Surface Differential Geometry, Movable Frame Method

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July 14, 2020

Gaussian Curvature

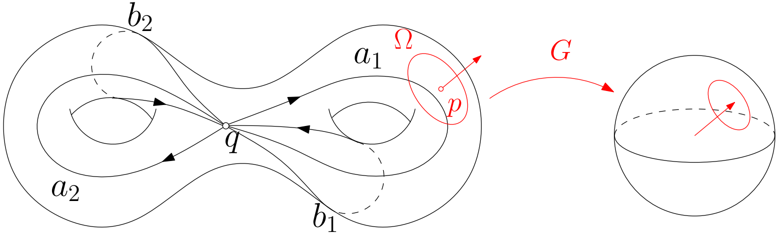

Figure: Gaussian curvature.

Gauss map: $\mathbf{r}(p) \mapsto \mathbf{n}(p)$,

$$K(p) := \lim_{\Omega \to \{p\}} \frac{|G(\Omega)|}{|\Omega|}$$

Gaussian Curvature

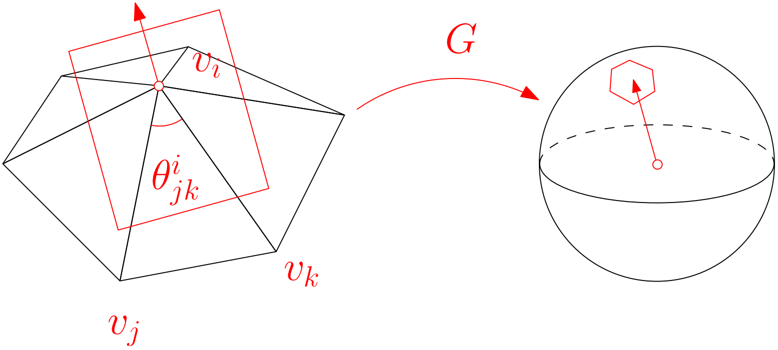

Figure: Discrete Gaussian curvature.

$$G(v_i) := \{\mathbf{n} \in \mathbb{S}^2 | \exists \text{Support plane with normal } \mathbf{n}\}.$$

Gaussian Curvature

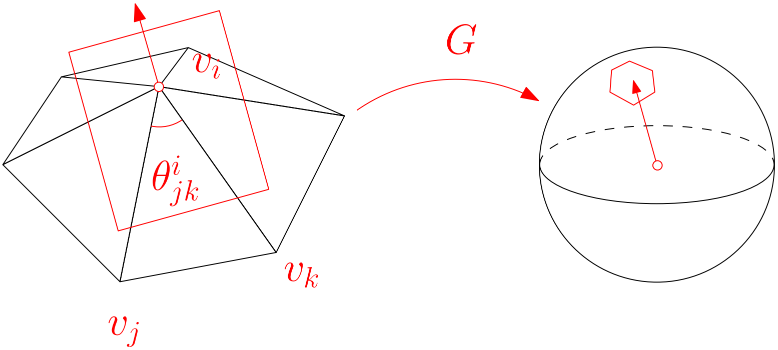

Figure: Discrete Gaussian curvature for convex vertex.

$$K(v_i) := |G(v_i)| = 2\pi - \sum_{jk} \theta^i_{jk}.$$

Gauss-Bonnet

For a closed oriented metric surface $(S, \mathbf{g})$,

$$\int_S K dA = 2\pi\chi(S).$$

For a closed oriented discrete polygonal surface M,

$$\sum_{v_i} K(v_i) = 2\pi\chi(M).$$

Gaussian Curvature

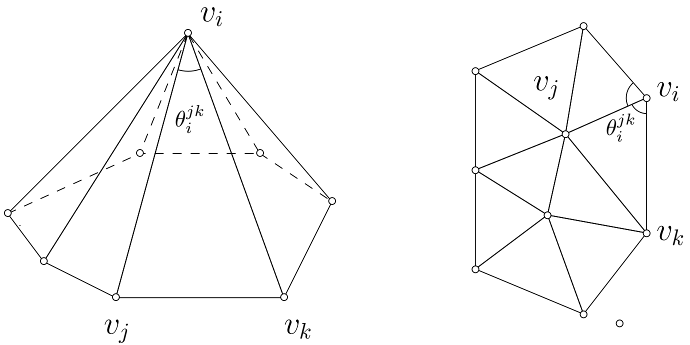

Figure: Discrete Gaussian curvature.

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \tag{1}$$

Gauss-Bonnet

Theorem (Discrete Gauss-Bonnet Theorem)

Given polyhedral surface $(S, V, \mathbf{d})$, the total discrete curvature is

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic number of S.

Proof.

We denote the polyhedral surface $M = (V, E, F)$, if M is closed, then

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jK} = 2\pi|V| - \pi|F|.$$

Since M is closed, $3|F| = 2|E|$,
$\chi(S) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F| = |V| - \frac{1}{2}|F|.$ □

Discrete Guass-Bonnet

continued.

Assume M has bounary ∂M. Assume the interior vertex set is V_0, boundary vertex set is V_1, then $|V| = |V_0| + |V_1|$; assume interior edge set is E_0, boundary edge set is E_1, then $|E| = |E_0| + |E_1|$. Furthermore, all boundaries are closed loops, hence boundry vertex number equals to the boundary edge number, $|V_1| = |E_1|$. Every interior edge is adjacent to two faces, every boundary edge is adjacent to one face, we have $3|F| = 2|E_0| + |E_1| = 2|E_0| + |v_1|$. We compute the Euler number

$$\chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1| = |V_0| + |F| - |E_0|,$$

by $|E_0| = 1/2(3|F| - |V_1|)$

$$\chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|$$

Discrete Guass-Bonnet

continued.

we have:

$$\begin{aligned}
\sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) &= \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk} \right) \\
&= 2\pi |V_0| + \pi |V_1| - \pi |F| \\
&= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right) \\
&= 2\pi \chi(M).
\end{aligned} \tag{2}$$

$\square$.

Movable Frame

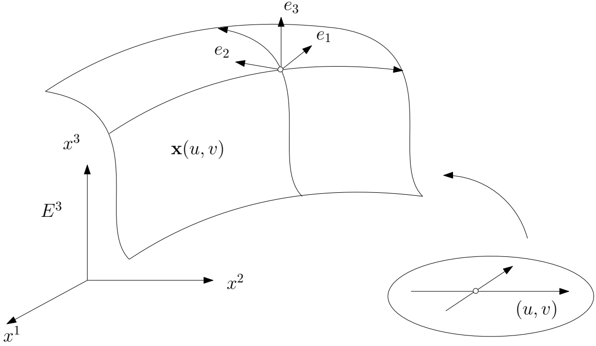

Figure: A parametric surface.

Orthonormal Movable frame

Movable Frame

Suppose a regular surface S is embedded in $\mathbb{R}^3$, a parametric representation is $\mathbf{r}(u,v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let $\mathbf{e}_3$ be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orhonormal frame field* of the surface.

Orthonormal Movalbe frame

Tangent Vector

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$. $d\mathbf{r}$ is orthogonal to the normal vector $\mathbf{e}_3$.

Motion Equation

$$d\mathbf{e}_i = \omega_{i1}\mathbf{e}_1 + \omega_{i2}\mathbf{e}_2 + \omega_{i3}\mathbf{e}_3,$$

where $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$. Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \quad 0 = d\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle + \langle \mathbf{e}_i, d\mathbf{e}_j \rangle$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

Motion Equation

Motion Equation

$$dr = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1\omega_1 + \omega_2\omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1\omega_{31} - \omega_2\omega_{32} = \omega_1\omega_{13} + \omega_2\omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \to \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \to d\mathbf{e}_3, \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 \to \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K\omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gaussian curvature

Weigarten Mapping

$\{\omega_1, \omega_2\}$ form the basis of the cotangent space, therefore ω_{13}, ω_{23} can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so $K = h_{11}h_{22} - h_{12}h_{21}$, the mean curvature $H = \frac{1}{2}(h_{11} + h_{22})$.

Gauss's theorem Egregium

Theorem (Gauss' Theorem Egregium)

The Gaussian curvature is intrinsic, solely determined by the first fundamental form.

Proof.

$$
\begin{aligned}
0 =& d^2\mathbf{e}_1 \\
=& d(\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) \\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge d\mathbf{e}_2 + d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge d\mathbf{e}_3 \\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) + \\
& d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge (\omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2) \\
=& (d\omega_{12} - \omega_{13} \wedge \omega_{32})\mathbf{e}_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})\mathbf{e}_3
\end{aligned}
$$

therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{23} = -K\omega_1 \wedge \omega_2.$$

Gauss's theorem Egregium

Lemma

$$\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2$$

Proof.

$$\begin{aligned}
0 &= d^2\mathbf{r} \\
&= d(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2\mathbf{e}_2 - \omega_w \wedge d\mathbf{e}_2 \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge (\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) + \\
&\quad d\omega_2\mathbf{e}_2 - \omega_2 \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) \\
&= (d\omega_1 - \omega_2 \wedge \omega_{21})\mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12})\mathbf{e}_2 + \\
&\quad -(\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23})\mathbf{e}_3.
\end{aligned}$$

Therefore $d\omega_1 = \omega_2 \wedge \omega_{21}$ and $d\omega_2 = \omega_1 \wedge \omega_{12}$. $\square$

Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvautre is given by

$$K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u.$$

Proof.

Let $(S,\mathbf{g})$ be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2+dy^2).$$

Then

$$\left\{\begin{array}{rcl}\omega_1 &=& e^u dx\\ \omega_2 &=& e^u dy\end{array}\right. \left\{\begin{array}{rcl}\mathbf{e}_1 &=& e^{-u}\frac{\partial}{\partial x}\\ \mathbf{e}_2 &=& e^{-u}\frac{\partial}{\partial y}\end{array}\right.$$

$\square$

Continued.

By direct computation,

$$
\begin{aligned}
d\omega_1 &= de^u \wedge dx & d\omega_2 &= de^u \wedge dy \\
&= e^u(u_x dx + u_y dy) \wedge dx & &= e^u(u_x dx + u_y dy) \wedge dy \\
&= e^u u_y dy \wedge dx & &= e^u u_x dx \wedge dy.
\end{aligned}
$$

therefore

$$
\begin{aligned}
\omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2 \\
&= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\
\omega_{12} &= -u_y dx + u_x dy.
\end{aligned}
$$

Gaussian Curvature

Continued.

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge_\omega 2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u} dx \wedge dy} = -\frac{1}{e^{2u}} \Delta u.$$

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$

□

Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1 - x^2 - y^2) - 2x(-2x)}{(1 - x^2 - y^2)^2} = \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}$$

similarly

$$u_{yy} = \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1 - x^2 - y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx} + u_{yy}) = -1.$$

$\square$

Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, *then*

$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}
\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda) \\
&= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\
&= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).
\end{aligned}$$

$\square$

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet)

Suppose M is a closed orientable C^2 surface, then

$$\int_M A dA = 2\pi\chi(M),$$

where dA is the area element of hte surface, $\chi(M)$ is the Euler characteristic number of M.

Proof.

Construct a smooth vector field v, with isolated zeros $\{p_1, p_2, \cdots, p_n\}$. Choose a small disk $D(p_i, \varepsilon)$. On the surface

$$\bar{M} = M \setminus \bigcup_{i=1}^{n} D(p_i, \varepsilon)$$

□

Gauss-Bonnet Theorem

Proof.

construct orthonormal frame $\{p, e_1, e_2, e_3\}$, where

$$e_1(p) = \frac{v(p)}{|v(p)|}, \quad e_3(p) = n(p).$$

The integration

$$\int_{\bar{M}} KdA = \int_{\bar{M}} K\omega_1 \wedge \omega_2 = -\int_{\bar{M}} d\omega_{12}$$

by Stokes theorem and Poincarère-Hopf theorem, we obtain

$$-\sum_{i=1}^{n} \int_{\partial D(p_i,\varepsilon)} \omega_{12} = 2\pi \sum_{i=1}^{n} \text{Index}(p_i, v) = 2\pi\chi(M).$$

Here by $\omega_{12} = \langle de_1, e_2 \rangle$, ω_{12} is the rotation speed of e_1. Let $\varepsilon \to 0$, the equation holds. □

Computing Geodesics

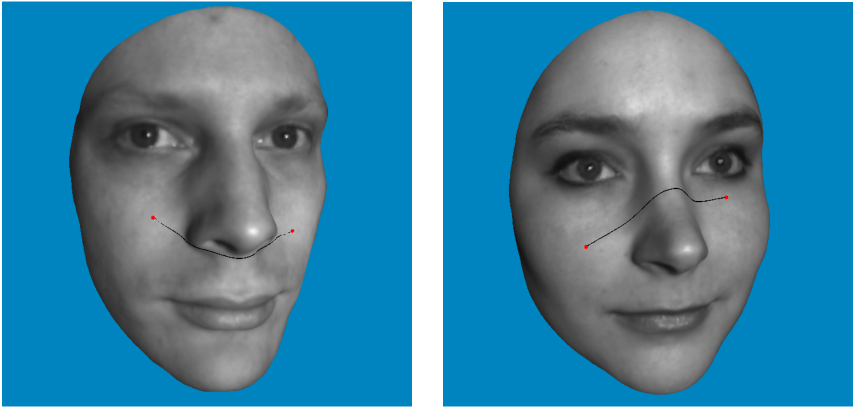

Figure: Geodesics.

Covariant Differential

Definition (Covariant Differentiation)

Covariant differentiation is the generalization of directional derivatives, satisfies the following properties: assume v and w are tangent vector fields on a surface, $f : S \to \mathbb{R}$ is a C^1 function, then

1. $D(v + w) = D(v) + D(w)$,
2. $D(fv) = df(v) + fDv$,
3. $D\langle v, w\rangle = \langle Dv, w\rangle + \langle v, Dw\rangle$.

By movable framework, the motion equation of the surface is

$$d\mathbf{e_1} = \omega_{12}\mathbf{e_2} + \omega_{13}\mathbf{e_3}, \quad d\mathbf{e_2} = \omega_{21}\mathbf{e_1} + \omega_{23}\mathbf{e_3},$$

We only keep tangential component, and delete the normal part to obtain covariant differential

$$D\mathbf{e_1} = \omega_{12}\mathbf{e_1}, \quad D\mathbf{e_2} = \omega_{21}\mathbf{e_1}.$$

Covariant Differential

Definition (Parallel transport)

Suppose S is a metric surface, $\gamma : [0,1] \to S$ is a smooth curve, $v(t)$ is a vector field along γ, if

$$\frac{Dv}{dt} \equiv 0,$$

then we say the vector field $v(t)$ is parallel transportation along γ.

Given a tangent vector field $v = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, then

$$\begin{aligned} Dv &= df_1\mathbf{e_1} + f_1 D\mathbf{e_1} + df_2\mathbf{e_2} + f_2 D\mathbf{e_2} \\ &= (df_1 - f_2\omega_{12})\mathbf{e_1} + (df_2 + f_1\omega_{12})\mathbf{e_2}. \end{aligned}$$

and

$$\frac{Dv}{dt} = \left(\frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt}\right)\mathbf{e_1} + \left(\frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt}\right)\mathbf{e_2}.$$

where $\frac{\omega_{12}}{dt} = \langle \omega_{12}, \dot{\gamma}\rangle$. If $\omega_{12} = \alpha dx + \beta dy$, then $\frac{\omega_{12}}{dt} = \alpha\dot{x} + \beta\dot{y}$.

Parallel Transport

Parallel Transport Equation

Therefore parallel vector field satisfies the ODE

$$\left\{\begin{array}{rcl} \frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt} & = & 0 \\ \frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt} & = & 0 \end{array}\right.$$

Given an intial condition $v(0)$, the solution uniquely exists.

Geodesic Curvature

Definition (Geodesic Curvature)

Assume $\gamma : [0,1] \to S$ is a C^2 curve on a surface S, s is the arc length parameter. Construct orthonormal frame field along the curve $\{\mathbf{e_1}, \mathbf{e_2}, e_3\}$, where $\mathbf{e_1}$ is the tangent vector field of γ, e_3 is the normal field of the surface,

$$k_g := \frac{D\mathbf{e_1}}{ds} = k_g \mathbf{e_2}$$

is called geodesic curvature vector,

$$k_g = \langle \frac{D\mathbf{e_1}}{ds}, \mathbf{e_2} \rangle$$

is called geodesic curvature.

Geodesic Curvature

Geodesic curvature, normal curvature

Given a spacial curve, its curvature vector satisfies

$$\frac{d^2\gamma}{ds^2} = k_g \mathbf{e_2} + k_n \mathbf{e_3},$$

where k_n is the normal curvature of the curve. The curvature of the curve, geodesic curvature and normal curvature satisfy

$$k^2 = k_g^2 + k_n^2.$$

Geodesic curvature k_g only depends on the Riemannian metric of the surface, is independent of the 2nd fundamental form. Therefore k_g is intrinsic, k_n is extrinsic.

Geodesic Curvature

We use isothermal parameter (u, v) of $(S, \mathbf{g})$, given a curve $\gamma(s)$ with arc length parameter s. Construct orthonormal frame $\{p; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\mathbf{e}_3$ is the normal field of S. The tangent vector of γ is $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$ is orthogonal to $\bar{\mathbf{e}}_1$ everywhere. The angle between $\bar{\mathbf{e}}_1$ and $\mathbf{e}_1$ is $\theta(s)$,

$$\begin{cases} \bar{\mathbf{e}}_1 &= \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \end{cases}$$

Direct computation

$$\begin{aligned} D\bar{\mathbf{e}}_1 &= D(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2) = d\cos\theta\mathbf{e}_1 + \cos\theta_1 D\mathbf{e}_1 + d\sin\theta\mathbf{e}_2 + \sin\theta D\mathbf{e}_2 \\ &= -\sin\theta d\theta\mathbf{e}_1 + \cos\theta\omega_{12}\mathbf{e}_2 + \cos\theta d\theta\mathbf{e}_2 - \sin\theta\omega_{12}\mathbf{e}_1 \\ &= -\sin\theta(d\theta + \omega_{12})\mathbf{e}_1 + \cos\theta(\omega_{12} + d\theta)\mathbf{e}_2 \end{aligned}$$

$$k_g = \langle \frac{D\bar{\mathbf{e}}_1}{ds}, \bar{\mathbf{e}}_2 \rangle = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

Geodesic Curvature

Under the isothermal coordinates, we have $\omega_{12} = -u_y dx + u_x dy$. Suppose on the parameter domain, the planar curve arc length is dt, then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} \\ &= \frac{d\theta}{dt}\frac{dt}{ds} + \frac{-u_y dx + u_x dy}{dt}\frac{dt}{ds} \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_{\mathbf{n}} u) \end{aligned}$$

where k is the curvature of the planar curve, n is the normal to the planar curve.

Geodesic Curvature

Lemma

Given a metric surface $(S, \mathbf{g})$, under conformal deformation, $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$, the geodesic curvature satisfies

$$k_{\bar{\mathbf{g}}} = e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda).$$

Proof.

$$\begin{aligned} k_{\mathbf{g}} &= e^{-(u+\lambda)}(k - \partial_{\mathbf{n}}(u+\lambda)) \\ &= e^{-\lambda}(e^{-u}(k - \partial_{\mathbf{n}}u) - e^{-u}\partial_{\mathbf{n}}\lambda) \\ &= e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda) \end{aligned}$$

□

Geodesics

Definition (geodesic)

Given a metric surface $(S, \mathbf{g})$, a curve $\gamma : [0,1] \to S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

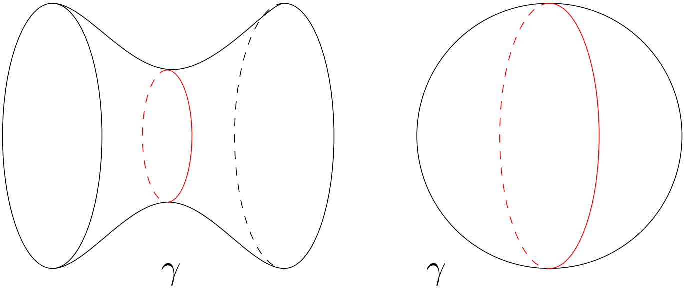

Figure: Stable and unstable geodesics.

Lemma (geodesic)

If γ is the shortest curve connecting p and q, then γ is a geodesic.

Proof.

Consider a family of curves, $F : (-\varepsilon, \varepsilon) \to S$, such that $F(0, t) = \gamma(t)$, and

$$F(s, 0) = p, F(s, 1) = q, \frac{\partial F(s, t)}{\partial s} = f(t)\mathbf{e_2}(t),$$

where $f : [0, 1] \to \mathbb{R}$, $f(0) = f(1) = 0$. Fix parameter s, curve $\gamma_s := F(s, \cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left|\frac{d\gamma_s(t)}{dt}\right|^2 dt, \quad \frac{\partial L(s)}{\partial s} = -\int_0^1 f k_{\mathbf{g}}(\tau) d\tau.$$

□

Geodesics

The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.