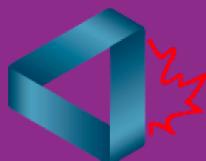




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Crux Mathematicorum with Mathematical Mayhem

Former Editors / Anciens Rédacteurs: Bruce L.R. Shawyer, James E. Totten, Václav Linek,
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Upcoming special issue in memory of Bruce Shawyer

We were saddened by the news of Bruce Shawyer's recent passing. Bruce contributed in many ways to the Canadian mathematics community, including as a coach for several Canadian teams at the International Mathematical Olympiad (IMO), organizing the IMO held in Canada in 1995, and serving as the Editor-in-Chief of *Crux Mathematicorum with Mathematical Mayhem* from 1996 to 2001. To honour his memory, we will have a special issue in fall 2022.

This issue will be dedicated to Bruce Shawyer in November 2022. This is a call for submissions. We encourage problem proposals, articles, expositions of Bruce's contributions to mathematics and math education, tributes to Bruce, and reminiscences.

Please circulate this call to others in the math community, particularly those who may have known Bruce Shawyer. If you would like to contribute to the issue, please send any materials to crux-editors@cms.math.ca by August 15th.

MATHEMATTIC

No. 32

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by May 1, 2022.



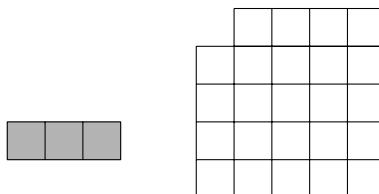
MA156. In a kindergarten, 17 children made an even number of postcards. Any group of 5 children made no more than 25 postcards while any group of 3 children made no less than 14 postcards. Determine the total number of postcards made.

MA157. The four sides and one diagonal of a quadrilateral have lengths 1, 2, 2.8, 5 and 7.5, not necessarily in that order. Determine which number was the length of the diagonal.

MA158. *Proposed by Aravind Mahadevan.*

A semi-circle is inscribed in ΔABC such that it is tangent to AB and AC and its diameter lies along the side BC . If $AB = 13$, $AC = 14$ and $BC = 15$, find the radius of the semi-circle. (Solvers may find Heron's formula for the area of a triangle with sides a , b , and c useful: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.)

MA159. A 5-by-5 square consists of 25 1-by-1 small squares. If one corner square is removed, prove that it is not possible to cover the rest of the squares by eight 3-by-1 rectangles as shown in the figure.



MA160. In a right triangle, the smallest height is one-quarter the length of the hypotenuse. Determine the measure, in degrees, of the smallest angle of this triangle.



Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2022**.*



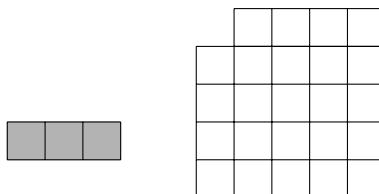
MA156. Dans une certaine garderie, les 17 enfants ont produit un nombre pair de cartes postales. Tout ensemble de 5 enfants a produit au plus 25 cartes postales, tandis que tout ensemble de 3 enfants a produit au moins 14 cartes postales. Déterminer le nombre total de cartes produites en cette garderie.

MA157. Les quatre côtés et une des diagonales d'un certain quadrilatère ont les longueurs 1, 2, 2.8, 5 et 7.5, mais pas nécessairement dans cet ordre. Déterminer lequel de ces nombres représente la diagonale de ce quadrilatère.

MA158. *Proposé par Aravind Mahadevan.*

Un demi cercle est inscrit dans $\triangle ABC$ de façon à ce qu'il soit tangent à AB et AC , puis que son diamètre se situe sur le côté BC . Si $AB = 13$, $AC = 14$ et $BC = 15$, déterminer le rayon de ce demi cercle. (Il sera possiblement utile de faire intervenir la formule de Héron pour la surface d'un triangle de côtés a , b et c : $A = \sqrt{s(s-a)(s-b)(s-c)}$, où $s = \frac{a+b+c}{2}$.)

MA159. Un carré de taille 5 par 5 consiste de 25 petits carrés de taille 1 par 1. On lui enlève un carré de coin, tel qu'illustré. Démontrer qu'il est impossible de pavier le grand carré modifié, à l'aide de huit tuiles rectangulaires de taille 3 par 1.



MA160. Dans un certain triangle rectangle, la plus courte altitude est de longueur égale au quart de la longueur de l'hypoténuse. Déterminer, en degrés, le plus petit angle de ce triangle.



MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(7), p. 326–328.



MA131. Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.

Originally from 2009 Alberta High School Mathematics Competition, Part II, problem 5.

We received 11 submissions of which 10 were correct and complete. We present the solution by Richard Hess, slightly edited.

Suppose that $k = a^3 + b^3$ for positive integers a and b and that $k \equiv 1 \pmod{3}$. In this case, write $k = 3m + 1$ for m a positive integer, and note that

$$k^k = k^{3m} \cdot k = k^{3m} \cdot (a^3 + b^3) = (ak^m)^3 + (bk^m)^3;$$

that is, k^k is a sum of two cubes.

Any choice of a and b that satisfies either $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$ or $a \equiv b \equiv 2 \pmod{3}$ results in $k \equiv 1 \pmod{3}$, so there are infinitely many choices of a and b that will satisfy the desired conditions.

For the case $a \equiv 1 \pmod{3}$ and $b \equiv 0 \pmod{3}$, we get the solutions $k = 28, 91, 217, 280$ and so on.

For the case $a \equiv b \equiv 2 \pmod{3}$, we get the solutions $k = 16, 133, 250, 520$ and so on.

MA132. *Proposed by Ed Barbeau.*

Determine all sets consisting of an odd number $2m + 1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the smallest $m + 1$ integers is equal to the sum of the largest m integers.

We received 8 solutions. We present the solution by Digby Smith, lightly edited.

Given the positive integer m , let k be a positive integer such that

$$k + (k + 1) + \cdots + (k + m) = (k + m + 1) + (k + m + 2) + \cdots + (k + 2m).$$

Rearranging gives

$$(m + 1)k + (1 + 2 + \cdots + m) = mk + (1 + 2 + \cdots + m) + m^2,$$

making $k = m^2$. Hence for each positive integer m there exists exactly one set satisfying the property, namely $\{m^2, m^2 + 1, \dots, m^2 + 2m\}$.

MA133. *Proposed by Nguyen Viet Hung.*

Find all pairs (x, y) of positive integers satisfying the equation

$$x^2 - 2x + 29 = 7^x y.$$

We received 9 solutions, of which we present the one by Mohammad Bakkar, slightly expanded by the editor.

Notice that the right hand side is divisible by 7. Since the left hand side can be rewritten as

$$x^2 - 2x + 29 = (x - 1)^2 + 28,$$

we see that $x \equiv 1 \pmod{7}$. Set $x = 7k + 1$ for some non-negative integer k . Then

$$(7k)^2 + 28 = 7^x \cdot y, \quad \text{or equivalently} \quad 7k^2 + 4 = 7^{x-1} \cdot y.$$

If $x > 1$, then the right hand side is divisible by 7, so we have to have $x = 1$. We obtain $x = 1, y = 4$ as the only solution.

MA134. If the perimeter of an isosceles right-angled triangle is 8, what is its area?

Originally from 2003 Manitoba Mathematical Contest, problem 3b.

We received 11 submissions, all of which were correct. We present the solution by Bala Venkataaraman, modified by the editor.

Let a be the side length of one of the triangle's legs. The hypotenuse therefore has length $\sqrt{2}a$. By assumption, $2a + \sqrt{2}a = 8 \Rightarrow a = \frac{8}{2+\sqrt{2}}$. As the area A of an isosceles right triangle is given by $A = \frac{a^2}{2}$, we have that $A = 16(3 - 2\sqrt{2})$.

MA135. 80 students responded to a survey about sports they played.

30 played basketball.

26 played rugby.

28 played hockey.

12 played basketball and rugby.

8 played hockey and rugby.

x played basketball and hockey only.

4 played all 3 sports.

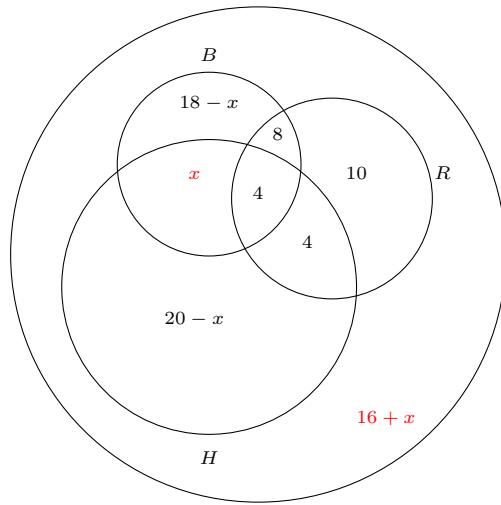
Twice as many played none of the 3 sports as played basketball and hockey only.

If a student is picked at random from the whole group, what is the probability that the student plays only 1 of the 3 sports?

Originally question 2 of The π Quiz 2017, Round 8 by the Irish Maths Teachers' Association.

We received 5 submissions of which 3 were correct and complete. We present the solution by Miguel Amengual Covas.

Let B , H , R represent all students of the group that played basketball, hockey, rugby, respectively. According to the following Venn diagram



the required probability is $\frac{(18 - x) + (20 - x) + 10}{80}$ i.e., $\frac{24 - x}{40}$. We are told also that $16 + x = 2x$, which gives $x = 16$.

Thus, the required probability is $\frac{24 - 16}{40} = 0.2$

MA136. *Sent in by Ed Barbeau, from correspondence with Harold Reiter.*

Solve the alphametic

$$SETA - ATES = EAST$$

where $S > E > T > A$ are digits in the 4-digit numbers.

We received 5 solutions. The following is by Huang Aaron.

From the equation

$$SETA - ATES = (S - A)(E - T - 1)(T - E + 9)(A - S + 10) = EAST$$

we obtain

$$\begin{aligned} S - A &= E \\ E - T - 1 &= A \\ T - E + 9 &= S \\ A - S + 10 &= T. \end{aligned}$$

Adding the first and fourth equation gives

$$E + T = 10.$$

Adding the second and third equation gives

$$A + S = 8.$$

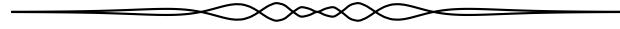
Observe that $E \geq 6$ and $S \leq 7$, we must have

$$S = 7$$

and

$$E = 6.$$

Hence $T = 4$ and $A = 1$ and the problem is solved.



From the bookshelf of . . .

Andy Liu

This new feature of MathemAttic brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to MathemAttic@cms.math.ca. Publishers are also welcome to send along books for possible review.

A Mixed Bag

by Raymond Smullyan

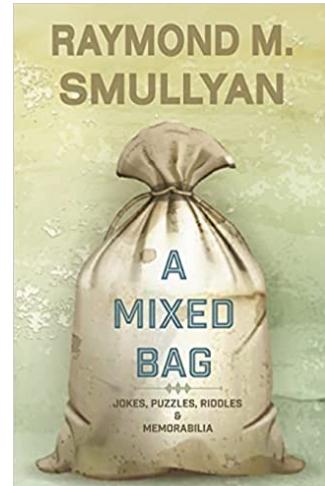
ISBN 978-0-9861445-7-8, softcover, 144+ pages

Published by Sagging Meniscus in 2016.

I call Raymond Smullyan, who passed away in 2017 at age 98, a 3M logician (explained later). He himself regarded Kurt Gödel as the greatest of all logicians. Indeed, Gödel's Incompleteness Theorem in 1931 was a major result in metamathematics that shook the foundation of the world of mathematics. Raymond had devoted his life to disseminating Gödel's idea to the general public in an ingenious, illuminating and invigorating manner.

This is primarily done through a sequence of eleven books of logic puzzles. In order of publication, they are:

- [1] *What is the Name of this Book?*
- [2] *The Lady or the Tiger?*
- [3] *Alice in Puzzleland.*
- [4] *To Mock a Mocking Bird.*
- [5] *Forever Undecided.*
- [6] *Satan, Cantor, and Infinity.*
- [7] *The Riddles of Scheherazade.*
- [8] *Logical Labyrinth.*
- [9] *King Arthur in Search of his Dog.*
- [10] *The Gödelian Puzzle Book.*
- [11] *The Magic Garden of George B.*



A Mixed Bag consists of largely a collection of personal reminiscences: how he got married to the musician Blanche, how he got hired by the prestigious Dartmouth College before he even had a bachelor's degree, how he got that degree from the University of Chicago on the strength of courses he never took but had taught, how he met Kurt Gödel at the Institute of Advanced Studies, and so on. Many of

the accounts reflected on his additional lives as a mathematician, a magician and a musician.

Humor was an essential component of Raymond's character. This book is spiced with many fantastic jokes. Here are some samples. They are chosen because they are short. There are much better ones.

1. Teacher: If your father has ten dollars and you ask for six, how many will he still have?
Kid: Ten.
Teacher: You don't know your math.
Kid: You don't know my father.
2. Teacher: There are three kinds of people — those who can count and those who can't.
Kid: Five out of four people can't count.
3. Patient: My trouble is that I believe I am a dog.
Doctor: Since when?
Patient: Since I was a *puppy*.
4. Patient: My trouble is that I am losing my memory.
Doctor: Since when?
Patient: Since when *what*?

What is a book by Raymond without puzzles? There are plenty of them. They may loosely be classified into two kinds. The first is what Raymond called monkey tricks. Here are two of them.

Puzzle 1. A certain man had great grandchildren, yet none of his grandchildren had any children! How is this possible?

Puzzle 2. A man was driving along a highway. His headlights were broken, there were no street lights on and there was no moon out. There was a pedestrian crossing the street about a hundred and fifty yards in front of him. The driver knew that the pedestrian was there and stopped his car in time to avoid hitting him. How did he know that the pedestrian was there?

The second is the more traditional logical-reasoning problems, even though Raymond had a knack of making traditions unconventional. Here are two of them.

Puzzle 3. You have a line of people. The first one in line is married and the last one is not. Prove that at least one married one is directly in front of an unmarried one.

Puzzle 4. In a certain flower garden, each flower was either red, yellow or blue. All three colors were actually represented. One statistician observed that whichever three flowers were picked, at least one was bound to be yellow. Another observed that whatever three flowers were picked, at least one was bound to be red. From these two observations, does it logically follow that given any three of the flowers, at least one is bound to be blue?

The answers to these puzzles are given in the book, which also has many stories, often with a philosophical bent. Here is one of them.

A monk came up the mountain to interview the Master, who asked him whether he came from the North or the South. “The South,” was the reply. “In that case,” said the Master, “have a cup of tea.” The next morning, another monk came up the mountain for an interview, and the Master likewise asked him whether he came from the North or the South. This time, the monk said he had come from the North. “In that case,” said the Master, “have a cup of tea.” Later on, the Master’s assistant said to him: “I don’t understand, Master; you told one that since he was from the South, he should have a cup of tea, and the other, that since he was from the North, he should have a cup of tea. How come?” The Master replied: “Have a cup of tea.”

The book ended with another such story.

A certain great Sage in the East was reputed to be the wisest man in the world. A philosopher heard about him and was anxious to meet him. It took him fifteen years to find him, but when he finally did, he asked him: “What is the best question that can be asked, and what is the best answer that can be given?” The great Sage replied: “The best question that can be asked is the question you have asked, and the best answer that can be given is the answer I am now giving.”

The review was provided by Andy Liu. Andy ran a Mathematical Circle for Edmonton upper elementary and junior high students from 1981 to 2012. He has given lectures to students in six continents. He has been the vice-president of the International Mathematics Tournament of the Towns since 1992. He had been involved in various capacities in the International Mathematical Olympiad from 1981 to 2016. He regularly attended the International Puzzle Party and the Gathering for Gardner from 1991 to 2018. Andy was involved with Crux Editorial Board as a Book Review editor from 1990 to 1998. He has authored eighteen mathematics books so far, and edited several others.

TEACHING PROBLEMS

No.15

Margo Kondratieva

Basic geometric configurations: Which one do you see?

Teaching Problems usually features problems that have been integrated into the teaching experiences of the contributors. This issue is a little different in that the experience with a problem in a math contest suggests its potential merit as one to be incorporated into teaching. The example draws forth a range of methods of solution as well as identifying links to different known geometrical results. Others involved in math contests may wish to share such examples from their experiences in future issues. Contributions to Teaching Problems are welcomed via mathemattic@cms.math.ca.

Why do people suggest different solutions to a given problem? George Pólya in his famous book, *How to solve it?*, proposed that in problem solving one starts with understanding and analysing what is given and what is to be found. Once a problem is understood the solver needs to design an action plan. For that, one looks at relevant facts that could connect the known and unknown data. Speaking about Euclidean geometry, some geometrical facts can be stored in our memory in the form of figures that illustrate a required property, possibly indicating a reason for it. I call such figures basic geometric configurations (BGCs). The choice of your solution depends on the BGCs that come to mind in association with the problem. Here is an illustration of how this works.

Consider a geometry problem from the *2021 Canadian Open Mathematics Challenge* (part B, number 3). Note, we will use the notation $[PQR]$ to denote the area of triangle PQR .

Two right triangles ΔAXY and ΔBXY have a common hypotenuse XY and side lengths (in units) $|AX| = 5$, $|AY| = 10$, and $|BY| = 2$. Sides AY and BX intersect at P . Determine the area (in square units) of ΔPXY .

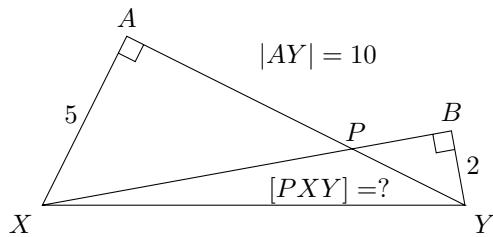


Figure 1: The problem.

The BGC “Pythagorean theorem” allows us to find one side in a right triangle if two other sides are known. Thus, we find the length $|XY| = \sqrt{125}$ from the right triangle XAY and then the length of $|XB| = 11$ from the right triangle XBY . In order to use the theorem one needs to focus on the relevant part of the figure and ignore the rest, as illustrated in Figure 2.

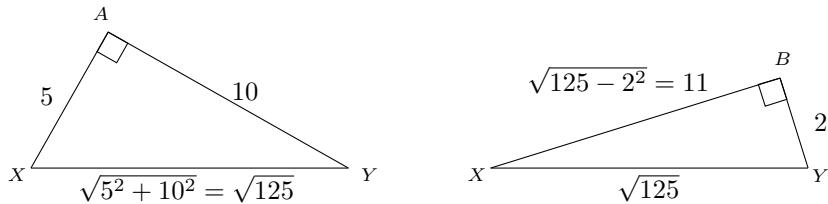


Figure 2: Applications of the BGC “Pythagorean theorem” to the cases of right triangles XAY and XBY .

Another well known BGC depicts the fact that the area of a triangle is equal to “a half of base times height”. Drop the height from P on XY . Call the foot D . Then we have

$$[PXY] = \frac{1}{2}|XY| \cdot |PD| = \frac{1}{2}|XP| \cdot |BY| = \frac{1}{2}|YP| \cdot |AX|$$

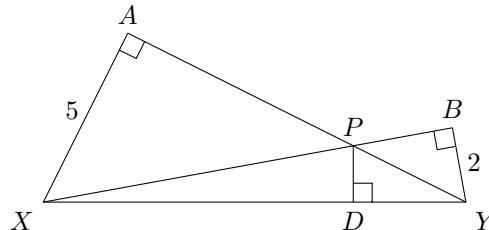


Figure 3: The heights and bases in the triangle PXY .

Since we know $|XY| = \sqrt{125} = 5\sqrt{5}$, $|AX| = 5$ and $|BY| = 2$, the problem then reduces to finding the length of either $|PD|$ or $|XP|$ or $|YP|$.

Below we consider six different solutions of the problem, each of which is related to a more sophisticated BGC. We start with presenting three methods of finding $|PD|$.

Method 1. Figure 4 depicts the following properties. In an acute triangle KLM , let the segments KK' , LL' and MM' be the heights and P be the orthocentre. Then we have:

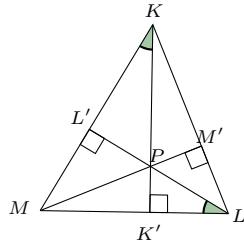


Figure 4a: BGCs depicting properties of the heights in acute and right triangles.

BGC 1: The ratio of two heights in an acute triangle is equal to the ratio of the lengths from the feet of these heights to the third vertex of the triangle:

$$\frac{|KK'|}{|LL'|} = \frac{|MK'|}{|ML'|}. \quad (1)$$

Proof: As seen in Figure 4a,

$$\tan(\angle L'MK') = \frac{|KK'|}{|MK'|} = \frac{|LL'|}{|ML'|}$$

which is equivalent to (1).

BGC 2: The product of a height and the distance from the orthocentre to the foot of this height is equal to the product of the distances from that foot to the two other vertices of the triangle:

$$|KK'| \cdot |PK'| = |MK'| \cdot |LK'|. \quad (2)$$

Proof: Relation (2) is equivalent to $\frac{|KK'|}{|MK'|} = \frac{|LK'|}{|PK'|}$. The latter is true because $\angle PKL' = \angle PLK'$, as indicated in Figure 4a, and so $\cot(\angle PKL') = \cot(\angle PLK')$.

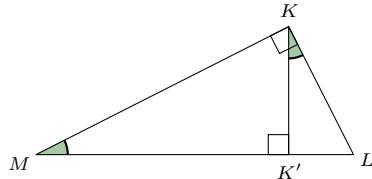


Figure 4b: BGCs depicting properties of the heights in right triangles.

In the special case when $\angle MKL = 90^\circ$, points P , L' , and M' coincide with K . BGC1 then becomes

$$\frac{|KK'|}{|LK|} = \frac{|MK'|}{|MK|}, \quad (1')$$

which is true due to the fact that $\angle LKK' = \angle KMK'$, as indicated in Figure 4b. Then BGC 2 reduces to the following fact.

BGC 2': If $\angle MKL = 90^\circ$, we have, from Figure 4b:

$$|KK'|^2 = |MK'| \cdot |LK'|. \quad (2')$$

Remark: This formula could be alternatively derived by noticing that MKK' and KLK' are similar right triangles, and so $\frac{|MK'|}{|KK'|} = \frac{|KK'|}{|LK'|}$.

The facts (1) and (2) become useful for solving our problem if we identify the points M, L, L', M' with the points X, Y, A, B respectively. By extending XA and YB we obtain the point of intersection K . Point $K' = D$ is the foot of the perpendicular dropped from P on XY .

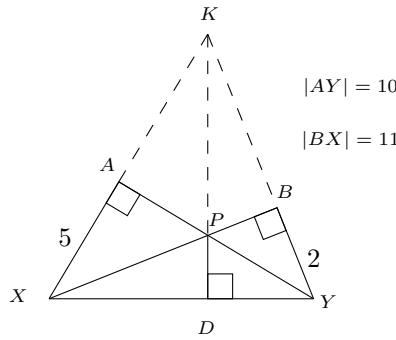


Figure 5: Setting up our problem for the application of BGC 1 and BGC 2.

These BGCs lead to the following solution of our problem.

Solution 1: Applying BGC 1 to Figure 5, we get:

$$\frac{|KD|}{|YA|} = \frac{|XD|}{|XA|} \Rightarrow \frac{|KD|}{10} = \frac{|XD|}{5}$$

and hence

$$|XD| = \frac{1}{2}|KD|.$$

Similarly,

$$\frac{|KD|}{|XB|} = \frac{|YD|}{|YB|} \Rightarrow |YD| = \frac{2}{11}|KD|.$$

Then,

$$|XD| + |YD| = \left(\frac{1}{2} + \frac{2}{11}\right)|KD| = \frac{15}{22}|KD|.$$

On the other hand,

$$|XD| + |YD| = |XY| = 5\sqrt{5},$$

so, $\frac{15}{22}|KD| = 5\sqrt{5}$. Therefore, $|KD| = \frac{22}{3}\sqrt{5}$, $|XD| = \frac{11}{3}\sqrt{5}$ and $|YD| = \frac{4}{3}\sqrt{5}$.

Now from BGC 2, $|PD| = \frac{|XD| \cdot |YD|}{|KD|} = \frac{2\sqrt{5}}{3}$. Therefore

$$[XPY] = \frac{1}{2}|XY| \cdot |PD| = \frac{25}{3}.$$

Method 2. The following fact relates the vertical segments depicted in Figure 6.

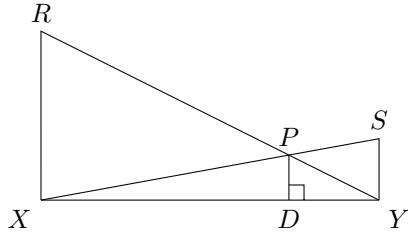


Figure 6: BGC 3 “two right triangles with common leg”

BGC 3: Let RXY and SXY be two right triangles for which leg XY is shared and the hypotenuses intersect at P . Let PD be perpendicular to XY . Then

$$\frac{1}{|PD|} = \frac{1}{|RX|} + \frac{1}{|SY|}. \quad (3)$$

Proof: Since PDY and RXY are similar right triangles, we have $\frac{|YD|}{|PD|} = \frac{|XY|}{|RX|}$. Since PDX and SYX are similar right triangles, we have $\frac{|XD|}{|PD|} = \frac{|XY|}{|SY|}$. Therefore,

$$\frac{|YD|}{|PD|} + \frac{|XD|}{|PD|} = \frac{|XY|}{|RX|} + \frac{|XY|}{|SY|}.$$

On the other hand,

$$\frac{|YD|}{|PD|} + \frac{|XD|}{|PD|} = \frac{|YD| + |DX|}{|PD|} = \frac{|XY|}{|PD|}.$$

Thus, we obtain

$$\frac{|XY|}{|RX|} + \frac{|XY|}{|SY|} = \frac{|XY|}{|PD|}.$$

Dividing through by $|XY|$ gives the required relation.

This BGC leads to the following solution of our problem.

Solution 2: Extend YA to intersect at R with the line perpendicular to XY through X , as shown. Likewise let S be the intersection of XB (extended beyond B) with the line perpendicular to XY through Y in figure 7.

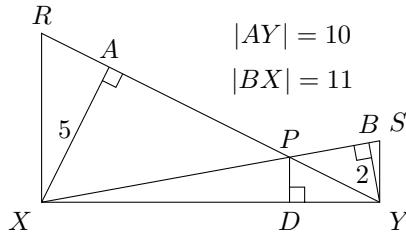


Figure 7: Application of BGC 3 in Solution 2.

In the right triangle RXY , the segment XA is the height dropped to the hypotenuse RY . From BGC 2' we have the relation $|XA|^2 = |RA| \cdot |AY|$. Thus, $|RA| = 5^2/10 = 5/2$. Applying the Pythagorean theorem to right triangle XAR yields

$$|RX| = \sqrt{25 + \frac{25}{4}} = \frac{5\sqrt{5}}{2} = \frac{25}{2\sqrt{5}}.$$

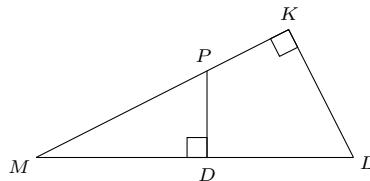
BGC 2' applied to the right triangle SXY gives $|YB|^2 = |SB| \cdot |BX|$, so $|SB| = 4/11$. Then from the right triangle YSB , $|SY| = \sqrt{4 + \frac{16}{121}} = \frac{10\sqrt{5}}{11} = \frac{50}{11\sqrt{5}}$.

Then, from BGC 3,

$$\frac{1}{|PD|} = \frac{2\sqrt{5}}{25} + \frac{11\sqrt{5}}{50} = \frac{3\sqrt{5}}{10}$$

so $|PD| = \frac{10\sqrt{5}}{15} = \frac{2\sqrt{5}}{3}$.

Method 3. This approach is based on a BGC depicting two right triangles that share an acute angle in the following way.



$$\Delta MPD \sim \Delta MLK$$

Figure 8: Similar right triangles BGC 4.

BGC 4: Drop a perpendicular PD from any point P on a leg of a right triangle KLM to its hypotenuse LM . Then the two right triangles KLM and DPM are similar.

BGC 4 leads to the following solution.

Solution 3: Let $|PD| = h$.

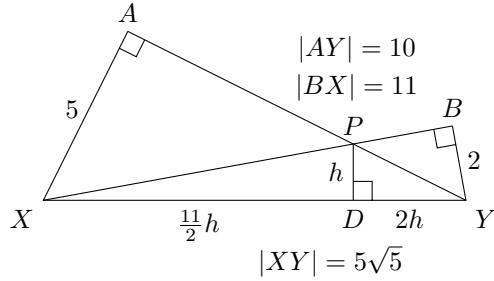


Figure 9: Application of BGC 4 in Solution 3.

Triangles XPD and XYB are similar by BGC 4. Thus,

$$\frac{|XD|}{|XB|} = \frac{|PD|}{|YB|} \Rightarrow \frac{|XD|}{11} = \frac{h}{2}$$

and hence $|XD| = \frac{11}{2}h$. Similarly, triangles YPD and YXA are similar by BGC 4 which yields,

$$\frac{|YD|}{|YA|} = \frac{|PD|}{|XA|} \Rightarrow \frac{|YD|}{10} = \frac{h}{5}$$

and therefore $|YD| = 2h$.

Then

$$|XD| + |YD| = \frac{11}{2}h + 2h = \frac{15}{2}h.$$

On the other hand,

$$|XD| + |YD| = |XY| = 5\sqrt{5},$$

therefore,

$$h = \frac{2}{15}5\sqrt{5} = \frac{2\sqrt{5}}{3} = |PD|.$$

One may identify a geometric configuration involving another pair of similar right triangles.

BGC 5: Given two lines intersecting at P , drop a perpendicular XA from any point X of one line on another one and a perpendicular YB from any point Y of the second line on the first one. Then the right triangles XAP and YBP are similar as shown in Figure 10. This is because $\angle APX = \angle BPY$ as vertical angles and $\angle AXP = \angle BYP$ as complementary to them.

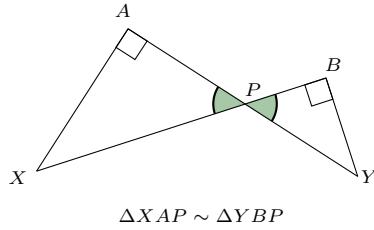


Figure 10: Similar right triangles BGC 5.

Recognition of this BGC may yet lead to different solutions that use distinct properties and relations in similar triangles. We present three of them below.

Solution 4: Triangles XAP and YBP are similar by BGC 5. Then, since the ratio of the sides is $\frac{|AX|}{|BY|} = \frac{5}{2}$ we can deduce that the ratio of the areas is $\frac{[XAP]}{[YBP]} = \left(\frac{5}{2}\right)^2$. Thus, $[XAP] = \frac{25}{4}[YBP]$.

Labelling by x, y the area of the triangles PXY and YBP respectively, as shown in Figure 11, we get the following expressions:

$$\begin{aligned}[XYA] &= [PXY] + [XAP] = x + \frac{25}{4}y \\ [XYB] &= [PXY] + [YBP] = x + y.\end{aligned}$$

On the other hand, $[XYA] = \frac{1}{2}|XA| \cdot |YA| = 25$ and $[XYB] = \frac{1}{2}|XB| \cdot |YB| = 11$.

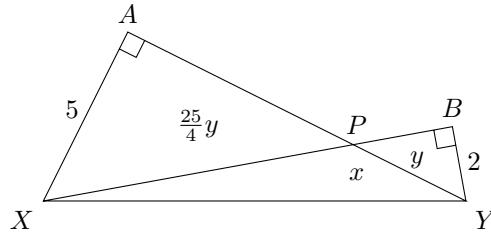


Figure 11: Application of BGC 5 in Solution 4

Thus, we obtain the system of equations

$$\begin{aligned}x + \frac{25}{4}y &= 25 \\ x + y &= 11\end{aligned}$$

Solving the system we get $x = \frac{25}{3}$ and $y = \frac{8}{3}$. Thus, the desired area is $\frac{25}{3}$ square units.

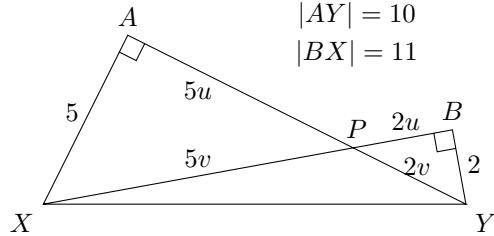
Solution 5:

Figure 12: Application of BGC 5 in Solution 5

Triangles XAP and YBP are similar by BGC 5. We have $\frac{|AP|}{|PB|} = \frac{|XP|}{|PY|} = \frac{5}{2}$, so $|AP| = 5u$, $|PB| = 2u$, $|XP| = 5v$, $|PY| = 2v$, for some numbers u and v . Thus,

$$\begin{aligned}|BX| &= 2u + 5v = 11 \\ |AY| &= 5u + 2v = 10\end{aligned}$$

Solving this system we obtain $u = \frac{4}{3}$ and $v = \frac{5}{3}$. Thus, $|XP| = \frac{25}{3}$.

Therefore, the desired area is $\frac{1}{2}|XP| \cdot |BY| = \frac{25}{3}$ square units.

Solution 6: Let $|AP| = x$. Then $|YP| = 10 - x$.

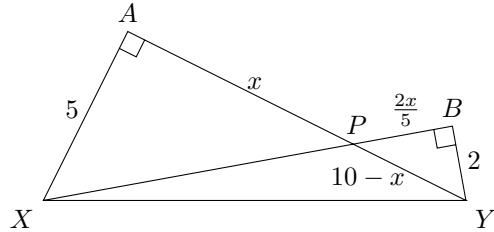


Figure 13: Application of BGC 5 in Solution 6

Triangles XAP and YBP are similar by BGC 5. We have

$$\frac{|PB|}{|AP|} = \frac{2}{5} \Rightarrow |PB| = \frac{2}{5}x \Rightarrow |PB|^2 = \frac{4x^2}{25}.$$

On the other hand, from the right triangle PBY ,

$$|PB|^2 = |PY|^2 - |YB|^2 = (10 - x)^2 - 4 = 96 - 20x + x^2.$$

Thus, $\frac{4x^2}{25} = 96 - 20x + x^2$ and so

$$21x^2 - 500x + 2400 = 0.$$

Solving this equation we obtain $x_1 = \frac{20}{3}$ and $x_2 = \frac{120}{7}$. We take the first root $x = \frac{20}{3}$ because of the restriction $|AP| = x < 10 = |AY|$. Then $|YP| = 10 - \frac{20}{3} = \frac{10}{3}$ and the area $[PXY] = \frac{1}{2}|YP| \cdot |AX| = \frac{25}{3}$ square units.

Closing comments

Considering the range of solution methods is there one that spoke to you as being most appealing? Was it an aspect of the presentation or the geometrical ideas in play or some other element that resonated with your problem-solving style?

Note that solutions 1, 2, and 3 require solving a linear equation, while solutions 4 and 5 require solving a system of linear equations and solution 6 requires solving a quadratic equation. Solution 3 is probably the most “economical” which gives an advantage in a competition setting, especially if calculators are not allowed. However, each solution highlights an interesting point of view and is valuable for making mathematical connections. Do you have yet another way of solving this problem?



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OLYMPIAD CORNER

No. 400

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by May 1, 2022.



OC566. Prove that if a and b are real numbers such that $a + b > 2$, then

$$(a - 1)x + b < x^2 < ax + (b - 1)$$

for infinitely many real numbers x .

OC567. In a group of people, there are some mutually friendly pairs. For a positive integer $k \geq 3$, we say that the group is k -good if every k people in the group can be seated around a round table so that every two neighbors are mutually friends. Prove that if the group is 6-good, then it is also 7-good.

OC568. Point K is marked inside a parallelogram $ABCD$. Point M is the midpoint of BC , point P is the midpoint of KM . Prove that if $\angle APB = \angle CPD = 90^\circ$, then $AK = DK$.

OC569. Let ABC be a triangle with $\angle A = 80^\circ$ and $\angle C = 30^\circ$. Let M be an internal point to triangle ABC such that $\angle MAC = 60^\circ$ and $\angle MCA = 20^\circ$. If N is the intersection point of lines BM and AC , prove that MN is the angle bisector of $\angle AMC$.

OC570. Let n be a positive integer. Assume that in the set $\{1, 2, \dots, n\}$ there are exactly M squarefree integers k such that $\left\lfloor \frac{n}{k} \right\rfloor$ is odd. Prove that M is odd.



Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2022**.*



OC566. Démontrer que si a et b sont des nombres réels tels que $a + b > 2$, alors

$$(a - 1)x + b < x^2 < ax + (b - 1)$$

pour un nombre infini de réels x .

OC567. Dans tout ensemble de personnes, il existe des amitiés. Pour un entier $k \geq 3$, on dit qu'un ensemble de personnes est *k-amical* si tout sous ensemble de k personnes peut siéger autour d'une table ronde de façon à ce chacun soit ami avec ses deux voisins. Démontrer que tout ensemble 6-amical est aussi 7-amical.

OC568. Le point K se trouve à l'intérieur d'un certain parallélogramme $ABCD$; M est le point milieu de BC , tandis que P est le point milieu de KM . Démontrer que si $\angle APB = \angle CPD = 90^\circ$, alors $AK = DK$.

OC569. Soit ABC un triangle tel que $\angle A = 80^\circ$ et $\angle C = 30^\circ$; soit aussi M un point à l'intérieur du triangle, tel que $\angle MAC = 60^\circ$ et $\angle MCA = 20^\circ$. Si N est le point d'intersection des lignes BM et AC , démontrer que MN est bissectrice de $\angle AMC$.

OC570. Soit n un entier positif et supposons que l'ensemble $\{1, 2, \dots, n\}$ contient exactement M entiers k sans facteur carré tels que $\left\lfloor \frac{n}{k} \right\rfloor$ est impair. Démontrer que M est impair.



OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(7), p. 338–339.

OC541. In a convex quadrilateral $ABCD$, suppose $\angle ABC = \angle ACD$ and $\angle ACB = \angle ADC$. Assume that the center O of the circle circumscribed to the triangle BCD is different from point A . Prove that triangle OAC is a right triangle.

*Originally from 2018 Czech-Slovakia Math Olympiad, 5th Problem, Category A,
Final Round.*

We received 9 correct solutions. We present two solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.

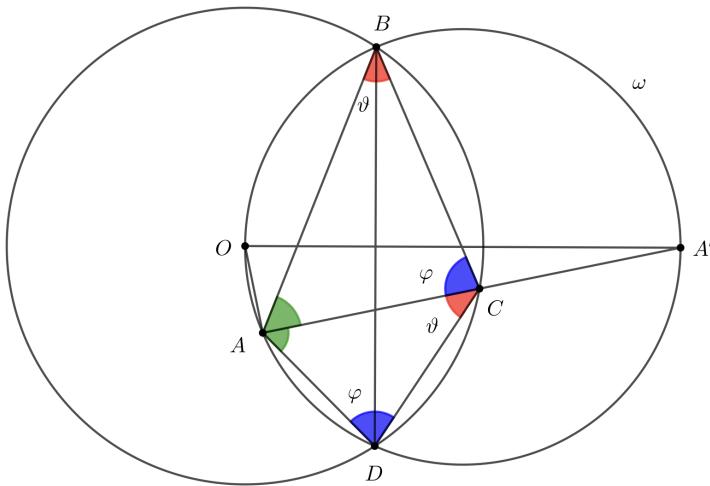
Let $\angle ABC = \angle ACD = \vartheta$ and $\angle ACB = \angle ADC = \varphi$. Then

$$\angle DAC = \angle CAB = 180^\circ - \vartheta - \varphi.$$

So

$$\angle DAB = 360^\circ - 2(\vartheta + \varphi) = 2(180^\circ - \angle BCD) = \angle DOB.$$

So A, O, B, D belong on the same circle, say ω . Let A' be the other point of intersection of AC with ω . Since AC is the angle bisector of $\angle DAB$, then A' belongs on the perpendicular bisector of BD . But so does O , as $OB = OD$. Thus OA' is a diameter of ω and so $OA \perp AC$ as required.



Solution 2, by Michel Bataille.

We embed the problem in the complex plane with origin at A . Let b, c, d be the affixes of B, C, D respectively, and let U , with affix u , be the point of intersection of the perpendicular bisector λ of BC and the line μ perpendicular to AC through A . We answer the problem by showing that U is also on the perpendicular bisector of CD (so that $O = U$ and $OA \perp AC$).

The equation of μ is $\frac{z}{c} + \frac{\bar{z}}{\bar{c}} = 0$, that is, $z\bar{c} + \bar{z}c = 0$. The equation of λ is $|z - b|^2 = |z - c|^2$, that is, $(z\bar{b} + \bar{z}b) - (z\bar{c} + \bar{z}c) = |b|^2 - |c|^2$. Solving the system of these two equations, an easy calculation gives

$$u = \frac{-c(|b|^2 - |c|^2)}{b\bar{c} - \bar{b}c}.$$

(We show below that $b\bar{c} - \bar{b}c \neq 0$).

From the hypotheses about the quadrilateral $ABCD$, a spiral similarity with centre A transforms B into C and C into D , hence, $c = \alpha b$ and $d = \alpha c$ for some nonzero complex number α . First, it follows that $b\bar{c} - \bar{b}c = |b|^2(\bar{\alpha} - \alpha) \neq 0$ since $b \neq 0$ (because $B \neq A$) and $\alpha \neq \bar{\alpha}$ (otherwise α is a real number and A, B, C would be collinear).

Second, we can write u as $u = \frac{c(|\alpha|^2 - 1)}{\bar{\alpha} - \alpha}$.

Finally, we have to check that $|u - c|^2 = |u - \alpha c|^2$, that is

$$-(u\bar{c} + \bar{u}c) + |c|^2 = -(\bar{\alpha}u\bar{c} + \alpha\bar{u}c) + |\alpha|^2|c|^2.$$

Now, recalling that $u\bar{c} + \bar{u}c = 0$, we obtain

$$\bar{\alpha}u\bar{c} + \alpha\bar{u}c = \bar{\alpha}c \frac{c(|\alpha|^2 - 1)}{\bar{\alpha} - \alpha} + \alpha c \frac{\bar{c}(|\alpha|^2 - 1)}{\alpha - \bar{\alpha}} = |c|^2(|\alpha|^2 - 1)$$

hence $|u - c|^2 = |u - \alpha c|^2$ holds and we are done.

OC542. Let x_1, x_2, \dots, x_n be positive integers. Assume that in their decimal representations no x_i "is an extension" of another x_j . For instance, 123 is an extension of 12, 459 is an extension of 4, but 134 is not an extension of 123. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 3.$$

Originally from 2018 Italy Math Olympiad, 3rd Problem, Final Round.

We received 6 correct solutions. We present two solutions.

Solution 1, by Oliver Geupel.

Let us say that the positive integer x *reduces* to the positive integer y , if $y = \lfloor x/10 \rfloor$. We write $x \rightarrow y$ if x reduces to y . Observe that the positive integer x "is

an extension" of the positive integer y if and only if there is a chain $x \rightarrow \dots \rightarrow y$ of reductions from x to y . We say that the finite set M of positive integers *reduces* to the set N of positive integers (and write $M \rightarrow N$) if N is obtained by the following construction: Put the greatest element a_1 of M and reduce it to, say, the positive integer b . Include b , and remove all elements a_1, a_2, \dots, a_m that reduce to b (i.e. $a_1 \rightarrow b, a_2 \rightarrow b, \dots, a_m \rightarrow b$), to obtain

$$N = (M \cup \{b\}) \setminus \{a_1, a_2, \dots, a_m\}. \quad (1)$$

We say that a finite set M of positive integers is *good* if no element of M is an extension of another element of M . Note that if M is good and $M \rightarrow N$, then N is also a good set. Call a good set M of positive integers *irreducible* if it does not reduce to any other set. It is clear that the irreducible sets are the subsets of $\{1, 2, \dots, 9\}$. For a finite set M of positive integers, let $f(M) = \sum_{x \in M} 1/x$. For every irreducible set M , it holds

$$\begin{aligned} f(M) &\leq \sum_{k=1}^9 \frac{1}{k} = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{4} + \frac{1}{5}\right) + \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) \\ &< 1 + 1 + \frac{1}{2} + \frac{3}{7} < 3. \end{aligned}$$

If M is good and $M \rightarrow N$, then we have with the notation (1):

$$\begin{aligned} f(M) &= f(N) + \frac{1}{a_1} + \dots + \frac{1}{a_m} - \frac{1}{b} \\ &< f(N) + \frac{1}{10b} + \frac{1}{10b+1} + \dots + \frac{1}{10b+9} - \frac{1}{b} < f(N). \end{aligned}$$

The set $M_1 = \{x_1, x_2, \dots, x_n\}$ reduces in a finite chain $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_k$ to an irreducible set M_k , where $f(M_1) < f(M_2) < \dots < f(M_k) < 3$.

Solution 2, by UCLan Cyprus Problem Solving Group.

Fix a digit k and suppose that for each m , there are d_m of the x_i 's which begin with k and have m digits. We may assume that the largest x_i which begins with k (if there is one) has N digits. The condition of the problem guarantees that for each N -digit number M beginning with k there is at most one x_i which is a prefix of M . Since an m -digit number is a prefix of exactly 10^{N-m} numbers with M -digits, we get that

$$d_1 \cdot 10^{N-1} + d_2 \cdot 10^{N-2} + \dots + d_N \leq 10^{N-1}$$

or equivalently

$$d_1 + \frac{d_2}{10} + \dots + \frac{d_N}{10^{N-1}} \leq 1.$$

This is actually an immediate application of Kraft's inequality but we chose to spell out the proof as it is not that known in Olympiad circles.

Now we observe that each m -digit number beginning with k is at least $10^m k$ and so contributes at most $\frac{1}{10^m k}$ in the required sum. So the total contribution of the x_i 's beginning with k is at most

$$\frac{d_1}{k} + \frac{d_2}{10k} + \cdots + \frac{d_N}{10^{N-1}k} \leq \frac{1}{k}$$

Therefore

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9} = 2 + \frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} < 2 + \frac{2}{4} + \frac{3}{7} < 3.$$

OC543. There are 50 cards in a box with the first 100 positive integers written on them. That is, the first card has number 1 on one side and number 2 on the other side, the second card has number 3 on one side and number 4 on the other, and so on up to the 50-th card which has number 99 on one side and 100 on the other side. Eliza takes four cards out of the box and calculates the sum of the eight numbers written on them. How many distinct sums can Eliza get?

Originally from 2018 Romania Math Olympiad, 4th Problem, Grade 5, District Round.

We received 5 submissions, of which 4 were correct and complete. We present the solution by Oliver Geupel.

We show that the answer is 185.

For $1 \leq n \leq 50$, the sum of the two numbers written on the card with number n is $(2n - 1) + 2n = 4n - 1$. Hence, the sum of the eight numbers written on four distinct cards with numbers i, j, k , and ℓ is $4(i + j + k + \ell - 1)$. We have

$$10 = 1 + 2 + 3 + 4 \leq i + j + k + \ell \leq 47 + 48 + 49 + 50 = 194.$$

Thus, Eliza cannot get more than $194 - 9 = 185$ distinct sums.

It remains to show that the sum $s = i + j + k + \ell$ can attain every integer value in the range from 10 to 194.

If $s = 4n$ where $3 \leq n \leq 48$, then we put $\{i, j, k, \ell\} = \{n - 2, n - 1, n + 1, n + 2\}$.

If $s = 4n + 1$ where $3 \leq n \leq 48$, we take $\{i, j, k, \ell\} = \{n - 2, n, n + 1, n + 2\}$.

If $s = 4n + 2$ where $2 \leq n \leq 48$, put $\{i, j, k, \ell\} = \{n - 1, n, n + 1, n + 2\}$.

Finally, if $s = 4n + 3$ where $2 \leq n \leq 47$, then $\{i, j, k, \ell\} = \{n - 1, n, n + 1, n + 3\}$ does the job.

OC544. Prove that if $n \geq 2$ is an integer, then there exist invertible matrices $A_1, A_2, \dots, A_n \in \mathcal{M}_2(\mathbb{R})$ with nonzero entries such that

$$A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1} = (A_1 + A_2 + \cdots + A_n)^{-1}.$$

Originally 2018 Romania Math Olympiad, 1st Problem, Grade 11, District Round.

We received 6 correct solutions. We present 2 solutions.

Solution 1, by Michel Bataille.

Let \mathcal{S} be the set of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d, a+b, c+d, ad-bc$ are nonzero real numbers and let $C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Note that $C^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = -C - I_2$ where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix.

Pick any A_1 in \mathcal{S} . It is readily checked that $A_2 = A_1 C$ and

$$A_1 + A_2 = A_1(I_2 + C) = -A_1C^{-1}$$

are in \mathcal{S} . Moreover, we have

$$(A_1 + A_2)^{-1} = -CA_1^{-1}$$

and

$$A_1^{-1} + A_2^{-1} = A_1^{-1} + C^{-1}A_1^{-1} = (I_2 + C^{-1})A_1^{-1} = -CA_1^{-1}.$$

Thus, A_1, A_2 answer the problem for $n = 2$.

We continue the proof by induction: assume that for some integer $n \geq 2$ we have A_1, A_2, \dots, A_n in \mathcal{S} such that $A_1 + A_2 + \dots + A_n \in \mathcal{S}$ and

$$A_1^{-1} + A_2^{-1} + \dots + A_n^{-1} = (A_1 + A_2 + \dots + A_n)^{-1}.$$

Consider $A_{n+1} = (A_1 + A_2 + \dots + A_n)C$. From the case $n = 2$ above, $A_{n+1} \in \mathcal{S}$ and

$$A_1 + A_2 + \dots + A_n + A_{n+1} = (A_1 + A_2 + \dots + A_n) + (A_1 + A_2 + \dots + A_n)C \in \mathcal{S}.$$

In addition, again from the case $n = 2$, we have

$$\begin{aligned} (A_1 + A_2 + \dots + A_n + A_{n+1})^{-1} &= (A_1 + A_2 + \dots + A_n)^{-1} + A_{n+1}^{-1} \\ &= A_1^{-1} + A_2^{-1} + \dots + A_n^{-1} + A_{n+1}^{-1}. \end{aligned}$$

This completes the induction step and the answer to the problem.

Solution 2, by Corneliu-Avram Manescu.

The given equality is equivalent to

$$(A_1 + A_2 + \dots + A_n)(A_1^{-1} + A_2^{-1} + \dots + A_n^{-1}) = I_2.$$

Let $A_1 = A, A_2 = \dots = A_n = B$. The condition becomes

$$(A + (n-1)B)(A^{-1} + (n-1)B^{-1}) = I_2.$$

Denote $X = BA^{-1}$ and multiply the two parentheses. It follows that

$$I_2 + (n-1)(X + X^{-1}) + (n-1)^2 I_2 = I_2,$$

that is,

$$X + X^{-1} + (n-1)I_2 = O_2,$$

i.e.

$$X^2 + (n-1)X + I_2 = O_2.$$

The matrix

$$X = \begin{pmatrix} 1 & n+1 \\ -1 & -n \end{pmatrix}$$

satisfies this equation because $\text{tr}(X) = -(n-1)$ and $\det(X) = 1$ by the Cayley-Hamilton theorem. Now, choose

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and then

$$B = XA = \begin{pmatrix} 1 & n+1 \\ -1 & -n \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} n+3 & n+2 \\ -n-2 & -n-1 \end{pmatrix}.$$

OC545. Solve in real numbers the system of equations

$$\begin{cases} x^2y + 2 = x + 2yz \\ y^2z + 2 = y + 2zx \\ z^2x + 2 = z + 2xy \end{cases}$$

Originally from 2018 Poland Math Olympiad, 3rd Problem, First Round.

We received 11 submissions of which 10 were correct and complete. We present the solution by Oliver Geupel.

A straightforward check shows that $(-1, -1, -1)$, $(1, 1, 1)$, and $(2, 2, 2)$ are solutions for (x, y, z) . We show that there are no other solutions. Let us refer to the given equations as to (1), (2), and (3).

Rearrange the terms to get

$$x(xy - 1) = 2(yz - 1), \quad y(yz - 1) = 2(zx - 1), \quad z(zx - 1) = 2(xy - 1).$$

Multiplying the terms on either sides, we arrive at

$$xyz(xy - 1)(yz - 1)(zx - 1) = 8(xy - 1)(yz - 1)(zx - 1).$$

This means that either $xy = 1$, or $yz = 1$, or $zx = 1$, or $xyz = 8$. We consider the cases in succession.

First, suppose that $xy = 1$. By (1), we obtain $x + 2 = x + 2yz$ and hence $yz = 1$. Similarly, $zx = 1$. Thus, $z^2 = yz \cdot zx/(xy) = 1$, i.e. $z = -1$ or $z = 1$. If $z = -1$, then we obtain $x = zx/z = -1$ and $y = xy/x = -1$. On the other hand, if $z = 1$, then we arrive at $x = zx/z = 1$ and $y = xy/x = 1$. The cases $yz = 1$ and $zx = 1$ are each similar by the cyclic structure of the given equations.

It remains to consider the case when $xyz = 8$. If two of the numbers x , y , and z are negative, say, $x < 0$ and $y < 0$, then we get $y^2z + 2 > 0 > y + 2zx$, a contradiction to (2). Therefore, x , y , and z are positive. Make the substitution $(a, b, c) = (2/x, 2/y, 2/z)$. Then, a , b , and c are positive real numbers such that $abc = 1$. By the AM-GM inequality, we have $3a^2 + a \geq 4a^{7/4}$ with similar inequalities in b and c , respectively. We rewrite the given system of equations as

$$\begin{cases} 4c + a = 1 + 4a^2 \\ 4a + b = 1 + 4b^2 \\ 4b + c = 1 + 4c^2. \end{cases}$$

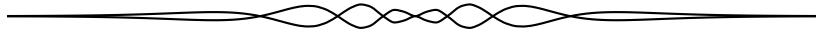
It follows that

$$4c = 1 + 4a^2 - a = (1 - a)^2 + 3a^2 + a \geq (1 - a)^2 + 4a^{7/4}.$$

Analogously, $4a \geq (1 - b)^2 + 4b^{7/4}$ and $4b \geq (1 - c)^2 + 4c^{7/4}$. We prove by contradiction that $a = b = c = 1$. Assuming the contrary, we obtain

$$64 = 4a \cdot 4b \cdot 4c > 4a^{7/4} \cdot 4b^{7/4} \cdot 4c^{7/4} = 64,$$

which is absurd. This shows that $a = b = c = 1$, i.e. $x = y = z = 2$.



The Last Problem: Demystified

Sam Hopkins

The June 2021 issue of *Crux Mathematicorum* included, on page 287, an intriguing problem entitled simply “The Last Problem.” Here it is:

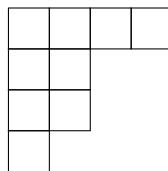
A $m \times n$ rectangular array is made up of the positive integers 1, 2, 3, ..., mn arranged in such a way that each row and each column is monotonically decreasing. In particular, mn must appear in the upper left corner and 1 in the lower right corner. An operator on the array is as follows. The number in the lower right corner is circled. Once any number is circled, the smaller of two of its neighbours, one immediately to the left in the same row and the other immediately above in the same column, is also circled. If there is only one such number, it is circled. In this way, a track of $m + n - 1$ circled numbers from the lower right to the upper left is obtained. Now the number in the lower right is transferred to the upper left position and the rest of the circled numbers are displaced one position along the track. The uncircled numbers are not moved. The same operation is then repeated, with the understanding that, once any number k is transferred from the lower right position to the upper left position, it is treated as though its magnitude were $mn + k$.

Prove or disprove:

- (a) After mn operations, each number in the array is restored to its initial position;
- (b) If i moves down on the j th move, then j moves down on the i th move;
- (c) If i moves right on the j th move, then j moves right on the i th move.

Here we aim to demystify this problem, and, if not exactly explain its solution, at least situate it in its proper mathematical context.

A *partition* of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$. For example, $\lambda = (4, 2, 2, 1)$ is a partition of 9. Associated to any partition is its *Young diagram*: the left- and top-justified array of boxes which has λ_i boxes in the i th row. The Young diagram of $(4, 2, 2, 1)$ is



A *standard Young tableau (SYT)* of shape λ is a filling of the Young diagram of λ with the numbers $1, 2, \dots, n$ so that these numbers are strictly increasing along rows and down columns. An SYT of shape $(4, 2, 2, 1)$ is

1	3	5	8
2	4		
6	9		
7			

Young diagrams and tableaux are named after Alfred Young (1873–1940), a British mathematician who pioneered the study of the group of permutations of a finite set. Much is known about SYTs because of their connection to algebra. For instance, there is a beautiful formula for the number of SYTs of given shape. The *hook* of a box u in a Young diagram consists of all boxes directly below, or directly to the right of, that box, including the box itself. The *hook length* of a box is the number of boxes in its hook. For example, the box with entry 2 in the above SYT has a hook length of 4. The celebrated *hook length formula* says that the number of SYTs of shape λ , a partition of n , is

$$\frac{n!}{\prod_u h(u)},$$

where the product runs over all boxes u of the Young diagram of λ , and $h(u)$ is the hook length of the box u . So for example there are $9!/(7 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 \cdot 1) = 216$ SYTs of shape $(4, 2, 2, 1)$. To learn more about tableaux in general, see Yong's short note [3] or Sagan's survey [1].

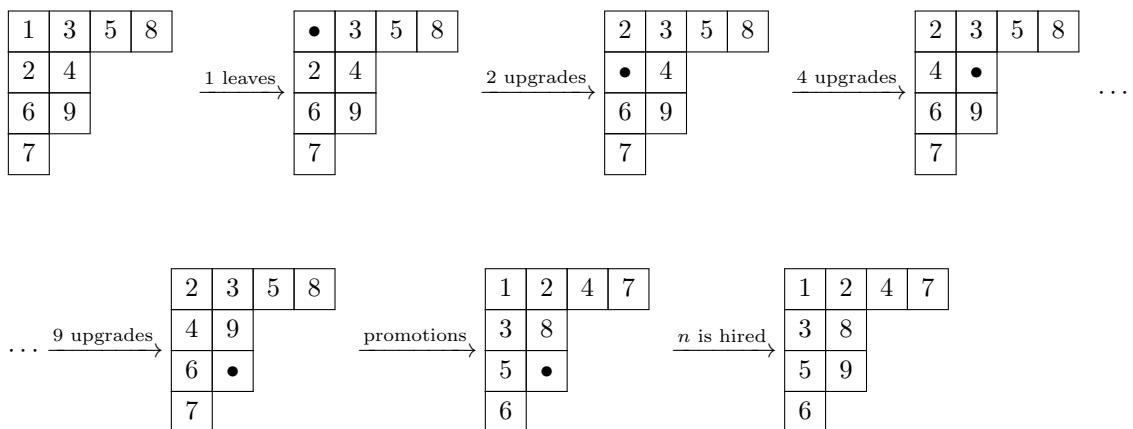
Our present interest in SYTs lies not in their enumeration but rather in a certain operation on them, which we now explain using a somewhat fanciful analogy.

We can view a Young diagram as a building whose rooms are the boxes of the diagram. (This presents some engineering challenges because the rows get longer towards the top, but never mind that.) If this building belongs to a hierarchical organization, like a company, then we can view a filling of the Young diagram with numbers $1, 2, \dots$, as an assignment of rooms to the person of rank 1, the person of rank 2, and so on. Suppose that to the left of our Young diagram office building is a beautiful ocean. Naturally, everyone in the building wants to have the best view of this ocean, and hence would always prefer to have a room as much to the left (to be closer to the ocean) and above (to have a higher viewpoint) as possible. So, in order to respect the pecking order, we might require room assignments to be such that every person has a lesser rank than the people in the rooms to their left and above them. Room assignments like this are precisely SYTs.

But now suppose that the CEO (the person of rank 1) leaves the company for a better opportunity elsewhere. Their departure creates an opening in a very desirable room. Those of lesser rank will fill this opening. Since the company does

not want people moving their stuff a long way across the building, only the people whose rooms are adjacent, either to the right or below, can compete to fill that open room. Of course, among these two, the room is awarded to the person of greater rank. They move from their current room to the more desirable one, and in doing so they create a new room opening, which is filled in the same manner: with the people currently adjacent to the right and below competing. In this way the departure of the CEO causes a series of room re-assignments, which eventually terminates with an undesirable room at the bottom-right of the building being emptied. Then, two final things happen to complete the corporate restructuring. First, everyone in the building gets a “promotion,” meaning that the person of rank 2 becomes rank 1, the person of rank 3 becomes rank 2, and so on. And second, a new intern, of the least rank n , gets hired to fill the empty room.

Here is an example of this procedure:



This entire operation, which takes one SYT of shape λ to another one, is in fact called *promotion*. The promotion operation on tableaux was introduced, together with another closely related operation called *evacuation*, by M.P. Schützenberger¹. The sliding process which goes into the definition of both promotion and evacuation was termed *jeu de taquin* by Schützenberger. “Jeu de taquin” literally translates to “teasing game,” but is the name in French for what is usually called the “15 Puzzle” in English. For an excellent introduction to promotion and evacuation, see Stanley’s survey [2].

Promotion is an invertible operation. To see this, we can imagine doing all of the steps backwards: firing the intern, demoting everyone, and forcing them into

¹The French mathematician Marcel-Paul Schützenberger (1920–1996) had a wide range of scientific interests: e.g., he obtained a doctorate in medicine in 1948; and in the 1960s he worked with the famous linguist Noam Chomsky on the analysis of formal languages. In algebraic combinatorics he is especially remembered for seminal contributions to the theory of tableaux, symmetric functions, and Schubert calculus.

worse rooms until there is a spot at the top for a new CEO. Hence, for any given tableau T there must be some number of times we can apply promotion to T that will get us back to T . But for most shapes, promotion behaves quite chaotically and it takes a long time for us to get back to where we started. For instance, we would need to apply promotion 60 times to our running example SYT of shape $(4, 4, 2, 1)$ in order to return to it. (In contrast, evacuation is always an involution, meaning if we apply it twice we get back to where we started.)

There are a very small number of partition shapes for which promotion behaves in an orderly fashion: see [2, §4]. These nice shapes include the *rectangle*²

$$a \times b := \overbrace{(b, b, \dots, b)}^{a \text{ times}}.$$

For any SYT of rectangular shape $a \times b$, if we apply promotion ab times we get back to where we started; this is Theorem 4.1(a) in [2]. Note that we might get back to where we started even before ab applications of promotion. For example, for the following SYT of shape 2×3 , we can apply promotion 3 times to return to it:

1	2	3	1	2	5	1	3	4	1	2	3
4	5	6	3	4	6	2	5	6	4	5	6

By now the reader may recognize that “The Last Problem” from the June 2021 issue of *Crux* precisely concerns the promotion operation applied to SYTs of rectangular shape³. Part (a) of the problem asks the reader to show that ab applications of promotion applied to an SYT T of shape $a \times b$ returns the initial tableau T . There is no really simple proof of this fact: it does follow from the fundamental properties of jeu de taquin as developed by Schützenberger, but it takes quite a while to develop this theory. We would be very impressed if any reader submitted a correct solution to this problem.

Parts (b) and (c) also follow from known properties of promotion: see Theorem 2.3 in [2], which relates the “principal chain” and “trajectory” of a tableau, and Theorem 4.1(a) of [2], which explains that evacuation for rectangular SYTs is 180° rotation plus swapping each number i for $n + 1 - i$.

Although we were not able to explain the full solution to “The Last Problem” in this short space, we hope that we have inspired the reader to learn more about tableaux and their fascinating properties, as well as the dynamical operations defined on them.

²Shapes that behave well under promotion also include the *staircase* $\delta_n = (n, n - 1, \dots, 1)$, though understanding promotion for the staircase is even more involved than for the rectangle.

³But note that the arrays there are 180° rotations of SYTs as we defined them here.

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2. R. P. Stanley, *Promotion and evacuation*, Electron. J. Combin., Volume 16 (2): paper 2.9, 24, 2009. Available online [here](#).
3. A. Yong, *What is . . . a Young tableau?*, Notices Amer. Math. Soc., 54 (2), pp. 240–241, 2–7. Available online [here](#).

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 1, 2022**.

4711. *Proposed by Sergey Sadov.*

Let ABC be a triangle with incenter I and excenters I_A, I_B, I_C . Prove that the centroid of the four-point system $\{I, I_A, I_B, I_C\}$ is the circumcenter of $\triangle ABC$.

4712. *Proposed by Michel Bataille.*

Let n be a positive integer and for $z \in \mathbb{C} - \{1, 2, \dots, n\}$ let

$$U_n(z) = \sum_{k=1}^n \frac{1}{k-z} \quad \text{and} \quad V_n(z) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{k}{(k-z)^2}.$$

Evaluate the ratio $\frac{U_n(z)}{V_n(z)}$ in closed form.

4713. *Proposed by András Szilárd.*

Let x, y be distinct positive real numbers. Prove that if there exists $n_1 \in \mathbb{N}$ such that $[nx]$ divides $[ny]$ for all natural numbers $n \geq n_1$, then x and y are integers ($[a]$ denotes the integer part of the real number a).

4714. *Proposed by András Szilárd.*

Let $0 < a < b$ be two numbers and let f be a function on $[a, b]$ that is twice differentiable, increasing and concave with nonnegative values. Prove that if $a \leq x_1 \leq x_2 \leq \dots \leq x_n \leq b$, then

$$\sum_{i=1}^n \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} + x_i} \geq 0,$$

where $x_{n+1} = x_1$ and $n \geq 3$.

4715. *Proposed by George Stoica.*

- a) Let A be a 3 by 3 matrix all of whose entries are complex numbers on the unit circle, and so that $\det(A) = 0$. Must A have two proportional rows or columns?
- b) What if A is 4 by 4?

4716. *Proposed by Michael Friday.*

The three roots of the cubic $x^3 + 4x^2 + 4x + 1 = 0$ are the slopes of the sides of a triangle. Find the slope of its Euler line.

4717. *Proposed by Toyesh Sharma.*

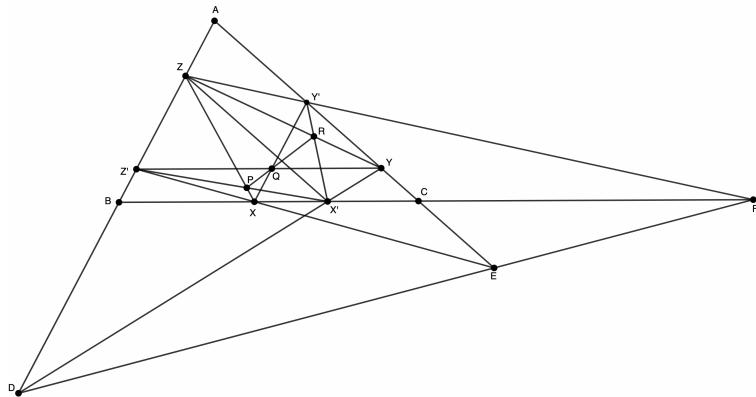
Find the value of the following integral:

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^4 y^3 z^2}{(x+y+z)(x^2+y^2+z^2)-(x^3+y^3+z^3)} dx dy dz.$$

4718. *Proposed by Pericles Papadopoulos.*

Let X, Y and Z be arbitrary points on the sides BC, AC and AB of $\triangle ABC$, respectively. The parallel through X to AB meets AC at Y' ; the parallel through Y to BC meets AB at Z' ; the parallel through Z to AC meets BC at X' . Assuming A, B, C, X, Y, Z are distinct, prove the following:

- a) points $D = X'Y \cap AB$, $E = XZ' \cap AC$ and $F = ZY' \cap BC$ are collinear;
- b) points $P = ZX \cap X'Z'$, $Q = Z'Y \cap XY'$ and $R = ZY \cap X'Y'$ are collinear.



4719. *Proposed by Neculai Stanciu, modified by the Editorial Board.*

We are given a triangle ABC with circumcenter O . For any point P_1 on the line CA define the following:

P_2 is the point where the line through P_1 perpendicular to OA intersects AB ,
 P_3 is the point where the line through P_2 perpendicular to OB intersects BC ,
 P_4 is the point where the line through P_3 perpendicular to OC intersects CA ,
 P_5 is the point where the line through P_4 perpendicular to OA intersects AB , and
 P_6 is the point where the line through P_5 perpendicular to OB intersects BC .

Prove that P_6P_1 is perpendicular to OC .

4720. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers with $a^2 + b^2 + c^2 = 12$. Prove that

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \geq 16.$$

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1 mai 2022**.*

4711. *Proposée par Sergey Sadov.*

Soit ABC un triangle et I le centre de son cercle inscrit ; soient aussi I_A, I_B et I_C les centres des cercles exinscrits. Démontrer que le centroïde de l'ensemble à 4 points $\{I, I_A, I_B, I_C\}$ est le centre du cercle circonscrit de $\triangle ABC$.

4712. *Proposée par Michel Bataille.*

Soit n un entier positif; pour $z \in \mathbb{C} - \{1, 2, \dots, n\}$, soient

$$U_n(z) = \sum_{k=1}^n \frac{1}{k-z} \quad \text{et} \quad V_n(z) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{k}{(k-z)^2}.$$

Déterminer le ratio $\frac{U_n(z)}{V_n(z)}$ en forme close.

4713. Proposé par András Szilárd.

Soient x, y des nombres réels positifs distincts. Démontrer que s'il existe $n_1 \in \mathbb{N}$ tel que $[nx]$ divise $[ny]$ pour tout nombre naturel $n \geq n_1$, alors x et y sont des entiers ($[a]$ dénote la partie entière du nombre réel a).

4714. *Proposée par András Szilárd.*

Soient $0 < a < b$ deux nombres réels et f une fonction sur $[a, b]$, deux fois différentiable, croissante, concave, puis à valeurs non négatives. Démontrer que si $a \leq x_1 \leq x_2 \leq \dots \leq x_n \leq b$, alors

$$\sum_{i=1}^n \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} + x_i} \geq 0,$$

où $x_{n+1} = x_1$ et $n \geq 3$.

4715. *Proposé par George Stoica.*

- a) Soit A une matrice de taille 3 par 3 formée de nombres complexes se trouvant sur le cercle unitaire, puis telle que $\det(A) = 0$. A doit-elle alors avoir deux rangées ou colonnes proportionnelles?
- b) Qu'en est-il si A est plutôt de taille 4 par 4?

4716. *Proposé par Michael Friday.*

Les trois racines de la cubique $x^3 + 4x^2 + 4x + 1 = 0$ sont les pentes d'un certain triangle. Déterminer la pente de la droite d'Euler de ce triangle.

4717. *Proposé par Toyesh Sharma.*

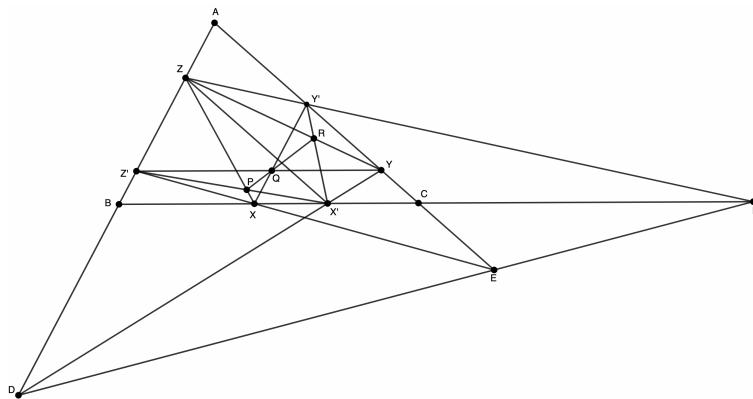
Déterminer la valeur de l'intégrale suivante:

$$\int_0^1 \int_0^1 \int_0^1 \frac{x^4 y^3 z^2}{(x+y+z)(x^2+y^2+z^2)-(x^3+y^3+z^3)} dx dy dz.$$

4718. *Proposé par Pericles Papadopoulos.*

Soient X , Y et Z des points quelconques sur les côtés BC , AC et AB de $\triangle ABC$, respectivement. La ligne parallèle à AB et passant par X rencontre AC en Y' ; la ligne parallèle à BC et passant par Y rencontre AB en Z' ; la ligne parallèle à AC et passant par Z rencontre BC en X' . Supposer que A, B, C, X, Y et Z sont distincts et démontrer les suivantes:

- a) les points $D = X'Y \cap AB$, $E = XZ' \cap AC$ et $F = ZY' \cap BC$ sont alignés;
- b) les points $P = ZX \cap X'Z'$, $Q = Z'Y \cap XY'$ et $R = ZY \cap X'Y'$ sont alignés.



4719. *Proposé par Neculai Stanciu, avec modification venant de l'éditeur.*

Soit le triangle ABC , où O dénote le centre du cercle circonscrit. Pour P_1 un point quelconque sur la ligne CA , on définit les points suivants:

P_2 est le point où la ligne passant par P_1 et perpendiculaire à OA intersecte AB ,
 P_3 est le point où la ligne passant par P_2 et perpendiculaire à OB intersecte BC ,
 P_4 est le point où la ligne passant par P_3 et perpendiculaire à OC intersecte CA ,
 P_5 est le point où la ligne passant par P_4 et perpendiculaire à OA intersecte AB ,
et

P_6 est le point où la ligne passant par P_5 et perpendiculaire à OB intersecte BC .

Démontrer que P_6P_1 est perpendiculaire à OC .

4720. *Proposé par George Apostolopoulos.*

Soient a , b et c des nombres réels positifs tels que $a^2 + b^2 + c^2 = 12$. Démontrer que

$$\frac{a^4}{\sqrt{a^3+1}} + \frac{b^4}{\sqrt{b^3+1}} + \frac{c^4}{\sqrt{c^3+1}} \geq 16.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(7), p. 352–356.



4661. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with the point $M \in BC$ such that

$$MC - MB = \frac{AC^2 - AB^2}{2BC}.$$

The centroids of triangles AMB and AMC are G_1 and G_2 , respectively. Prove that A, G_1, M, C are concyclic points if and only if A, B, M, G_2 are also concyclic points.

We received 14 solutions. We present 4 of them here.

Solution 1, by Michel Bataille.

Let $BC = a, CA = b, AB = c$, as usual, and let $MB = u, MC = v$ (so that $u + v = a$). In barycentric coordinates relative to (A, B, C) , we have $M = (0, v, u)$ and $G_1 = (a : a + v : u)$ (since $3G_1 = A + B + M$) and $G_2 = (a : v : a + u)$. We know that the circle Γ_1 through A, M, C has an equation of the form

$$a^2yz + b^2zx + c^2xy = (x + y + z)(\alpha x + \beta y + \gamma z).$$

Expressing that $A(1 : 0 : 0), M(0 : v : u), C(0 : 0 : 1)$ are on this circle, we obtain that the equation of Γ_1 is

$$a^2yz + b^2zx + c^2xy = auy(x + y + z).$$

Similarly, the equation of the circle Γ_2 through A, M, B is

$$a^2yz + b^2zx + c^2xy = avz(x + y + z).$$

Thus, for $i = 1, 2$, G_i is on the circle Γ_i if and only if $\delta_i = 0$ where

$$\delta_1 = au(a+v) + b^2u + c^2(a+v) - 3au(a+v), \quad \delta_2 = av(a+u) + b^2(a+u) + c^2v - 3av(a+u).$$

From the hypothesis, we have $2a(v - u) = b^2 - c^2$, hence

$$\delta_2 - \delta_1 = a(b^2 - c^2 - 2a(v - u)) = 0$$

and therefore $\delta_1 = 0$ if and only if $\delta_2 = 0$, that is, $G_1 \in \Gamma_1$ if and only if $G_2 \in \Gamma_2$, as desired.

Solution 2, by Prithwijit De.

Denote by a, b, c the lengths of BC , CA , and AB respectively. Let $X_1 = AG_1 \cap BC$ and $X_2 = AG_2 \cap BC$. If A, G_1, M, C are concyclic points then

$$X_1M.X_1C = X_1G_1.X_1A = \frac{X_1A^2}{3} \Rightarrow 4X_1A^2 = 3.MB.(2a - MB).$$

We also have

$$2c^2 + 2AM^2 = 4X_1A^2 + MB^2 \Rightarrow c^2 + AM^2 = MB.(3a - MB).$$

If A, B, M, G_2 are concyclic points then analogously we obtain

$$b^2 + AM^2 = MC.(3a - MC).$$

But

$$b^2 + AM^2 = MC.(3a - MC) \Leftrightarrow c^2 + AM^2 = MB.(3a - MB)$$

because

$$(b^2 + AM^2) - (c^2 + AM^2) = b^2 - c^2$$

and

$$MC.(3a - MC) - MB.(3a - MB) = (MC - MB).(2a) = b^2 - c^2,$$

which shows that

$$b^2 + AM^2 - MC.(3a - MC) = c^2 + AM^2 - MB.(3a - MB).$$

The equivalence of concyclicity of $\{A, G_1, M, C\}$ and $\{A, B, M, G_2\}$ occurs when

$$b^2 + AM^2 - MC.(3a - MC) = c^2 + AM^2 - MB.(3a - MB) = 0.$$

Solution 3, by Theo Koupelis.

Let a, b, c be the lengths of the sides of $\triangle ABC$. Without loss of generality let $b \geq c$ so that $MC \geq MB$. Let D be the foot of the perpendicular from A to BC , and let F, E, N be the midpoints of BM, CM, BC , respectively. From the given condition we get

$$AC^2 = AB^2 + BC^2 + BC \cdot (2MC - 2MB - BC),$$

and from the law of cosines we have

$$AC^2 = AB^2 + BC^2 - 2BC \cdot BD.$$

Therefore,

$$BC = 2BN = 2(MC - MB + BD)$$

and thus

$$MB - BD = MC - BN = MC - CN.$$

That is, M is the midpoint of DN .

We will now show that $FM \cdot FC = FG_1 \cdot FA = \frac{1}{3}AF^2$ (that is, A, G_1, M, C are concyclic) if and only if $EM \cdot EB = EG_2 \cdot EA = \frac{1}{3}AE^2$ (that is, A, B, M, G_2 are concyclic). We have:

$$\begin{aligned} FM \cdot FC &= \frac{1}{3}AF^2 \iff 3 \cdot \frac{BM}{2} \cdot \left(\frac{BM}{2} + CM \right) = AD^2 + \left(\frac{BM}{2} - \frac{DN}{2} \right)^2 \\ &\iff 2BM \cdot (BM + DN) = 4AD^2 + DN^2 - 6CM \cdot BM, \end{aligned}$$

and similarly

$$\begin{aligned} EM \cdot EB &= \frac{1}{3}AE^2 \iff 3 \cdot \frac{CM}{2} \cdot \left(\frac{CM}{2} + BM \right) = AD^2 + \left(\frac{CM}{2} + \frac{DN}{2} \right)^2 \\ &\iff 2CM \cdot (CM - DN) = 4AD^2 + DN^2 - 6CM \cdot BM. \end{aligned}$$

But $CM = \frac{1}{2}(a+DN)$ and $BM = \frac{1}{2}(a-DN)$, and thus $CM \cdot (CM - DN) = BM \cdot (BM + DN)$. Therefore A, G_1, M, C are concyclic points if and only if A, B, M, G_2 are also concyclic points.

Expressing the segments CM, BM, DN and AD in terms of a, b, c , we find that the above condition is equivalent to $12a^4 - 8(b^2 + c^2)a^2 + (b^2 - c^2)^2 = 0$.

Solution 4, by Miguel Amengual Covas.

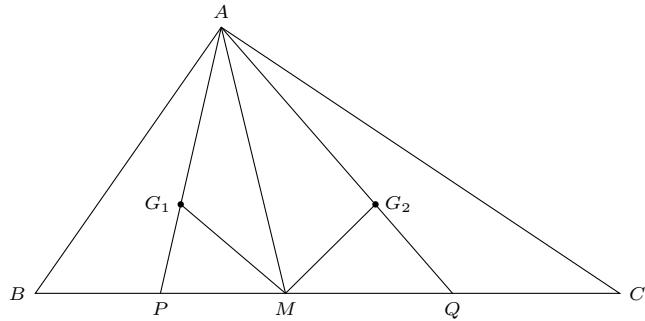
Let $a = BC$, $b = CA$, $c = AB$. We put $BM = x$. Then $MC = a - x$ and the given equality becomes $a - 2x = \frac{b^2 - c^2}{2a}$ which we rewrite as

$$4ax - c^2 = 2a^2 - b^2$$

Subtracting $x(a+x)$ from each side gives

$$3ax - c^2 - x^2 = 2a^2 - b^2 - x(a+x). \quad (1)$$

Let AG_1 and AG_2 (extended) meet BC at P and Q , respectively.



Points A, G_1, M, C are cyclic if and only if

$$PG_1 \cdot PA = PM \cdot PC,$$

that is,

$$\left(\frac{1}{3}PA\right) \cdot PA = \frac{x}{2} \left(a - \frac{x}{2}\right), \quad (2)$$

where, in $\triangle ABM$, $PA^2 = \frac{1}{4}(2(AB^2 + AM^2) - BM^2) = \frac{1}{4}(2(c^2 + AM^2) - x^2)$.

Substituting for PA^2 into (2) and solving for AM^2 yields

$$AM^2 = 3ax - c^2 - x^2. \quad (3)$$

Similarly, points A, B, M, G_2 are cyclic if and only if

$$QG_2 \cdot QA = QM \cdot QB,$$

that is,

$$\left(\frac{1}{3}QA\right) \cdot QA = \left(\frac{a-x}{2}\right) \left(\frac{a+x}{2}\right), \quad (4)$$

where, in $\triangle AMC$,

$$QA^2 = \frac{1}{4}(2(AM^2 + AC^2) - MC^2) = \frac{1}{4}(2(AM^2 + b^2) - (a-x)^2).$$

Substituting for QA^2 into (4) and solving for AM^2 yields

$$AM^2 = 2a^2 - b^2 - x(a+x). \quad (5)$$

Since, by hypothesis, (1) holds, then (3) and (5) hold simultaneously, and the conclusion follows.

4662. Proposed by Michel Bataille.

Let A and B be complex $p \times p$ matrices such that $AB = BA$ and $A^3B = A$ and let m, n be integers with $m \geq n \geq 1$ and $m \neq 2n$. Show that A^mB^n is equal to a power of A or a power of AB .

We received 14 submissions and they were all correct. We present the solution by the majority of solvers.

First, it is easy to verify by induction that $A^{2k+1}B^k = A$ for any non-negative integer k . Next we proceed according to the sign of $m - 2n$:

- If $m > 2n$, then $m = 2n + 1 + \ell$ for some $\ell \geq 0$. We have

$$A^mB^n = A^\ell A^{2n+1}B^n = A^{\ell+1} = A^{m-2n}.$$

- If $m < 2n$, then $m = n + \ell$ for some $0 \leq \ell \leq n - 1$. We have

$$A^m B^n = A^{m-2\ell-1} (A^{2\ell+1} B^\ell) B^{n-\ell} = A^{m-2\ell} B^{n-\ell} = (AB)^{n-\ell} = (AB)^{2n-m}.$$

Editor's Comment. As Eagle Problem Solvers pointed out, the assumption that A and B are complex $p \times p$ matrices is not crucial. More generally, the proof given above holds for any two elements A and B of a semigroup with the given properties.

4663. Proposed by Vijay Dasari.

Let M be any point in the plane of an acute triangle ABC with sides a, b, c . Prove that

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq 1,$$

with equality when M is the orthocenter.

We received 16 solutions, all of which were correct. We present the solution by Mohamed Amine Ben Ajiba.

Since ABC is an acute triangle, we have

$$b^2 + c^2 - a^2 \geq 0, \quad c^2 + a^2 - b^2 \geq 0, \quad \text{and} \quad a^2 + b^2 - c^2 \geq 0.$$

By Bergström's inequality, we have

$$\begin{aligned} & \frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \\ &= \frac{(aAM)^2}{(ab)^2 + (ca)^2 - a^4} + \frac{bBM^2}{(bc)^2 + (ab)^2 - b^4} + \frac{cCM^2}{(ca)^2 + (bc)^2 - c^4} \\ &\geq \frac{(aAM + bBM + cCM)^2}{2[(ab)^2 + (bc)^2 + (ca)^2] - (a^4 + b^4 + c^4)}. \end{aligned}$$

From the formula

$$2[(ab)^2 + (bc)^2 + (ca)^2] - (a^4 + b^4 + c^4) = 16F^2,$$

where F is the area of triangle ABC , we obtain

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq \left(\frac{aAB + bBM + cCM}{4F} \right)^2.$$

However we know that (see below)

$$aAB + bBM + cCM \geq 4F, \tag{1}$$

with equality exactly when M is the orthocenter. Therefore

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq 1,$$

with equality exactly when M is the orthocenter.

For the proof of (1), let A_1, B_1, C_1 be the feet of the altitudes of ABC , and let A', B', C' be the projections of M onto its sides. We then have $AM + A'M \geq AA_1$ or

$$aAM \geq aAA_1 - aA'M = 2F - 2[BMC],$$

where $[X]$ is the area of X . Similarly, we have

$$bBM \geq 2F - 2[CMA] \text{ and } cCM \geq 2F - 2[AMB].$$

Adding these inequalities and using the relation

$$[AMB] + [BMC] + [CMA] = F,$$

we obtain

$$aAB + bBM + cCM \geq 4F.$$

Equality holds when M is the orthocenter of triangle ABC .

4664. Proposed by Marian Cucoanes and Lorian Saceanu.

Let $ABCDEF$ be a convex cyclic hexagon that respects the following rules:

- a) The lines AD, BE, CF are concurrent;
- b) $(1/3)(AF + BC + DE) = AB = CD = EF$.

Prove that $ABCDEF$ is a regular hexagon.

Almost all of the 10 submissions we received used an approach similar to that of our featured solution by Anay Aggarwal.

Denote by O the point where the chords AD, BE, CF concur. In triangles ABO and EDO we have

$$\angle ABO = \angle ABE = \angle ADE = \angle ODE,$$

while the (vertical angles) at O satisfy $\angle AOB = \angle EOD$. Consequently, $\Delta ABO \sim \Delta EDO$ and

$$\frac{AB}{ED} = \frac{AO}{EO}.$$

Analogously,

$$\frac{CD}{AF} = \frac{CO}{AO} \quad \text{and} \quad \frac{EF}{CB} = \frac{EO}{CO}.$$

Hence

$$\frac{AB \cdot CD \cdot EF}{CB \cdot AF \cdot DE} = 1.$$

Consequently, our assumption $AB = CD = EF = \frac{AF+BC+DE}{3}$ implies that

$$\frac{AF + BC + DE}{3} = \sqrt[3]{AB \cdot CD \cdot EF} = \sqrt[3]{CB \cdot AF \cdot DE}.$$

However, the AM-GM inequality tells us that

$$\frac{AF + BC + DE}{3} \geq \sqrt[3]{AF \cdot BC \cdot DE},$$

with equality if and only if $AF = BC = DE$. Thus, all six sides of the hexagon are equal and, because it is inscribed in a circle, it is regular.

4665. *Proposed by Daniel Sitaru.*

Find

$$\lim_{n \rightarrow \infty} \left(\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(1 + \sin^n x)} dx \right).$$

We received 25 submissions, all of which are correct. We present the similar solutions by Brian Bradie, UCLan Cyprus Solving Group, and Eagle Problem Solvers.

Since

$$\frac{\cos x}{(\sin x)(1 + \sin^n x)} = \frac{\cos x}{\sin x} - \frac{(\sin^{n-1} x)(\cos x)}{1 + \sin^n x},$$

we have

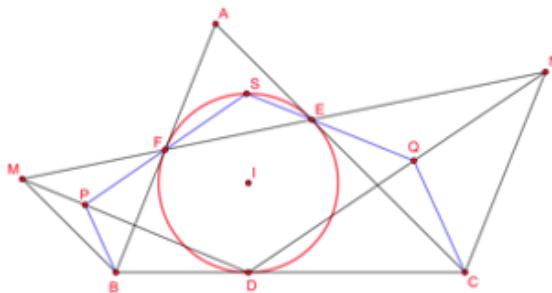
$$\begin{aligned} \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(1 + \sin^n x)} dx &= \left(\ln \sin x - \frac{1}{n} \ln(1 + \sin^n x) \right) \Big|_{\pi/6}^{\pi/2} \\ &= -\frac{1}{n} \ln 2 - \ln \frac{1}{2} + \frac{1}{n} \ln \left(1 + \frac{1}{2^n} \right). \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \left(\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x(1 + \sin^n x)} dx \right) = -\ln 1/2 = \ln 2.$$

4666. *Proposed by Dong Luu.*

Let ABC be a triangle and let the circle I be tangent to BC , CA and AB at points D , E and F , respectively. Let M , N be the points on the line EF such that BM is parallel to AC and CN is parallel to AB . Let P and Q be points on DM and DN , respectively such that BP is parallel to CQ . Denote by S the intersection point of PF and QE . Prove that S lies on the circle I .



We received 12 solutions. We present the solution by Jason Fang, edited.

In the following, $\angle A$, $\angle B$ and $\angle C$ denote the angles of $\triangle ABC$.

In $\triangle AFE$, $AF = AE$ and hence

$$\angle AFE = \angle AEF = \frac{\angle B + \angle C}{2}.$$

From $CN \parallel AF$ we get $\triangle CNE \sim \triangle AFE$, and so

$$\angle NCE = \angle A \text{ and } \angle CNE = \angle CEN = \frac{\angle B + \angle C}{2};$$

also $CN = CE$. From $CE = CD$ it follows that $CN = CD$; since

$$\angle NCD = \angle NCE + \angle ECD = \angle A + \angle C$$

we get

$$\angle CND = \frac{\angle B}{2} \text{ and hence} \quad (1)$$

$$\angle DNE = \angle CNE - \angle CND = \frac{\angle C}{2}. \quad (2)$$

We repeat the first part of the argument with B , M and E instead of C , N and F (starting from $BM \parallel AE$) to get

$$\angle DMB = \frac{\angle C}{2}; \quad (3)$$

we also note that $\triangle BMF \sim \triangle CNE$.

Combining (2) and (3) we get $\angle QNE = \angle PMB$. As angles whose sides are parallel lines, we also have $\angle PBM = \angle QCE$. It follows that the similarity transformation which maps $\triangle BMF$ to $\triangle CNE$ maps P to the isogonal conjugate of Q . Hence $\angle PFM = \angle QEC$. From $\triangle FSE$, using this observation and the fact that opposite angles are equal, we calculate

$$\begin{aligned} \angle FSE &= 180^\circ - (\angle SEF + \angle SFE) = 180^\circ - (\angle QEN + \angle PFM) \\ &= 180^\circ - (\angle QEN + \angle QEC) \\ &= 180^\circ - \angle NEC \\ &= 180^\circ - \angle AEF. \end{aligned}$$

Finally, note that $\angle AEF = \angle FDE$ (since the intercepted arc on circle I is the same for both angles). Hence $\angle FSE = 180^\circ - \angle FDE$, so $FSED$ is a cyclic quadrilateral. Therefore, S lies on the circle I .

Editor's comment. It was brought to the attention of the editors that this problem also appeared as M2672 in Issue 10 (2021) of the Russian magazine *Kvant*. We take this opportunity to remind our readers that problems submitted to any journal should be original and not actively under consideration at another publication.

4667. *Proposed by Conar Goran.*

Let $x_1, \dots, x_n > 0$ be real numbers and $s = \sum_{i=1}^n x_i$. Prove

$$\prod_{i=1}^n x_i^{x_i} \leq \left(\frac{1}{s} \sum_{i=1}^n x_i^2 \right)^s.$$

When does equality occur?

We received 8 submissions, all correct. Most solutions are similar to one another and use either the weighted AM-GM inequality or the Jensen's inequality. We present two such solutions.

Solution 1, by Benjamin Braiman.

Since each x_i is positive and $\sum_{i=1}^n \frac{x_i}{s} = 1$, we have by the weighted AM-GM inequality that

$$\prod_{i=1}^n x_i^{\frac{x_i}{s}} \leq \sum_{i=1}^n \left(\frac{x_i}{s} \right) x_i = \frac{1}{s} \sum_{i=1}^n x_i^2,$$

from which it follows that

$$\prod_{i=1}^n x_i^{x_i} \leq \left(\frac{1}{s} \sum_{i=1}^n x_i^2 \right)^s.$$

Equality holds if and only if all x_i s are equal.

Solution 2, by Brian Bradie.

By Jensen's inequality we have

$$\sum_{i=1}^n \frac{x_i}{s} \ln x_i \leq \ln \left(\sum_{i=1}^n \frac{x_i^2}{s} \right),$$

so

$$\ln \left(\prod_{i=1}^n x_i^{\frac{x_i}{s}} \right) \leq \ln \left(\sum_{i=1}^n \frac{x_i^2}{s} \right).$$

Exponentiating both sides then yields

$$\prod_{i=1}^n x_i^{\frac{x_i}{s}} \leq \frac{1}{s} \sum_{i=1}^n x_i^2, \quad \text{or} \quad \prod_{i=1}^n x_i^{x_i} \leq \left(\frac{1}{s} \sum_{i=1}^n x_i^2 \right)^s,$$

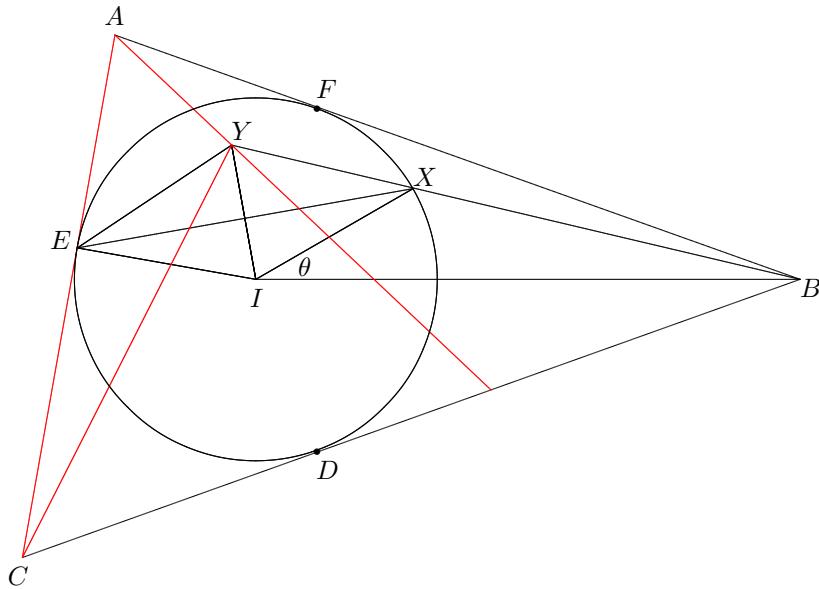
with equality when $x_1 = x_2 = \dots = x_n$.

4668. *Proposed by Jiahao Chen.*

Let Γ be the inscribed circle of triangle ABC , and I is the center of Γ . Suppose Γ touches BC , CA and AB at D , E and F , respectively. Let X be an arbitrary

point on the smaller arc DF , and the line perpendicular to XE passing through I intersects line BX in point Y . Show that IY is the external angle bisector of the angle AYC .

We received 5 submissions: two were complete and essentially correct, and another two correctly proved that the line IY was an angle bisector but did not address the matter of whether it was internal or external; the fifth was computer aided so it is hard to determine its status. We feature the solution by Marie-Nicole Gras, with the final step modified by the editor.



We shall prove that, more generally, IY is the internal bisector of $\angle AYC$ when X is on the arc DF that contains E , and the external bisector on the other (smaller) arc DF . Let r be the inradius; we denote by $\angle A$, $\angle B$ and $\angle C$ the value of $\angle BAC$, $\angle CBA$ and $\angle ACB$, respectively. Since I is the incenter, we have the equalities:

$$\begin{aligned}\beta &:= \angle BIF = \frac{\pi}{2} - \frac{\angle B}{2}; \\ \alpha &:= \angle BIE = \frac{\pi}{2} - \frac{\angle B}{2} + \pi - \angle A = \frac{\pi}{2} + \frac{\angle B}{2} + \angle C.\end{aligned}$$

We introduce cartesian coordinates with the origin at I and axis \overrightarrow{Ix} on \overrightarrow{IB} . We put $\theta = \angle(\overrightarrow{IB}, \overrightarrow{IX})$; since X is on the smaller arc DF , we have $-\frac{\pi}{2} < \beta < \theta < \pi - \beta < \frac{\pi}{2}$; we find

$$\begin{aligned}B &\sim \left(\frac{r}{\cos \beta}, 0 \right), \\ E &\sim (r \cos \alpha, r \sin \alpha), \quad F \sim (r \cos \beta, r \sin \beta), \quad D \sim (r \cos \beta, -r \sin \beta), \\ X &\sim (r \cos \theta, r \sin \theta).\end{aligned}$$

I) Coordinates of Y .

Equation of line BX is

$$(r \sin \theta)x - \left(r \cos \theta - \frac{r}{\cos \beta}\right)y = r^2 \frac{\sin \theta}{\cos \beta}.$$

Line ℓ , passing through I and perpendicular to the chord XE , has parametric equation:

$$x = t \cos \frac{\theta + \alpha}{2}, \quad y = t \sin \frac{\theta + \alpha}{2},$$

where t is given by

$$\begin{aligned} & (r \sin \theta)t \cos \frac{\theta + \alpha}{2} - \left(r \cos \theta - \frac{r}{\cos \beta}\right)t \sin \frac{\theta + \alpha}{2} = r^2 \frac{\sin \theta}{\cos \beta} \\ \iff & t \left(\sin \theta \cos \frac{\theta + \alpha}{2} - \cos \theta \sin \frac{\theta + \alpha}{2} + \frac{1}{\cos \beta} \sin \frac{\theta + \alpha}{2} \right) = r \frac{\sin \theta}{\cos \beta} \\ \iff & t \left(\sin \frac{\theta - \alpha}{2} + \frac{1}{\cos \beta} \sin \frac{\theta + \alpha}{2} \right) = r \frac{\sin \theta}{\cos \beta} \\ \iff & t \left(\sin \frac{\theta - \alpha}{2} \cos \beta + \sin \frac{\theta + \alpha}{2} \right) = r \sin \theta. \end{aligned}$$

We put

$$w = \sin \frac{\theta - \alpha}{2} \cos \beta + \sin \frac{\theta + \alpha}{2};$$

then, since lines BX and ℓ are concurrent, $w \neq 0$, and $t = \frac{r \sin \theta}{w}$. The coordinates (z, z') of Y are

$$Y \sim \left(\frac{r \sin \theta}{w} \cos \frac{\theta + \alpha}{2}, \frac{r \sin \theta}{w} \sin \frac{\theta + \alpha}{2} \right).$$

II) Equation of line AY .

Vertex A is the intersection of lines tangent to the incircle at points F and E ; then, its coordinates (a, a') are given by the system

$$\begin{cases} a \cos \beta + a' \sin \beta = r \\ a \cos \alpha + a' \sin \alpha = r; \end{cases}$$

the determinant is $\sin(\alpha - \beta) = \sin(\pi - \angle A) \neq 0$. Then, the coordinates of A are

$$\begin{aligned} a &= r \frac{\sin \alpha - \sin \beta}{\sin(\alpha - \beta)} = r \frac{2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}}{2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha - \beta}{2}} = r \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}} \\ a' &= r \frac{\cos \beta - \cos \alpha}{\sin(\alpha - \beta)} = r \frac{2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}}{2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha - \beta}{2}} = r \frac{\sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}. \end{aligned}$$

It follows that the equation of line AY is

$$\begin{aligned} & \left(\frac{r \sin \theta}{w} \sin \frac{\theta + \alpha}{2} - \frac{r}{\cos \frac{\alpha - \beta}{2}} \sin \frac{\alpha + \beta}{2} \right) x - \left(\frac{r \sin \theta}{w} \cos \frac{\theta + \alpha}{2} - \frac{r}{\cos \frac{\alpha - \beta}{2}} \cos \frac{\alpha + \beta}{2} \right) y \\ &= \frac{r^2 \sin \theta}{w \cos \frac{\alpha - \beta}{2}} \sin \frac{\theta - \beta}{2} \end{aligned}$$

or

$$\begin{aligned} ux - vy &= r \sin \theta \sin \frac{\theta - \beta}{2}, \quad \text{with} \\ u &= \sin \theta \cos \frac{\alpha - \beta}{2} \sin \frac{\theta + \alpha}{2} - w \sin \frac{\alpha + \beta}{2}, \\ v &= \sin \theta \cos \frac{\alpha - \beta}{2} \cos \frac{\theta + \alpha}{2} - w \cos \frac{\alpha + \beta}{2}. \end{aligned}$$

III) Distance d from I to line AY .

Since the coordinates of I are $(0,0)$, we have

$$d^2 = \frac{\sin^2 \theta \sin^2 \frac{\theta - \beta}{2} r^2}{u^2 + v^2} = \frac{\sin^2 \theta \sin^2 \frac{\theta - \beta}{2} \sin^2 \frac{\theta + \beta}{2} r^2}{(u^2 + v^2) \sin^2 \frac{\theta + \beta}{2}}.$$

We compute the denominator:

$$\begin{aligned} u^2 + v^2 &= \sin^2 \theta \cos^2 \frac{\alpha - \beta}{2} + w^2 - 2w \sin \theta \cos \frac{\alpha - \beta}{2} \cos \frac{\theta - \beta}{2}, \\ 4(u^2 + v^2) \sin^2 \frac{\theta + \beta}{2} &= \sin^2 \theta \left(2 \sin \frac{\theta + \beta}{2} \cos \frac{\alpha - \beta}{2} \right)^2 + 4 \sin^2 \frac{\theta + \beta}{2} w^2 \\ &\quad - 2w \sin \theta \left(2 \sin \frac{\theta + \beta}{2} \cos \frac{\alpha - \beta}{2} \right) \left(2 \sin \frac{\theta + \beta}{2} \cos \frac{\theta - \beta}{2} \right). \end{aligned}$$

We notice that

$$\begin{aligned} 2 \sin \frac{\theta + \beta}{2} \cos \frac{\alpha - \beta}{2} &= \sin \frac{\theta + \alpha}{2} + \sin \left(\frac{\theta - \alpha}{2} + \beta \right) \\ &= \sin \frac{\theta + \alpha}{2} + \sin \frac{\theta - \alpha}{2} \cos \beta + \cos \frac{\theta - \alpha}{2} \sin \beta \\ &= w + \cos \frac{\theta - \alpha}{2} \sin \beta; \\ 2 \sin^2 \frac{\theta + \beta}{2} &= 1 - \cos(\theta + \beta) = 1 - \cos \theta \cos \beta + \sin \theta \sin \beta; \\ 2 \sin \frac{\theta + \beta}{2} \cos \frac{\theta - \beta}{2} &= \sin \theta + \sin \beta. \end{aligned}$$

We deduce that

$$\begin{aligned}
4(u^2 + v^2) \sin^2 \frac{\theta + \beta}{2} &= \sin^2 \theta \left(w + \cos \frac{\theta - \alpha}{2} \sin \beta \right)^2 + 2(1 - \cos \theta \cos \beta + \sin \theta \sin \beta) w^2 \\
&\quad - 2w \sin \theta \left(w + \cos \frac{\theta - \alpha}{2} \sin \beta \right) (\sin \theta + \sin \beta) \\
&= \sin^2 \theta \left(w^2 + \cos^2 \frac{\theta - \alpha}{2} \sin^2 \beta \right) + 2(1 - \cos \theta \cos \beta) w^2 \\
&\quad - 2w \sin \theta \left(w \sin \theta + \cos \frac{\theta - \alpha}{2} \sin^2 \beta \right) \\
&\quad + \left[2w \sin^2 \theta \cos \frac{\theta - \alpha}{2} + 2w^2 \sin \theta - 2w \sin \theta \left(w + \cos \frac{\theta - \alpha}{2} \sin \theta \right) \right] \sin \beta,
\end{aligned}$$

and the coefficient of $\sin \beta$ in the brackets is equal to zero.

IV) Line IY is the bisector of $\angle AYC$.

All calculations relating to vertex C can be deduced from those obtained with A by replacing β by $-\beta$. Note that the expression for d is invariant if we replace β by $-\beta$. It follows that I is equidistant from lines AY and CY ; consequently, IY is a bisector of one pair of the angles defined by those two lines. It remains to prove that it is the external bisector of $\angle AYC$ when X is on the smaller arc DF , and the internal bisector when on the larger. The lines AY and CY divide the plane into four quadrants; as usual, we call the *first quadrant* that which is defined by $\angle AYC$ — that is, the first quadrant is the region bounded by the rays YA and YC . The *second quadrant* is bounded by the rays YC and $-YA$, and so on. The point I is in the first or third quadrants (making YI the internal bisector of $\angle AYC$) if the rotation that takes the ray IA to the ray IY is in the same direction as the rotation that takes the ray IY to the ray IC . The point I is in the second or fourth quadrants if those two rotations are in opposite directions. The sense of the rotations is given by the sign of the determinants $\begin{vmatrix} a & a' \\ z & z' \end{vmatrix}$ and $\begin{vmatrix} z & z' \\ c & c' \end{vmatrix}$, namely,

$$\begin{aligned}
az' - a'z &= \frac{r}{\cos \frac{\alpha - \beta}{2}} \frac{r \sin \theta}{w} \left(\sin \frac{\theta + \alpha}{2} \cos \frac{\alpha + \beta}{2} - \sin \frac{\alpha + \beta}{2} \cos \frac{\theta + \alpha}{2} \right) \\
&= \frac{r^2 \sin \theta}{w \cos \frac{\alpha - \beta}{2}} \sin \frac{\theta - \beta}{2},
\end{aligned}$$

and (by replacing β by $-\beta$),

$$zc' - z'c = -(cz' - c'z) = -\frac{r^2 \sin \theta}{w \cos \frac{\alpha + \beta}{2}} \sin \frac{\theta + \beta}{2}.$$

Consider their product

$$(az' - a'z)(zc' - z'c) = \frac{r^4 \sin^2 \theta}{w^2 \cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}} \sin \frac{\beta - \theta}{2} \sin \frac{\beta + \theta}{2}.$$

Since $\beta = \frac{\pi}{2} - \frac{\angle B}{2}$ and $\alpha = \frac{\pi}{2} + \frac{\angle B}{2} + \angle C$, we have

$$\begin{aligned}\frac{\alpha - \beta}{2} &= \frac{B + C}{2} = \frac{\pi}{2} - \frac{A}{2} \\ \frac{\alpha + \beta}{2} &= \frac{\pi + C}{2} = \frac{\pi}{2} + \frac{C}{2};\end{aligned}$$

We deduce that $\cos \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2} < 0$, which clearly does not depend on θ .

When $-\beta < \theta < \beta$ (where X is on the small arc DF as in the original statement of the problem), we have

$$\sin \frac{\beta - \theta}{2} \sin \frac{\beta + \theta}{2} = \frac{1}{2} (\cos \theta - \cos \beta) > 0.$$

It follows that $(az' - a'z)(zc' - z'c) < 0$, and I is in the second or fourth quadrant, whence YI is the external bisector of $\angle AYC$. Otherwise, when $\theta > \beta$ or $\theta < -\beta$, $(az' - a'z)(zc' - z'c) > 0$, I is in the first or third quadrant, and YI is the internal angle bisector.

Editor's comments. Note that when X is at D or F the angle AYC is undefined (because Y is at A or C); when $X = E$, "external" has no meaning because $\angle AYC = 180^\circ$; and when X is at either point where BI meets the incircle, then $Y = I$ and the line YI is undefined. When X is permitted to be an arbitrary point of the incircle different from those five points, the problem of proving that the line YI bisects $\angle AYC$ is relatively easy; the hard work comes in showing that it is external if and only if X is on the smaller arc DF ; this is apparently not so easy.

4669. Proposed by Warut Suksompong.

For a given positive integer n , a $4n \times 4n$ table is partitioned into $16n^2$ unit squares, each of which is coloured in one of 4 given colours. A set of four cells is called *colourful* if the centers of the cells form a rectangle with sides parallel to the sides of the table, and the cells are coloured in all four different colours. Determine the maximum number of colourful sets.

We received 2 correct solutions. There was an additional incorrect solution and a solution that conjectured the correct answer on the basis of some examples. We present a solution based on that of UC Lan Cyprus Problem Solving Group.

The maximum number is $24n^4$. Denote the colours by 1, 2, 3, 4.

In a given row, suppose that there are b_k cells with colour k , so that

$$b_1 + b_2 + b_3 + b_4 = 4n.$$

Then the number of pairs of cells in that row with different colours is

$$\begin{aligned}
 & b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 \\
 &= \frac{(b_1 + b_2 + b_3 + b_4)^2 - (b_1^2 + b_2^2 + b_3^2 + b_4^2)}{2} \\
 &= \frac{(4n)^2 - (b_1^2 + b_2^2 + b_3^2 + b_4^2)}{2} \leq \frac{16n^2 - (1/4)(b_1 + b_2 + b_3 + b_4)^2}{2} \\
 &= \frac{16n^2 - 4n^2}{2} = 6n^2.
 \end{aligned}$$

Hence the number of pairs of cells lying in the same row of the table with different colours is not greater than $(4n)(6n^2) = 24n^3$.

Fix a pair (i, j) of indices and let there be m_{ij} pairs of cells in the same row, one in each of columns i and j , with different colours. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be the number of pairs for which the colours are, respectively, $(1, 2), (3, 4), (1, 3), (2, 4), (1, 4), (2, 3)$. Then the number of colourful sets in these two columns is

$$\begin{aligned}
 a_1a_2 + a_3a_4 + a_5a_6 &\leq \left(\frac{a_1 + a_2}{2}\right)^2 + \left(\frac{a_3 + a_4}{2}\right)^2 + \left(\frac{a_5 + a_6}{2}\right)^2 \\
 &\leq \left(\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{2}\right)^2 = \frac{m_{ij}^2}{4}.
 \end{aligned}$$

The total number of colorful sets does not exceed

$$\frac{1}{4} \sum \{m_{ij}^2 : 1 \leq i, j \leq 4n\}.$$

The quantities m_{ij} are subject to the constraints $m_{ij} \leq 4n$ for each (i, j) and $\sum m_{ij} \leq 24n^3$.

In obtaining an upper bound for $\sum m_{ij}$, we can systematically use the following procedure for replacing pairs $\{u, v\}$ of summands, where $u \geq v$, by pairs $\{u + e, u - e\}$ where $e > 0$, thus increasing the sum since

$$(u + e)^2 + (v - e)^2 = (u^2 + v^2) + 2e(u - v) + 2e^2 > u^2 + v^2.$$

This can be done to lead us to a sum where the entries add up to $24n^3$, and $6n^2$ entries are equal to $4n$ with the rest equal to 0. Thus $\sum m_{ij}^2 \leq (6n^2)(4n)^2 = 4(24n^4)$ and the number of colourful sets does not exceed $24n^4$. It remains to find a configuration that realizes this bound.

We can construct a maximal table in this way. Let A_k be a $n \times n$ table with each cell coloured k . These are put together to form a $(4n) \times (4n)$ table:

$$\begin{array}{cccc}
 A_1 & A_2 & A_3 & A_4 \\
 A_2 & A_1 & A_4 & A_3 \\
 A_3 & A_4 & A_1 & A_2 \\
 A_4 & A_3 & A_2 & A_1
 \end{array}$$

There are $\binom{4}{2}n^2$ ways of picking the rows containing the cells of a colourful set. For each choice, there are $4n^2$ ways of picking the columns. Hence there are $24n^4$ colourful sets in all.

Editor's comment. The proposer had a similar solution, with an alternative argument for the bound on $\sum b_i$:

$$\begin{aligned} & b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 \\ &= \binom{4n}{2} - \sum_{i=1}^4 \binom{b_i}{2} = \binom{4n}{2} + \frac{1}{2} \sum_{i=1}^4 b_i - \frac{1}{2} \sum_{i=1}^4 b_i^2 \\ &\leq 2n(4n-1) + \frac{1}{2}(4n) - \frac{1}{8}(b_1+b_2+b_3+b_4)^2 = 8n^2 - \frac{1}{8}(16n^2) = 6n^2. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} & b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 \\ &= \frac{1}{2}[b_1(b_2+b_3+b_4) + b_2(b_3+b_4+b_1) + b_3(b_4+b_1+b_2) + b_4(b_1+b_2+b_3)] \\ &= \frac{1}{2} \sum_{i=1}^4 b_i(4n-b_i) = 2n \sum_{i=1}^4 b_i - \frac{1}{2} \sum_{i=1}^4 b_i^2 \leq 8n^2 - 2n^2 = 6n^2. \end{aligned}$$

For the maximal configuration, the proposer considered the $4n \times 4n$ array consisting of n^2 4×4 tables identically coloured as follows:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

where successive quartets of rows or columns are the same. There are

$$\binom{4n}{2} - 4\binom{n}{2} = 6n^2$$

choices of pairs of distinct rows from which colourful sets can be selected. Now consider a distinct pair of rows each consisting of n quartets of colours. There are 4 ways of selecting a colourful set if all its elements are in the same corresponding quartet in the two rows. There are 8 ways of selecting a colourful set if its left two elements lie in one quartet and the right two in another quartet in the two rows. Hence the total number of colourful sets involving these two rows is

$$4n + 8\binom{n}{2} = 4n^2.$$

Therefore the total number of colourful sets in this array is $24n^4$.

4670. *Proposed by Nguyen Viet Hung.*

Let a, b, c be real numbers such that $(a+b)(b+c)(c+a) \neq 0$. Prove that

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2 + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 1.$$

We received 20 submissions of which 14 were correct and complete. We present the solution by Michel Bataille utilizing an approach shared in most submissions.

With $x = \frac{b}{b+c}$, $y = \frac{c}{c+a}$, $z = \frac{a}{a+b}$, the inequality to be proved becomes

$$x^2 + y^2 + z^2 + 4xyz \geq 1.$$

Since $1-x = \frac{c}{b+c}$, $1-y = \frac{a}{c+a}$, $1-z = \frac{b}{a+b}$, we have

$$xyz = (1-x)(1-y)(1-z) \left(= \frac{abc}{(a+b)(b+c)(c+a)}\right).$$

Hence,

$$4xyz = 2 - 2(x+y+z) + 2(xy+yz+zx),$$

and by adding $x^2 + y^2 + z^2$ on both sides we obtain

$$x^2 + y^2 + z^2 + 4xyz = 1 + ((x+y+z) - 1)^2 \geq 1.$$

Editor's comment. Oliver Geupel's proof included an example in which equality is achieved.

