A Chebyshev Technique for Solving Nonlinear Optimal Control Problems

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Abstract—**This paper introduces a numerical technique for solving nonlinear optimal control problems. The state and control variables are expanded in the Chebyshev series, and an algorithm is provided for approximating the system dynamics, boundary conditions, and performance index. Application of this method results in the transformation of differential and integral expressions into systems of algebraic or transcendental expressions in the Chebyshev coefficients. The optimum condition is obtained by applying the method of constrained extremum. For linear-quadratic optimal control problems, the state and control variables are determined by solving a set of linear equations in the Chebyshev coefficients. Applicability is illustrated with the minimum-time and maximum-radius orbit transfer problems.**

I. Introduction

Up to now enormous effort has been spent on the development of computational techniques for solving optimal control problems [1]–[8], mostly using Bellman's dynamic programming [9] and Pontryagin's maximum principle method [10]. Other contributions [11]–[18] describe techniques which resemble the Ritz–Galerkin procedure for variational problems [19]–[20].

The purpose of this paper is to present an alternative algorithm for solving complicated nonlinear, multivariable, constrained optimal control problems. We extend the work of Urabe [21]–[22], among others, on the numerical computation of solutions to multipoint boundary value problems for ordinary differential equations via expansions in the Chebyshev series to optimal control problems. Earlier, polynomial approximation of either the state alone [11], [16], [17], the control alone [14], [15], or both the state and the control [12], [13] has been suggested. Limitations of the work include restrictions to linear-quadratic problems, use of power series representations, elimination of the control, etc. Compared to these approaches we present a general numerical technique to solve nonlinear optimal control problems by parameterizing both the state and control variables. Considering the important advantages of the use of Chebyshev polynomials in numerical analysis with regard to minimax principles and least-squares techniques [23]–[26], our approach is based upon the expansion of both the state and control functions in the Chebyshev series with unknown coefficients. The differential and integral expressions which arise in the system dynamics and the performance index, the initial or boundary conditions, or even general multipoint boundary conditions—which would complicate severely most numerical techniques—are converted into algebraic or transcendental equations in the unknown coefficients. In this way, the optimal control problem is replaced by a parameter optimization problem which consists of the minimization or maximization of the performance index subject to algebraic or transcendental constraints. We can then replace the constrained extremum problem by an unconstrained extremum problem by applying the method of Lagrange [27] or a penalty function technique [28]–[30]. Eventually, unknown system parameters which have to be optimized can be determined within this procedure.

The technique has been thoroughly tested on problems of all kinds, and shows very accurate results [31]–[33]. Vlach [24] stated that, of all ultraspherical polynomials, the Chebyshev polynomials of the first class can uniformly approximate a much broader class of functions. Although we recognize that for some special examples other orthogonal polynomials could possibly perform slightly better, we find the Chebyshev polynomials computationally much more attractive. We have also compared our results to those obtained by a spline approximation with collocation [13], and our results were better with fewer unknowns. Paraskevopoulos has confirmed our belief in the Chebyshev polynomials by several experiments on linear time-invariant systems [34], [35] which demonstrated advantages compared to expansions in terms of Walsh, block-pulse, Hermite, Laguerre, and Legendre functions.

We will illustrate the method by practical examples for which either an exact solution from Pontryagin's maximum principle or an approximate solution reported in the literature was available.

II. The Chebyshev Approach

We assume that the system dynamics are described by the following ordinary differential equation:

$$\frac{dX}{d\tau}=F(X, U, \tau), \qquad 0 \leqslant \tau \leqslant T \text{ (specified or free)} \tag{1}$$

with initial condition

$$X(0)=X_0 \tag{2}$$

where $X(\tau)$ is the s-dimensional state vector, and $U(\tau)$ is the r-dimensional control vector. The objective is to determine, over all piecewise differentiable control vector functions, the control vector $U(\tau)$ which minimizes or maximizes the performance index I given by

$$I=H[X(T), T]+\int_0^T G(X, U, \tau)\, d\tau. \tag{3}$$

The vector function F and the scalar functions H and G are generally nonlinear, and are assumed to be continuously differentiable with respect to their arguments. Without loss of generality we will, for the sequel, assume that $s = r = 1$. The time transformation

$$\tau=\frac{T}{2}(1+t) \tag{4}$$

is introduced in order to use the Chebyshev polynomials of the first class, $\{T_n(t) = \cos(n \cos^{-1} t)\}$, defined on the interval $t \in$

Manuscript received September 30, 1986; revised July 21, 1987. Paper recommended by Associate Editor, W. J. Rugh.

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IEEE Log Number 8719055.

0018-9286/88/0400-0333$01.00

$(-1, 1)$. It follows that (1)–(3) are replaced by

$$\frac{dx}{dt}=F(x, u, t, T), \qquad -1 \leqslant t \leqslant 1, \tag{5}$$

$$x(-1)=x_{-1}=X_0, \tag{6}$$

$$I=h[x(1), T]+\int_{-1}^{1} g(x, u, t, T)\, dt. \tag{7}$$

By expanding the state and control in a Chebyshev series of order m, we determine the following approximate solution:

$$x_m(t)=\frac{1}{2}a_0T_0(t)+\sum_{n=1}^{m} a_nT_n(t),$$

$$u_m(t)=\frac{1}{2}b_0T_0(t)+\sum_{n=1}^{m} b_nT_n(t) \tag{8}$$

where $\alpha \equiv (a_0, a_1, \cdots, a_m)$ and $\beta \equiv (b_0, b_1, \cdots, b_m)$ are unknown. For simplicity we have used the same degree of expansion for the state and control, where the choice of m depends on the required accuracy. The approximation errors for $x(t)$ and $u(t)$ are $\Sigma_{n=m+1}^{\infty} a_nT_n(t)$ and $\Sigma_{n=m+1}^{\infty} b_nT_n(t)$, respectively. If we increase the number of terms, the approximation will improve and will tend to the exact solution. However, there is a certain limit beyond which increasing m will not result in any improvement, and, on the contrary, will cause degradation of performance (due to roundoff errors). It is evident that, due to the use of orthogonal functions for representing the state and the control, increasing m does not result in an appreciable change in the low-order coefficients.

A. The Performance Index Approximation

Substituting the approximations for the state and control from (8) into (7), setting $\{C_n(\alpha, T)\}$ and $\{B_n(\alpha, \beta, T)\}$ to be the Chebyshev coefficients of $h[x_m(t), T]$ and $g(x_m(t), u_m(t), t, T)$, respectively, and using the properties that [25]

$$T_n(1)=1, \qquad (n=0, 1, \cdots) \tag{9}$$

and

$$\int_{-1}^{1} q(t)\, dt=q_0-\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}q_n \quad \text{if } q(t)=\frac{1}{2}q_0T_0(t)+\sum_{n=1}^{\infty} q_nT_n(t) \tag{10}$$

we obtain the following approximation for I:

$$J(\alpha, \beta, T)=\frac{1}{2}C_0(\alpha, T)+\sum_{n=1}^{\infty} C_n(\alpha, T) +B_0(\alpha, \beta, T)-\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}B_n(\alpha, \beta, T). \tag{11}$$

For computational reasons the infinite series are truncated at order N. Then the coefficients $\{C_n\}$ and $\{B_n\}$ can be calculated very accurately by means of the following approximation formula [25], [26]:

$$C_n(\alpha, T)=\frac{2}{K}\sum_{i=1}^{K} h[x_m(\cos\theta_i), T]\cos n\theta_i,$$

$$B_n(\alpha, \beta, T)=\frac{2}{K}\sum_{i=1}^{K} g[x_m(\cos\theta_i), u_m(\cos\theta_i), \cos\theta_i, T]\cos n\theta_i,$$

$$(n=0, 1, \cdots, N),\ K>N,\ \theta_i=\frac{2i-1}{K}\cdot\frac{\pi}{2}. \tag{12}$$

Generally, J is nonlinear in α and β.

B. Approximation of the System Dynamics

The system dynamics are replaced by the following approximation:

$$\frac{dx_m}{dt}=f_M(x_m(t), u_m(t), t, T) \tag{13}$$

where

$$f_M(x_m(t), u_m(t), t, T)=\frac{1}{2}A_0(\alpha, \beta, T)T_0(t) +\sum_{n=1}^{M} A_n(\alpha, \beta, T)T_n(t) \tag{14}$$

with

$$A_n(\alpha, \beta, T)=\frac{2}{K}\sum_{i=1}^{K} f(x_m(\cos\theta_i), u_m(\cos\theta_i), \cos\theta_i, T)\cos n\theta_i,$$

$$(n=0, 1, \cdots, M),\ K>M,\ \theta_i=\frac{2i-1}{K}\cdot\frac{\pi}{2}.$$

The left-hand side of (13) is a polynomial of degree $m-1$, while the right-hand side is a polynomial of degree M. If the function f is nonlinear, we choose $M = m-1$, after Urabe [21], [22], but if f is linear we have found—from many experiments on several problems, but without theoretical proof—the choice $M = m$ is better. Applying the Chebyshev balance principle to (13), that is, equating the coefficients of same-order Chebyshev polynomials, yields

$$a_0'=A_0,\ a_1'=A_1, \cdots, a_{m-1}'=A_{m-1},$$

$$0=A_m, \cdots, 0=A_M \tag{15}$$

where $\{a_n'\}$ represents the Chebyshev coefficients of dx_m/dt.

A classical property for Chebyshev series [25] expresses the relationship between $\{a_n\}$ and $\{a_n'\}$

$$a_{n-1}'-a_{n+1}'-2na_n=0, \qquad (n=1, 2, \cdots) \tag{16}$$

which, with (15), gives the following relationships between α, β, and T:

$$F_{n-1}(\alpha, \beta, T) \equiv A_{n-1}(\alpha, \beta, T)-A_{n+1}(\alpha, \beta, T)-2na_n=0, \quad (n=1, 2, \cdots, m),$$

$$F_{n-1}(\alpha, \beta, T) \equiv A_{n-1}(\alpha, \beta, T)-A_{n+1}(\alpha, \beta, T)=0, \quad (n=m+1, \cdots, M+1),$$

$$\text{with } A_{M+1}(\alpha, \beta, T)=A_{M+2}(\alpha, \beta, T)=0. \tag{17}$$

Equations (17) will replace the system dynamics (5).

Another property of Chebyshev polynomials is [25]

$$T_n(-1)=(-1)^n, \qquad (n=1, 2, \cdots) \tag{18}$$

and therefore the initial condition (6) is replaced by

$$F_{M+1}(\alpha) \equiv \frac{1}{2}a_0+\sum_{n=1}^{m}(-1)^na_n-x_{-1}=0. \tag{19}$$

C. The Chebyshev Coefficients for $x(t)$ and $u(t)$

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find α, β, and T (if free) so that $J(\alpha, \beta, T)$ is minimal, or maximal, subject to the constraints (17) and (19). Many mathematical programming techniques can be used to solve this constrained extremum problem. The solution proposed by Lagrange is to form an unconstrained problem by appending the constraints to the performance index by means of Lagrange multipliers. If we define

$$L(\alpha, \beta, \lambda, T)=J(\alpha, \beta, T)+\sum_{\nu=0}^{M+1} \lambda_\nu F_\nu(\alpha, \beta, T) \tag{20}$$

where $\lambda \equiv (\lambda_0, \lambda_1, \cdots, \lambda_{M+1})$ represents the unknown Lagrange multipliers, then the necessary conditions for stationarity are given by

$$\frac{\partial L}{\partial a_\mu}=0, \frac{\partial L}{\partial b_\mu}=0, \qquad (\mu=0, 1, \cdots, m),$$

$$\frac{\partial L}{\partial T}=0 \qquad \text{if } T \text{ is free,}$$

$$F_\nu=0, \quad (\nu=0, 1, \cdots, M+1). \tag{21}$$

Hence, the determining equations for the unknowns are

$$\frac{\partial J}{\partial a_\mu}+\sum_\nu \lambda_\nu \frac{\partial F_\nu}{\partial a_\mu}=0, \qquad (\mu=0, 1, \cdots, m), \tag{22}$$

$$\frac{\partial J}{\partial b_\mu}+\sum_\nu \lambda_\nu \frac{\partial F_\nu}{\partial b_\mu}=0, \tag{23}$$

$$\frac{\partial J}{\partial T}+\sum_\nu \lambda_\nu \frac{\partial F_\nu}{\partial T}=0 \qquad \text{if } T \text{ is free,} \tag{24}$$

$$F_\nu=0, \qquad (\nu=0, 1, \cdots, M+1). \tag{25}$$

Sufficient conditions for a local minimum (maximum) are the stationarity conditions (22)–(25), and the convexity condition expressing the positive (negative) definiteness of a certain quadratic form [27].

Starting values for α, β, (and T) are usually obtained from some physical insight in the problem, or by applying the proposed method for very low order. Once these initial values are given, starting values for λ can be obtained by selecting any $M + 2$ equations from (22)–(24), and solving the resulting linear system for λ. The iterative Newton method, or a modified version for better convergence [26], can now be applied to solve (22)–(25).

Remark: The constrained optimization problem could be converted into an unconstrained problem by using a penalty function technique [28]–[30], thus avoiding an increase in the dimensionality of the problem. The unconstrained problem could also be solved by a derivative-free method [36] instead of the Newton method.

III. Additional Constraints on $x(t)$ and/or $u(t)$

A. Equality Constraints

Additional boundary conditions are replaced by their approximate version. For example,

$$x(1)=x_1 \tag{26}$$

is replaced by $x_m(1) = x_1$ giving

$$\frac{1}{2} a_0+\sum_{n=1}^{m} a_n-x_1=0 \tag{27}$$

which is then considered as another constraint, completing the system of constraints (17) and (19).

Constraints of the more general form

$$\psi(x(1))+\int_{-1}^{1} \phi(x, u, t)\, dt=0 \tag{28}$$

or even general multipoint boundary conditions, which could severely encumber many numerical methods, can be treated according to the approach applied to (7).

Equality constraints of the form

$$C(u(t), t)=0, \; C(x(t), u(t), t)=0 \text{ or } S(x(t), t)=0 \tag{29}$$

can be treated as (5): $x_m(t)$ and/or $u_m(t)$ are substituted, $C(\cdot)$ and/or $S(\cdot)$ are expanded in the Chebyshev series, and the Chebyshev balance principle is applied. The resulting constraints then have to be adjoined to the other constraints. Another possibility is to apply the penalty function technique, which involves replacing the function $g(\cdot)$ from (7) by $g(\cdot) + kC^2(\cdot)$, or $g(\cdot) + kS^2(\cdot)$, with k positive (negative) in the minimization (maximization) problem.

B. Inequality Constraints

Constraints of the form

$$C(u(t), t) \leqslant 0, \; C(x(t), u(t), t) \leqslant 0 \text{ or } S(x(t), t) \leqslant 0 \tag{30}$$

can be handled by introducing slack variables [7], [37], [38]. For a scalar constraint of the type

$$C(x(t), u(t), t) \leqslant 0, \qquad -1 \leqslant t \leqslant 1 \tag{31}$$

we define a slack variable $y(t)$ by

$$C(x(t), u(t), t)=-y^2(t) \tag{32}$$

which means that the inequality constraint (31) is replaced by the equality constraint (32). If we now expand the unknown $y(t)$ in a Chebyshev series with coefficients $\{c_n\}$, and apply the technique for equality constraints, we obtain additional constraint relations in $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. The coefficients $\{c_n\}$ can then be determined by adding

$$\frac{\partial L}{\partial c_\mu}=0 \tag{33}$$

to the equations (22)–(25).

Another possibility, which has been applied to linear-quadratic optimal control problems [12], [13], consists of substituting $x_m(t)$ and $u_m(t)$ from (8) into (31) and calculating (31) at a number of points t_i, $-1 = t_0 < t_1 < \cdots < t_{K'} = 1$. This yields

$$D_\nu(\alpha, \beta) \leqslant 0, \quad (\nu=0, 1, \cdots, K') \tag{34}$$

and the problem then is to find α, β, and T in order to minimize or maximize J such that $\{F_\nu = 0\}$ and $\{D_\nu \leqslant 0\}$ are satisfied.

Sirisena used piecewise polynomials [14], [16] to solve the state/costate differential equations, and introduced a transformation technique to handle problems with saturation-type control constraints, and boundary constraints on the state variables [15]. This could be applied here as well.

Remark: If the control variable should appear to be discontinuous (e.g., bang-bang problems) the method outlined above should be modified slightly. For such problems we recommend dividing the original interval $\tau(0, T)$ into several subintervals depending upon the number of discontinuities. The instants at which the discontinuities occur are treated as additional unknowns, and every subinterval is transformed into $t \in (-1, 1)$ on which the proposed technique is then applied. Expressing the continuity of the state variables at the unknown instants gives additional constraint relations. This has been applied to the problem of the double integrator with the constraint $-1 \leqslant u \leqslant 1$ for which the bang-bang solution for $u(t)$ and the solution for $x(t)$ were successfully calculated.

IV. Theoretical and Computational Considerations

As has been proved for variational problems [19], we can show for optimal control problems that increasing the order of the Chebyshev series results in a decreasing (increasing) value for the performance index for a minimization (maximization) problem [33]. However, a theoretical error estimation for the performance index and the state and control variables has yet to be established.

There is no rule for determining the parameter values m, K, $\cdots$ in order to reach a given accuracy. Urabe [21], [22] has described a method to determine very accurately the numerical solution of nonlinear ordinary differential equations, and has shown how to study the existence and uniqueness problem of an exact solution near the calculated Chebyshev approximation, and how to estimate the error on the approximation. However, we believe it will be very hard to apply Urabe's results to the optimal control problem, and we suggest, for engineering purposes, a somewhat different view of the error estimation problem. Substitution of the calculated $u_m(t)$ into (5) gives

$$\frac{dx}{dt}=f(x, u_m, t, T), \qquad -1 \leqslant t \leqslant 1. \tag{35}$$

Numerical integration of (35) is possible for given initial or terminal conditions. If we compare the so-obtained solution $\bar{x}(t)$ to the calculated Chebyshev approximation $x_m(t)$ using the uniform norm, we can define a practical and easy-to-use error estimate for the dynamical equations as

$$\epsilon_{\text{dyn}}=\max_{-1\leqslant t\leqslant 1} \|\bar{x}(t)-x_m(t)\| \tag{36}$$

where $\|\cdots\|$ represents the Euclidean norm. A similar treatment has been used by Taylor and Constantinides [39]. We therefore suggest to use either ϵ_{dyn} or $|J(\alpha_{m+1}, \beta_{m+1}) - J(\alpha_m, \beta_m)|$, or both, to decide whether the computed solution is close enough to the optimal solution. We found from numerous tests on linear and nonlinear, one- and multidimensional problems (with two-point boundary constraints, terminal time given or free, ...) that, once m has been chosen, good values for N and K can be found as follows: $N = 1.5$ m and $K = 3$ m. However, these choices for N and K are not critical at all and larger values increase the computational time, but do not generally further improve the accuracy.

V. Examples

A. Problem Treated by Feldbaum

The object is to find the optimal control $U(\tau)$ which minimizes

$$I=\frac{1}{2}\int_0^1 (X^2+U^2)\, d\tau \tag{37}$$

such that

$$\frac{dX}{d\tau}=-X+U, \qquad 0 \leqslant \tau \leqslant 1 \tag{38}$$

and

$$X(0)=1 \tag{39}$$

are satisfied. Transforming τ to the t-interval $(-1, 1)$, the exact solution to this linear-quadratic problem can be found by applying Pontryagin's maximum principle [40]. To get better insight into the proposed method, we first determine analytically the Chebyshev approximation of third order ($m = 3$). Choosing $M = m = 3$, the unknowns $\alpha \equiv (a_0, a_1, a_2, a_3)$ and $\beta \equiv (b_0, b_1, b_2, b_3)$ must satisfy the constraints

$$\begin{aligned}
F_0 &\equiv \frac{1}{2}(-a_0+b_0)-\frac{1}{2}(-a_2+b_2)-2a_1=0,\\
F_1 &\equiv \frac{1}{2}(-a_1+b_1)-\frac{1}{2}(-a_3+b_3)-4a_2=0,\\
F_2 &\equiv \frac{1}{2}(-a_2+b_2)-6a_3=0,\\
F_3 &\equiv \frac{1}{2}(-a_3+b_3)=0,\\
F_4 &\equiv \frac{1}{2}a_0-a_1+a_2-a_3-1=0.
\end{aligned} \tag{40}$$

With $h = 0$ and $g = (x^2 + u^2)/4$, the approximate performance index to be minimized is

$$J=\frac{1}{4}\left(\frac{1}{2}a_0^2+\frac{2}{3}a_1^2+\frac{14}{15}a_2^2+\frac{34}{35}a_3^2-\frac{2}{3}a_0a_2-\frac{4}{5}a_1a_3 +\frac{1}{2}b_0^2+\frac{2}{3}b_1^2+\frac{14}{15}b_2^2+\frac{34}{35}b_3^2-\frac{2}{3}b_0b_2-\frac{4}{5}b_1b_3\right). \tag{41}$$

Equations (22), (23), and (25) give 13 determining equations for α, β, and λ. It is to be emphasized that the determining equations are linear, and the following solution is readily obtained:

$$\begin{aligned}
x_3(t)&=0.574210\,T_0(t)-0.351882\,T_1(t)+0.066769\,T_2(t)-0.007140\,T_3(t),\\
u_3(t)&=-0.172393\,T_0(t)+0.182266\,T_1(t)-0.018907\,T_2(t)-0.007140\,T_3(t),\\
&\lambda_0=0.083045,\ \lambda_1=-0.062183,\ \lambda_2=0.034404,\\
&\lambda_3=-0.022262,\ \lambda_4=-0.385863,\\
&J=0.192931.
\end{aligned} \tag{42}$$

Since the convexity conditions are satisfied [33], this Chebyshev approximation offers at least a local minimum.

We can now expand the exact solutions for $x(t)$ and $u(t)$ in the Chebyshev series. A comparison between the results for the third-order Chebyshev approximation and the exact solution shows that the error in the Chebyshev coefficients is of the order of 10^{-2} while for the performance index an agreement of about 5 decimal figures is obtained. Higher order Chebyshev approximations have been computed with very high precision on a CDC Cyber 170/750. The results for $m = 3$, $m = 5$, $m = 7$, and $m = 9$ are reported in Table I. The results gradually tend to the exact results as we systematically proceed to the higher order approximations. The Chebyshev approximation of ninth order is a very accurate approximation of the exact solution. The largest deviation in the coefficients is smaller than 2×10^{-10}, and there is an agreement of 10 decimal figures for J.

B. Minimum Time Orbit Transfer Problem

One of the best known trajectory optimization examples in the literature [3], [7], [39], [41]–[44], is the problem of minimizing the transfer time of a constant low-thrust ion rocket between the orbits of Earth and Mars. This involves the determination of the thrust angle history for which no exact solution is known. The

TABLE I
THE FELDBAUM PROBLEM

m	N	K	J
3	5	10	0.192931
5	8	16	0.1929094
7	10	20	0.19290930
9	15	30	0.1929092981
exact solution for J : 0.1929092981			

problem is simplified here in that the orbits of both Earth and Mars are assumed to be circular and coplanar, and the gravitational attractions of the two planets on the rocket are neglected. Since the assumption of constant thrust implies constant fuel consumption rate, minimum time corresponds to minimum fuel consumption. If we choose the state variables X_1, X_2, X_3, respectively, as the distance of the rocket from the Sun, and the radial and tangential velocities, then the transfer is governed by the following time-varying equations [43]

$$\frac{dX_1}{d\tau}=X_2, \tag{43}$$

$$\frac{dX_2}{d\tau}=\frac{X_3^2}{X_1}-\frac{\gamma}{X_1^2}+\frac{R_0\sin U}{m_0+\dot{m}\tau}, \qquad 0\leqslant\tau\leqslant T\,(\text{free}), \tag{44}$$

$$\frac{dX_3}{d\tau}=-\frac{X_2X_3}{X_1}+\frac{R_0\cos U}{m_0+\dot{m}\tau}. \tag{45}$$

In these expressions, γ characterizes the gravitational attraction from the Sun, R_0 is the constant low-thrust magnitude, U is the control angle measured from the local horizontal, m_0 is the initial mass, and $\dot{m}$ is the constant propellant consumption rate. Using normalized values, we have $\gamma = 1$, $R_0 = 0.1405$, $m_0 = 1$, and $\dot{m} = -0.07487$. The time-unit is 58.18 days, according to the applied normalization. At the beginning of the maneuver the rocket is assumed to have velocity and radial position corresponding to the orbit of Earth, while at the end, the velocity and radial position correspond to the orbit of Mars. This leads to the following boundary conditions:

$$\begin{gathered}X_1(0)=1.0,\ X_2(0)=0.0,\ X_3(0)=1.0\\ X_1(T)=1.525,\ X_2(T)=0.0,\ X_3(T)=0.8098\end{gathered} \tag{46}$$

where T is the unknown final time to be minimized. Thus, the performance index is simply $I = T$. After transformation to $t \in (-1, 1)$, the state variables x_1, x_2, x_3, the control variable u, and the right-hand sides of the differential equations—which will be denoted by f_1, f_2, and f_3—are expanded in the Chebyshev series with coefficients $\{a_{1,n}\}$, $\{a_{2,n}\}$, $\{a_{3,n}\}$, $\{b_n\}$, $\{A_{1,n}\}$, $\{A_{2,n}\}$, and $\{A_{3,n}\}$, respectively. The function f_1 is linear in x_2 and its series is truncated at order $M_1 = m$, while f_2 and f_3 are both nonlinear in x_1, x_2, x_3, and u and their series are therefore truncated at order $M_2 = m - 1$. We obtain the following constraints:

$$\begin{aligned}&A_{1,0}-A_{1,2}-2a_{1,1}=0,\\ &A_{1,1}-A_{1,3}-4a_{1,2}=0,\\ &\cdots\\ &A_{1,m-2}-A_{1,m}-2(m-1)a_{1,m-1}=0,\\ &A_{1,m-1}-2ma_{1,m}=0,\\ &A_{1,m}=0,\\ &\frac{1}{2}a_{1,0}+\sum_{n=1}^{m}(-1)^n a_{1,n}-1.0=0,\\ &\frac{1}{2}a_{1,0}+\sum_{n=1}^{m}a_{1,n}-1.525=0.\end{aligned} \tag{47}$$

Analogously,

$$\begin{aligned}&A_{2,0}-A_{2,2}-2a_{2,1}=0,\\ &A_{2,1}-A_{2,3}-4a_{2,2}=0,\\ &\cdots\\ &A_{2,m-2}-2(m-1)a_{2,m-1}=0,\\ &A_{2,m-1}-2ma_{2,m}=0,\\ &\frac{1}{2}a_{2,0}+\sum_{n=1}^{m}(-1)^n a_{2,n}=0,\\ &\frac{1}{2}a_{2,0}+\sum_{n=1}^{m}a_{2,n}=0\end{aligned} \tag{48}$$

and

$$\begin{aligned}&A_{3,0}-A_{3,2}-2a_{3,1}=0,\\ &A_{3,1}-A_{3,3}-4a_{3,2}=0,\\ &\cdots\\ &A_{3,m-2}-2(m-1)a_{3,m-1}=0,\\ &A_{3,m-1}-2ma_{3,m}=0,\\ &\frac{1}{2}a_{3,0}+\sum_{n=1}^{m}(-1)^n a_{3,n}-1.0=0,\\ &\frac{1}{2}a_{3,0}+\sum_{n=1}^{m}a_{3,n}-0.8098=0.\end{aligned} \tag{49}$$

Adjoining these constraints, $\{F_\nu\}$, to the performance index by means of Lagrange multipliers gives

$$L(\alpha,\beta,\lambda,T)=T+\sum_{\nu=0}^{3m+6}\lambda_\nu F_\nu(\alpha,\beta,T). \tag{50}$$

The determining equations are then

$$\begin{gathered}\sum_\nu\lambda_\nu\frac{\partial F_\nu}{\partial a_{1,\mu}}=0,\ \sum_\nu\lambda_\nu\frac{\partial F_\nu}{\partial a_{2,\mu}}=0,\ \sum_\nu\lambda_\nu\frac{\partial F_\nu}{\partial a_{3,\mu}}=0,\\ \sum_\nu\lambda_\nu\frac{\partial F_\nu}{\partial b_\mu}=0,\qquad(\mu=0,1,\cdots,m),\\ 1+\sum_\nu\lambda_\nu\frac{\partial F_\nu}{\partial T}=0,\\ F_\nu=0,\qquad(\nu=0,1,\cdots,3m+6).\end{gathered} \tag{51}$$

These nonlinear equations are solved with the iterative Newton method. The starting value for the unknown final time is taken to be $\bar{T} = 3.00$. Hence, if we assume that $X_1(\tau)$ behaves linearly between the specified initial and terminal point, the starting values for $\{a_{1,n}\}$ can be determined. From (43) we derive the starting values for $\{a_{2,n}\}$, and if we neglect the left-hand side, and the third term in (44), we obtain $X_3^2(\tau) = 1/X_1(\tau)$ from which some starting values for $\{a_{3,n}\}$ can be derived. Finally, starting values for $\{b_n\}$ can be determined by assuming that $U(\tau)$ varies linearly from 45° to 315°. The starting values for λ are determined as described above. For the case $m = 7$, $K = 20$, the results after 6

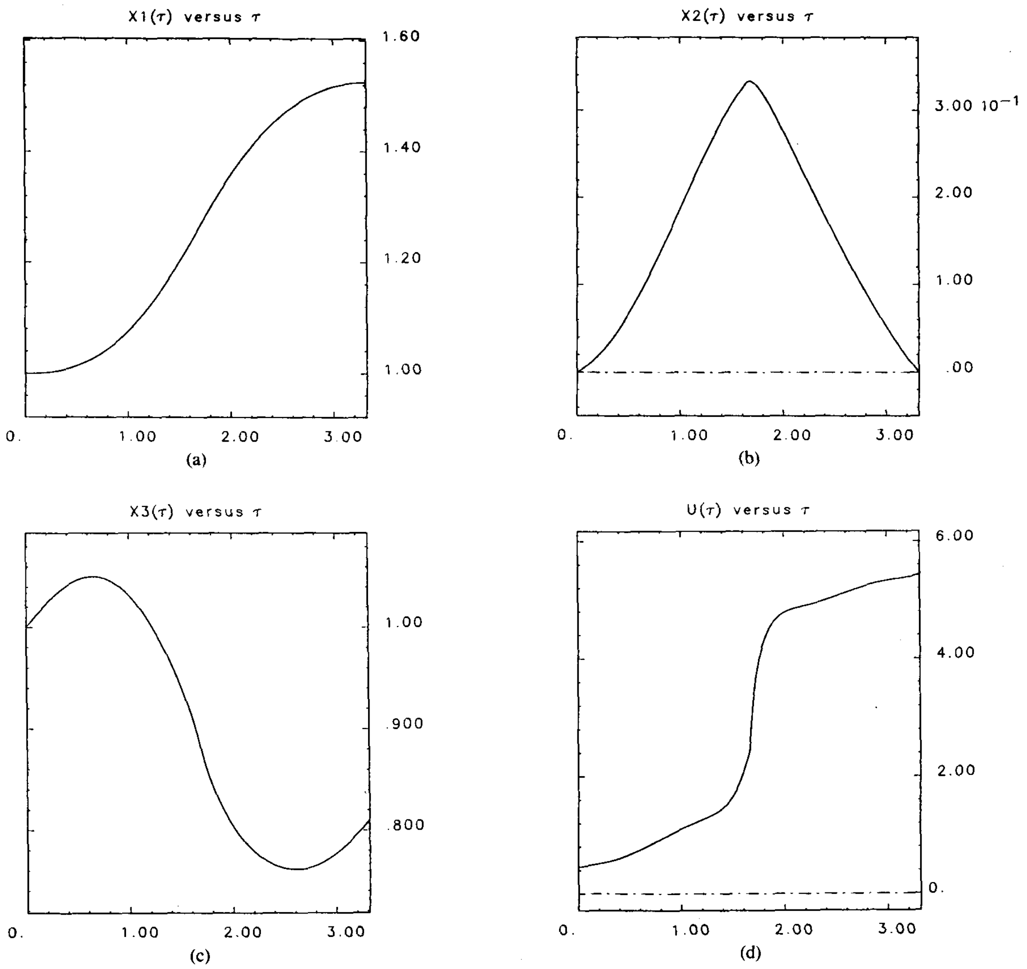

Fig. 1. State and control variables for the orbit transfer problem.

iterations or 9.08 s execution time are that the minimum time is 3.33069, while SFNU $\equiv \Sigma_\nu |F_\nu| = 0.10 \times 10^{-7}$. The sum of the absolute values of the corrections on the unknowns, SDEL, is 0.24×10^{-6}. The boundary conditions (46) are accurately satisfied, with errors smaller than 10^{-13}, and the dynamical error $\epsilon_{\rm dyn}$ is 0.58×10^{-2}. Thus, (43)–(45) are satisfied quite well. If we increase K from 20 to 30, the execution time increases from 9.08 to 10.97 s, but the first 6 decimals in the results do not change. For $m = 9$ and $K = 30$ the minimum time becomes 3.32263 while SFNU and SDEL are 0.24×10^{-7} and 0.54×10^{-6}, respectively. The boundary-condition errors remain smaller than 10^{-13}, and $\epsilon_{\rm dyn}$ decreases to 0.32×10^{-2}. Finally, for $m = 11$ and $K = 40$ we have $T = 3.31874$, SNFU $= 0.25 \times 10^{-9}$, SDEL $= 0.51 \times 10^{-8}$, boundary-condition errors smaller than 10^{-13}, and $\epsilon_{\rm dyn} = 0.11 \times 10^{-2}$. The state and control variables obtained for $m = 11$ are displayed in Fig. 1, and the corresponding optimal trajectory is shown in Fig. 2. From Fig. 2 we see that the thrust has a positive radial component during the first half of the transfer, and then rapidly reverses itself to have a negative radial component in the second half. This accelerating and slowing-down effect of the thrust is also seen in Fig. 1(b) ($X_2(\tau)$ versus τ).

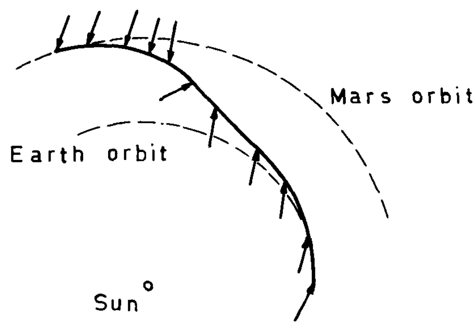

Fig. 2. Optimal transfer trajectory.

This problem has also been successfully solved by Moyer and Pinkham [42] using several trajectory optimization techniques such as steepest descent (first and second variation), and generalized Newton–Raphson, in addition to penalty function techniques. Falb and de Jong [3] examined this problem by means of the quasi-linearization method following three different procedures to convert the two-point boundary value problem (TPBVP) with variable final time into a TPBVP with fixed final time. One of these procedures, which replaces the minimum time problem by a sequence of radius maximization problems, and adjusts the final time iteratively so that the maximum radius coincides with the required value of the radius at the end of the maneuver, has also been applied by McGill and Kenneth [43]. Hontoir and Cruz [44] used a refined shooting method, and Taylor and Constantinides [39] applied the epsilon method of Balakrishnan [45]. Their results are compared to those obtained by our technique in Table II. Some of these techniques analytically removed the control variable and therefore their results cannot be compared to ours on a fair basis. Nevertheless, we obtain, globally, the best results. Although Moyer and Pinkham obtain a smaller final time, the differential equations and the boundary conditions are not satisfied very accurately (error of 0.1 and 0.05 percent). At the final time the velocity of the rocket should agree with the velocity corresponding to the trajectory with radius $X_1(T)$, and, as this is not the case for Moyer and Pinkham, the rocket will not remain on the calculated trajectory.

C. Maximum Radius Orbit Transfer Problem

Given a constant low-thrust rocket operating for a given length of time T, we wish to find the thrust angle history to transfer this rocket from the orbit of Earth to the largest possible circular orbit. Dyer and McReynolds [5] examined this problem using the discrete successive sweep algorithm. As they did, we choose the final time T to be 3.32. The system dynamics are described by (43)–(45), but the boundary conditions are now

$$X_1(0)=1.0,\ X_2(0)=0.0,\ X_3(0)=1.0,$$
$$X_2(T)=0.0,\ X_3(T)=(X_1(T))^{-1/2}. \quad (51)$$

Here $X_1(T)$ is free and is to be maximized, hence $I = X_1(T)$. Again, we obtain the constraints (47)–(49) except that, since there is no end condition on X_1, the last equation of (47) vanishes, and the last equation of (49) must be replaced by

$$\frac{1}{2}a_{3,0}+\sum_{n=1}^{m}a_{3,n}-\left(\frac{1}{2}a_{1,0}+\sum_{n=1}^{m}a_{1,n}\right)^{-1/2}=0. \quad (52)$$

The performance index approximation is now

$$J=\frac{1}{2}a_{1,0}+\sum_{n=1}^{m}a_{1,n}. \quad (53)$$

Applying a similar procedure as above, we obtain the following results. For $m = 7, K = 20$, we have $X_1(T) = 1.52126$ and $\epsilon_{\text{dyn}} = 0.57 \times 10^{-2}$; for $m = 9$, $K = 30$, we have $X_1(T) = 1.52407$ and $\epsilon_{\text{dyn}} = 0.32 \times 10^{-2}$; and for $m = 11$, $K = 40$, we have $X_1(T) = 1.52545$ and $\epsilon_{\text{dyn}} = 0.11 \times 10^{-2}$. The results from Dyer and McReynolds are compared to our results in Table III. Note that for $m = 11$ our $X_1(T)$ is 0.36 percent larger, and our boundary conditions are perfectly satisfied (errors smaller than 10^{-13}) compared to the 0.6 percent error for Dyer and McReynolds.

TABLE II
RESULTS FOR THE MINIMUM-TIME ORBIT TRANSFER PROBLEM

methods	T	max. error boundary conditions
Moyer, Pinkham		
- gradient		
first }variation	3.317	0.1 %
second	3.317	0.05 %
- gener. Newton-Raphson	3.3207	-
Falb, de Jong	3.3193	-
Hontoir, Cruz	3.3194	-
Taylor, Constantinides	3.3819	0
Chebyshev method		
m = 7	3.33069	$< 10^{-13}$
m = 9	3.32263	$< 10^{-13}$
m = 11	3.31874	$< 10^{-13}$

TABLE III
RESULTS FOR THE MAXIMUM-RADIUS ORBIT TRANSFER PROBLEM

methods	$X_1(T)$	max. error boundary conditions
Dyer, McReynolds	1.52	0.6 %
Chebyshev method		
m = 7	1.521261	$< 10^{-13}$
m = 9	1.524071	$< 10^{-13}$
m = 11	1.525450	$< 10^{-13}$

VI. CONCLUSIONS

The aim of the proposed method is the determination of the optimal control and state vector by a direct method of solution based upon Chebyshev series expansions. Thus, the optimal control problem reduces to a problem of solving a system of nonlinear algebraic or transcendental equations, which is much easier than the numerical integration of the nonlinear TPBVP derived from Pontryagin's maximum principle method.

Although the amount of work to develop a general computer program seems quite large (depending upon the technique which is used to solve the constrained extremum problem), the computational time compares very favorably to other techniques such as the gradient and quasi-linearization methods. Our solutions for standard examples are much more accurate, and the errors on the boundary conditions are negligible. Contrary to some other techniques the proposed method needs no previous elimination of the control variable, which for complicated multidimensional nonlinear control problems will generally not be possible. The solution for x and u can be stored in terms of their Chebyshev coefficients, and can be continuously generated at any time during the interval.

The application of the technique to multivariable nonlinear problems with various constraints on state and/or control has been illustrated by a linear-quadratic problem and two orbit transfer problems. The method has also been applied successfully to other problems, for example the brachistochrone problem; rocket motion in a gravitational field where the control consists of attaining a maximum range at a given time T, the height being specified at that time; the problem of Roitenberg [46] in which the state and control exhibit oscillatory behavior; and on parameter estimation problems for systems with input and output measurements [47].

The (modified) technique has also been investigated for problems with discontinous control functions, and excellent results were obtained for the minimum-time state-transfer problem for a double integrator where the control is restricted to the amplitude range $-1 \leqslant u \leqslant 1$ and the resulting control strategy is of the bang-bang type.

Since both the state and control are parameterized, the authors believe that problems with, e.g., state variable inequality constraints can be treated more easily than with other methods [16], [48]. Preliminary results for such problems seem to confirm these suggestions.

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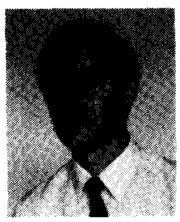

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