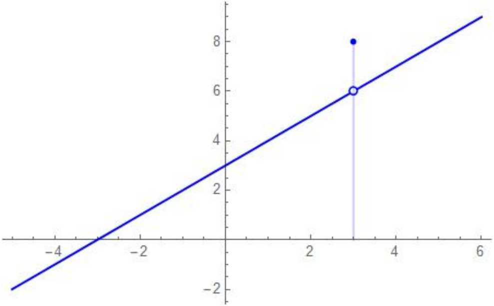
Why Limits & Derivatives in Political Science?

- We use derivatives heavily in optimization problems.
  - Commonly, in game theory.
  - I use them in my decision theoretic model for my dissertation.
- As such, derivatives can come in handy, when we answer questions like the following:
  - What is the optimal level of military expenditure for a dictator? (when balancing between external security threats and the risk of a strong military taking over the government)
  - How do political parties determine the location and intensity of pre-election violence?
- Limits come up everyone- in stats, in game theory, in formal proofs, etc.
  - Derivatives are actually limits!

Definition of a Limit (Non-Formal)

- Suppose we have a function $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{, if } x \neq 3 \\ 8 & \text{, if } x = 3 \end{cases}$



- As the values of x approach 3 from either side, the values of $y = f(x)$ approach 6. Notationally, $\lim_{x \to 3} f(x) = 6$.

Definition of a Limit (Formal)

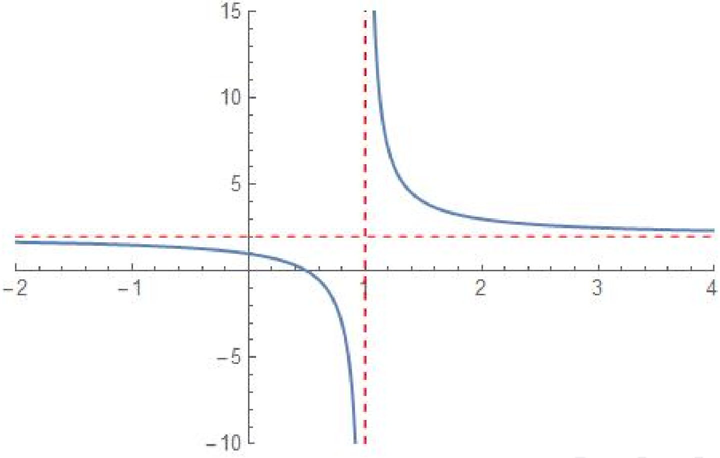
- For a function $f : A \to \mathbb{R}$, a real number L is said to be a limit of f at c if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L < \varepsilon|$.[1] In English, please?
- L is the limit of $f(x)$ at c; when the value of $f(x)$ nears L as x approaches c.
- Right-handed Limit: The value of $f(x)$ when approaching c from the right-hand side of the x-axis.
  - $\lim\limits_{x \to c^+} f(x) = L$
- Left-handed Limit: The value of $f(x)$ when approaching c from the left-hand side of the x-axis.
  - $\lim\limits_{x \to c^-} f(x) = L$
- We say that $f(x)$ has a limit, when $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = L$.[2]

[1] Bartle & Sherbert, p. 104

[2] Ryan, p. 84

Examples of Right-Handed and Left-Handed Limits

- Let $f(x) = \dfrac{1}{x-1} + 2$. Find the following limits by looking at the graph. Does this function have a limit at any point c?
  - $\lim\limits_{x\to 1^+} = ?$, $\lim\limits_{x\to 1^-} = ?$, $\lim\limits_{x\to +\infty} = ?$, $\lim\limits_{x\to -\infty} = ?$



Definition of Continuity at a Point

A function f(x) is **continuous at point** c, if and only if all the three conditions are satisfied.

1. $f(a)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(a)$

A function f(x) is **discontinuous at point** c, if it fails to be continuous at c.

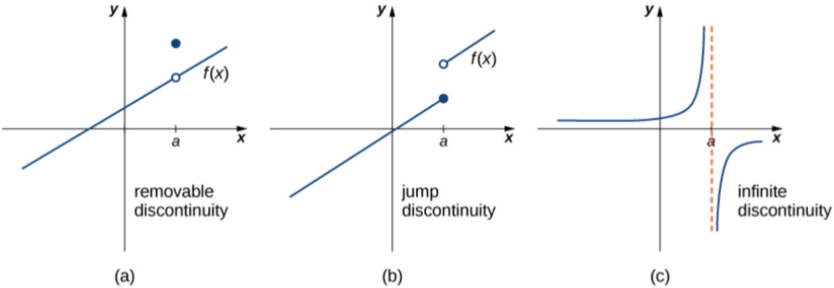
Types of Discontinuity at a Point

Definition

If $f(x)$ is discontinuous at a, then

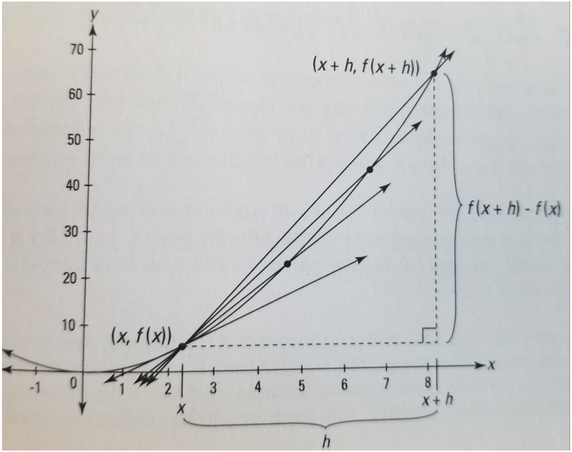
1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Types of Discontinuity at a Point



Definition of the Derivative

The division of $f(x+h) - f(x)$ by h would give us the slope of the big triangle you see below. Now, imagine that h approaches 0 ($h \to 0$). This formula would then give us the slope of the tangent line at $(x, f(x))$.



Definition of Derivative

The following formulas are often used as the definition of derivatives.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

If you are confused about the second definition, the Khan Academy *video* could help you with getting the intuition.

Derivative Example

Let us get the derivative for $f(x) = x^2$ using equation (1).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) \\ &= 2x + 0 = 2x \end{aligned}$$

Exercise: Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$. Find the slope of tangent line at $x = 3$. In other words, $g'(3) =?$ Solve the question, using equation (1).

Derivative Rule for Polynomials

- You may have realized/known, there is an easier way to solve the derivatives for both of our previous examples where $f(x) = x^2$ and $g(x) = x^3$.
- Before that let me introduce the other notation for derivatives:
  - $f'(x) = \frac{df}{dx}$ or $f'(x) = \frac{d}{dx}f(x)$
- In line with the new notation, the shortcut for derivatives of polinomials would go as follows:
  - $\frac{d}{dx}(ax^b) = bax^{b-1} \Rightarrow \frac{d}{dx}(x^3) = 3x^2$

Derivative Rules

The Chain Rule

With complicated derivatives, we use the chain rule: $\dfrac{df}{dx} = \dfrac{df}{du} \cdot \dfrac{du}{dx}$.

Example

$\dfrac{d}{dx}(3x-2)^{20} =$? You can assign $u(x) = 3x - 2 \Rightarrow \dfrac{df}{du} = 20 \cdot (3x-2)^{19}$

Since $\dfrac{du}{dx} = 3 \Rightarrow \dfrac{df}{dx} = [20 \cdot (3x-2)^{19}] \cdot 3 = 60 \cdot (3x-2)^{19}$

Exercise

$\dfrac{d}{dx}\sqrt{2x^2+1} =$?

Derivative Rules

The Product Rule

When taking derivative of multiplication of two functions, e.g. $f(x)$ and $g(x)$, we use the product rule: $\frac{d}{dx}(fg) = g \cdot \frac{df}{dx} + f \cdot \frac{dg}{dx}$ We can also write it with our less intimidating notation, using $'$ sign. $(fg)' = f' \cdot g + g' \cdot f$

Example

Let $f(x) = (x^2 - 1)$ and $g(x) = 4x^3$. Then, $(fg)' = ?$.
$f'(x) = 2x$ and $g'(x) = 12x^2$. Following the product rule;
$(fg)' = f' \cdot g + g' \cdot f = 2x \cdot 4x^3 + 12x^2 \cdot (x^2 - 1)$

Exercises

$\frac{d}{dx}(2x+3)^8(x^2-1)^5 = ?$, $\frac{d}{dx}(5x^2-2)\sqrt{8x+3} = ?$
Hint: Use both the chain and product rules!

Derivative Rules

The Quotient Rule

When differentiating division of two functions, we use the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}.$$ More simply; $$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$$

Example

Let $\frac{d}{dx}\frac{x^2+4}{x-3} = ?$ Using the quotient rule; we have

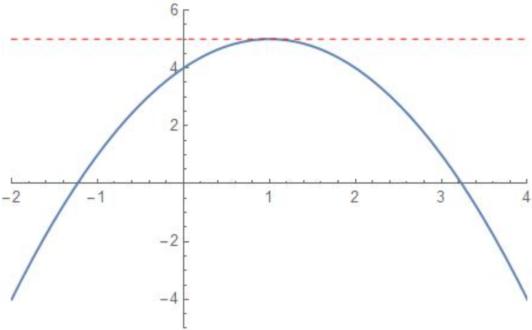
$$\frac{2x \cdot (x-3) - 1 \cdot \left(x^2+4\right)}{(x-3)^2} = 1 - \frac{13}{(x-3)^2}$$

Exercise

$$\frac{d}{dx}\frac{\sqrt{x}}{3x-4} = ?$$

Back to Optimization

- We use derivatives to find maximum and minimum points of functions. i.e., where their slope is zero.
- For instance, $f(x) = -(x-1)^2 + 5$. How would we find the maximum point of this function? Hint: Take the first derivative, make it equal to zero, and solve for x and $f(x)$.



- Answer: (1, 5)

Why Does This All Matter?- An Example

Positive Political Theory

In formal theory, game theory, decision theoretic models, we use derivatives to solve optimization problems very often.

Research Question

Where do political parties initiate the highest level of pre-election violence in order to win the elections?

A Formal Model

Imagine there are 2 districts indexed by $i \in 1, 2$, and 2 actors: Government (G) and Opposition (O). G can lower the turnout (t) for the opposition by initiating violence (v). Since government has limited resources, it must choose wisely how much violence it should commit in which district.

Why Does This All Matter?- An Example

Government's Utility Function

Imagine $\alpha_i \in (0,1)$ is the initial support G has in each district. And, that violence is a costly activity, where the cost function is $k(v)$. Government wants to maximize its total vote share in the elections;

$$U_G = \sum_{i=1}^{2} \frac{\overbrace{\alpha_i \cdot t_{G,i}(v_i)}^{G's\ Vote\ Share}}{\underbrace{\alpha_i \cdot t_{G,i}(v_i)}_{G's\ Vote\ Share} + \underbrace{(1-\alpha_i) \cdot t_{O,i}(v_i)}_{O's\ Vote\ Share}} - \underbrace{\sum_{i=1}^{2} k(v_i)}_{Total\ Cost}$$

Solution by Derivatives

Although this example may seem complicated, the idea is very simple. We take the derivative of government's utility function, equate it to 0, then find the optimizing levels of violence for each district. Hence, we answer an important research question, and hope for publication!