Functions and Intro to Linear Algebra Practice Questions

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Functions

Basics of functions

Rewrite the following arguments into functions (IV $\rightarrow$ DV, $y = a + bx$)

1. *Answer:* cognitive/political sophistication $\rightarrow$ the ability to discern information credibility

2. *Answer:* vicarious experiences $\rightarrow$ differences perspectives of various racial and ethnic groups

3. *Answer* European settler mortality $\rightarrow$ institutional quality $\rightarrow$ economic performance

Linear equations on graphs

Find the intercept and slope of the following graph. Can you write the function in mathematical form?

Answer: $y = \frac{1}{3}x + 3$

Nonlinear functions

Simplify

1. $x^0 = 1$
2. $x^{-2} \times x^3 = x$
3. $y^7 y^6 y^5 y^4 = y^{22}$
4. $(b \cdot b \cdot b) \times c^{-3} = b^3 \cdot c^{-3}$
5. $\left(\frac{1}{27b^3}\right)^{1/3} = \frac{1}{3b}$
6. $\sqrt{x} \times \sqrt[5]{x} = x^{\frac{7}{10}}$
7. $\ln(5) \cdot \ln(45) = 2\ln 3 \cdot (\ln 5)^2$
8. $\ln(x^5 y^3) - \ln \frac{x^3}{y} = \ln x^2 y^4$
9. $\ln(3x) - 2\ln(x+2) = \ln \frac{3x}{(x+2)^2}$

Linear functions

Rewrite the following by taking the log of both sides. Is the result a linear function?

1. $y = \alpha + x_1^{\beta_1} + \beta_2 x_2 + \beta_3 x_3 \rightarrow$ Not Linear
2. $y = \alpha \times x_1^{\beta_1} \times x_2^{\beta_2} \times x_3^{\beta_3} \rightarrow$ Linear
3. $y = \alpha \times x_1^{\beta_1} \times \frac{x_2^{\beta_2}}{x_3^{\beta_3}} \rightarrow$ Linear

Logarithms

Explain how one unit change in the right-hand side variable (L and $\ln(L)$, respectively) leads to changes in the left-hand side variable (U) for each equation.

- $U = 4 + 2L$
- $U = 4 + 2\ln(L)$

Answer: With logarithms, we will have a proportional change in the dependent variable with a unit change in the independent variable. We can write the changes in the log term in the following way:

$$\Delta \ln(L) = \ln L_t - \ln L_{t-1} = \ln \frac{L_t}{L_{t-1}}$$

Inverse Function

When a function is defined as $f(x) : A \to B$, the inverse function is $f^{-1}(x) : B \to A$. For example, if $f(x) = 2x+3$, then its inverse is $f^{-1}(x) = \frac{x-3}{2}$. Find the inverse of the following functions.

1. $f(x) = 8x - 5$ $\quad f(x)' = \frac{x+5}{8}$
2. $f(x) = 2/(x-4)$ $\quad f(x)' = \frac{2}{x} + 4$
3. $f(x) = \sqrt{(x+3)}$ $\quad f(x)' = x^2 - 3$

Finding equilibrium

1. Find equilibrium P^*, Q^*

$$\begin{aligned} 1.Q_1 &= 53 - 3P_1 \\ Q_2 &= 6P_2 - 10 \\ P* &= 7, Q* = 32 \end{aligned} \qquad \begin{aligned} 2.Q_1 &= 30 - 2P_1 \\ Q_2 &= -5 + 5P_2 \\ P* &= 5, Q* = 20 \end{aligned}$$

Notation

Find solutions to the following questions

1. $C = \{2^1, 2^3, 2^4\},\ c_i \in C.\quad \prod_{i=1}^{n} c_i = 2^8$

2. When $x \in \{3,4,5,6\}$, and $f(x) = x+3$,
   $\max_x f(x) = 9$
   $\arg\max_x f(x) = 6$

3. $x \in \mathbb{R}, f(x) = 19 - (x-2)^2$.
   $\max_x f(x) = 19$
   $\arg\max_x f(x) = 2$

Simplify the following:

$$\begin{aligned}\log(\prod_{i=1}^{n} x_i) &= \log(x_1 \cdot x_2 \cdot x_3 \cdots x_n) \\ &= \sum_{i=1}^{n} \log x_i\end{aligned}$$

Show that in general

$$\sum_{i=1}^{m} \prod_{j=1}^{n} x_i y_j \neq \prod_{j=1}^{n} \sum_{i=1}^{m} x_i y_j$$

$$\begin{aligned}\sum_{i=1}^{m} \prod_{j=1}^{n} x_i y_j &= \sum_{i=1}^{m} x_i y_1 \cdot x_i y_2 \cdot x_i y_3 \cdots x_i y_n \\ &= x_1 y_1 \cdot x_1 y_2 \cdot x_1 y_3 \cdots x_1 y_n + x_2 y_1 \cdot x_2 y_2 \cdot x_2 y_3 \cdots x_2 y_n + \cdots \\ &+ x_m y_1 \cdot x_m y_2 \cdot x_m y_3 \cdots x_m y_n\end{aligned}$$

$$\begin{aligned}\prod_{j=1}^{n} \sum_{i=1}^{m} x_i y_j &= \prod_{j=1}^{n} x_1 y_j + x_2 y_j + \cdots + x_m y_j \\ &= (x_1 y_1 + x_2 y_1 + \cdots + x_m y_1)(x_1 y_2 + x_2 y_2 + \cdots + x_m y_2) \cdots (x_1 y_n + x_2 y_n + \cdots + x_m y_n)\end{aligned}$$

Intro to Linear Algebra I

Basic Vector Algebra

Define $s = 3, t = 1, u = [2, 4, 8], v = [9, 7, 5]$.

1. $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 9 + 4 \cdot 7 + 8 \cdot 5$
2. Calculate $\mathbf{u} \cdot \mathbf{u}' = 2^2 + 4^2 + 8^2$
3. Calculate $(s + t)(\mathbf{u} + \mathbf{v}) = [44, 44, 52]$.

1. Practice: calculate vector norms

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad ||v|| = \sqrt{3^2 + 7^2} \qquad \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad ||v|| = \sqrt{5^2 + 3^2}$$

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \qquad ||v|| = \sqrt{3^2 + 3^2 + 3^2} \begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix} \qquad ||v|| = \sqrt{5^2 + 2^2 + 6^2}$$

2. Which vector is longer, $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$? $\begin{bmatrix} 5 \\ -2 \\ 6 \end{bmatrix}$ is longer.

Matrix calculations

1. Given the following matrices, perform the calculations below. (Some of the calculations cannot be performed.)

$$A = \begin{bmatrix} 5 & 1 & 2 \\ 6 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 5 \\ -2 & -3 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -5 & 3 \\ -3 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, E = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

   a) $A + B = \begin{bmatrix} 8 & 5 & 7 \\ 4 & -1 & 9 \end{bmatrix}$

   b) $A - C$ Two matrices are not conformable.

   c) $A + 5B = \begin{bmatrix} 20 & 21 & 27 \\ -4 & -13 & 33 \end{bmatrix}$

   d) $3A = \begin{bmatrix} 15 & 3 & 6 \\ 18 & 6 & 9 \end{bmatrix}$

   e) $B^T - C = \begin{bmatrix} 2 & -4 \\ 9 & -6 \\ 8 & 5 \end{bmatrix}$

   f) BA Two matrices are not conformable.

   g) $AC = \begin{bmatrix} -6 & 15 \\ -13 & 21 \end{bmatrix}$

   h) $DB = \begin{bmatrix} 4 & -1 & 23 \\ 6 & 7 & 38 \end{bmatrix}$

   i) $DE = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

2. Matrix transposition. Find the transpose of the following matrices.

   a) $\begin{bmatrix} 5 & 1 & 2 \\ 6 & 2 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & 6 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

   b) $\begin{bmatrix} 15 & 16 \\ 9 & 4 \\ 9 & 3 \end{bmatrix}' = \begin{bmatrix} 15 & 9 & 9 \\ 16 & 4 & 3 \end{bmatrix}$

   c) $\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & 6 \\ 1 & 3 \end{bmatrix}$

3. Given the following matrices and their dimensions, calculate the dimensions after matrix multiplication. $\underset{3\times 2}{A}, \underset{2\times 4}{B}, \underset{5\times 2}{C}, \underset{4\times 3}{D}$

    a) $AB : 3 \times 4$

    b) $BD : 2 \times 3$

    c) BC : not conformable

    d) $DA : 4 \times 2$

    e) $CA^T 5 \times 3$

    f) $B^T C^T : 4 \times 5$

    g) $D^T B^T : 3 \times 2$

Determinant and Inverse of a matrix

1. Show that when $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $\quad A \cdot A^{-1} = A^{-1} \cdot A = I$

$$\begin{aligned} A^{-1} &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ A \cdot A^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Linear systems of equations

1. Rewrite the following linear systems of equations into matrix form. (No need to solve them.)

$$\begin{aligned} x + y + 2z &= 2 \\ 3x - 2y + z &= 1 \\ y - z &= 3 \end{aligned} \qquad \begin{aligned} 2x + 3y - z &= -8 \\ x + 2y - z &= 2 \\ -x - 4y + z &= -6 \end{aligned} \qquad \begin{aligned} x - y + 2z &= 2 \\ 4x + y - 2z &= 10 \\ x + 3y + z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & -1 \\ -1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 0 \end{bmatrix}$$

2. For the following expression on the relationship between political blame and regional political variables, simplify it in matrix algebra form.

$$\begin{aligned} Y_i = {} & \beta_0 + \beta_1 CHANGELIV + \beta_2 BLAMECOMM + \beta_3 INCOME \\ & + \beta_4 FARMER + \beta_5 OWNER + \beta_6 BLUESTATE \\ & + \beta_7 WHITESTATE + \beta_8 FORMMCOMM + \beta_9 AGE \\ & + \beta_{10} SQAGE + \beta_{11} SEX + \beta_{12} SIZEPLACE \\ & + \beta_{13} EDUC + \beta_{14} FINHS + \beta_{15} ED * HS \\ & + \beta_{16} RELIG + \beta_{17} NATION + E_i, \text{for } i\text{=1 to } n \end{aligned}$$

$$\Rightarrow \mathbf{y} = \mathbf{X}\beta$$

During the math camp week, you have been exposed to linear regression with different angles. The following set of questions brings you back to the very first set of classes on functions and intro to linear algebra. Our goal is to derive a coefficient in linear regression by hand. We have the following hypothetical data:

Table 1: Oil revenue and a dictator's survival

Country	Oil Revenue (billion USD)	Dictator in Power (years)
Kazakhstan	1	2
Iran	3	3
Libya	2	1

We can write the X matrix as follows, including an intercept:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}, \text{ and the y vector as, } \mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Now remember the equation for calculating a coefficient, $\hat{\mathbf{b}}$, is, $\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$

1. Calculate $\hat{\mathbf{b}}$.

2. Our goal is to find a and b in the following equation: $Survival = a + b \cdot Oil$. You calculated them from the previous question. Re-write the equation with the values from $\hat{\mathbf{b}}$.

Solution

1. Calculate $\hat{\mathbf{b}}$.

$$\begin{aligned}\hat{\mathbf{b}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\ &= \left(\begin{bmatrix}1 & 1\\1 & 3\\1 & 2\end{bmatrix}'\begin{bmatrix}1 & 1\\1 & 3\\1 & 2\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1\\1 & 3\\1 & 2\end{bmatrix}'\begin{bmatrix}2\\3\\1\end{bmatrix} \\ &= \begin{bmatrix}1\\0.5\end{bmatrix}\end{aligned}$$

2. Our goal is to find a and b in the following equation: $Survival = a + b \cdot Oil$. You calculated them from the previous question. Re-write the equation with the values from $\hat{\mathbf{b}}$.

$$\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\0.5\end{bmatrix}$$

$$Survival = 1 + 0.5 \cdot Oil$$