de Rham Cohomology, Hodge Decomposition

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Exterior Differential

Insight

The homology of a manifold is the difference between the closed loops and the boundary loops.

The cohomology of a manifold is the difference between the curl free vector fields and the gradient vector fields.

Insight

Consider a planar vector field defined on $\mathbb{C} \setminus \{0\}$,

$$\mathbf{v}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

direct computation $\nabla \times \mathbf{v}(x, y) = 0$.

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = 0.$$

But choose the unit circle

$$\oint_\gamma \omega = \oint_\gamma d \ \tan^{-1} \frac{y}{x} = 2\pi$$

therefore $\mathbf{v}$ is not a gradient field. Namely, $d\theta$ locally is integrable, globally not.

Smooth Manifold

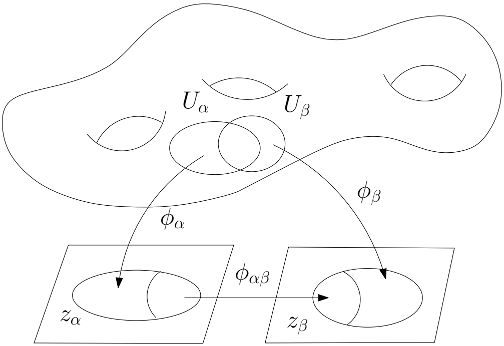


Figure: A manifold.

Smooth Manifold

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.

Tangent Space

Definition (Tangent Vector)

A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.

Push forward

Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from M to N, $\gamma : (-\epsilon, \epsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = \mathbf{v} \in T_pM$, then $\phi \circ \gamma$ is a curve on N, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(\mathbf{v}) = (\phi \circ \gamma)'(0) \in T_{\phi(p)}N,$$

as the push-forward tangent vector of $\mathbf{v}$ induced by ϕ.

differential forms

Definition (Differential 1-form)

The tangent space T_pM is an n-dimensional vector space, its dual space T_p^*M is called the cotangent space of M at p. Suppose $\omega \in T_p^*M$, then $\omega : T_pM \to \mathbb{R}$ is a linear function defined on T_pM, ω is called a differential 1-form at p.

A differential 1-form field has the local representation

$$\omega(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \omega_i(x^1, x^2, \cdots, x^n) dx_i,$$

where $\{dx_i\}$ are the differential forms dual to $\{\frac{\partial}{\partial x_j}\}$, such that

$$\langle dx_i, \frac{\partial}{\partial x_i} \rangle = dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

High order exterior forms

Definition (Tensor)

A tensor Θ of type (m,n) on a manifold M is a correspondence that associates to each point $p \in M$ a multi-linear map

$$\Theta_p : T_pM \times T_pM \times \cdots \times T_p^*M \cdots \times T_p^*M \to \mathbb{R},$$

where the tangent space T_pM appears m times and cotangent space T_p^*M appears n times.

Definition (exterior m-form)

An exterior m-form is a tensor ω of type $(m,0)$, which is skew symmetric in its arguments, namely

$$\omega_p(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \cdots, \xi_{\sigma(m)}) = (-1)^{\sigma} \omega_p(\xi_1, \xi_2, \cdots, \xi_m)$$

for any tangent vectors $\xi_1, \xi_2, \cdots, \xi_m \in T_pM$ and any permutation $\sigma \in S_m$, where S_m is the permutation group.

differential forms

Differential Form

The local representation of ω in $(x^1, x^2, \cdots, x^m)$ is

$$\omega = \sum_{1\leq i_1<i_2<\cdots<i_m\leq n} \omega_{i_1 i_2\cdots i_m} dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_m} = \omega_I dx^I,$$

ω_I is a function of the reference point p, ω is said to be differentiable, if each ω_I is differentiable.

Wedge product

Definition (Wedge product)

A coordinate free representation of wedge product of m_1-form ω_1 and m_2-form ω_2 is defined as $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{m_1+m_2})$ equals

$$\sum_{\sigma \in S_{m_1+m_2}} \frac{(-1)^\sigma}{m_1! m_2!} \omega_1 \left(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(m_1)}\right) \omega_2 \left(\xi_{\sigma(m_1+1)}, \cdots, \xi_{\sigma(m_1+m_2)}\right)$$

Wedge product

Give k differential 1-forms, their exterior wedge product is given by:

$$\omega_1 \wedge \omega_2 \cdots \omega_k(v_1, v_2, \cdots, v_k) = \begin{vmatrix} \omega_1(v_1) & \omega_1(v_2) & \dots & \omega_1(v_k) \\ \omega_2(v_1) & \omega_2(v_2) & \dots & \omega_2(v_k) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_k(v_1) & \omega_k(v_2) & \dots & \omega_k(v_k) \end{vmatrix}$$

Exterior is anti-symmetric, suppose $\sigma \in S_k$ is a permutation, then

$$\omega_{\sigma(1)} \wedge \omega_{\sigma(2)} \wedge \cdots \wedge \omega_{\sigma(k)} = (-1)^\sigma \omega_1 \wedge \omega_2 \wedge \cdots \wedge \sigma_k.$$

Pull back

Definition (Pull back)

Suppose $\phi : M \to N$ is a differentiable map from M to N, ω is an m-form on N, then the pull-back $\phi^*\omega$ is an m-form on M defined by

$$(\phi^*\omega)_p(\xi_1, \cdots, \xi_m) = \omega_{\phi(p)}(\phi_*\xi_1, \cdots, \phi_*\xi_m), p \in M$$

for $\xi_1, \xi_2, \cdots, \xi_m \in T_pM$, where $\phi_*\xi_j \in T_{\phi(p)}N$ is the push forward of $\xi_j \in T_pM$.

Integration

Integration in Euclidean space

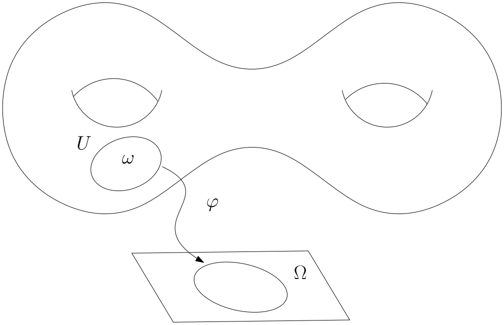
Suppose that $U \subset \mathbb{R}^n$ is an open set,

$$\omega = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

then

$$\int_U \omega = \int_U f(x)dx^1 dx^2 \cdots dx^n.$$

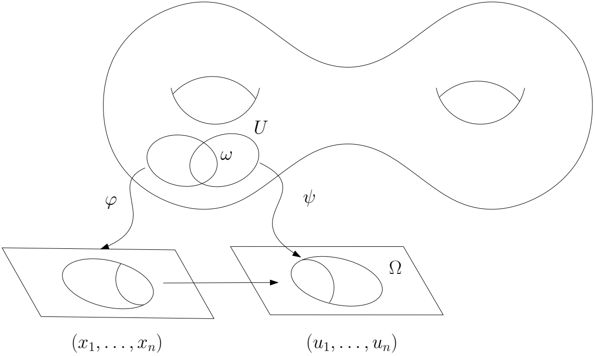
Integration



Suppose $U \subset M$ is an open set of a manifold M, a chart $\phi : U \to \Omega \subset \mathbb{R}^n$, then

$$\int_U \omega = \int_\Omega (\phi^{-1})^* \omega.$$

Integration



Integration is independent of the choice of the charts. Let $\psi : U \to \psi(U)$ be another chart, with local coordinates $(u_1, u_2, \cdots, u_n)$, then

$$\int_{\phi(U)} f(x) dx^1 dx^2 \cdots dx^n = \int_{\psi(U)} f(x(u)) det \left(\frac{\partial x^i}{\partial u^j} \right) du^1 du^2 \cdots du^n.$$

Integration on Manifolds

consider a covering of M by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and choose a partition of unity $\{f_i\}$, $i \in I$, such that $f_i(p) \geq 0$,

$$\sum_i f_i(p) \equiv 1, \forall p \in M.$$

Then $\omega_i = f_i \omega$ is an n-form on M with compact support in some U_α, we can set the integration as

$$\int_M \omega = \sum_i \int_M \omega_i.$$

Exterior Derivative

Exterior Derivative of a Function

Suppose $f : M \to \mathbb{R}$ is a differentiable function, then the exterior derivative of f is a 1-form,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i.$$

Exterior Derivative of Differential Forms

The exterior derivative of an m-form on M is an $(m+1)$-form on M defined in local coordinates by

$$d\omega = d(\omega_I dx^I) = (d\omega_I) \wedge dx^I,$$

where $d\omega_I$ is the differential of the function ω_I.

Exterior Derivative

The exterior derivative of a differential 1-form is given by:

$$d\left(\sum \omega_i dx_i\right) = \sum_{i,j} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j}\right) dx_i \wedge dx_j,$$

that of a differential k-form

$$d(\omega_1 \wedge \omega_2 \cdots \wedge \omega_k) = \sum (-1)^{i-1} \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge d\omega_i \wedge \omega_{i+1} \wedge \cdots \wedge \omega_k.$$

Stokes Theorem

Theorem (Stokes)

let M be an n-manifold with boundary ∂M and ω be a differentialble $(n-1)$-form with compact support on M, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

Stokes Theorem

Theorem

Suppose Σ is a differential manifold, then we have

$$d^k \circ d^{k-1} = 0.$$

Proof.

Assume ω is a $k-1$ differential form, D is a $k+1$ chain, from Stokes theorem, we have

$$\int_D d^k \circ d^{k-1}\omega = \int_{\partial_k D} d^{k-1}\omega = \int_{\partial_{k-1}\circ\partial_k} \omega = 0,$$

since $\partial_{k-1} \circ \partial_k$. □

de Rham Cohomology

Let $\Omega^k(\Sigma)$ be the sapce of all differential k-forms, $d^k : \Omega^k(\Sigma) \to \Omega^{k+1}(\Sigma)$ be exterior differential operator.

Definition (Closed form)

k-form $\omega \in \Omega^k(\Sigma)$ is called a closed form, if $d^k\omega = 0$, namely $\omega \in \text{Ker} d^k$.

Definition (Exact Form)

k-differential form $\omega \in \Omega^k(\Sigma)$ is called exact form, if there is a $\tau \in \Omega^{k-1}(\Sigma)$, such that $\omega = d^{k-1}\tau$, namely $\omega \in \text{Img}\, d^{k-1}$.

Since $d^k \circ d^{k-1} = 0$, exact forms are closed, $\text{Img}\, d^{k-1} \subset \text{Ker}\, d^k$.

de Rham Cohomology

Definition (de Rham Cohomology)

Assume Σ is a differntial manifold, then de Rham complex is

$$\Omega^0(\Sigma,\mathbb{R}) \xrightarrow{d^0} \Omega^1(\Sigma,\mathbb{R}) \xrightarrow{d^1} \Omega^2(\Sigma,\mathbb{R}) \xrightarrow{d^2} \Omega^3(\Sigma,\mathbb{R}) \xrightarrow{d^3} \cdots$$

$$H^k_{dR}(\Sigma,\mathbb{R}) := \frac{\text{Ker } d^k}{\text{Img } d^{k-1}}$$

Theorem

The de Rham cohomology group $H^m_{dR}(M)$ is isomorphic to the cohomology group $H^m(M,\mathbb{R})$

$$H^m_{dR}(M) \cong H^m(M,\mathbb{R}).$$

Hodge Operator

Hodge Star Operator - First Definition

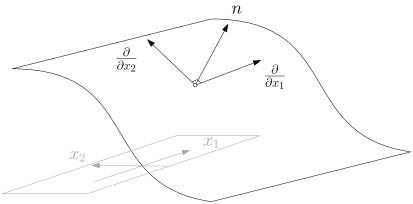
Suppose M is a Riemannian manifold, we can locally find oriented orthonormal basis of vector fields, and choose parameterization, such that

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right\}$$

form an oriented orthonormal basis. let

$$\{dx_1, dx_2, \cdots, dx_n\}$$

be the dual 1-form basis.



Hodge star operator

Definition (Hodge Star Operator)

The Hodge star opeartor $^* : \Omega^k(M) \to \Omega^{n-k}(M)$ is a linear operator

$$^*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n.$$

Hodge Star Operator

Let $\sigma = (i_1, i_2, \cdots, i_n)$ be a permutation, then the hoedge star operator

$$*\,(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n}.$$

L^2 norm

Definition

Let $\eta, \zeta \in \Omega^k(M)$ are two k-forms on M, then the norm is defined as

$$(\eta, \zeta) = \int_M \eta \wedge {}^*\zeta.$$

$\Omega^k(M)$ is a Hilbert space.

Hodge Star Operator - Second Equivalent Definition

Given a Riemannian manifold $(M, \mathbf{g})$, $\mathbf{g} = (g_{ij})$, which gives the inner product in the tangent space $T_p(M)$,

$$g_{ij} = \langle \partial_i, \partial_j \rangle_{\mathbf{g}}.$$

its inverse matrix is (g^{ij}), satisfies

$$\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k.$$

Riemannian metric

Definition (Dual Inner Product)

Given a n dimensional Riemannian manifold $(M, \mathbf{g})$, the dual inner product $\langle , \rangle_{\mathbf{g}} : T_p^*(M) \times T_p^*(M) \to \mathbb{R}$, $\forall \omega, \eta \in T_p^*(M)$, $\omega = \sum_{i=1}^n \omega_i dx^i$, $\eta = \sum_{i=1}^n \eta_i dx^i$, then

$$\langle \omega, \eta \rangle_{\mathbf{g}} = \sum_{i,j=1}^n g^{ij} \omega_i \eta_j .$$

Riemannian metric

Orthonormal Basis

Let $\{\theta_1, \theta_2, \cdots, \theta_n\}$ is a set of orthonormal basis

$$\langle \theta_i, \theta_j \rangle_{\mathbf{g}} = \delta_i^j.$$

Basis of $\Omega^k(M)$

We use $\{\theta_i\}$ to construct the basis of $\Omega^k(M)$,

$$\Omega^k(M) := \text{Span}\{\theta_{i_1} \wedge \theta_{i_2} \wedge \cdots \wedge \theta_{i_k} | i_1 < i_2 < \cdots < i_k\}.$$

Dual Inner Product

We define dual inner product $\langle , \rangle_{\mathbf{g}} : \Omega^k(M) \times \Omega^k(M)$ as follows:

$$\langle \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}, \theta_{j_1} \wedge \cdots \wedge \theta_{j_k} \rangle = \delta_{i_1 \cdots i_k}^{j_1 \cdots j_k}.$$

Riemannian Volume Element

Riemannian volume Element

Let $G = \det(g_{ij})$, then in the local coordinates, the Riemannian volume element is defined as

$$\omega_{\mathbf{g}} = \sqrt{G} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Definition (Hodge Star Operator)

$^* : \Omega^k(M) \to \Omega^{n-k}(M)$,

$$\omega \wedge {}^*\eta = \langle \omega, \tau \rangle_{\mathbf{g}} \omega_{\mathbf{g}}.$$

Therefore

$$^*(1) = \omega_{\mathbf{g}}, \quad {}^*\omega_{\mathbf{g}} = 1.$$

Inner Product

Definition (Inner Product)

Let $(M, \mathbf{g})$ be a n dimensional Riemannian manifold, ζ and η are differential k-forms, $0 \leq k \leq n$, then ζ and η inner product is defined as

$$(\zeta, \eta) := \int_M \zeta \wedge {}^*\eta = \int_M \langle \zeta, \eta \rangle_{\mathbf{g}} \omega_{\mathbf{g}}$$

Hodge Star Operator on Surface - Type I

Suppose $(S, \mathbf{g})$ is a surface with a Riemannian metric, with isothermal coordinates (u, v), the metric is

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2),$$

Then

$$\frac{\partial}{\partial x_1} = e^{-\lambda}\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial x_1} = e^{-\lambda}\frac{\partial}{\partial v},$$

And

$$dx_1 = e^{\lambda}du, \quad dx_2 = e^{\lambda}dv.$$

Hodge Star

$$^*dx_1 = dx_2, \quad ^*du = dv; \quad ^*dx_2 = -dx_1, \quad ^*dv = -du;$$

$$^*(1) = dx_1 \wedge dx_2 = e^2 du \wedge dv, ^*(dx_1 \wedge dx_2) = 1.$$

Hodge Star Operator on Surface - Type II

Suppose $(S, \mathbf{g})$ is a surface with a Riemannian metric, with isothermal coordinates (u, v), the metric is

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2),$$

surface area element is

$$\omega_{\mathbf{g}} = e^{2\lambda(u,v)} du \wedge dv.$$

Given 1-forms $\omega = \omega_1 du + \omega_2 dv$ and $\tau = \tau_1 du + \tau_2 dv$, its wedge product is

$$\omega \wedge \tau = (\omega_1 \tau_2 - \omega_2 \tau_1) du \wedge dv.$$

Inner product is

$$\langle \omega, \tau \rangle_{\mathbf{g}} = e^{-2\lambda(u,v)}(\omega_1 \tau_1 + \omega_2 \tau_2).$$

Hodge Star Operator on Surface

$$(\omega_1 du + \omega_2 dv) \wedge {}^*du = \langle \omega, du \rangle_{\mathbf{g}} \omega_{\mathbf{g}} = e^{-2\lambda} \omega_1 e^{2\lambda} du \wedge dv,$$

This shows ${}^*du = dv$, similarly ${}^*dv = -du$.

$${}^*(\omega_1 du + \omega_2 dv) = \omega_1 dv - \omega_2 du.$$

Hence ${}^{**}\omega = -\omega$.

Electronic Field

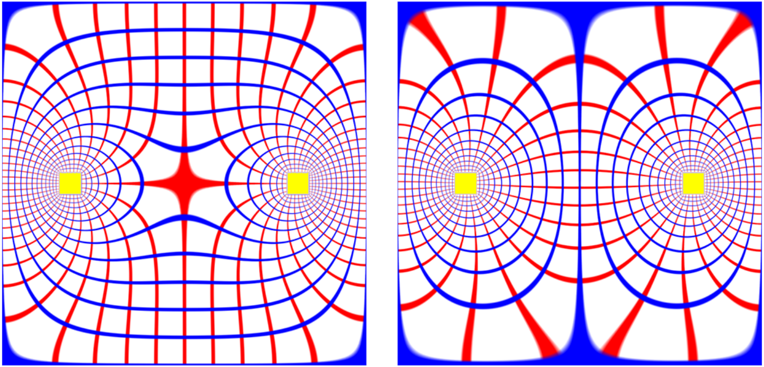


Figure: Hodge star operator.

Hodge Decomposition

Codifferential operator

Definition

The codifferential operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined as

$$\delta = (-1)^{k+1+k(n-k)} * d^*,$$

where d is the exterior derivative.

Lemma

The codifferential is the adjoint of the exterior derivative, in that

$$(\delta\zeta, \eta) = (\zeta, d\eta).$$

Laplace Operator

Definition (Laplace Operator)

The Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$,

$$\Delta = d\delta + \delta d.$$

Lemma

The Laplace operator is symmetric

$$(\Delta\zeta, \eta) = (\zeta, \Delta\eta)$$

and non-negative

$$(\Delta\eta, \eta) \geq 0.$$

Proof.

$$(\Delta\zeta, \eta) = (d\zeta, d\eta) + (\delta\zeta, \delta\eta).$$

□

Harmonic Forms

Definition (Harmonic forms)

Suppose $\omega \in \Omega^k(M)$, then ω is called a k-harmonic form, if

$$\Delta\omega = 0.$$

Lemma

ω is a harmonic form, if and only if

$$d\omega = 0, \delta\omega = 0.$$

Proof.

$$0 = (\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

$\square$

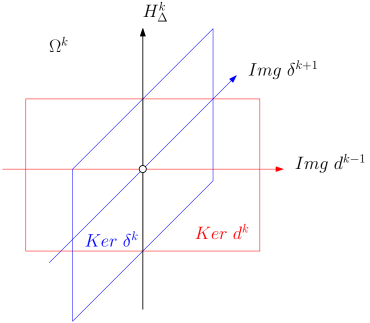
Hodge Decomposition

Definition (Harmonic form group)

All harmoic k-forms form a group, denoted as $H^k_\Delta(M)$.

Theorem (Hodge Decomposition)

$$\Omega_k = imgd^{k-1} \bigoplus img\delta^{k+1} \bigoplus H^k_\Delta(M).$$



Hodge Decomposition

Proof.

$(imgd^{k-1})^{\perp} = \{\omega \in \Omega^k(M) | (\omega, d\eta) = 0, \forall \eta \in \Omega^{k-1}(M)\}$, because $(\omega, d\eta) = (\delta\omega, \eta)$, so $(imgd^{k-1})^{\perp} = ker\delta^k$. similarly, $(img\delta^{k+1})^{\perp} = kerd^k$. Because $imgd^{k-1} \subset kerd^k$, $img\delta^{k+1} \subset ker\delta^k$, therefore $imgd^{k-1} \perp img\delta^{k+1}$,

$$\Omega^k = imgd^{k-1} \oplus img\delta^{k+1} \oplus (imgd^{k-1} \oplus img\delta^{k+1})^{\perp}$$

$(imgd^{k-1} \oplus img\delta^{k+1})^{\perp} = (imgd^{k-1})^{\perp} \cap (img\delta^{k+1})^{\perp} = ker\delta^k \cap kerd^k = H^k_{\Delta}$. □

Hodge Decomposition

suppose $\omega \in kerd^k$, then $\omega \perp img\delta^{k+1}$, then $\omega = \alpha + \beta$, $\alpha \in imgd^{k-1}$, $\beta \in H^k_\Delta(M)$, define project $h : kerd^k \to H^k_\Delta(M)$,

Theorem

Suppose ω is a closed form, its harmonic component is $h(\omega)$, then the map:

$$h : H^k_{dR}(M) \to H^k_\Delta(M).$$

is isomorphic.

Each cohomologous class has a unique harmonic form.

Harmonic 1-forms

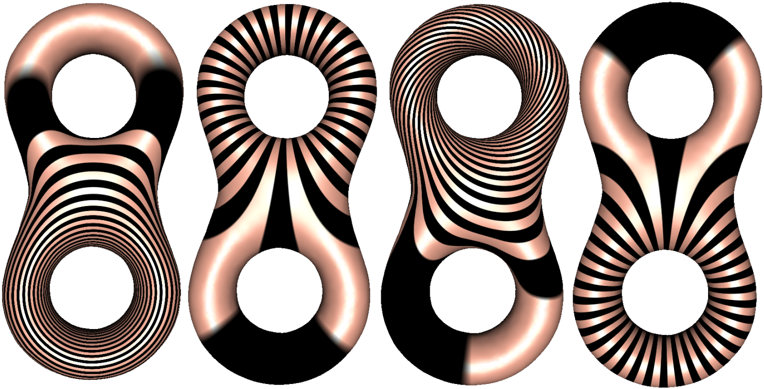


Figure: Harmonic 1-form group basis.