# Assignment - Matheus Douglas MATOS MARCONDES

November 25, 2019

```
In [1]: import numpy as np
        import numpy.random as rand
        import matplotlib.pyplot as plt

        from sklearn.datasets import make_blobs, make_moons, make_circles
        from sklearn.model_selection import train_test_split
        from sklearn.preprocessing import OneHotEncoder
```

## 1 Perceptrons and Back propagation

## 2 Submit to moodle by the 25th November - Please submit a PDF version of your notebook

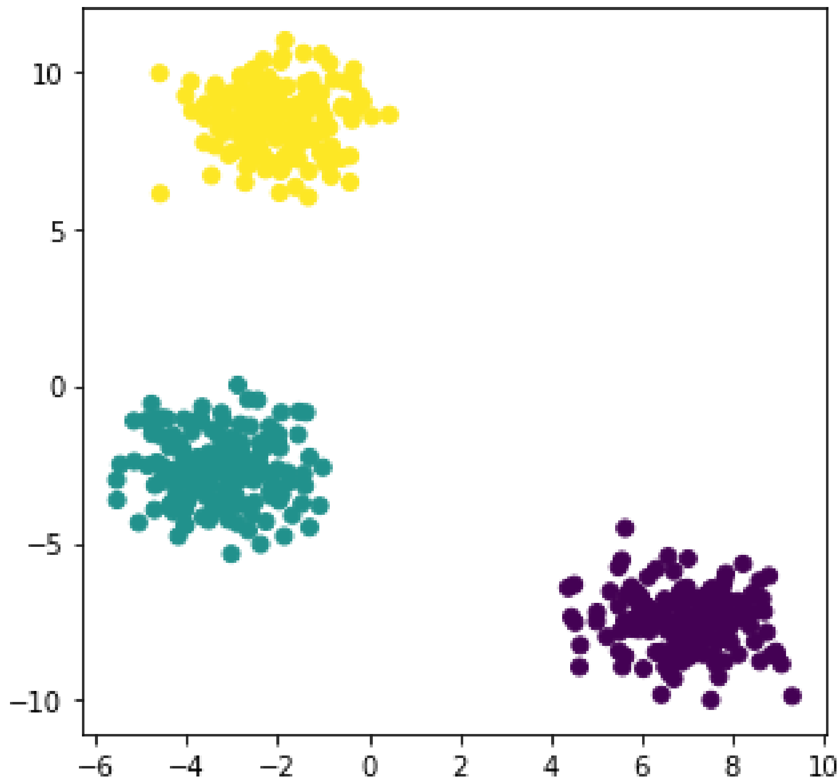
### 2.1 One layer perceptron - Linear separation

We start by generating *blobs*. In this setting data is linearly separable, we can thus use a single layer perceptron.

$$\hat{y} = W^{\top}x + b$$
$$\hat{P}(y_j = 1) = Softmax(\hat{y})_j$$

With $W \in M_{P,N}(\mathbb{R})$ and $b$ a vector of size $N$. $P, N$ the dimension of the input and the number of classes.

```
In [2]: X, Y = make_blobs(n_samples=500, n_features=2)
        plt.figure(figsize=(5,5))
        plt.scatter(X[:,0], X[:,1], c=Y)
        plt.show()
        Y = OneHotEncoder().fit_transform(Y.reshape(-1,1)).toarray()
```




```
In [3]: X_train, X_test, Y_train, Y_test = train_test_split(X, Y, random_state=42,
                                                            test_size=0.4)
```

### 2.1.1 Task 1: Implement softmax and the forward pass of a single layer percepton.

```
In [4]: def softmax(y):
            """
            Takes the output of a layer as input and returns a probability distribution
            input:
                y (np.array)

            returns:
                y (np.array)
            """
            aux = y.reshape(-1)
            aux = np.exp(aux)
            somme = aux.sum()
            aux = aux / somme

            return aux.reshape(-1,1)

        def forward_one_layer(W, b, x):
```

```python
    """
    Computes the forward pass of a single layer perceptron
    input:
        W (np.array): (INPUT_SHAPE, N_CLASSES) The weight matrix of the perceptron
        b (np.array): (N_CLASSES, 1) The bias matrix of the perceptron
        x (np.array): (INPUT_SHAPE, 1) The input of the perceptron

    returns:
        (np.array) (N_CLASSES, 1)
    """
    return W.T @ x + b
```

The loss typically associated with a classification problem is the cross entropy loss:

$$loss = -\log(\hat{P}(y_j = 1))$$

#### 2.1.2 Question 1 (2 points): Derive the gradients of loss with respect to $W$ and $b$.

$$\nabla_W loss = ?$$

$$\nabla_b loss = ?$$

***Answer:*** We can write the loss function as:

$$\begin{aligned}
loss &= -\log\left(\hat{P}(y_j = 1)\right) \\
&= -\log\left(Softmax(\hat{y})_j\right) \\
&= -\log\left(\frac{e^{y_j}}{\sum_{k=1}^{C} e^{y_k}}\right) \\
&= \log\left(\sum_{k=1}^{C} e^{y_k}\right) - \log e^{y_j} \\
&= \log\left(\sum_{k=1}^{C} e^{y_k}\right) - y_j
\end{aligned}$$

where $y_k = W_{(k,\cdot)}^T x + b_k = \sum_{i=1}^{C} w_{i,k} x_i + b_k$.

Then, we have:

- For $k \neq j$:

$$\begin{aligned}
\frac{\partial loss}{\partial W_{i,k}} &= \frac{e^{y_k}}{\sum_{p=1}^{C} e^{y_p}} x_i \\
&= Softmax(\hat{y})_k x_i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial loss}{\partial b_k} &= \frac{e^{y_k}}{\sum_{p=1}^{C} e^{y_p}} \\
&= Softmax(\hat{y})_k
\end{aligned}$$

- For $k = j$:

$$\begin{aligned}
\frac{\partial loss}{\partial W_{i,j}} &= \frac{e^{y_j}}{\sum_{p=1}^{C} e^{y_p}} x_i - x_i \\
&= x_i \left(Softmax(\hat{y})_j - 1\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial loss}{\partial b_j} &= \frac{e^{y_j}}{\sum_{p=1}^{C} e^{y_p}} - 1 \\
&= Softmax(\hat{y})_j - 1
\end{aligned}
$$

Hence, we have:

$$
\begin{aligned}
\nabla_W loss &= \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \left( \begin{bmatrix} Softmax(\hat{y})_1 & \cdots & Softmax(\hat{y})_C \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 1_{jth} & 0 & \cdots \end{bmatrix} \right) \\
&= x \left( Softmax(\hat{y})^T - y^T \right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_b loss &= \begin{bmatrix} Softmax(\hat{y})_1 \\ \vdots \\ Softmax(\hat{y})_C \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1_{jth} \\ 0 \\ \vdots \end{bmatrix} \\
&= Softmax(\hat{y}) - y
\end{aligned}
$$

### 2.1.3 Task 2: Implement the gradients

```
In [5]: def compute_grads_one_layer(softmaxed, x, y):
            """
            inputs:
                softmaxed (np.array): (N_CLASSES, 1)
                y (np.array): (N_CLASSES, 1)
                x (np.array): (INPUT_SHAPE, 1)

            returns:
                d_W (np.array): (INPUT_SHAPE, N_CLASSES) Gradient of the loss with
                                                          respect to the weight matrix
                d_b (np.array): (N_CLASSES, 1) Gradient of the loss with respect to
                                               the bias matrix
            """
            d_W = x @ (softmaxed - y).T
            d_b = softmaxed - y

            return d_W, d_b

        def compute_loss(softmaxed, y):
            """
            inputs:
                softmaxed (np.array): (N_CLASSES, 1)
                y (np.array): (N_CLASSES, 1)

            returns:
```

```python
        (float)
    """
    return float( -np.log( softmaxed.T @ y ) )
```

### 2.1.4 Question 2 (1 points): As a sanity check, we want to compare the gradients we calculated to approximated gradients. How could we do this ?

**Answer:** In order to check if the gradients are corrects, we can calculate approximate gradients and compare the gradients to them. The approximate gradients can be calculated according to the partial derivative definition:

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

We calculate then for each element of the gradient of the function.
*Example:* For the *loss* with respect to $W$, we have:

$$\nabla_W loss_{(i,j)} = \lim_{h \to 0} \frac{loss(W+H) - loss(W)}{h}$$

where $H = (h_{m,n})$, with $h_{m,n} = \begin{cases} 0, & (m,n) \neq (i,j) \\ h, & (m,n) = (i,j) \end{cases}$.

### 2.1.5 Task 3: Implement the approx gradient function for the weight matrix

```python
In [6]: def approx_grad_W(W, b, x, y, h=0.0001):
            """
            Approximates the gradient with respect to W
            input:
                W (np.array): (INPUT_SHAPE, N_CLASSES) The weight matrix of the perceptron
                b (np.array): (N_CLASSES, 1) The bias matrix of the perceptron
                x (np.array): (INPUT_SHAPE, 1) The input of the perceptron
                y  (np.array): (N_CLASSES, 1) The input of the perceptron
                h (float): variation

            returns:
                d_W_approx (np.array): (INPUT_SHAPE, N_CLASSES)
            """
            # The approx gradient is (loss(W_(i,j)+h, b) - loss(W_(i,j), b)) / h
            y_0 = W.T @ x + b
            softmaxed = softmax(y_0)

            result = np.zeros(W.shape)
            for i in range(W.shape[0]):
                for j in range(W.shape[1]):
                    h_matrix = np.zeros(W.shape)
                    h_matrix[i][j] = h
                    W_h = W + h_matrix
                    y_0_h = W_h.T @ x + b
                    softmaxed_h = softmax(y_0_h)
```

```
                result[i][j] = (
                    compute_loss(softmaxed_h, y) - compute_loss(softmaxed, y)  ) / h

        return result
```

The following function trains the perceptron

```
In [7]: def train_one_layer(X_train, Y_train, X_test, Y_test, lr,
                            n_it=1000, test_freq=10, random_seed=42):

            INPUT_SHAPE = X_train.shape[1]
            N_CLASSES = Y_train.shape[1]

            # Initialise metrics lists
            loss = []
            acc = []
            test_acc = []
            approx_grads = []
            grads = []

            # Initialisation of the weigths
            np.random.seed(random_seed)
            b = rand.normal(size=(N_CLASSES, 1))
            W = rand.normal(size=(INPUT_SHAPE, N_CLASSES))

            # Shuffling data
            indexes = rand.randint(X_train.shape[0], size=n_it)
            # training loop
            for it, i in enumerate(indexes):
                x = X_train[i,:].reshape(-1,1)
                y = Y_train[i,:].reshape(-1,1)

                # Forward passs
                softmaxed = forward_one_layer(W, b, x)
                # Back propagation
                d_W, d_b = compute_grads_one_layer(softmaxed, x, y)
                W -= lr * d_W
                b -= lr * d_b

                # Recording approximate gradients
                grads.append(d_W)
                approx_grads.append(approx_grad_W(W, b, x, y))

                # Metrics recording
                loss.append(compute_loss(softmaxed, y))
                acc.append(np.argmax(softmaxed) == np.argmax(y))

                # Test loop
```

```
            if it % test_freq == 0:
                acc_temp = []
                for i in range(X_test.shape[0]):
                    x = X_train[i,:].reshape(-1,1)
                    y = Y_train[i,:].reshape(-1,1)
                    softmaxed = forward_one_layer(W, b, x)
                    acc_temp.append(np.argmax(softmaxed) == np.argmax(y))

                test_acc.append(np.mean(acc_temp))

        return W, b, loss, acc, test_acc, np.stack(grads, -1), np.stack(approx_grads, -1)
```

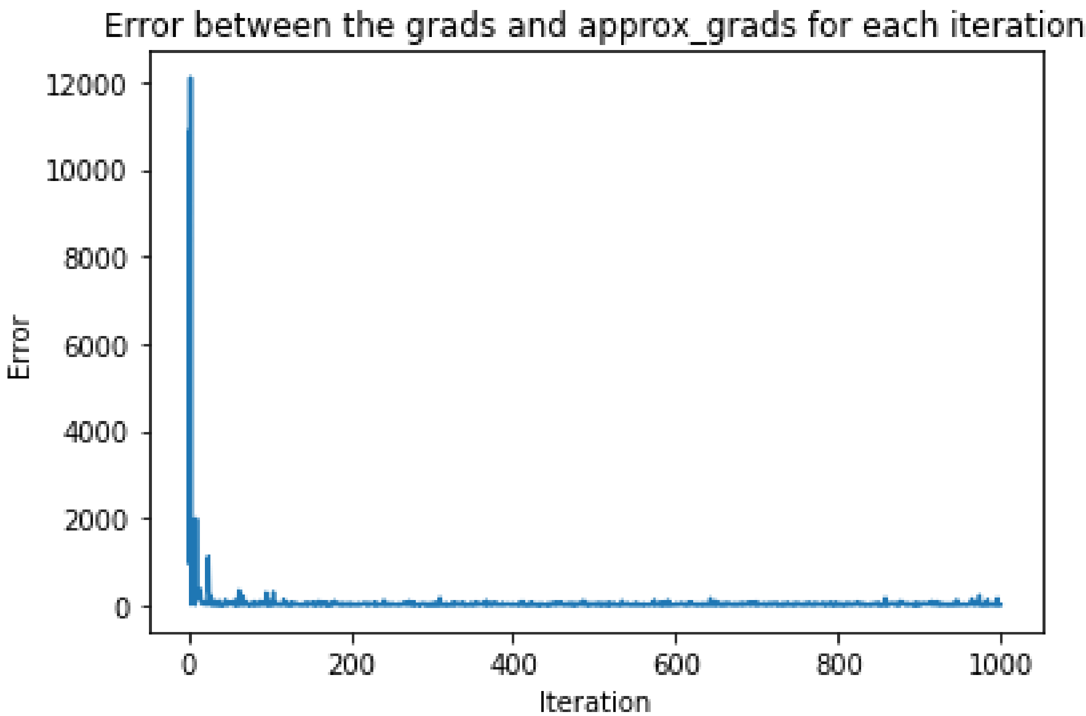
### 2.1.6 Task 4 (4 points): Using the previous function train the model. By producing an appropriate set of plots validate that it trained correctly and that you computed the correct gradients. Remember that a plot should be self explanatory.

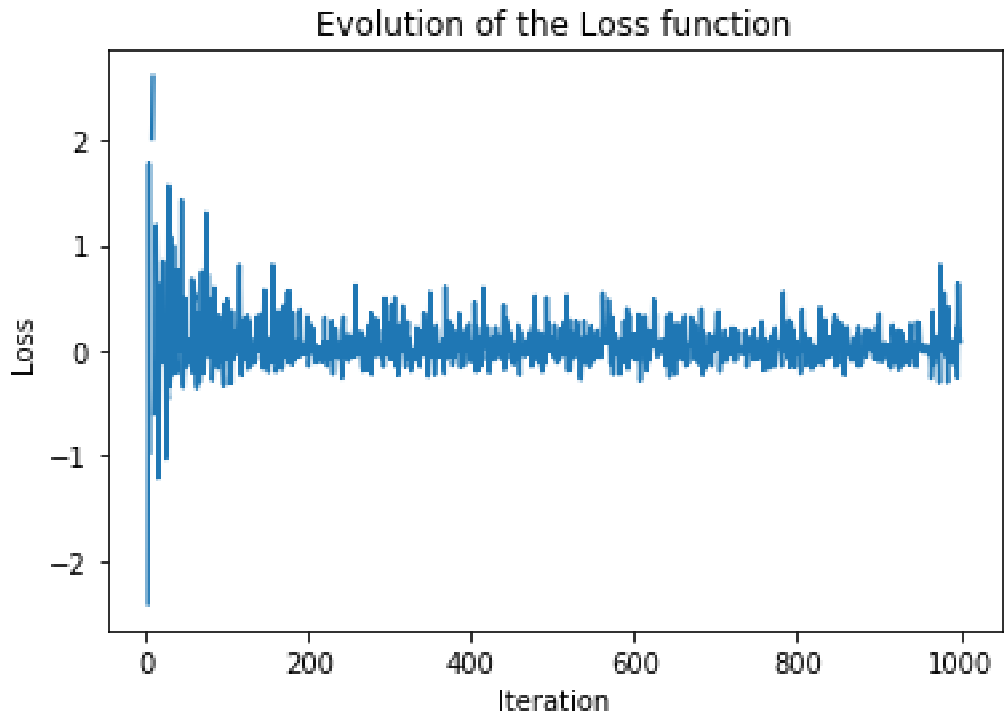
In [8]:

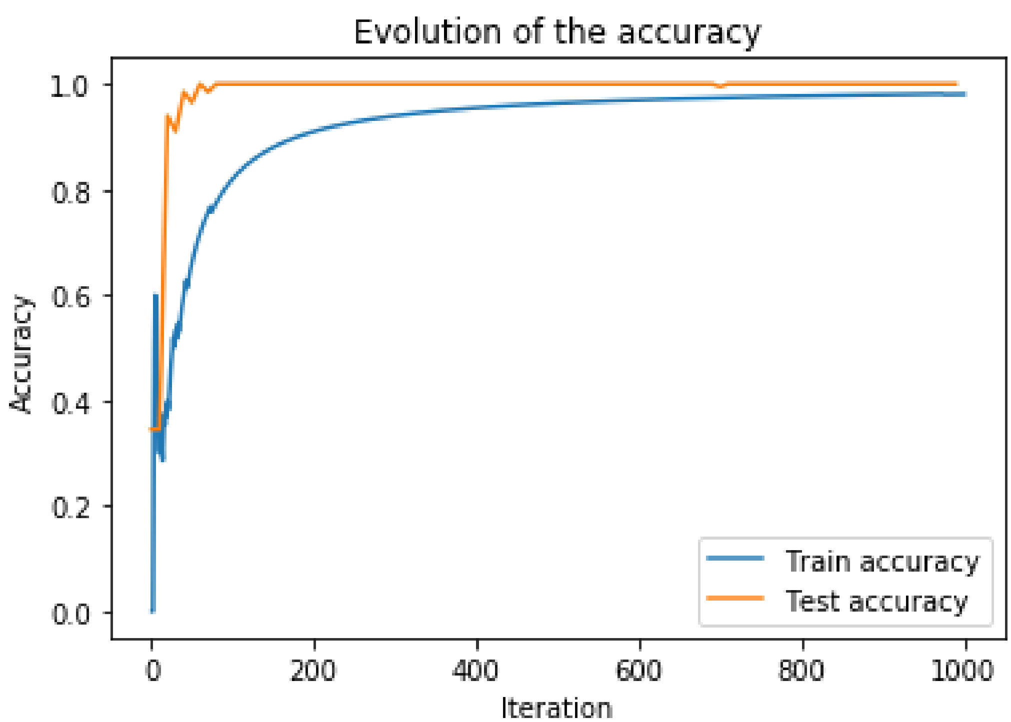
```
W, b, loss, acc, test_acc, grads, approx_grads = train_one_layer(X_train, Y_train,
                                                                 X_test, Y_test, 0.01)

iteration = [t+1 for t in range(1000)]

# Comparing the values between grads and approx_grads
error = []
for it in range(grads.shape[2]):
    aux = 0;
    for i in range(grads.shape[0]):
        for j in range(grads.shape[1]):
            aux += (grads[i][j][it] - approx_grads[i][j][it])**2
    error.append(aux)

plt.plot(iteration, error)
plt.xlabel("Iteration")
plt.ylabel("Error")
plt.title("Error between the grads and approx_grads for each iteration")
plt.show()

# evolution of loss function
plt.plot(iteration, loss)
plt.xlabel("Iteration")
plt.ylabel("Loss")
plt.title("Evolution of the Loss function")
plt.show()

# Training data accuracy
freq = 10
acc_true = [a/b for a,b in zip(np.array(acc).cumsum(), range(1,len(acc)+1))]
plt.plot(iteration, acc_true, label="Train accuracy")
plt.plot([t*freq for t in range(100)], test_acc, label="Test accuracy")
```

```
plt.xlabel("Iteration")
plt.ylabel("Accuracy")
plt.title("Evolution of the accuracy")
plt.legend()
plt.show()
```

```
C:\Users\Matheus Douglas\Anaconda3\lib\site-packages\ipykernel_launcher.py:26: RuntimeWarning:
```




Evolution of the Loss function
Loss
Iteration
2
1
0
−1
−2
0
200
400
600
800
1000

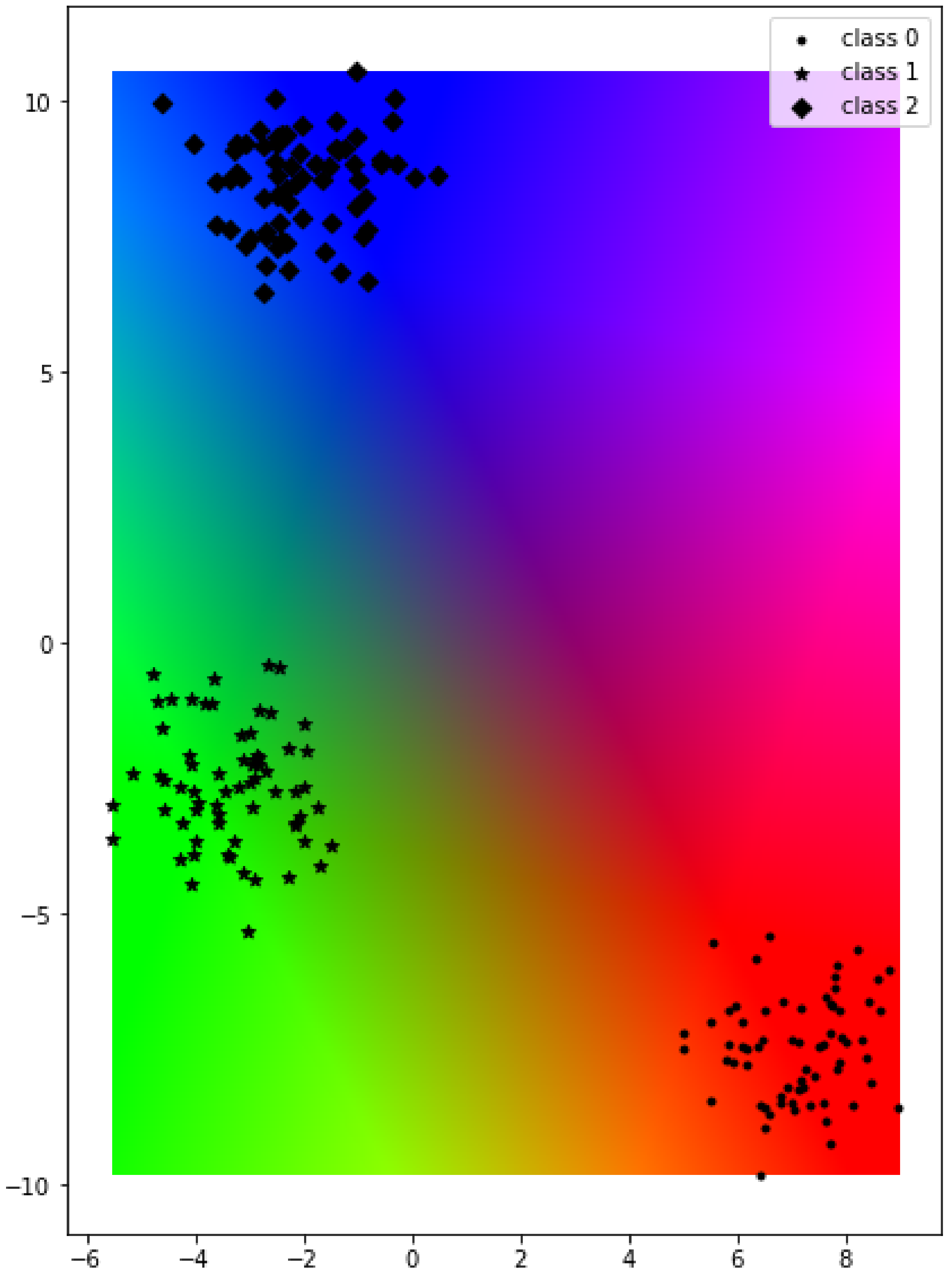

Evolution of the accuracy
Accuracy
Iteration
1.0
0.8
0.6
0.4
0.2
0.0
0
200
400
600
800
1000
Train accuracy
Test accuracy

## 3 Non linearity

Now that the perceptron is trained, we can visualize its decision function.

```
In [9]: def plot_decision(X, Y, forward, figure=None):
            """Plots the decision function of a perceptron with respect to a
               forward function

            input:
                X, Y (np.array): Test data
                forward (function): only accepts x as input
                                     (Ex: lambda x: forward_one_layer(W, b, x))
                figure (plt.figure): optional, usefull if you dont want to generate
                                      any new figure, in the case of suplots.
            """
            markers=[".", "*", "D"]
            low0, high0 = np.min(X[:,0]), np.max(X[:,0])
            low1, high1 = np.min(X[:,1]), np.max(X[:,1])
            data = np.zeros((100,100,Y.shape[1]))
            for i1, x1 in enumerate(np.linspace(low0,high0,100)):
                for i2, x2 in enumerate(np.linspace(low1,high1,100)):
                    x = np.array([x1, x2]).reshape(-1, 1)
                    softmaxed = forward(x)
                    data[i2, i1, :] = softmaxed.reshape(-1)
            if Y.shape[1] < 3:
                data = data[:,:,0]

            if figure is None:
                plt.figure(figsize=(10,10))

            plt.imshow(data, extent=(low0,high0,low1,high1), origin='lower',
                       interpolation='gaussian')
            for c in range(Y.shape[1]):
                plt.scatter(X[np.argmax(Y, 1) == c, 0], X[np.argmax(Y, 1) == c, 1], c='k',
                            marker=markers[c], label="class %i" % c)

            plt.legend()
            plt.show()

In [10]: plot_decision(X_test, Y_test, lambda x: forward_one_layer(W, b, x))
```

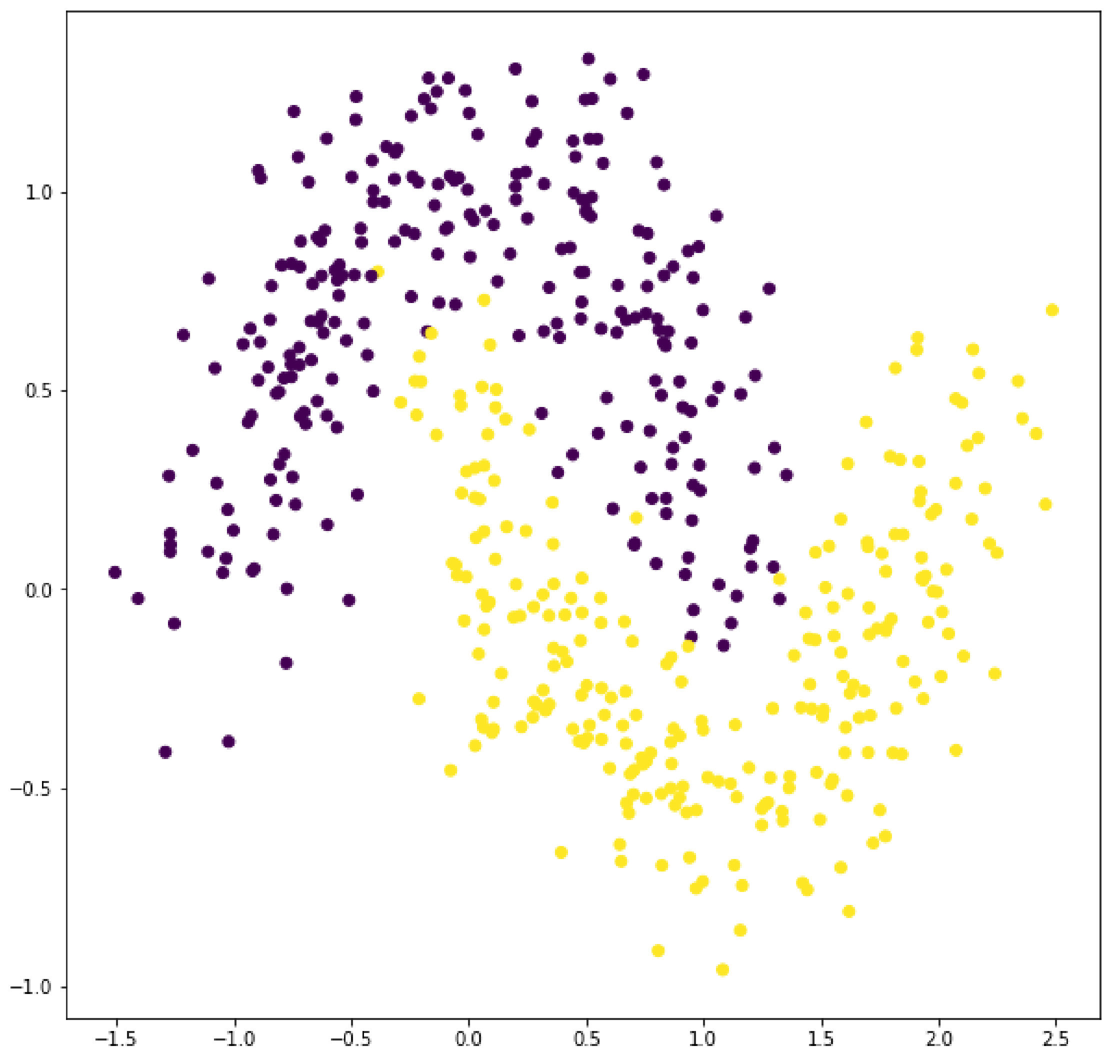
```
Clipping input data to the valid range for imshow with RGB data ([0..1] for floats or [0..255]
```




The data we used was linearly separable, but what if it is not ?

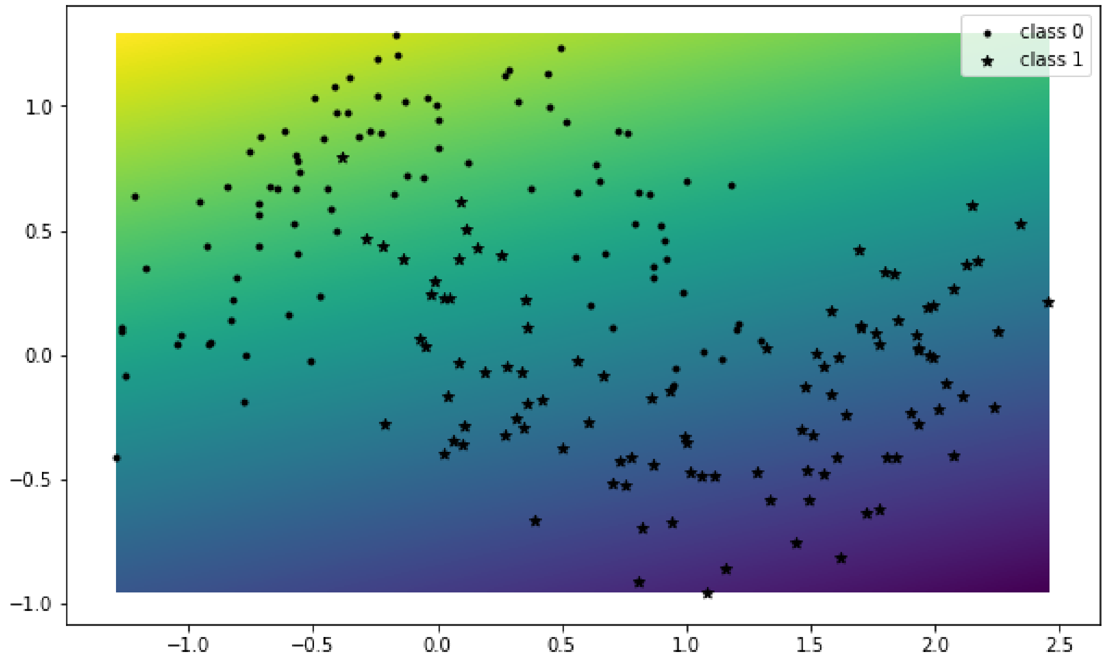
### 3.0.1 Task 5 (1 point): Using the following data train a new perceptron and visualize its decision function.

```
In [11]: X, Y = make_moons(n_samples=500, noise=.2)
         plt.figure(figsize=(10,10))
         plt.scatter(X[:,0], X[:,1], c=Y)
         plt.show()
         Y = OneHotEncoder().fit_transform(Y.reshape(-1,1)).toarray()

         X_train, X_test, Y_train, Y_test = train_test_split(X, Y, random_state=42,
                                                             test_size=0.4)
```




```
In [12]: W, b, loss, acc, test_acc, grads, approx_grads = train_one_layer(X_train, Y_train,
                                                                          X_test, Y_test, 0.01)
         plot_decision(X_test, Y_test, lambda x: forward_one_layer(W, b, x))
```

```
C:\Users\Matheus Douglas\Anaconda3\lib\site-packages\ipykernel_launcher.py:26: RuntimeWarning:
```




## 4 Two Layer perceptron - Feed forward neural network

In neural networks, non linearity comes from having at least one hidden layer and a non linear activation function such as the *ReLU* function.

$$h = ReLU(W_h^\top x + b_h)$$

$$\hat{y} = W_o^\top h + b_o$$

$$\hat{P}(y_j = 1) = Softmax(\hat{y})_j$$

With $W_h \in M_{N,H}(\mathbb{R})$ and $W_o \in M_{C,H}(\mathbb{R})$ and $b^h$, $b^o$ vectors of corresponding dimensions. $H$ is the number of hidden units.

The *ReLU* funciton is defined as:

$$ReLU(x) = \begin{cases} 0 \text{ if } x < 0 \\ x \text{ otherwise} \end{cases}$$

#### 4.0.1 Question 3 (2 points): Derive the gradients with respect to $W_h$ and $b_h$.

**Answer:**

We have that:

$$\begin{aligned}
ReLU(x) &= \max(0, x) \\
ReLU'(x) &= \frac{dReLU}{dx}(x) \\
&= \mathbb{1}_{\{x>0\}} \\
&= \mathbb{1}_{\{ReLU(x)>0\}} \\
&= ReLU'(ReLU(x))
\end{aligned}$$

Also,

$$\begin{aligned}
h_i &= ReLU\left(W_{h(i,\cdot)}^T x + b_{h(i)}\right) \\
&= ReLU\left(\textstyle\sum_k w_{h(k,i)} x + b_{h(i)}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial h_i}{\partial W_{h(u,v)}} &= \begin{cases} 0 & , \quad i \neq v \\ x_u ReLU'\left(\sum_k w_{h(k,i)} x_k + b_{h(i)}\right) & , \quad i = v \end{cases} \\
&= \begin{cases} 0 & , \quad i \neq v \\ x_u ReLU'(h_i) & , \quad i = v \end{cases}
\end{aligned}$$

$$\frac{\partial h_i}{\partial b_{h(v)}} = \begin{cases} 0 & , \quad i \neq v \\ ReLU'(h_i) & , \quad i = v \end{cases}$$

Then, for the $\hat{y}$, we have:

$$\begin{aligned}
\hat{y}_i &= W_{0(i,\cdot)}^T h + b_{0(i)} \\
&= \textstyle\sum_k w_{0(k,i)} h_k + b_{0(i)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \hat{y}_i}{\partial W_{h(u,v)}} &= \sum_k w_{0(k,i)} \frac{\partial h_k}{\partial W_{h(u,v)}} \\
&= w_{0(v,i)} \frac{\partial h_v}{\partial W_{h(u,v)}} \\
&= w_{0(v,i)} x_u ReLU'(h_v)
\end{aligned}$$

$$\frac{\partial \hat{y}_i}{\partial b_{h(v)}} = w_{0(v,i)} ReLU'(h_v)$$

Therefore, the gradients of the *loss* with respect to $W_h$ and $b_h$ are:

$$
\begin{aligned}
loss &= \log\left(\sum_k e^{y_k}\right) - y_j \\
\frac{\partial loss}{\partial W_{h(u,v)}} &= \frac{\sum_k e^{y_k}\frac{\partial y_k}{\partial W_{h(u,v)}}}{\sum_k e^{y_k}} - \frac{\partial y_j}{\partial W_{h(u,v)}} \\
&= \sum_k Softmax(\hat{y})_k \frac{\partial y_k}{\partial W_{h(u,v)}} - \frac{\partial y_j}{\partial W_{h(u,v)}} \\
&= \sum_k Softmax(\hat{y})_k w_{0(v,k)} x_u ReLU'(h_v) - \\
& \quad w_{0(v,j)} x_u ReLU'(h_v) \\
&= x_u ReLU'(h_v) W_{0(v,\cdot)} \left[Softmax(\hat{y}) - y\right] \\
\frac{\partial loss}{\partial b_{h(v)}} &= ReLU'(h_v) W_{0(v,\cdot)} \left[Softmax(\hat{y}) - y\right]
\end{aligned}
$$

We can note that:

$$
\nabla_{W_h} loss = x \nabla_{b_h} loss^T = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \begin{bmatrix} \nabla_{b_h} loss_1^T & \cdots & \nabla_{b_h} loss_h^T \end{bmatrix}
$$

#### 4.0.2 Task 5 (5 points): Based on the previous implementation complete the following functions and train a 2 layers perceptron.

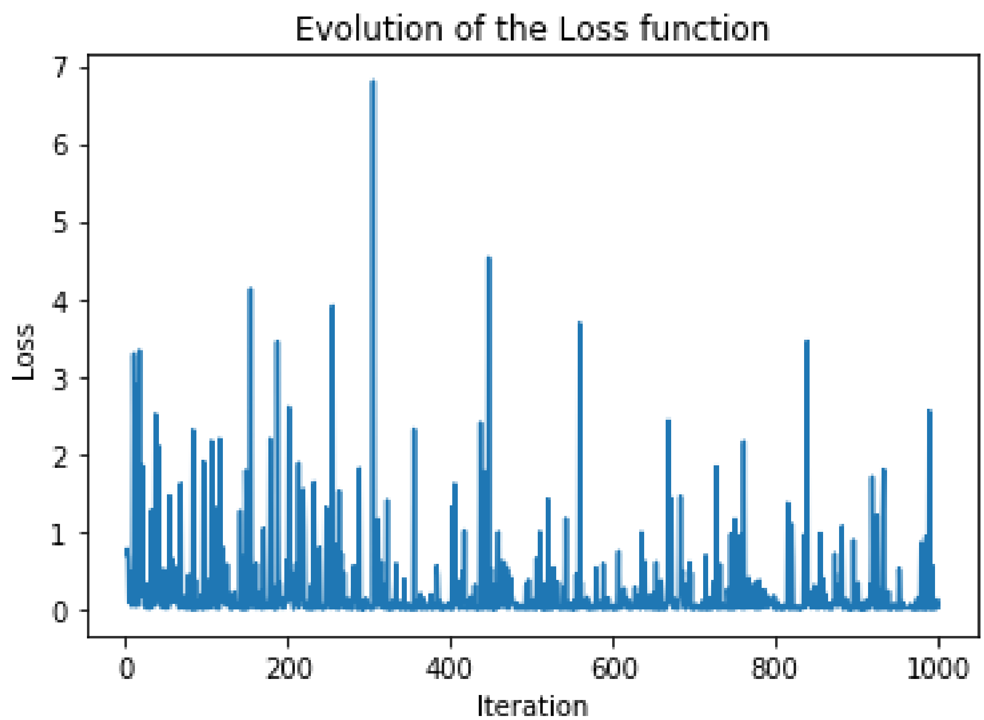
In [13]:
```
def relu(x):
    """
    input:
        x (np.array)

    returns:
        x (np.array)
    """
    aux = x.reshape(-1)
    result = np.zeros(len(aux))
    for i in range(len(result)):
        result[i] = max(0,aux[i])
    return result.reshape(-1,1)

def d_relu(x):
    """Computes the derivative of the relu
    input:
        x (np.array)

    returns:
        x (np.array)
    """
    aux = x.reshape(-1)
    result = np.zeros(len(aux))
```

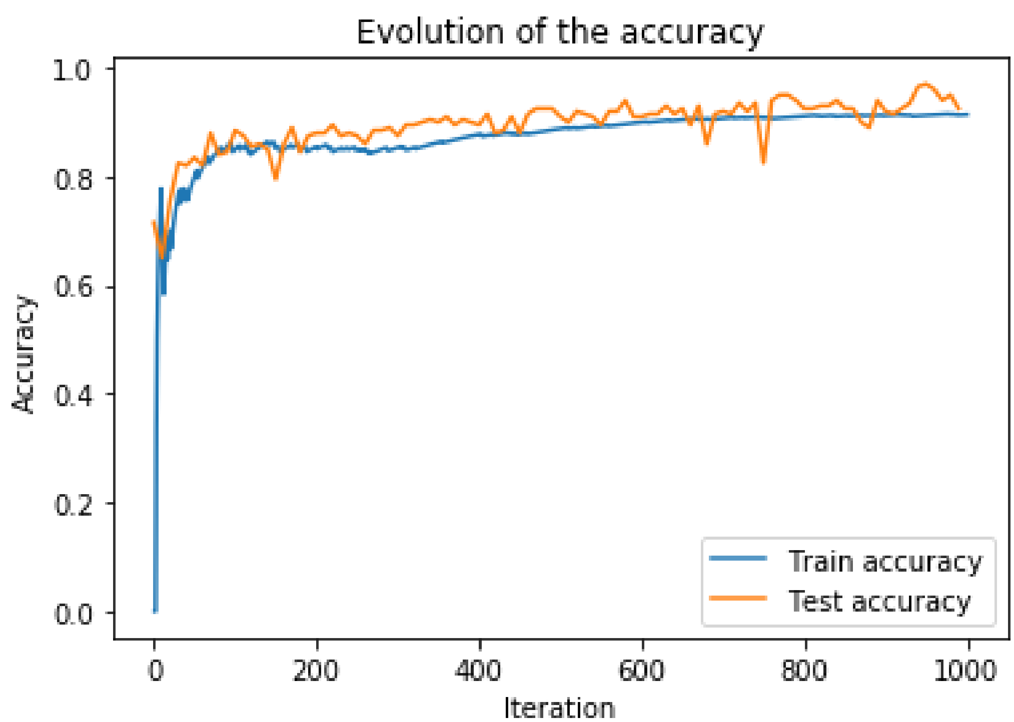
```python
    for i in range(len(result)):
        if aux[i] > 0:
            result[i] = 1
        else:
            result[i] = 0
    return result.reshape(-1,1)

def forward_two_layers(Wo, bo, Wh, bh, x):
    """Forward pass of a teo layer perceptron with relu activation
    input:
        Wh (np.array): (INPUT_SHAPE, HIDDEN_SHAPE) The weight matrix of the
                                                   hidden layer
        Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) The weight matrix of the
                                                 output layer
        bh (np.array): (HIDDEN_SHAPE, 1) The bias matrix of the hidden layer
        bo (np.array): (N_CLASSES, 1) The bias matrix of the output layer
        x (np.array): (INPUT_SHAPE, 1) The input of the perceptron

    returns:
        softmaxed (np.array): (N_CLASSES, 1) the output of the network after
                                             final activation
        hidden (np.array): (HIDDEN_SHAPE, 1) the output of the hidden layer
                                             after activation
        out (np.array): (N_CLASSES, 1) the output of the network before
                                       final activation
    """

    hidden = relu(Wh.T @ x + bh) # h
    out = Wo.T @ hidden + bo     # y_hat
    softmaxed = softmax(out)    # Softmax(y_hat)

    return softmaxed, hidden, out

def compute_grads_two_layers(hidden, softmaxed, Wo, x, y):
    """Forward pass of a teo layer perceptron with relu activation
    input:
        hidden (np.array): (HIDDENT_SHAPE, 1) the output of the hidden layer
                                              after activation
        softmaxed (np.array): (N_CLASSES, 1) the output of the network after
                                             final activation
        Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) The weight matrix of the
                                                 output layer
        x (np.array): (INPUT_SHAPE, 1) The input of the perceptron
        y (np.array): (N_CLASSES, 1) Ground truth class

    returns:
        d_Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) Gradient with respect
                         to the weight matrix of the output layer
```

```python
        d_bo (np.array): (N_CLASSES, 1) Gradient with respect to the bias matrix
                                        of the output layer
        d_Wh (np.array): (INPUT_SHAPE, HIDDEN_SHAPE) Gradient with respect to the
                        weight matrix of the hidden layer
        d_bh (np.array): (HIDDEN_SHAPE, 1) Gradient with respect to the bias
                                           matrix of the hidden layer
    """
    d_Wo, d_bo = compute_grads_one_layer(softmaxed, hidden, y)

    d_bh = np.zeros(hidden.shape)
    for i in range(d_bh.shape[0]):
        aux = d_relu(hidden)
        d_bh[i,0] = aux[i,0] * ( Wo[i,:] @ (softmaxed - y).reshape(-1) )

    d_Wh = np.outer(x, d_bh)

    return d_Wo, d_bo, d_Wh, d_bh

def train_two_layer(X_train, Y_train, X_test, Y_test, lr, n_hidden,
                    n_it=1000, test_freq=10, random_seed=42):

    INPUT_SHAPE = X_train.shape[1]
    N_CLASSES = Y_train.shape[1]

    # Initialise metrics lists
    loss = []
    acc = []
    test_acc = []
    approx_grads = []
    grads = []

    # Initialisation of the weigths
    np.random.seed(random_seed)
    bh = rand.normal(size=(n_hidden, 1))
    Wh = rand.normal(size=(INPUT_SHAPE, n_hidden))
    bo = rand.normal(size=(N_CLASSES, 1))
    Wo = rand.normal(size=(n_hidden, N_CLASSES))

    # Shuffling data
    indexes = rand.randint(X_train.shape[0], size=n_it)
    # training loop
    for it, i in enumerate(indexes):
        x = X_train[i,:].reshape(-1,1)
        y = Y_train[i,:].reshape(-1,1)

        # Forward passs
        softmaxed, hidden, out = forward_two_layers(Wo, bo, Wh, bh, x)
```

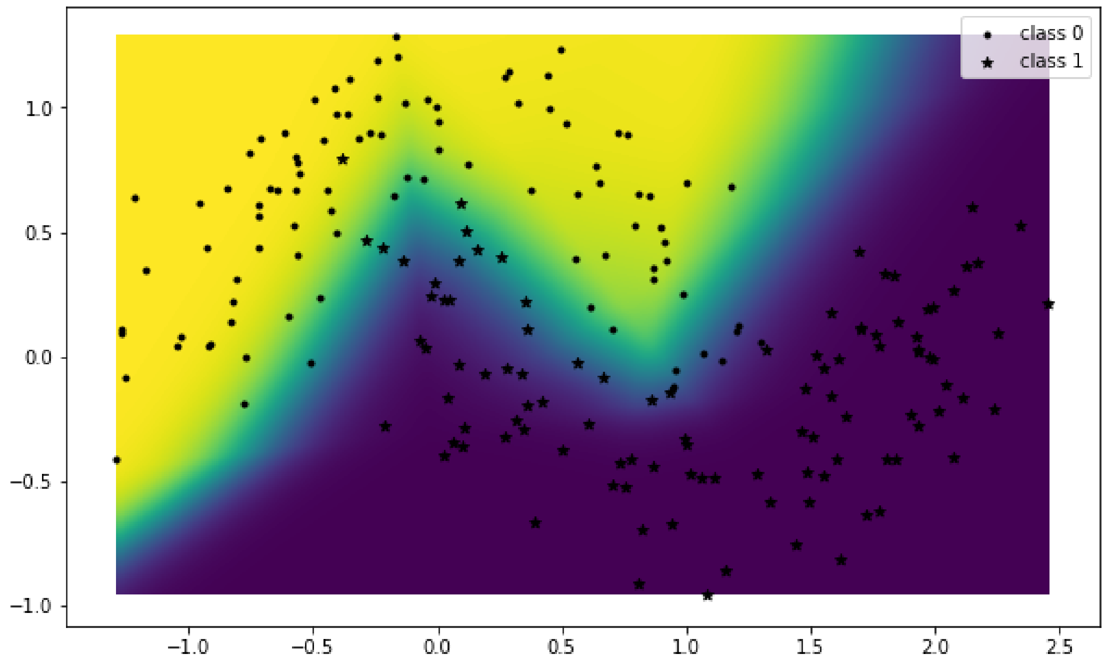
```
            # Back propagation
            d_Wo, d_bo, d_Wh, d_bh = compute_grads_two_layers(hidden, softmaxed,
                                                              Wo, x, y)
            bh -= lr * d_bh
            Wh -= lr * d_Wh
            bo -= lr * d_bo
            Wo -= lr * d_Wo

            # Metrics recording
            loss.append(compute_loss(softmaxed, y))
            acc.append(np.argmax(softmaxed) == np.argmax(y))

            # Test loop
            if it % test_freq == 0:
                acc_temp = []
                for i in range(X_test.shape[0]):
                    x = X_train[i,:].reshape(-1,1)
                    y = Y_train[i,:].reshape(-1,1)
                    softmaxed, hidden, out = forward_two_layers(Wo, bo, Wh, bh, x)
                    acc_temp.append(np.argmax(softmaxed) == np.argmax(y))

                test_acc.append(np.mean(acc_temp))

        return Wo, bo, Wh, bh, loss, acc, test_acc
```

In [14]:
```
Wo, bo, Wh, bh, loss, acc, test_acc = train_two_layer(X_train, Y_train,
                                                      X_test, Y_test, 0.1, 16)

iteration = [t+1 for t in range(1000)]

# evolution of loss function
plt.plot(iteration, loss)
plt.title("Evolution of the Loss function")
plt.xlabel("Iteration")
plt.ylabel("Loss")
plt.show()

# Training data accuracy
freq = 10
acc_true = [a/b for a,b in zip(np.array(acc).cumsum(), range(1,len(acc)+1))]
plt.plot(iteration, acc_true, label="Train accuracy")
plt.plot([t*freq for t in range(100)], test_acc, label="Test accuracy")
plt.xlabel("Iteration")
plt.ylabel("Accuracy")
plt.title("Evolution of the accuracy")
plt.legend()
plt.show()
```


Evolution of the Loss function
Loss
Iteration
0
1
2
3
4
5
6
7
200
400
600
800
1000


Evolution of the accuracy
Accuracy
Iteration
Train accuracy
Test accuracy
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000

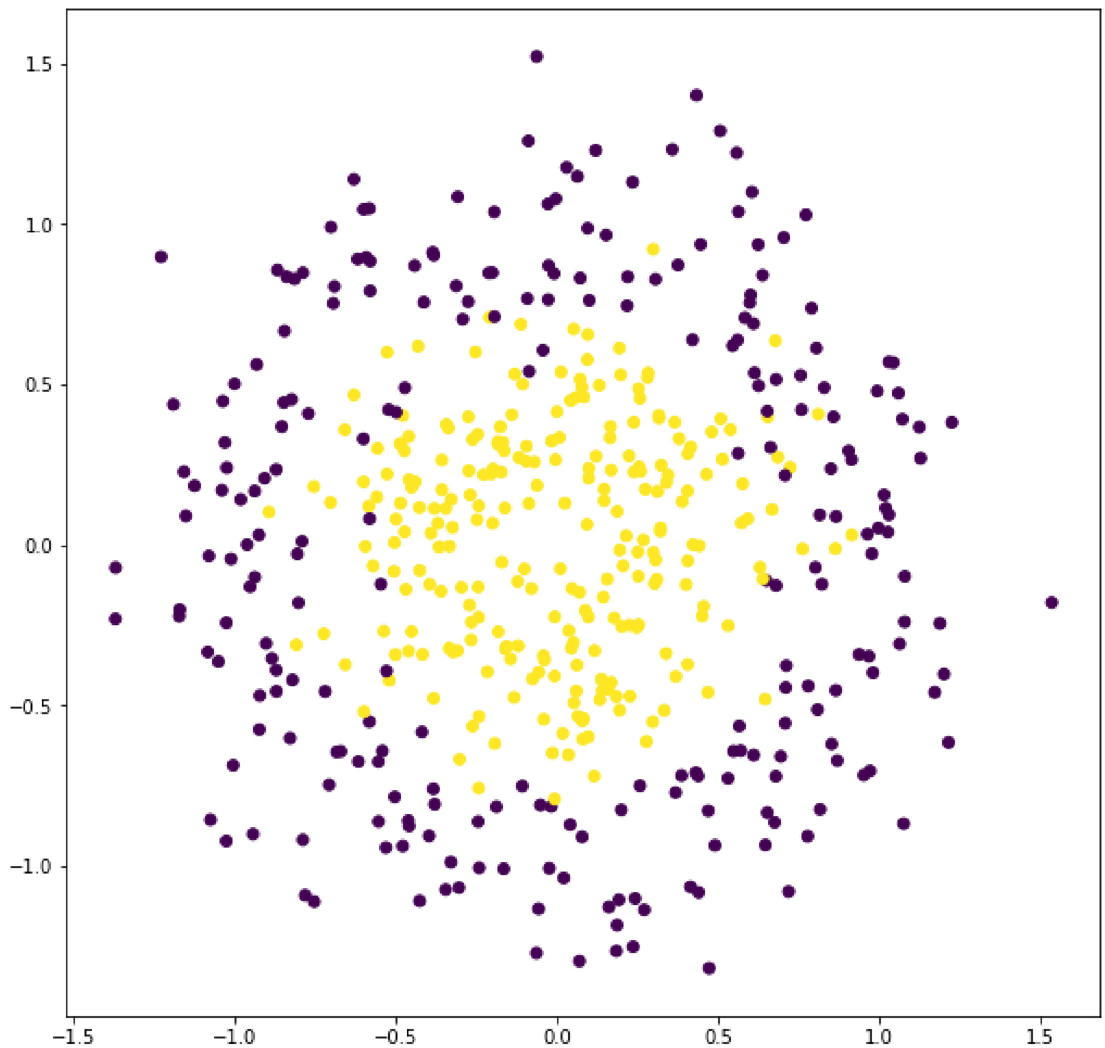
```
In [15]: plot_decision(X_test, Y_test, lambda x: forward_two_layers(Wo, bo, Wh, bh, x)[0])
```




#### 4.0.3 Task 7 (3 points): The non linearity of the decision function is conditionned by the number of units, visualize this effect. What can you comment on the smoothness of the boundary ? (Be carefull, to visualize this you need to properly train the networks)

To better visualize this let's generate new data.

```
In [16]: X, Y = make_circles(n_samples=500, noise=.2, factor=0.4)
         plt.figure(figsize=(10,10))
         plt.scatter(X[:,0], X[:,1], c=Y)
         plt.show()
         Y = OneHotEncoder().fit_transform(Y.reshape(-1,1)).toarray()

         X_train, X_test, Y_train, Y_test = train_test_split(X, Y, random_state=42,
                                                             test_size=0.4)
```




**Answer:**

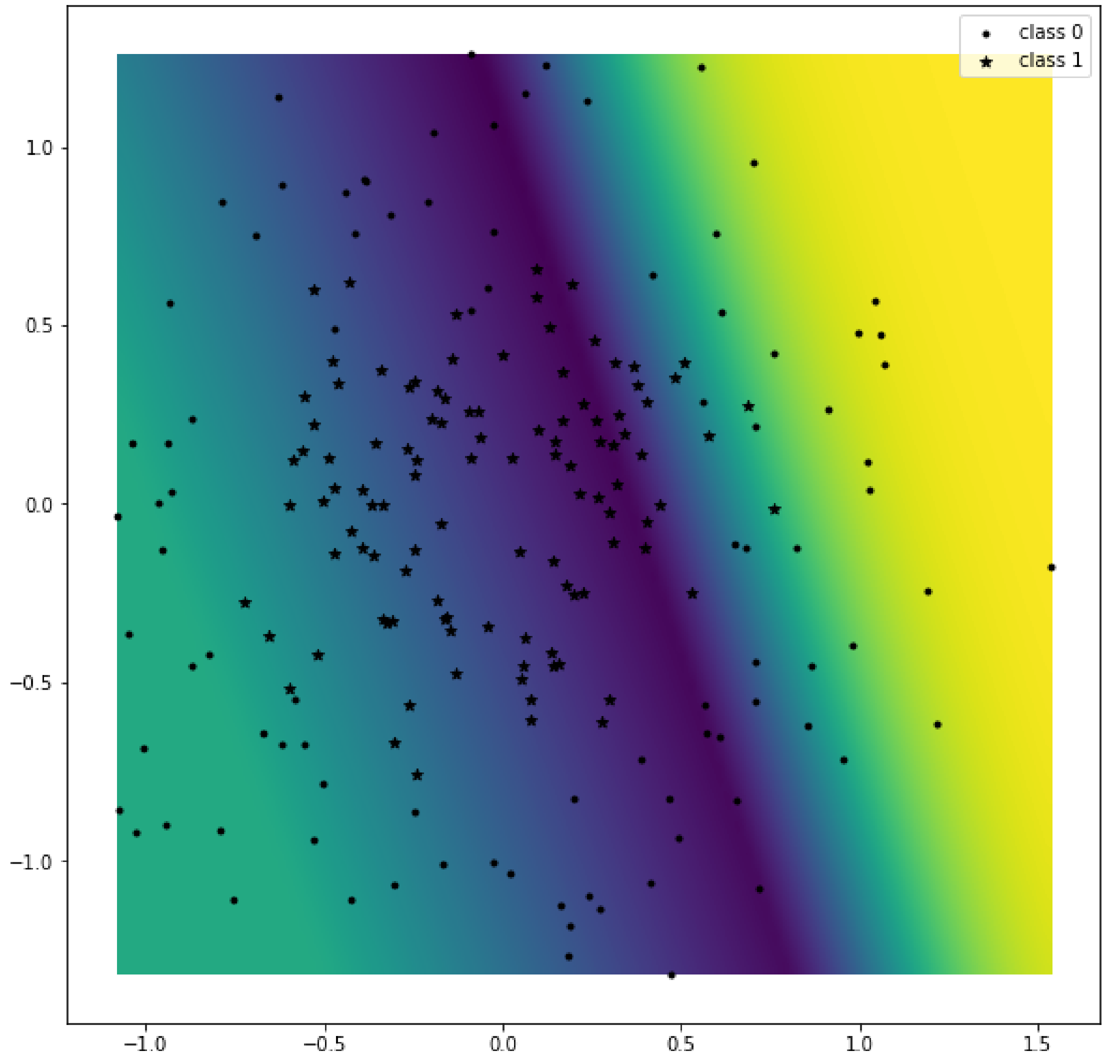
We can see below the effect of the number of units of perceptrons in the hidden layer to the decision function.

According to the results below, we can clearly realize that the boundary of the decision function become more well defined and smoother with the increase of the units.

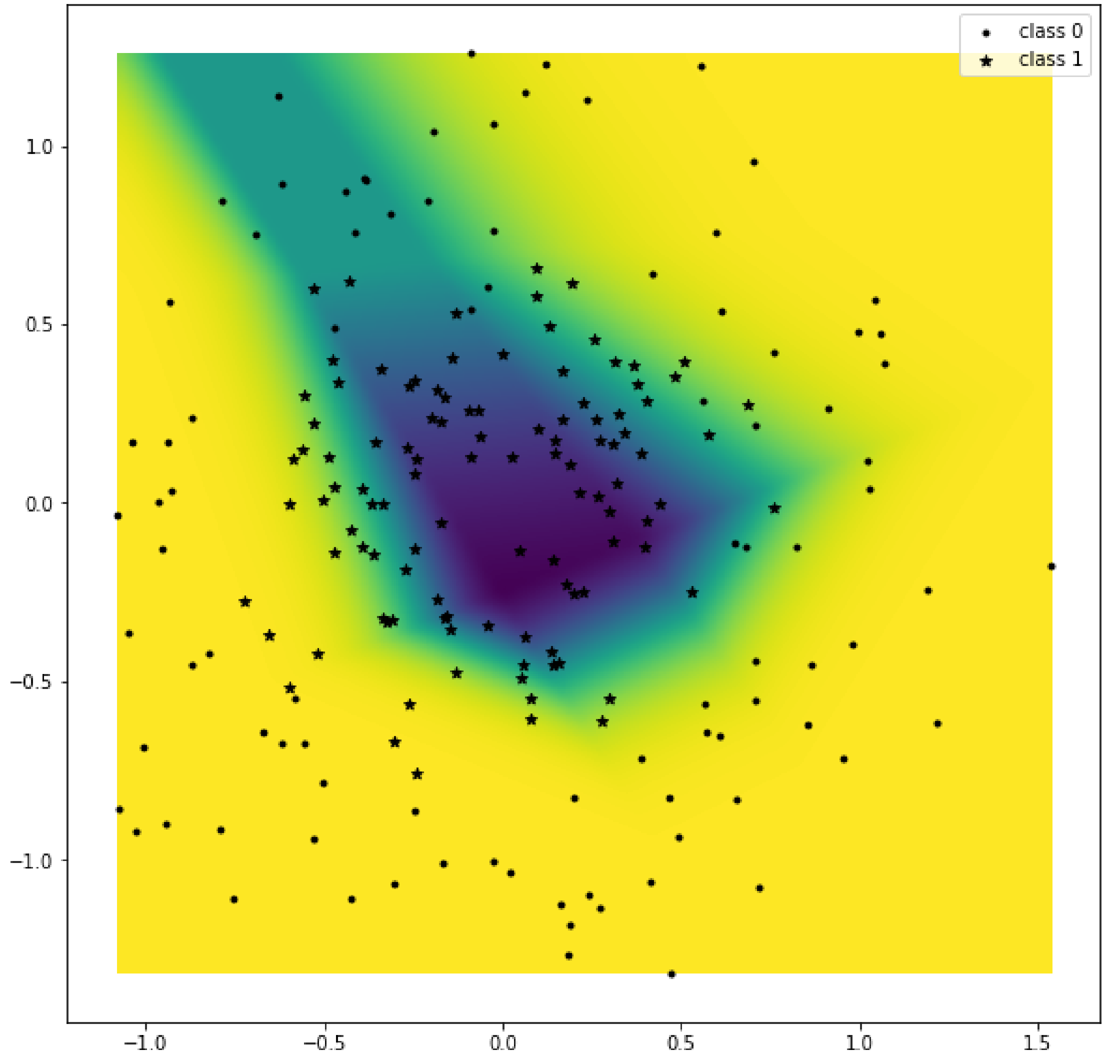
Using 2 units, the decision function looks linear. Using 5 units, the boundary is barely circular, but it is not well defined. With 10, 15, 20 and 50 units, the circular characteristic of the decision function is noticed, but they have different degrees of smoothness. And the more we use units of perceptrons in the hidden layer, the smoother the boundary becomes.

```
In [17]: values = [2, 5, 10, 15, 20, 50]
         for units_hidden in values:
             print("UNITS =", units_hidden)
             Wo, bo, Wh, bh, loss, acc, test_acc = train_two_layer(X_train, Y_train, X_test,
                                                                   Y_test, 0.1, units_hidden)
             plot_decision(X_test, Y_test, lambda x: forward_two_layers(Wo, bo, Wh, bh, x)[0])
```

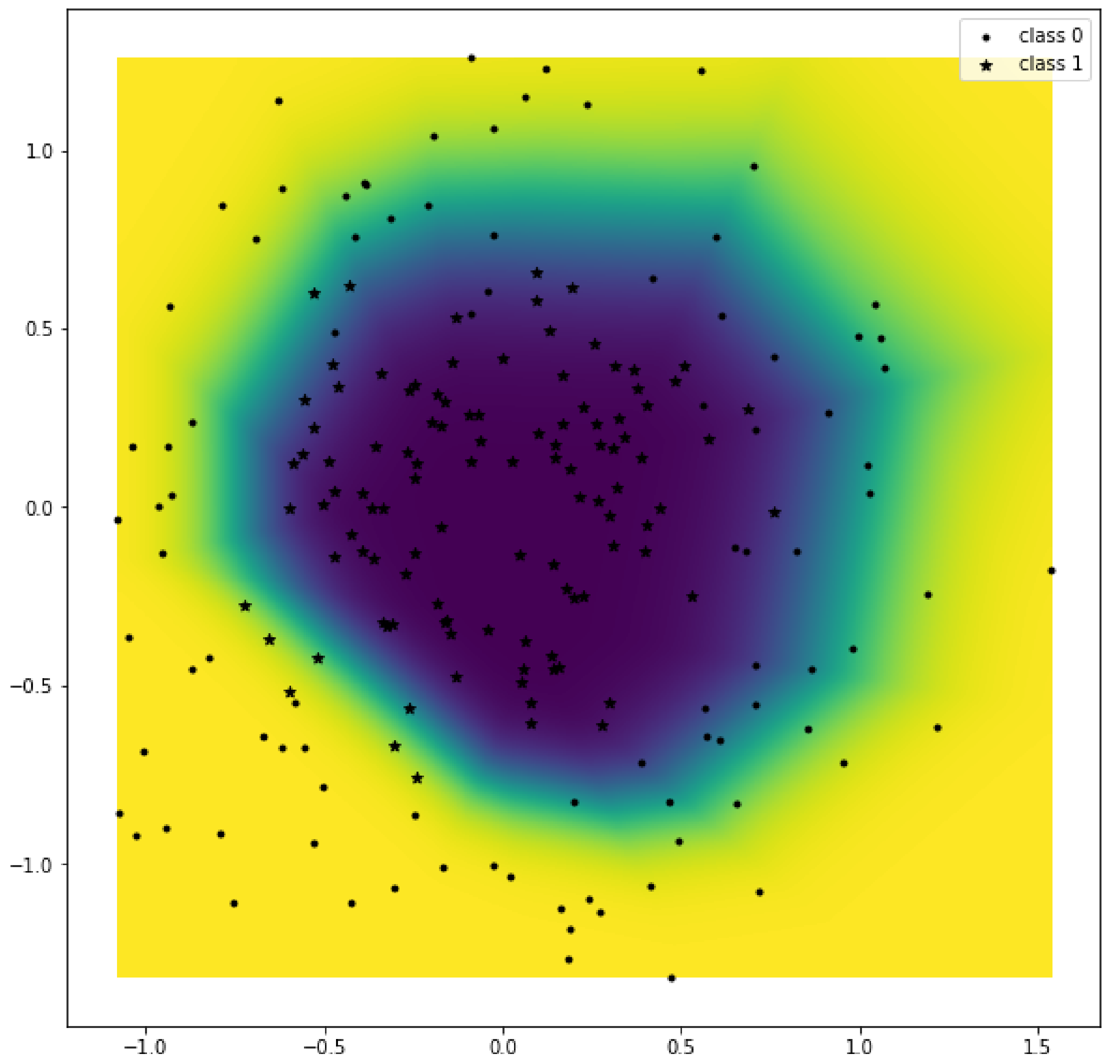
UNITS = 2



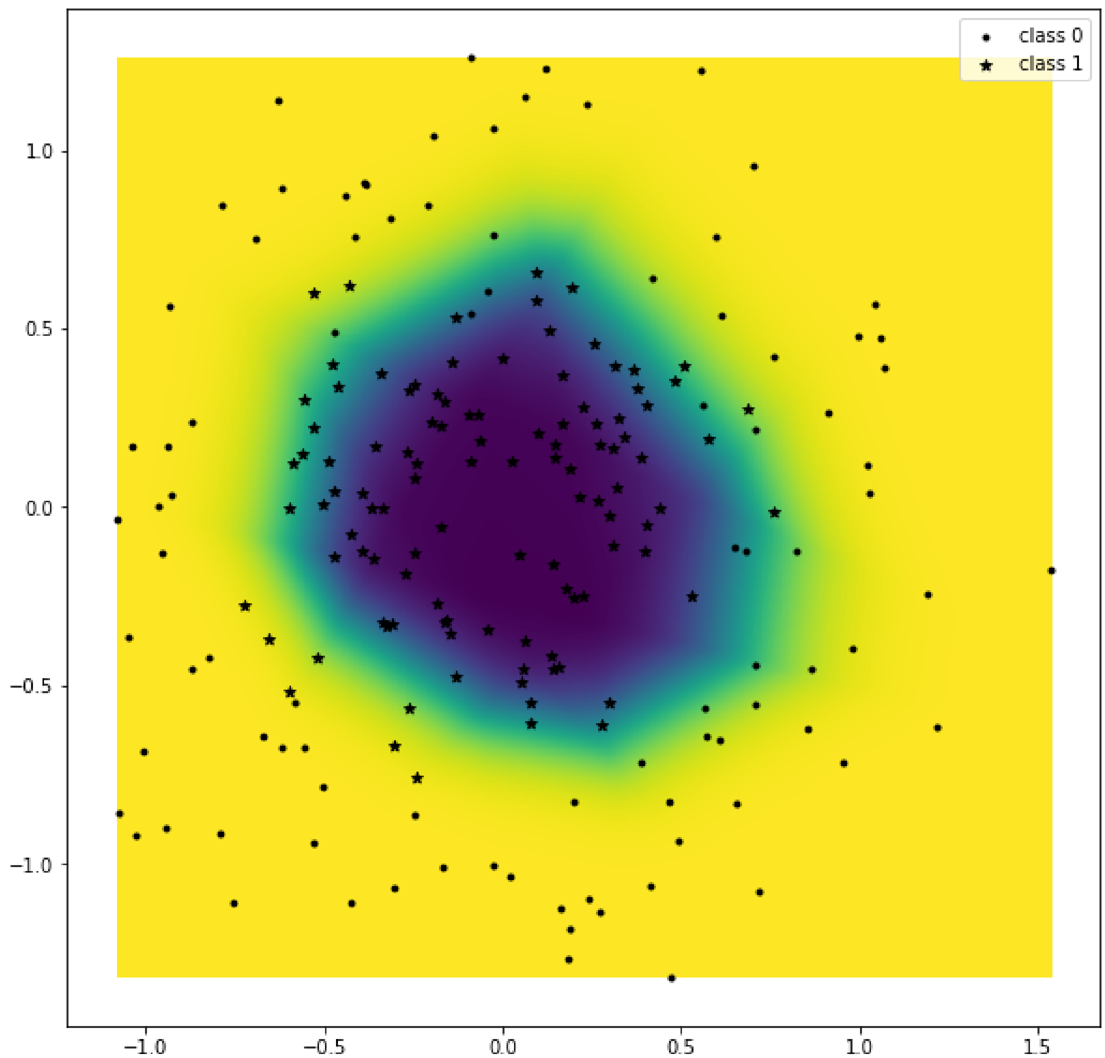

UNITS = 5



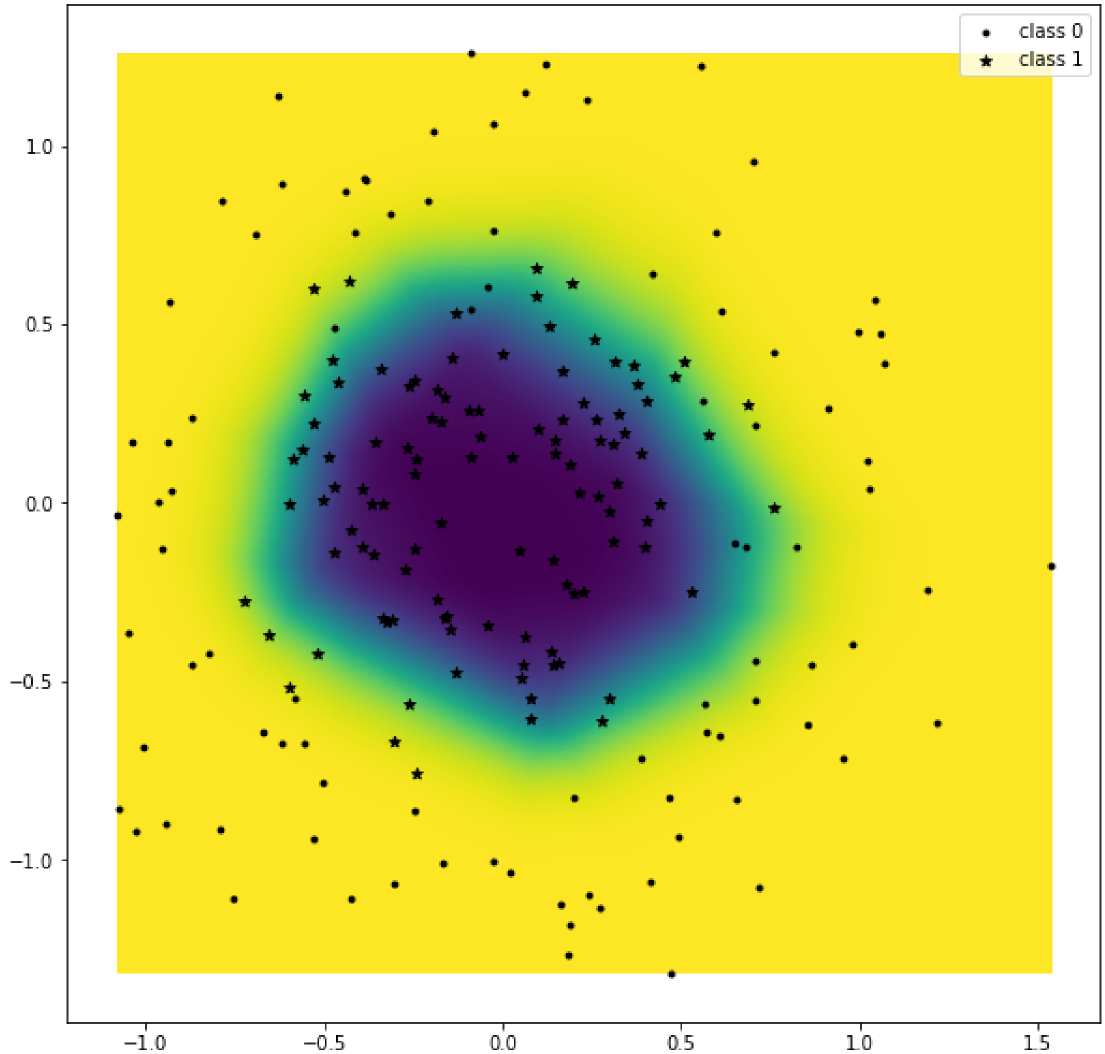

UNITS = 10



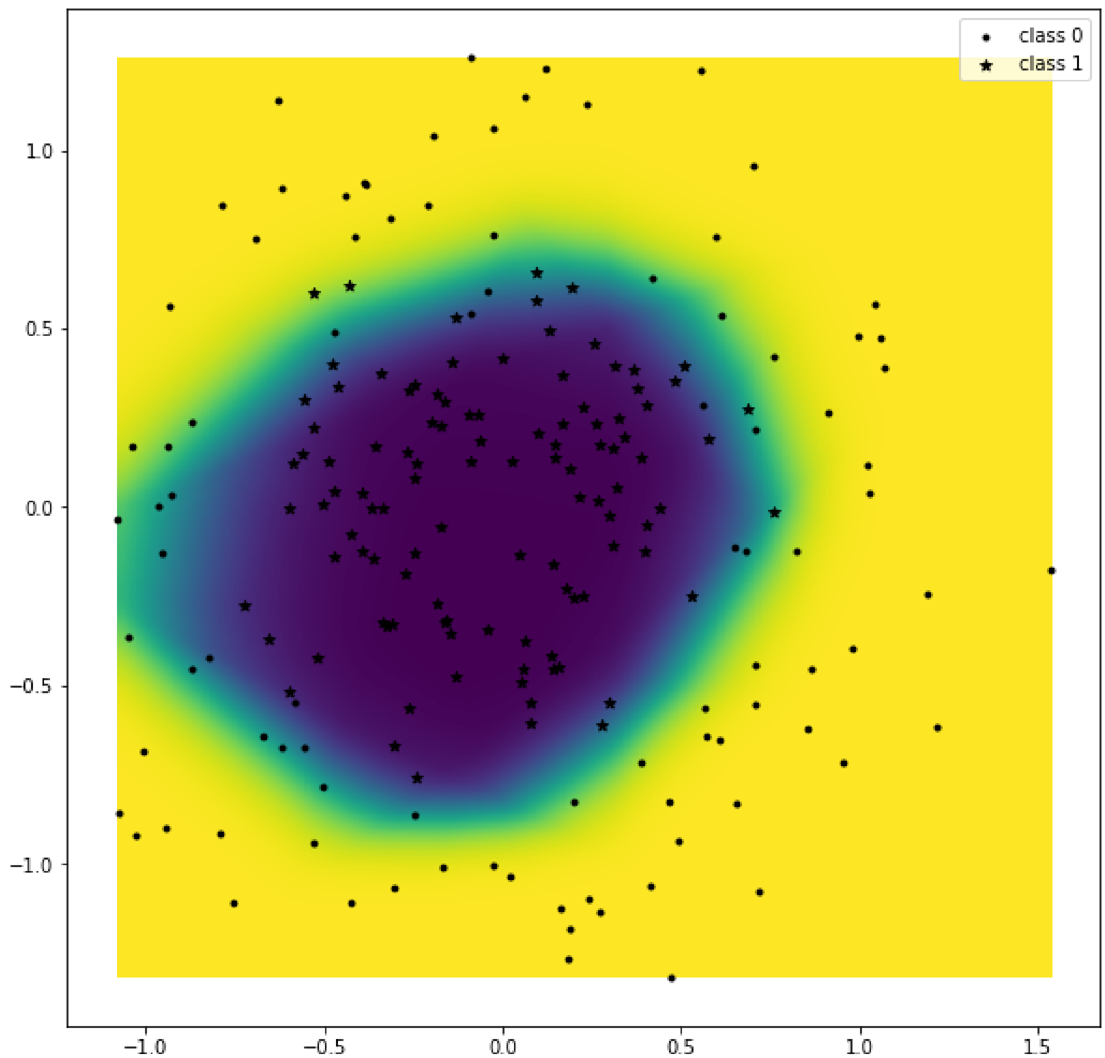

```
UNITS = 15
```




UNITS = 20




UNITS = 50




#### 4.0.4 Task 8 (2 points): To have a smooth boundary we need to change the activation. Change the activation from relu to tanh, and visualize the result.

If we use the activation to tanh, we have that:

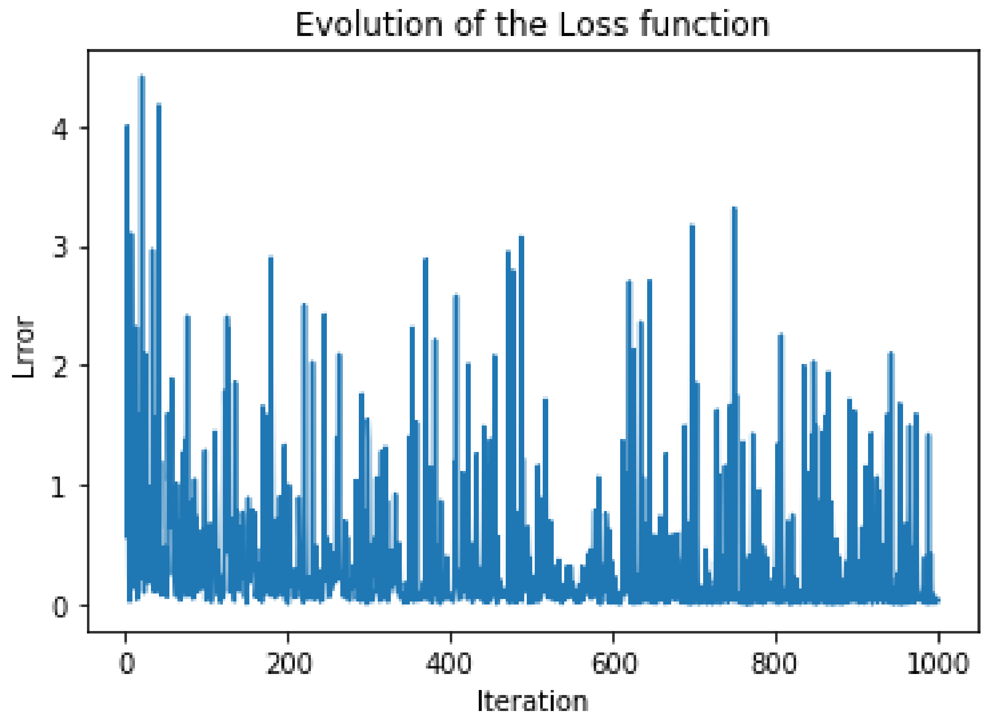
$$
\begin{aligned}
\tanh(x) &= \frac{e^{2x}-1}{e^{2x}+1} \\
\tanh'(x) &= 1-\tanh^2(x) \\
h_i &= \tanh\left(W_{h(i,\cdot)}^T x + b_{h(i)}\right) \\
&= \tanh\left(\sum_k w_{h(k,i)} x + b_{h(i)}\right) \\
\frac{\partial h_i}{\partial W_{h(u,v)}} &= \begin{cases} 0 & , \quad i \neq v \\ x_u \tanh'\left(\sum_k w_{h(k,i)} x_k + b_{h(i)}\right) & , \quad i = v \end{cases} \\
&= \begin{cases} 0 & , \quad i \neq v \\ x_u(1-h_i^2) & , \quad i = v \end{cases} \\
\frac{\partial h_i}{\partial b_{h(v)}} &= \begin{cases} 0 & , \quad i \neq v \\ 1-h_i^2 & , \quad i = v \end{cases} \\
\frac{\partial loss}{\partial W_{h(u,v)}} &= x_u(1-h_v^2) W_{0(v,\cdot)} \left[Softmax(\hat{y}) - y\right] \\
\frac{\partial loss}{\partial b_{h(v)}} &= (1-h_v^2) W_{0(v,\cdot)} \left[Softmax(\hat{y}) - y\right]
\end{aligned}
$$

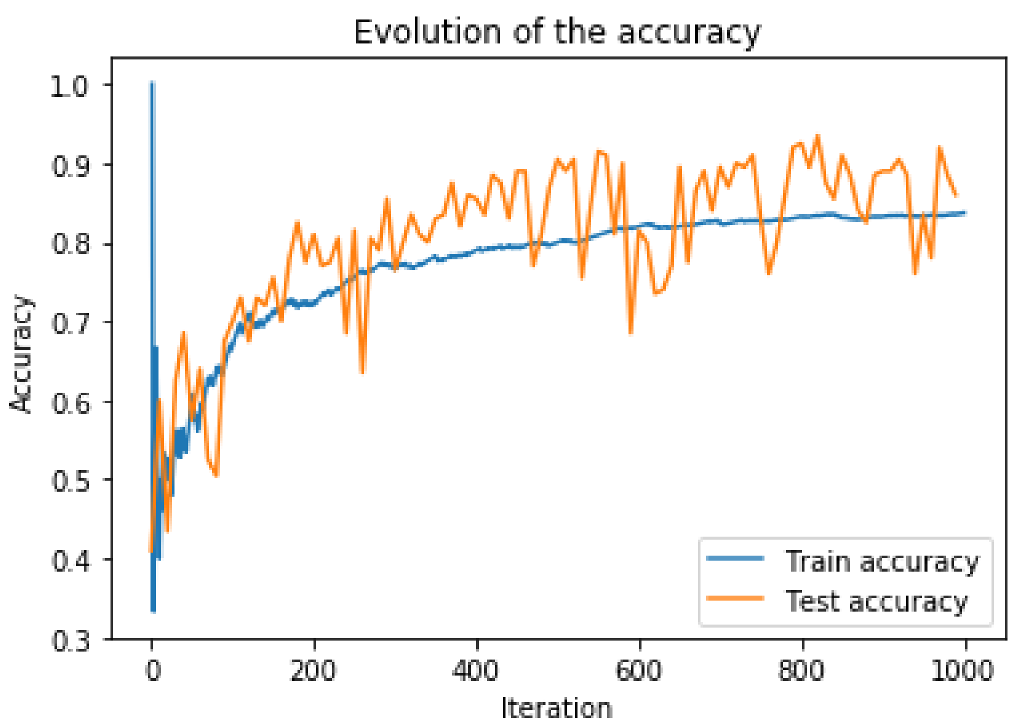
In [18]:
```
def tanh(x):
    """
    input:
        x (np.array)

    returns:
        x (np.array)
    """
    aux = x.reshape(-1)
    result = (np.exp(2*aux) - 1)/(np.exp(2*aux) + 1)
    return result.reshape(-1,1)

def d_tanh_hidden(hidden):
    """Computes the derivative of the tanh
    input:
        x (np.array)

    returns:
        x (np.array)
    """
    return 1 - hidden**2

def forward_two_layers(Wo, bo, Wh, bh, x):
    """Forward pass of a teo layer perceptron with tanh activation
```

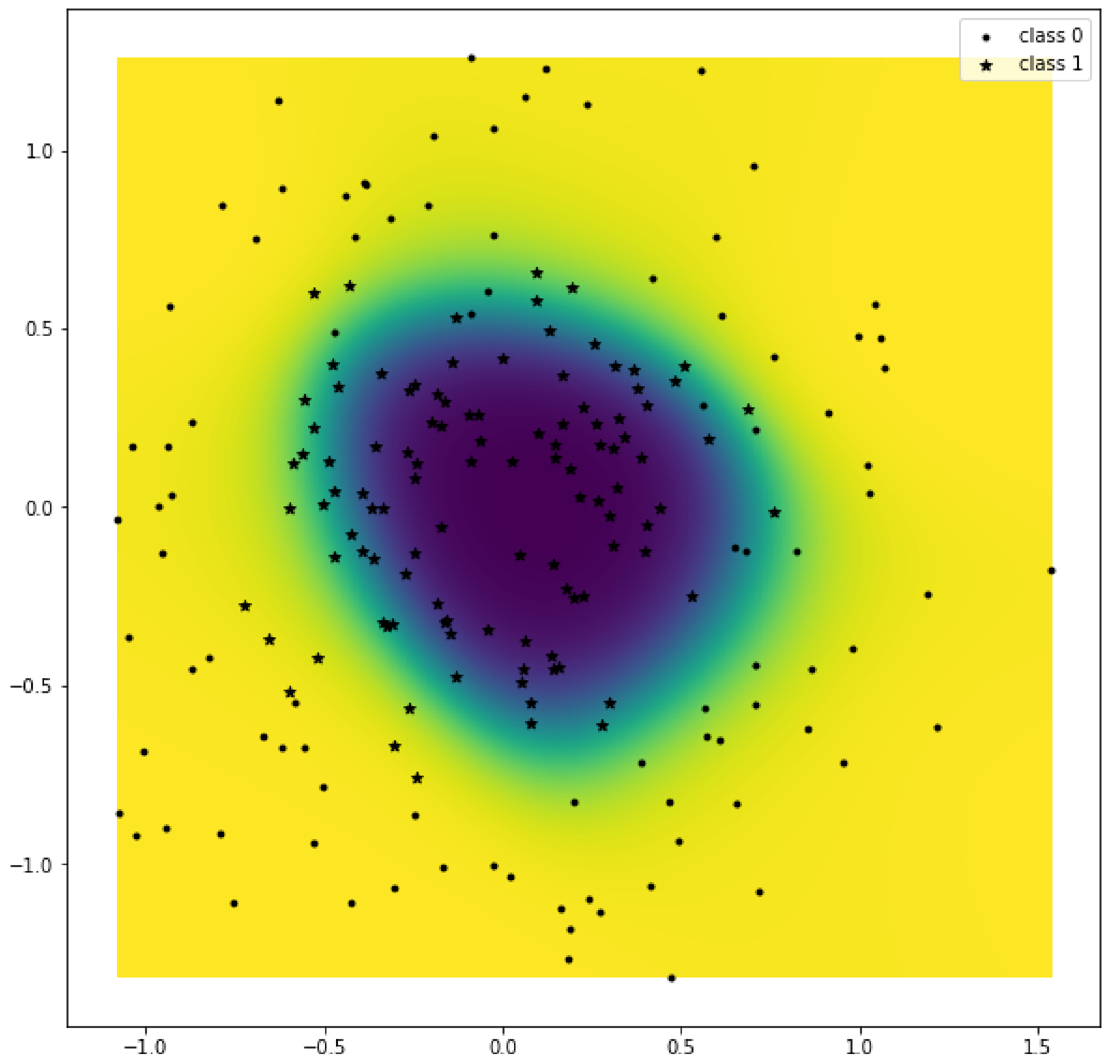
```python
    input:
        Wh (np.array): (INPUT_SHAPE, HIDDEN_SHAPE) The weight matrix of the
                                                   hidden layer
        Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) The weight matrix of the
                                                 output layer
        bh (np.array): (HIDDEN_SHAPE, 1) The bias matrix of the hidden layer
        bo (np.array): (N_CLASSES, 1) The bias matrix of the output layer
        x (np.array): (INPUT_SHAPE, 1) The input of the perceptron

    returns:
        softmaxed (np.array): (N_CLASSES, 1) the output of the network after
                                             final activation
        hidden (np.array): (HIDDEN_SHAPE, 1) the output of the hidden layer
                                             after activation
        out (np.array): (N_CLASSES, 1) the output of the network before
                                       final activation
    """

    hidden = tanh(Wh.T @ x + bh) # h
    out = Wo.T @ hidden + bo     # y_hat
    softmaxed = softmax(out)    # Softmax(y_hat)

    return softmaxed, hidden, out

def compute_grads_two_layers(hidden, softmaxed, Wo, x, y):
    """Forward pass of a teo layer perceptron with tanh activation
    input:
        hidden (np.array): (HIDDENT_SHAPE, 1) the output of the hidden layer
                                              after activation
        softmaxed (np.array): (N_CLASSES, 1) the output of the network after
                                             final activation
        Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) The weight matrix of the
                                                 output layer
        x (np.array): (INPUT_SHAPE, 1) The input of the perceptron
        y (np.array): (N_CLASSES, 1) Ground truth class

    returns:
        d_Wo (np.array): (HIDDEN_SHAPE, N_CLASSES) Gradient with respect
                        to the weight matrix of the output layer
        d_bo (np.array): (N_CLASSES, 1) Gradient with respect to the bias matrix
                                        of the output layer
        d_Wh (np.array): (INPUT_SHAPE, HIDDEN_SHAPE) Gradient with respect to the
                        weight matrix of the hidden layer
        d_bh (np.array): (HIDDEN_SHAPE, 1) Gradient with respect to the bias
                                           matrix of the hidden layer
    """
    d_Wo, d_bo = compute_grads_one_layer(softmaxed, hidden, y)
```

```
    d_bh = np.zeros(hidden.shape)
    for i in range(d_bh.shape[0]):
        aux = d_tanh_hidden(hidden)
        d_bh[i,0] = aux[i,0] * ( Wo[i,:] @ (softmaxed - y).reshape(-1) )

    d_Wh = np.outer(x, d_bh)

    return d_Wo, d_bo, d_Wh, d_bh
```

In [19]:
```
Wo, bo, Wh, bh, loss, acc, test_acc = train_two_layer(X_train, Y_train,
                                                      X_test, Y_test, 0.1, 16)

iteration = [t+1 for t in range(1000)]

# evolution of loss function
plt.plot(iteration, loss)
plt.title("Evolution of the Loss function")
plt.xlabel("Iteration")
plt.ylabel("Lrror")
plt.show()

# Training data accuracy
freq = 10
acc_true = [a/b for a,b in zip(np.array(acc).cumsum(), range(1,len(acc)+1))]
plt.plot(iteration, acc_true, label="Train accuracy")
plt.plot([t*freq for t in range(100)], test_acc, label="Test accuracy")
plt.xlabel("Iteration")
plt.ylabel("Accuracy")
plt.title("Evolution of the accuracy")
plt.legend()
plt.show()

plot_decision(X_test, Y_test, lambda x: forward_two_layers(Wo, bo, Wh, bh, x)[0])
```


Evolution of the Loss function
Lrror
Iteration
0
1
2
3
4
0
200
400
600
800
1000


Evolution of the accuracy
Accuracy
Iteration
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
200
400
600
800
1000
Train accuracy
Test accuracy


class 0
class 1
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5