Examples of Bayes' Theorem in Practice

1. *The Monty Hall Game Show Problem*

Question: In a TV Game show, a contestant selects one of three doors; behind one of the doors there is a prize, and behind the other two there are no prizes. After the contestant selects a door, the game-show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the other unopened door, or STICK to their original choice.

Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the prize the same whether they STICK or SWITCH? (Assume that the host selects a door to open, from those available, with equal probability).

[For detailed discussion, see, for example, *www.io.com/˜kmellis/monty.html.*]

Answer: Without loss of generality, let events A, B, C correspond to the prize being behind the selected, opened, and remaining door respectively, and let H_B denote the event that the host opens door B. Want to compare $\mathbf{P}(A|H_B)$ (STICK) with $\mathbf{P}(C|H_B)$ (SWITCH). Now $\mathbf{P}(A) = \mathbf{P}(B) = \mathbf{P}(C) = 1/3$, and we are given that $\mathbf{P}(H_B|A) = 1/2$, $\mathbf{P}(H_B|B) = 0$ and $\mathbf{P}(H_B|C) = 1$. Then the general version of Bayes' theorem gives

$$\mathbf{P}(A|H_B) = \frac{\mathbf{P}(H_B|A)\mathbf{P}(A)}{\mathbf{P}(H_B)} = \frac{\mathbf{P}(H_B|A)\mathbf{P}(A)}{\mathbf{P}(H_B|A)\mathbf{P}(A) + \mathbf{P}(H_B|B)\mathbf{P}(B) + \mathbf{P}(H_B|C)\mathbf{P}(C)} = \frac{\frac{1}{2}\frac{1}{3}}{\frac{1}{2}\frac{1}{3} + 0\frac{1}{3} + 1\frac{1}{3}} = \frac{1}{3}$$

so $\mathbf{P}(A|H_B) = 1/3$, and similarly $\mathbf{P}(C|H_B) = 2/3$. so it is advantageous to SWITCH.

2. *Question:* A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5 % of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the following probabilities:

(a) that the test result will be positive;

(b) that, given a positive result, the person is a sufferer;

(c) that, given a negative result, the person is a non-sufferer;

(d) that the person will be misclassified.

Answer: Let $T \equiv$ "Test positive", $S \equiv$ "Sufferer", $M \equiv$ "Misclassified." Then $\mathbf{P}(T|S) = 0.95$, $\mathbf{P}(T|S') = 0.10$, $\mathbf{P}(S) = 0.005$. Hence

(a) $\mathbf{P}(T) = \mathbf{P}(T|S)\mathbf{P}(S) + \mathbf{P}(T|S')\mathbf{P}(S') = (0.95 \times 0.005) + (0.1 \times 0.995) = 0.10425.$

(b) $\mathbf{P}(S|T) = \dfrac{\mathbf{P}(T|S)\mathbf{P}(S)}{\mathbf{P}(T|S)\mathbf{P}(S) + \mathbf{P}(T|S')\mathbf{P}(S')} = \dfrac{0.95 \times 0.005}{(0.95 \times 0.005) + (0.1 \times 0.995)} = 0.0455.$

(c) $\mathbf{P}(S'|T') = \dfrac{\mathbf{P}(T'|S')\mathbf{P}(S')}{\mathbf{P}(T')} = \dfrac{0.9 \times 0.995}{1 - 0.10425} = 0.9997.$

(d) $\mathbf{P}(M) = \mathbf{P}(T \cap S') + \mathbf{P}(T' \cap S) = \mathbf{P}(T|S')\mathbf{P}(S') + \mathbf{P}(T'|S)\mathbf{P}(S) = 0.09975.$