

Mathcamp Day 2: Integrals

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Section 1: Introduction

Section 2: Concept of Integral

Section 3: Fundamental Theorem of Calculus and Definite Integral

Section 4: Integrals and Distributions

Section 5: References

Section 1: Introduction

Introduction: Some ideas

As a political science graduate student...

- Why do we care about math?

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$$\begin{aligned}f(\underline{y}|\theta) &= \theta^{\sum y_i} (1-\theta)^{n-\sum y_i} & f(\theta) &= \int_{\{0 \leq \theta \leq 1\}} \\f(\theta|\underline{y}) &= \frac{f(\underline{y}|\theta)f(\theta)}{\int_{\{0 \leq \theta \leq 1\}} f(\underline{y}|\theta)f(\theta)d\theta} = \frac{\theta^{\sum y_i} (1-\theta)^{n-\sum y_i} \int_{\{0 \leq \theta \leq 1\}} \\&= \frac{\theta^{\sum y_i} (1-\theta)^{n-\sum y_i} \int_{\{0 \leq \theta \leq 1\}} \\&= \frac{\Gamma(\sum y_i + 1) \Gamma(n - \sum y_i + 1)}{\Gamma(n + 2)} \int_0^1 \frac{\Gamma(n + 2)}{\Gamma(\sum y_i + 1) \Gamma(n - \sum y_i + 1)} \theta^{\sum y_i} (1-\theta)^{n-\sum y_i} d\theta \\&= \frac{\Gamma(n + 2)}{\Gamma(\sum y_i + 1) \Gamma(n - \sum y_i + 1)} \theta^{\sum y_i} (1-\theta)^{n-\sum y_i} \int_{\{0 \leq \theta \leq 1\}}\end{aligned}$$

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SANTA CRUZ

3:08 / 3:31

Section 1: Introduction

- Why do we care about learning integral?
- What is the learning objective in today's math camp?
- How do you participate in the morning sessions?

Introduction: Link to Day 1

A brief overview of functions and derivatives

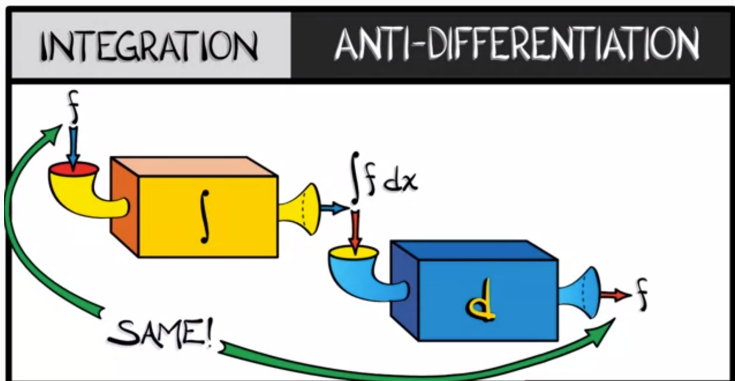


Figure 1: Integration and differentiation

For further references see Calculus: Single Variable Part 3 - Integration

Section 2: Concept of Integral

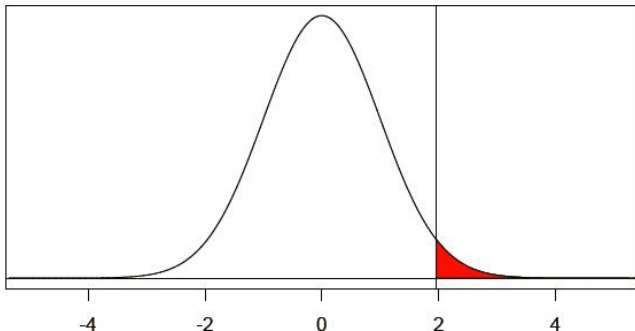
Concept of Integrals

Area under curve:

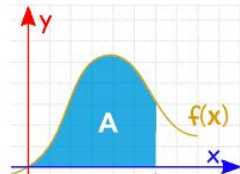
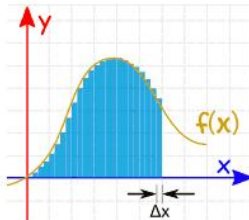
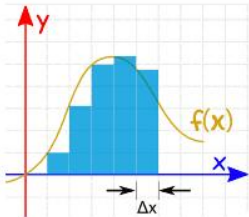
Let's take the standard normal curve as an example.

What if we want to know the area below the curve when $x \geq 1.96$?

We'll come back to this example later.



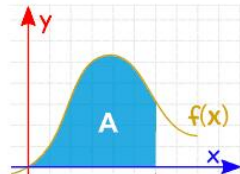
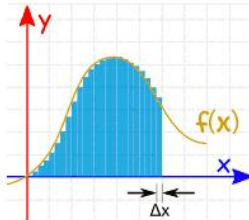
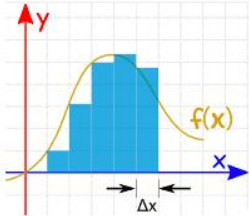
Concept of Integrals



- i. We could calculate the function at a few points and add up slices of width Δx .

^aReference: Math is fun: Introduction to Integration

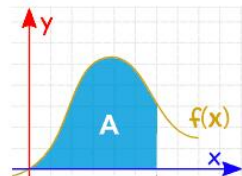
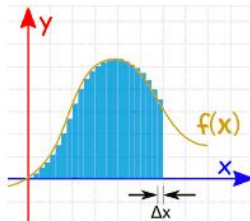
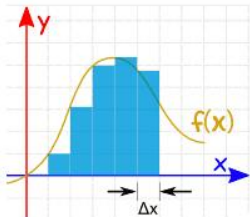
Concept of Integrals



- i. We could calculate the function at a few points and add up slices of width Δx .
- ii. We can make Δx a lot smaller and add up many small slices.

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Concept of Integrals



- i. We could calculate the function at a few points and add up slices of width Δx .
- ii. We can make Δx a lot smaller and add up many small slices.
- iii. As the slices approach zero in width, the answer approaches the true answer. ^a

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Concept of Integral

The way to find the area **under a curve** is to take the integral.

What is an integral?

- The area under the curve $f(x)$ for some range of $x = (a, b)$ is defined as “the definite integral for f from a to b ”.

$$\int_a^b f(x)dx$$

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. $[a, b]$.

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- Forms: indefinite integrals and definite integrals. A definite integral has a boundary, e.g. $[a, b]$.
- Useful illustration in Coursera (7 minutes)

Concept of Integral

The integral is the inverse function of the derivative:

$$F(x) = \int_a^b f(x) dx$$

$$f(x) = \frac{dF(x)}{dx}$$

Then, for any x ,

$$F(x) = \int_a^b f(x) dx = \int_a^b \frac{dF(x)}{dx} dx = F(x)$$

$$\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^b f(x) dx = f(x)$$

In other words, $F(x)$ is the anti-derivative of $f(x)$.

Concept of Integral

Too abstract?

Some examples...

What's an integral of $2x$?

- We know that the derivative of x^2 is
- An integral of $2x$ is
- We will write $\int 2x dx = x^2 + C$

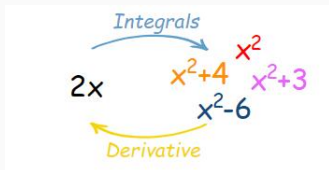
Power rule:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Concept of (Indefinite) Integrals

Why plus C?

It is the "Constant of Integration". It is there because of all the functions whose derivative is $2x$:



- The derivative of $x^2 + 4$ is $2x$, and the derivative of $x^2 + 99$ is also $2x$, and so on! Because the derivative of a constant is **zero**.
- So when we reverse the operation (to find the integral) we only know $2x$, but there could have been a constant of any value.

Concept of (Indefinite) Integrals

The anti-derivative, also called the indefinite integral, is not unique.

- There are multiple anti-derivatives for each function (for each C).
- This shifts the curve up or down the y -axis
- With information, such as a point the function passes through, you can solve for c .

Concept of (Indefinite) Integrals: Some Classic Anti-derivative Formulas

antiderivative = indefinite integral

$$\int 1 dx = x + c$$

$$\int k dx = kx + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} dx = \log x + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\log a} + c$$

Concept of (Indefinite) Integrals

Pause.....Questions?

Recall Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Practices:

i $\int 8x dx = \dots\dots$

ii $\int 6x^2 dx = \dots\dots$

iii $\int x^3 dx = \dots\dots$

iv $\int e^x dx = \dots\dots$

Section 3: Fundamental Theorem of Calculus and Definite Integral

Fundamental Theorem of Calculus and Definite Integral

A **deep** connection between derivatives and integrals makes integration much easier.

The definite integral, again, is defined by a specific range:

Fundamental Theorem of Calculus

Suppose F is differentiable on $[a, b]$ and that its derivative, f , is integrable. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

- This formula helps you to find area under the curve $f(x)$ for some range of x from a to b .
- Simply find an antiderivative F , evaluate at b , evaluate at a , and take $F(b) - F(a)$.

Fundamental Theorem of Calculus and Definite Integral: Uniform Distribution

Example I:

Uniform Distribution Suppose $f : \mathcal{R} \rightarrow \mathcal{R}$, with

$$f(x) = 1 \text{ if } x \in [0, 1]$$

$$f(x) = 0 \text{ otherwise}$$

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$$\int_0^{1/2} f(x) dx = \int_0^{1/2} 1 dx$$

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We will call $f(x) = 1$ the **uniform distribution**.

Fundamental Theorem of Calculus and Definite Integral: Area Under a Line

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Area Under a Line

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Fundamental Theorem of Calculus and Definite Integral: Integration Facts

If $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ and f_1, f_2 are integrable on $[a, b]$, then

- Consider the interval $[a, b]$ and $c \in [a, b]$. Then,

$$\begin{aligned}\int_c^c f_1'(x) dx &= f_1(c) - f_1(c) = 0 \\ \int_a^b f_1'(x) dx &= \int_a^c f_1'(x) dx + \int_c^b f_1'(x) dx \\ &= (f_1(c) - f_1(a)) + (f_1(b) - f_1(c)) \\ &= f_1(b) - f_1(a)\end{aligned}$$

Section 4: Integrals and Distributions

- What is distribution, and why should we care?

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- What is probability mass function (PMF)? What is probability density function (PDF)? What is the difference between the two?

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- What is distribution, and why should we care?
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- What is probability mass function (PMF)? What is probability density function (PDF)? What is the difference between the two?
 1. Intuitively, PDF is the probability that a continuous X drawn in a region that is indefinitely small at the limit of a shrinking process.
 2. Formally, $Pr(X \in [a, b]) = \int_a^b f(x)dx$. The probability of being in some interval $[a, b]$ is the integral of the PDF over that region.

Integrals and Distributions

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Integrals and Distributions

- What is cumulative distribution function (CDF) for both discrete and continuous variables?
 1. Intuitively, if we pick a number, and we want to know the probability that a random draw from a population that produces a value less than that specified number, we sum the individual probabilities of each value below the specified value.
 2. For discrete variable is

$$Pr(Y \leq y) = \sum_{i \leq y} p(i)$$

3. For continuous variable, we replace the sum with an integral to get the equation for the CDF:

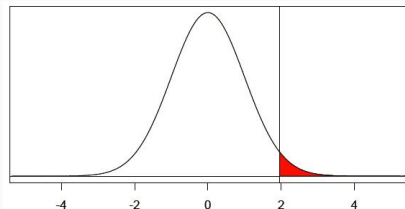
$$Pr(X \leq x) = F(x) = \int_{-\infty}^x f(t)dt$$

4. $0 \leq \text{CDF} \leq 1$.

Integrals and Distributions

Back to the normal distribution that started this off:

What if we want to know the area below the curve when $x \geq 1.96$?



To find the area, we need to integrate the function:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}}$$

To find the area, we need to integrate the PDF function of the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

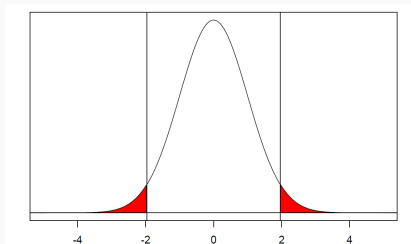
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$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

What's the next step to calculate the area under the curve when $x \geq 1.96$?

Integrals and Distributions

The area under the curve when $x \geq 1.96$ is 0.025. The function is symmetric, so the area when $x \leq -1.96$ is also 0.025. Together it is 0.05.



P-values are integrals under the normal curve. This is where the famous P-value coming from.

One of the most important **assumption** is the continuous variable X follows a normal distribution.

Section 5: References

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Alicia Uribe-McGuire. Math Camp Day 1. University of Illinois at Urbana-Champaign. August 22, 2017.

Justin Grimmer. Math Camp Day 4. University of Chicago. August 31, 2017.

Moore, Will H. A Mathematics Course for Political and Social Research. Princeton, NJ: Princeton University Press, 2013. Ch. 3, 10, 11.

Math is Fun: Introduction to Integration

Calculus: Single Variable Part 3 - Integration