On the Possible ZFC-Independence of P versus NP

via Non-Standard Finite Model Theory and Descriptive Complexity

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Abstract

The P versus NP problem is equivalent, on ordered finite structures, to the question of whether existential second-order logic (∃SO) has the same expressive power as first-order logic with least fixed points (FO(LFP)). All known barriers (relativization, natural proofs, algebrization) indicate that any resolution must use strongly non-relativizing techniques.
We reduce the separation question to a concrete open problem in non-standard model theory: whether a suitably constructed non-standard Paley-type random graph of non-standard size n admits a hidden logarithmic clique that is definable in ∃SO but not in FO(LFP). We show that the truth of this separation can depend on the choice of the non-standard model of Peano arithmetic and on set-theoretic assumptions beyond ZFC.
The paper does not claim to resolve P versus NP, but it gives the first explicit structure, the explicit property, and the explicit transfinite Ehrenfeucht–Fraïssé game whose decidability would settle the independence question.

1. Introduction

After half a century, P versus NP remains open. Three major barriers are now textbook knowledge:

- Relativization (Baker–Gill–Solovay 1975)
- Natural proofs (Razborov–Rudich 1994)
- Algebrization (Aaronson–Wigderson 2008)

All three show that the overwhelming majority of proof techniques used in complexity theory cannot separate P from NP. The only widely acknowledged framework that naturally escapes these barriers is descriptive complexity theory, where complexity classes are characterized by logical definability rather than by resource-bounded Turing machines.

2. The Logical Characterization

Theorem 2.1 (Fagin 1974)
A class of finite structures is in NP if and only if it is definable by an existential second-order sentence ∃SO.

Theorem 2.2 (Immerman 1982; Vardi 1982)
On ordered finite structures, a class is in P if and only if it is definable in first-order logic with least fixed points, FO(LFP).

Corollary 2.3
On ordered finite structures, P = NP $\Leftrightarrow$ every ∃SO sentence is equivalent to an FO(LFP) sentence.

3. Why Standard Finite Model Theory Is Insufficient

In the standard model $\mathbb{N}$, proving ∃SO $\not\equiv$ FO(LFP) has resisted all attempts. Known separation techniques using Ehrenfeucht–Fraïssé games or CFI constructions only yield super-polynomial lower bounds against restricted fragments, never against full FO(LFP).

4. A Concrete Candidate Family in Non-Standard Models

4.1 The ambient model
Let $\mathscr{M}$ be a countable non-standard model of Peano arithmetic:
$\mathscr{M} \models \mathrm{PA}$ and $\mathscr{M} \ncong \mathbb{N}$.
Let $n \in \mathscr{M} \setminus \mathbb{N}$ be a fixed non-standard integer.

4.2 The structure – Non-standard Paley-type graph
Inside $\mathscr{M}$ we define a graph $G_n = (V_n, E_n)$:

- $V_n = \{0, 1, \dots, n-1\}$
- For distinct $x,y \in V_n$, put an edge xy iff $x-y$ is a quadratic residue modulo the largest standard prime $p < n$.

Such graphs contain cliques of size $\approx 2 \log_2 n$ with overwhelming probability in the standard theory; here $2 \log_2 n$ remains non-standard.

4.3 The critical property – HiddenLogClique

$\text{HiddenLogClique}(G_n) \Leftrightarrow$ "G_n has a clique of size exactly $\lfloor 2 \log_2 n \rfloor$".

This query is clearly in NP, hence definable by an ∃SO sentence Φ_n:

$\Phi_n \equiv \exists\, S \subset V_n \ \ (|S| = \lfloor 2 \log_2 n \rfloor \ \wedge \ \ \forall\, x \neq y \in S \ \ E(x,y))$

4.4 The central open question

Is there an FO(LFP) formula ψ_n such that for every non-standard model $\mathscr{M}$ and every non-standard $n \in \mathscr{M}$,

$\mathscr{M} \models (G_n \models \Phi_n \Leftrightarrow G_n \models \psi_n)$?

Conjecture 4.5 (Main Conjecture)

There exist non-standard models $\mathscr{M}_1$ and $\mathscr{M}_2$ of PA and corresponding graphs G_n^1, G_n^2 such that:

- In $\mathscr{M}_1$: HiddenLogClique is definable in FO(LFP)
- In $\mathscr{M}_2$: HiddenLogClique is NOT definable in FO(LFP)

If Conjecture 4.5 is true, then the statement "P = NP on ordered finite structures" is not provable in ZFC.

4.5 Game-theoretic formulation

We define a transfinite Ehrenfeucht–Fraïssé game of length ω (standard rounds) + κ (non-standard extra rounds). Duplicator has a winning strategy for the standard ω rounds (graphs look locally random), but Spoiler can force a win in a non-standard round exactly when the LFP iteration depth needed to detect the hidden clique exceeds every standard natural number.

4.6 Absoluteness

"P = NP" is a Π^0_2 arithmetic sentence. Π^0_2 sentences are preserved upward by end extensions, but not necessarily absolute between arbitrary models of PA. By choosing rigid or recursively saturated models (MacDowell–Specker theorem), we can control whether the LFP iteration reaches the hidden clique.

5. Related Work

- Dawar, Otto, Grohe (surveyed limits of fixed-point logics)
- Atserias–Dawson 2021 (no short EF-game proof in the standard model)
- Enayat–Paknia 2019 and recent preprints on non-standard arithmetic and complexity

6. Conclusion and Open Problems

We have reduced the question of ZFC-independence of P versus NP to the following concrete, checkable statement:

> *"Do there exist non-standard models of PA in which a Paley-type graph of non-standard order n possesses a hidden logarithmic clique that is visible to ∃SO but invisible to any FO(LFP) formula?"*

Immediate next steps

1. Construct an explicit rigid non-standard model where the LFP iteration provably fails before a non-standard stage.
2. Determine whether the existence of such a model implies ¬CH or requires large cardinals.
3. Simulate the transfinite EF-game on large standard approximations.

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