CHAPTER 7

MODELLING AND CONTROL OF NONHOLONOMIC MECHANICAL SYSTEMS

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Abstract

The goal of this chapter is to provide tools for analyzing and controlling nonholonomic mechanical systems. This classical subject has received renewed attention because nonholonomic constraints arise in many advanced robotic structures, such as mobile robots, space manipulators, and multifingered robot hands. Nonholonomic behavior in robotic systems is particularly interesting, because it implies that the mechanism can be completely controlled with a reduced number of actuators. On the other hand, both planning and control are much more difficult than in conventional holonomic systems, and require special techniques. We show first that the nonholonomy of kinematic constraints in mechanical systems is equivalent to the controllability of an associated control system, so that integrability conditions may be sought by exploiting concepts from nonlinear control theory. Basic tools for the analysis and stabilization of nonlinear control systems are reviewed and used to obtain conditions for partial or complete nonholonomy, so as to devise a classification of nonholonomic systems. Several kinematic models of nonholonomic systems are presented, including examples of wheeled mobile robots, free-floating space structures and redundant manipulators. We introduce then the dynamics of nonholonomic systems and a procedure for partial linearization of the corresponding control system via feedback. These points are illustrated by deriving the dynamical models of two previously considered systems. Finally, we discuss some general issues of the control problem for nonholonomic systems and present open-loop and feedback control techniques, illustrated also by numerical simulations.

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7.1 Introduction

Consider a mechanical system whose configuration can be described by a vector of *generalized coordinates* $q \in \mathcal{Q}$. The configuration space $\mathcal{Q}$ is an n-dimensional smooth manifold, locally diffeomorphic to an open subset of $I\!R^n$. Given a trajectory $q(t) \in \mathcal{Q}$, the *generalized velocity* at a configuration q is the vector $\dot{q}$ belonging to the tangent space $T_q(\mathcal{Q})$.

In many interesting cases, the system motion is subject to constraints that may arise from the structure itself of the mechanism, or from the way in which it is actuated and controlled. Various classifications of such constraints can be devised. For example, constraints may be expressed as equalities or inequalities (respectively, *bilateral* or *unilateral* constraints) and they may explicitly depend on time or not (*rheonomic* or *scleronomic* constraints).

In the discussion below, one possible—by no means exhaustive—classification is considered. In particular, we will deal only with bilateral scleronomic constraints. A treatment of nonholonomic unilateral constraints can be found, for example, in [1].

Motion restrictions that may be put in the form

$$h_i(q) = 0, \qquad i = 1, \ldots, k < n, \tag{7.1}$$

are called *holonomic*[1] *constraints*. For convenience, the functions $h_i : \mathcal{Q} \mapsto I\!R$ are assumed to be smooth and independent. A system whose constraints, if any, are all holonomic, is called a *holonomic system*.

The effect of constraints like (7.1) is to confine the attainable system configurations to an $(n{-}k)$-dimensional smooth submanifold of $\mathcal{Q}$. A straightforward way to deal with holonomic constraints is provided by the Implicit Function theorem, that allows one to solve eq. (7.1) in terms of $n-k$ generalized coordinates, so as to eliminate the remaining k variables from the problem. In general, this procedure has only local validity and may introduce algebraic singularities. More conveniently, the configuration of the system can be described by properly defining $n-k$ *new* coordinates on the restricted submanifold, that characterize the actual *degrees of freedom* of the system. The study of the motion of this reduced system is completely equivalent to the original one. For simulation purposes, an alternative approach is to keep the constraint equations as such and use a Differential-Algebraic Equation (DAE) system solver.

Holonomic constraints are typically introduced by mechanical interconnections between the various bodies of the system. For example, prismatic and revolute joints commonly used in robotic manipulators are a source of such constraints. If we consider a fixed-base kinematic chain composed of n rigid links connected by elementary joints, the composite configuration space of the system is $(I\!R^3 \times SO(3))^n$. Since each joint imposes five constraints, the number of degrees of freedom is $6n - 5n = n$. We mention that it is possible to design robotic manipulators with joints that are not holonomic, as proposed in [2].

[1]'Holonomic' comes from the greek word ὅλος that means 'whole', 'integer'.

System constraints whose expression involves generalized coordinates and velocities in the form

$$a_i(q, \dot{q}) = 0, \qquad i = 1, \ldots, k < n,$$

are referred to as *kinematic constraints.* These will limit the admissible motions of the system by restricting the set of generalized velocities that can be attained at a given configuration. In mechanics, such constraints are usually encountered in the *Pfaffian form*

$$a_i^T(q)\dot{q} = 0, \quad i = 1, \ldots, k < n, \qquad \text{or} \qquad A^T(q)\dot{q} = 0, \tag{7.2}$$

that is, linear in the generalized velocities. The vectors $a_i : \mathcal{Q} \mapsto I\!R^n$ are assumed to be smooth and linearly independent.

Of course, the holonomic constraints (7.1) imply the existence of kinematic constraints expressed as

$$\frac{\partial h_i}{\partial q}\dot{q} = 0, \qquad i = 1, \ldots, k.$$

However, the converse is not necessarily true: it may happen that the kinematic constraints (7.2) are not integrable, i.e., cannot be put in the form (7.1). In this case, the constraints and the mechanical system itself are called *nonholonomic.*

The presence of nonholonomic constraints limits the system mobility in a completely different way if compared to holonomic constraints. To illustrate this point, consider a single Pfaffian constraint

$$a^T(q)\dot{q} = 0. \tag{7.3}$$

If constraint (7.3) is holonomic, then it can be integrated as

$$h(q) = c,$$

where $\partial h / \partial q = a^T(q)$ and c is an integration constant. In this case, the system motion is confined to a particular *level surface* of h, depending on the initial condition through the value of $c = h(q_0)$.

Assume instead that constraint (7.3) is nonholonomic. Then, even if the *instantaneous* mobility of the system is restricted to an $(n-1)$-dimensional space, it is still possible to reach any configuration in $\mathcal{Q}$. Correspondingly, the number of degrees of freedom is reduced to $n-1$, but the number of generalized coordinates cannot be reduced. This conclusion is general: for a mechanical system with n generalized coordinates and k nonholonomic constraints, although the generalized velocities at each point are confined to an $(n-k)$-dimensional subspace, *accessibility of the whole configuration space is preserved.*

The following is a classical instance of nonholonomic system.

Example. Consider a disk that rolls without slipping on a plane, as shown in Fig. 7.1, while keeping its midplane vertical. Its configuration is completely described by four variables: the position coordinates (x, y) of the point of contact with the ground in a fixed frame, the angle θ characterizing the disk orientation with respect to the x axis, and the angle ϕ between a chosen radial axis on the disk and the vertical axis.

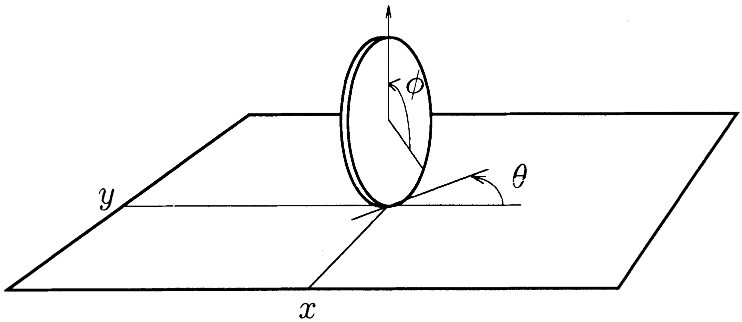

Figure 7.1: Generalized coordinates of a rolling disk

Due to the no-slipping constraint, the system generalized velocities cannot assume arbitrary values. In particular, denoting by ρ the disk radius, they must satisfy the constraints

$$\dot{x} - \rho \cos\theta \, \dot{\phi} = 0 \tag{7.4}$$
$$\dot{y} - \rho \sin\theta \, \dot{\phi} = 0, \tag{7.5}$$

thereby expressing the condition that the velocity of the disk center lies in the midplane of the disk.

The above kinematic constraints are not integrable and, as a consequence, there is no limitation on the configurations that may be attained by the disk. In fact, the disk may be driven from a configuration $(x_1, y_1, \theta_1, \phi_1)$ to a configuration $(x_2, y_2, \theta_2, \phi_2)$ through the following motion sequence:

1. Roll the disk so to bring the contact point from (x_1, y_1) to (x_2, y_2) along any curve of length $\rho \cdot (\phi_2 - \phi_1 + 2k\pi)$, where k is an arbitrary nonnegative integer.

2. Rotate the disk around the vertical axis from θ_1 to θ_2.

This confirms that the two constraints imposed on the motion of the rolling disk are nonholonomic. ∎

It should be clear from the discussion so far that, in the presence of kinematic constraints, it is essential to decide about their integrability. We shall address this problem in the following section.

7.2 Integrability of Constraints

Let us start by considering the case of a *single* Pfaffian constraint

$$a^T(q)\dot{q} = \sum_{j=1}^{n} a_j(q)\dot{q}_j = 0. \tag{7.6}$$

For constraint (7.6) to be integrable, there must exist a (nonvanishing) *integrating factor* $\gamma(q)$ such that

$$\gamma(q)a_j(q) = \frac{\partial h}{\partial q_j}(q), \qquad j = 1, \ldots, n, \tag{7.7}$$

for some function $h(q)$. The converse also holds: if there exists a $\gamma(q)$ such that $\gamma(q)a(q)$ is the gradient vector of some function $h(q)$, then constraint (7.6) is integrable. By using Schwarz's theorem, the integrability condition (7.7) may be replaced by

$$\frac{\partial(\gamma a_k)}{\partial q_j} = \frac{\partial(\gamma a_j)}{\partial q_k}, \qquad j, k = 1, \ldots, n, \tag{7.8}$$

which do not involve the unknown function $h(q)$. Note that conditions (7.8) imply that *linear* kinematic constraints (i.e., with constant a_j's) are always integrable.

Example. For the following differential constraint in $I\!R^3$

$$\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 = 0,$$

the integrability conditions (7.8) become

$$\begin{aligned} \frac{\partial \gamma}{\partial q_2} &= \gamma + \frac{\partial \gamma}{\partial q_1} q_1 \\ \frac{\partial \gamma}{\partial q_3} &= \frac{\partial \gamma}{\partial q_1} \\ \frac{\partial \gamma}{\partial q_3} q_1 &= \frac{\partial \gamma}{\partial q_2}. \end{aligned}$$

By substituting the second and third equations into the first one, it is possible to see that the only solution is $\gamma \equiv 0$. Hence, the constraint is not integrable. ∎

When dealing with multiple kinematic constraints in the form (7.2), the nonholonomy of each constraint considered separately is not sufficient to infer that the whole set of constraints is nonholonomic. In fact, it may still happen that $p \leq k$ independent linear combinations of the constraints

$$\sum_{i=1}^{k} \gamma_{ji}(q) a_i^T(q)\dot{q}, \qquad j = 1, \ldots, p,$$

are integrable. In this case, there exist p independent functions $h_j(q)$ such that

$$\text{span}\left\{\frac{\partial h_1}{\partial q}(q), \ldots, \frac{\partial h_p}{\partial q}(q)\right\} \subset \text{span}\left\{a_1^T(q), \ldots, a_k^T(q)\right\}, \quad \forall q \in \mathcal{Q},$$

and the system configurations are restricted to the $(n-p)$-dimensional manifold identified by the level surfaces of the h_j's, i.e.,

$$\{q \in \mathcal{Q} \colon h_1(q) = c_1, \ldots, h_p(q) = c_p\},$$

on which motion is started.

In the particular case $p = k$, the set of differential constraints is completely equivalent to a set of holonomic constraints; hence, it is itself holonomic.

Example. The two constraints

$$\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 = 0$$

and

$$\dot{q}_1 + \dot{q}_2 + q_1\dot{q}_3 = 0$$

are not integrable separately (in particular, the first is the nonholonomic constraint of the previous example). However, when taken together, by simple manipulations they can be put in the form

$$\begin{aligned} \dot{q}_1 + (q_1 + 1)\dot{q}_2 &= 0 \\ \dot{q}_1 + (q_1 + 1)\dot{q}_3 &= 0, \end{aligned}$$

that is trivially integrable, giving

$$\begin{aligned} q_2 + \log(q_1 + 1) &= c_1 \\ q_2 - q_3 &= c_2, \end{aligned}$$

where the c_i's are constants. ∎

If $1 \leq p < k$, the constraint set (7.2) is nonholonomic according to the foregoing definition. However, to emphasize that a subset of set (7.2) is integrable, we will refer to this situation as *partial nonholonomy*, as opposed to *complete nonholonomy* ($p = 0$).

Example. Consider the following three constraints in $I\!\!R^6$

$$A_1^T(q) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + A_2^T \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} = 0, \tag{7.9}$$

with

$$A_1(q) = \begin{bmatrix} \frac{\sqrt{3}}{2}\cos q_3 - \frac{1}{2}\sin q_3 & \sin q_3 & -\frac{1}{2}\sin q_3 - \frac{\sqrt{3}}{2}\cos q_3 \\ \frac{1}{2}\cos q_3 + \frac{\sqrt{3}}{2}\sin q_3 & -\cos q_3 & \frac{1}{2}\cos q_3 - \frac{\sqrt{3}}{2}\sin q_3 \\ \ell & \ell & \ell \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}.$$

This set of constraints is not holonomic, but it is partially integrable. In fact, by adding them up, we obtain

$$\dot{q}_3 + \frac{r}{3\ell}(\dot{q}_4 + \dot{q}_5 + \dot{q}_6) = 0,$$

that can be integrated as

$$q_3 = -\frac{r}{3\ell}(q_4 + q_5 + q_6) + c.$$

The set of constraints (7.9) characterizes the kinematics of an *omnidirectional* symmetric three-wheeled mobile robot [3]. In particular, q_1 and q_2 are the Cartesian coordinates of the robot center with respect to a fixed frame, q_3 is the orientation of the vehicle, while q_4, q_5, and q_6 measure the rotation angle of the three wheels. Also, r is the wheel radius and ℓ is the distance from the center of the robot to the center of each wheel. The partial integrability of the constraints indicates that the vehicle orientation is a function of the rotation angles of the wheels, and thus, may be eliminated from the problem formulation. ∎

At this stage, the question of integrability of multiple kinematic constraints is not obvious. However, integrability criteria can be obtained on the basis of a different viewpoint, that is introduced in the remainder of this section.

The set of k Pfaffian constraints (7.2) defines, at each configuration q, the admissible generalized velocities as those contained in the $(n-k)$-dimensional nullspace of matrix $A^T(q)$. Equivalently, if $\{g_1(q), \ldots, g_{n-k}(q)\}$ is a basis for this space, all the feasible trajectories for the mechanical system are obtained as solutions of

$$\dot{q} = \sum_{j=1}^{m} g_j(q)u_j = G(q)u, \qquad m = n - k, \tag{7.10}$$

for arbitrary $u(t)$. This may be regarded as a *nonlinear control system* with state vector $q \in I\!R^n$ and control input $u \in I\!R^m$. In particular, system (7.10) is *driftless*, namely $\dot{q} = 0$, when no input is applied. Moreover, from a mechanical point of view, it is *underactuated*, since there are less inputs than generalized coordinates $(m < n)$.

The choice of $G(q)$ in eq. (7.10) is not unique and, accordingly, the components of u will assume different meanings. In general, one can choose the columns g_j so that the corresponding u_j has a direct physical interpretation (see Section 7.5). Furthermore, the input vector u may have no relationship with the true *controls* of the mechanical system, that are, in general, forces or torques, depending on the actuation. For this reason, eq. (7.10) is referred to as the *kinematic model* of the constrained system.

To decide about the holonomy/nonholonomy of a set of kinematic constraints, it is convenient to study the *controllability* properties of the associated kinematic model. In fact:

1. If eq. (7.10) is controllable, given two arbitrary points q_1 and q_2 in $\mathcal{Q}$, there exists a choice of $u(t)$ that steers the system from q_1 to q_2. Equivalently, there exists a trajectory $q(t)$ from q_1 to q_2 that satisfies the kinematic constraints (7.2). As a consequence, the latter do not restrict the accessibility of the whole configuration space $\mathcal{Q}$, and thus, they are completely nonholonomic.

2. If eq. (7.10) is not controllable, the above reasoning does not hold and the kinematic constraints imply a loss of accessibility of the system configuration space. Hence, the underlying constraints are partially or completely integrable, depending on the dimension ν ($< n$) of the accessible region. In particular:

   2a. If $\nu > m$, the loss of accessibility is not maximal, meaning that eq. (7.2) is only partially integrable. According to our definition, the system is partially nonholonomic.

   2b. If $\nu = m$, the accessibility loss is maximal, and the whole set (7.2) is integrable. Hence, the system is holonomic.

We have already adopted this viewpoint in establishing the nonholonomy of the rolling disk in Section 7.1. In particular, the controllability of the corresponding kinematic system was proved *constructively*, i.e., by exhibiting a reconfiguration strategy. However, to effectively make use of this approach, it is necessary to have practical controllability conditions to verify for the nonlinear control system (7.10).

For this purpose, we shall review tools from control theory based on differential geometry. These tools apply to general nonlinear control systems

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j.$$

As we shall see later, the presence of the drift term $f(x)$ characterizes kinematic constraints in a more general form than eq. (7.2), as well as the dynamical model of nonholonomic systems.

7.3 Tools from Nonlinear Control Theory

The analysis of nonlinear control systems requires many concepts from differential geometry. To this end, the introductory definitions and a fundamental result (Frobenius' theorem) are briefly reviewed. Then, we recall different kinds of nonlinear controllability and their relative conditions, that will be used in the next section to characterize nonholonomic constraints. Finally, the basic elements of the stabilization problem for nonlinear systems are introduced. For a complete treatment, the reader is referred to [4] and [5], and to the references therein.

7.3.1 Differential Geometry

For simplicity, we will work with vectors $x \in I\!R^n$, and denote the tangent space of $I\!R^n$ at x by $T_x(I\!R^n)$. A smooth *vector field* $g : I\!R^n \mapsto T_x(I\!R^n)$ is a smooth mapping assigning to each point $x \in I\!R^n$ a tangent vector $g(x) \in T_x(I\!R^n)$. If $g(x)$ is used to define a differential equation as

$$\dot{x} = g(x),$$

the *flow* $\phi_t^g(x)$ of the vector field g is the mapping that associates to each point x the solution at time t of the differential equation evolving from x at time 0, or

$$\frac{d}{dt}\phi_t^g(x) = g(\phi_t^g(x)).$$

It is possible to show that the family of mappings $\{\phi_t^g\}$ is a one-parameter (viz. t) group of local diffeomorphisms under the composition operation. Hence

$$\phi_{t_1}^g \circ \phi_{t_2}^g = \phi_{t_1+t_2}^g.$$

For example, in linear systems it is $g(x) = Ax$ and the flow is the linear operator $\phi_t^g = e^{At}$.

Given two smooth vector fields g_1 and g_2, we note that the composition of their flows is generally *non-commutative*, that is

$$\phi_t^{g_1} \circ \phi_s^{g_2} \neq \phi_s^{g_2} \circ \phi_t^{g_1}.$$

Moreover, the new vector field $[g_1, g_2]$ whose coordinate-dependent expression is

$$[g_1, g_2](x) = \frac{\partial g_2}{\partial x} g_1(x) - \frac{\partial g_1}{\partial x} g_2(x)$$

is called the *Lie bracket* of g_1 and g_2. Two vector fields g_1 and g_2 are said to *commute* if $[g_1, g_2] = 0$. To appreciate the relevance of the Lie bracket operation, consider the differential equation

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{7.11}$$

associated with the two vector fields g_1 and g_2. If the two inputs u_1 and u_2 are never active at the same instant, the solution of eq. (7.11) is obtained by composing the flows relative to g_1 and g_2. In particular, consider the input sequence

$$u(t) = \begin{cases} u_1(t) = +1, u_2(t) = 0, & t \in [0, \varepsilon), \\ u_1(t) = 0, u_2(t) = +1, & t \in [\varepsilon, 2\varepsilon), \\ u_1(t) = -1, u_2(t) = 0, & t \in [2\varepsilon, 3\varepsilon), \\ u_1(t) = 0, u_2(t) = -1, & t \in [3\varepsilon, 4\varepsilon), \end{cases} \tag{7.12}$$

where ε is an infinitesimal interval of time. The solution of the differential equation at time 4ε is obtained by following the flow of g_1, then g_2, then $-g_1$, and finally $-g_2$ (see Fig. 7.2). By computing $x(\varepsilon)$ as a series expansion about $x_0 = x(0)$ along g_1, $x(2\varepsilon)$ as a series expansion about $x(\varepsilon)$ along g_2, and so on, one obtains

$$\begin{aligned} x(4\varepsilon) &= \phi_\varepsilon^{-g_2} \circ \phi_\varepsilon^{-g_1} \circ \phi_\varepsilon^{g_2} \circ \phi_\varepsilon^{g_1}(x_0) \\ &= x_0 + \varepsilon^2 \left(\frac{\partial g_2}{\partial x} g_1(x_0) - \frac{\partial g_1}{\partial x} g_2(x_0) \right) + O(\varepsilon^3), \end{aligned}$$

'a calculation which everyone should do once in his life' (R. Brockett). Note that, when g_1 and g_2 commute, no net motion is obtained as a result of the input sequence (7.12).

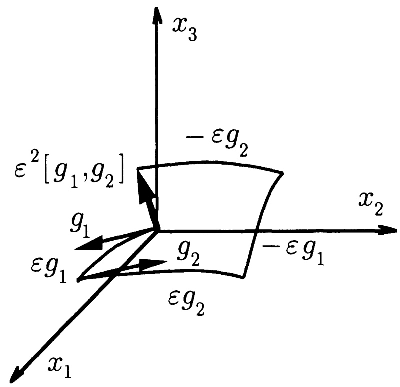

Figure 7.2: Lie bracket motion

The above computation shows that, at each point, infinitesimal motion is possible not only in the directions contained in the span of the input vector fields, but also in the directions of their Lie brackets. This is peculiar to the nonlinearity of the input vector fields in the *driftless* control system (7.11). Similarly, one can prove that, by using more complicated input sequences, it is possible to obtain motion in the direction of higher-order brackets, such as $[g_1, [g_1, g_2]]$ (see [6]).

Similar constructive procedures for characterizing admissible motions can be devised also for control systems with a drift vector field f, with the bracket operations involving a mix of f and g_i's.

Example. For a linear single-input system

$$\dot{x} = Ax + bu,$$

with drift $f(x) = Ax$ and input vector field $g(x) = b$, motion can be obtained in the direction of the (repeated) Lie brackets

$$\begin{aligned} -[f, g] &= Ab \\ [f, [f, g]] &= A^2 b \\ -[f, [f, [f, g]]] &= A^3 b \\ &\vdots \end{aligned}$$

a well-known result. ∎

The *Lie derivative* of $\alpha : I\!\!R^n \mapsto I\!\!R$ along g is defined as

$$L_g \alpha(x) = \frac{\partial \alpha}{\partial x} g(x).$$

The most important properties of Lie brackets, which are useful in computations, are

$$
\begin{array}{ll}
[f,g] = -[g,f] & \text{(skew-symmetry)} \\
[f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0 & \text{(Jacobi identity)} \\
[\alpha f, \beta g] = \alpha\beta[f,g] + \alpha(L_f\beta)g - \beta(L_g\alpha)f & \text{(chain rule)}
\end{array}
$$

with $\alpha, \beta \colon I\!R^n \mapsto I\!R$. The vector space $\mathcal{V}(I\!R^n)$ of smooth vector fields on $I\!R^n$, equipped with the Lie bracket as a product, is called a *Lie algebra.*

The smooth *distribution* Δ associated with the m smooth vector fields $\{g_1, \ldots, g_m\}$ is the map that assigns to each point $x \in I\!R^n$ a linear subspace of its tangent space, i.e.,

$$\Delta(x) = \text{span}\{g_1(x), \ldots, g_m(x)\} \subset T_x(I\!R^n).$$

Hereafter, we shall use the shorthand notation

$$\Delta = \text{span}\{g_1, \ldots, g_m\}.$$

Moreover, Δ is said to be *nonsingular* if $\dim \Delta(x) = r$, constant for all x. In this case, r is called the *dimension* of the distribution. Moreover, Δ is *involutive* if it is closed under the Lie bracket operation:

$$[g_i, g_j] \in \Delta, \quad \forall g_i, g_j \in \Delta.$$

The *involutive closure* $\bar{\Delta}$ of a distribution Δ is its closure under the Lie bracket operation. Hence, Δ is involutive if and only if $\bar{\Delta} = \Delta$.

A nonsingular distribution Δ on $I\!R^n$ of dimension k is *completely integrable* if there exist $n-k$ independent functions h_i such that

$$L_{g_j} h_i(x) = \frac{\partial h_i}{\partial x} g_j(x) = 0, \quad \forall x \in I\!R^n, \forall g_j \in \Delta, \quad i = 1, \ldots, n-k. \tag{7.13}$$

In this case, each hypersurface defined by the level surfaces

$$\{x \in I\!R^n \colon h_1(x) = c_1, \ldots, h_{n-k}(x) = c_{n-k}\}$$

is called an *integral manifold* of Δ. Equation (7.13) indicates that $\Delta(x)$ coincides with the tangent space to its integral manifold at x. As $c_1, \ldots, c_{n-k}$ vary, the level surfaces of the h_i's yield a *foliation* of $I\!R^n$. The surface obtained for a particular choice of the c_i's is called a *leaf.*

The following result gives a necessary and sufficient condition for the complete integrability of a distribution.

Theorem 1 (Frobenius). *A nonsingular distribution is completely integrable if and only if it is involutive.*

The proof of this theorem, for which we refer to [4], is particularly interesting in the sufficiency part, that is constructive in nature.

Example. Consider $\Delta = \text{span}\{g_1, g_2\}$, with

$$g_1(x) = \begin{bmatrix} x_2 \\ 0 \\ 1 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} x_3 \\ 1 \\ 0 \end{bmatrix}.$$

The dimension of Δ is 2 at any point $x \in I\!R^3$. A simple computation shows that $[g_1, g_2] = 0$, so that Δ is involutive and hence integrable. Indeed, it induces a foliation of $I\!R^3$ in the form

$$x_1 - x_2 x_3 = c,$$

with $c \in I\!R$. ■

Note that the distribution generated by a single vector field is always involutive and, therefore, integrable.

7.3.2 Controllability

Consider a nonlinear control system in the form

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j, \tag{7.14}$$

which is called *affine* in the inputs u_j. The state vector x belongs to $I\!R^n$, and each component of the control input $u \in I\!R^m$ takes values in the class of piecewise-constant functions $\mathcal{U}$ over time. For convenience, the drift vector field f is assumed to be smooth, together with the input vector fields g_j's. Denote by $x(t, 0, x_0, u)$ the unique solution of eq. (7.14) at time $t \geq 0$, corresponding to given input function $u(\cdot)$ and initial condition $x(0) = x_0$.

The control system (7.14) is *controllable* if, for any choice of x_1, $x_2 \in I\!R^n$, there exists a finite time T and an input u: $[0, T] \rightarrow \mathcal{U}$ such that $x(T, 0, x_1, u) = x_2$. Unfortunately, general criteria for verifying this natural form of controllability do no exist. For this reason, other structural characterizations of system (7.14) have been proposed, that are related to the previous definition.

Given a neighborhood V of x_0, denote by $\mathcal{R}^V(x_0, \tau)$ the set of states ξ for which there exists u: $[0, \tau] \rightarrow \mathcal{U}$ such that $x(\tau, 0, x_0, u) = \xi$ and $x(t, 0, x_0, u) \in V$ for $t \leq \tau$. In words, $\mathcal{R}^V(x_0, \tau)$ is the set of states *reachable at time* τ from x_0 with trajectories contained in V. Also, define

$$\mathcal{R}^V_T(x_0) = \bigcup_{\tau \leq T} \mathcal{R}^V(x_o, \tau),$$

which is the set of states *reachable within time* T from x_0 with trajectories contained in the neighborhood V.

The control system (7.14) is called

1. *locally accessible from* x_0 if, for all neighborhoods V of x_0 and all T, $\mathcal{R}_T^V(x_0)$ contains a non-empty open set Ω;

2. *small-time locally controllable from* x_0 if, for all neighborhoods V of x_0 and all T, $\mathcal{R}_T^V(x_0)$ contains a non-empty neighborhood of x_0.

To recognize the difference between these two concepts, observe the following

Example. Consider the control system in $I\!R^2$

$$\dot{x} = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

For any initial condition, the first state component x_1 may only increase while the second can move in any direction. Hence, $\mathcal{R}_T^V(x_0)$ contains a non-empty open set Ω, but does not contain any neighborhood of $x_0 = (x_{01}, x_{02})$. As a consequence, the control system is locally accessible from all points, but is not small-time locally controllable nor controllable. ∎

Note that:

- The previous definitions are local in nature. They may be globalized by saying that system (7.14) is *locally accessible*, or *small-time locally controllable*, if it is such for any x_0 in $I\!R^n$.

- Small-time local controllability implies local accessibility as well as controllability, while local accessibility does not imply controllability in general, as shown by the previous example. However, *if no drift vector is present*, then local accessibility implies controllability.

The interest of the concepts of local accessibility and small-time local controllability comes from the existence of algebraic tests characterizing such properties.

Define the *accessibility algebra* $\mathcal{C}$ of the control system (7.14) as the smallest subalgebra of $\mathcal{V}(I\!R^n)$ that contains $f, g_1, \ldots, g_m$. Note that, by definition, all the (repeated) Lie brackets of these vector fields also belong to $\mathcal{C}$. The *accessibility distribution* $\Delta_\mathcal{C}$ of system (7.14) is defined as

$$\Delta_\mathcal{C} = \text{span}\{v | v \in \mathcal{C}\},$$

i.e., the involutive closure of the distribution associated with $f, g_1, \ldots, g_m$.

The computation of $\Delta_\mathcal{C}$ may be organized as an iterative procedure:

$$\Delta_\mathcal{C} = \text{span}\left\{v | v \in \Delta_i, \forall i \geq 1\right\},$$

with

$$\begin{aligned} \Delta_1 &= \text{span}\{f, g_1, \ldots, g_m\} \\ \Delta_i &= \Delta_{i-1} + \text{span}\{[g, v] | \, g \in \Delta_1, v \in \Delta_{i-1}\}, \quad i \geq 2. \end{aligned}$$

The above procedure stops after κ steps, where κ is the smallest integer such that $\Delta_{\kappa+1} = \Delta_\kappa = \Delta_C$. Since $\dim \Delta_C \leq n$ necessarily, it follows that one stops after at most $n - m$ steps.

The accessibility distribution may be used for verifying local accessibility as indicated by a basic result, namely,

Theorem 2 (Chow). *If the* accessibility rank condition

$$\dim \Delta_C(x_0) = n$$

holds, then the control system (7.14) is locally accessible from x_0. If the accessibility rank condition holds for all $x \in I\!R^n$, the system is locally accessible. Conversely, if system (7.14) is locally accessible, then $\dim \Delta_C(x) = n$ *holds in an open and dense subset of $I\!R^n$.*

In particular, if the vector fields of the system are *analytic*, the accessibility rank condition is necessary and sufficient for local accessibility. If Chow's theorem is applied to a driftless control system

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i, \tag{7.15}$$

it provides a sufficient condition for controllability. The same is true for systems with drift, if $f(x)$ is such that

$$f(x) \in \operatorname{span}\{g_1(x), \ldots, g_m(x)\}, \qquad \forall x \in I\!R^n.$$

Moreover, if system (7.15) is controllable, then its *dynamic extension*

$$\dot{x} = \sum_{i=1}^{m} g_i(x) v_i \tag{7.16}$$

$$\dot{v}_i = u_i, \qquad i = 1, \ldots, m, \tag{7.17}$$

is also controllable [7]. The converse is trivially true.

As for small-time local controllability, only a *sufficient* condition exists, based on the following concept [8]: Consider a vector field $v \in \Delta_C$ obtained as a (repeated) Lie bracket of the system vector fields, and denote by $\delta^0(v), \delta^1(v), \ldots, \delta^m(v)$ the number of occurrences of $g_0 = f, g_1, \ldots, g_m$, respectively, in v. Define the *degree* of the bracket v as $\sum_{i=0}^{m} \delta^i(v)$.

Theorem 3 (Sussmann). *Assume that* $\dim \Delta_C(x_0) = n$ *and that, for any bracket $v \in \Delta_C$ such that $\delta^0(v)$ is odd, and $\delta^1(v), \ldots, \delta^m(v)$ are even, v may be written as a linear combination of brackets of lower degree. Then, system (7.14) is small-time locally controllable from x_0.*

Some remarks are offered as a conclusion.

- Assume that the accessibility distribution $\Delta_{\mathcal{C}}$ has constant dimension $\nu < n$ everywhere. Then, on the basis of Frobenius' theorem it is possible to show that, for any x_0, T and V, $\mathcal{R}_T^V(x_0)$ is contained in a ν-dimensional integral manifold of $\Delta_{\mathcal{C}}$; besides, $\mathcal{R}_T^V(x_0)$ contains itself a non-empty set of dimension ν (see [5, Prop. 3.12, p. 81]). Like in Chow's theorem, to reverse this statement it is necessary that $\dim \Delta_{\mathcal{C}}(x) = \nu$ in an open and dense subset of $I\!R^n$.

- The term 'local' may be discarded from the foregoing definitions if the system is *analytic*, since the requirement that the trajectories stay in a neighborhood of x_0 can be omitted.

- In the linear case
$$\dot{x} = Ax + \sum_{j=1}^{m} b_j u_j = Ax + Bu$$
all the previous definitions are global and collapse into the classical linear controllability concept. In particular, the accessibility rank condition at $x_0 = 0$ corresponds to
$$\text{rank}\,[\,B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B\,] = n, \tag{7.18}$$
the well-known Kalman necessary and sufficient condition for controllability.

7.3.3 Stabilizability

The *stabilization* problem for the control system (7.14) consists in finding a *feedback* control law of the form

$$u = \alpha(x) + \beta(x)v, \quad u, v \in I\!R^m,$$

so as to make a closed-loop equilibrium point x_e or an admissible trajectory $x_e(t)$ *asymptotically stable*. The adoption of feedback control laws is particularly suited for motion control, to counteract the presence of disturbances, initial errors or modeling inaccuracies. For point stabilization, x_e is typically an equilibrium point for the open-loop system, i.e., $f(x_e) = 0$. Indeed, for the driftless control system (7.15), any point is an open-loop equilibrium point. As for the tracking case, it is necessary to ensure that the trajectories to be stabilized are admissible for the system. This is of particular importance in the case of nonholonomic systems, for which the kinematic constraints preclude the possibility of following a generic trajectory. In the discussion below, we shall refer only to the point-stabilization case. A detailed presentation of stabilization results can be found in [9].

In linear systems, controllability implies asymptotic (actually, exponential) stabilizability by *smooth* state feedback. In fact, if condition (7.18) is satisfied, there exist choices of K such that the linear control

$$u = K(x_e - x)$$

makes x_e asymptotically stable.

For nonlinear systems, this implication does not hold. Local results may be obtained looking at the *approximate linearization* of system (7.14) in a neighborhood of x_e

$$\dot{\delta x} = \frac{\partial f}{\partial x}(x_e)\delta x + \sum_{j=1}^{m} g_j(x_e)u_j = A_e\delta x + B_e\delta u, \tag{7.19}$$

with $\delta x = x - x_e$ and $\delta u = u - u_e = u$. In particular, if system (7.19) is controllable, the nonlinear system (7.14) may be locally stabilized at x_e by a smooth feedback. This condition is sufficient but far from being necessary, as illustrated by

$$\dot{x} = x^2 u.$$

Although its linearization at $x = 0$ is identically zero, this system may be smoothly stabilized by setting $u = -x$.

On the other hand, if system (7.19) has unstable uncontrollable eigenvalues, then smooth (actually, even C^1) stabilizability is not possible, not even locally. As usual, the critical case is encountered when the approximate linearization has uncontrollable eigenvalues with zero real part. In this case, nothing can be concluded on the basis of the linear approximation, except that exponential stabilization cannot be achieved (see, for example,[10, Prop. 5.3, p. 110]).

However, in some cases one can use *necessary* conditions for the existence of a C^1 stabilizing feedback to gain insight into the critical case. The following topological result [11] is particularly useful to this aim.

Theorem 4 (Brockett). *If the system*

$$\dot{x} = \varphi(x, u)$$

admits a C^1 feedback $u = u(x)$ that makes x_e asymptotically stable, then the image of the map

$$\varphi : I\!R^n \times \mathcal{U} \mapsto I\!R^n$$

contains some neighborhood of x_e.

Example. We want to investigate the stabilizability of the equilibrium point $x = 0$ for the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2. \end{aligned}$$

By applying Chow's theorem, it is easy to see that the system is controllable. As for smooth stabilizability, nothing can be concluded by looking at the linearization in a neighborhood of $x = 0$, due to the presence of one uncontrollable zero eigenvalue. However, by noticing that no point of the form

$$\begin{pmatrix} 0 \\ 0 \\ \varepsilon \end{pmatrix}, \qquad \varepsilon \neq 0,$$

is in the image of φ, Brockett's theorem allows to infer that the stabilization of $x = 0$ by C^1 feedback is not possible. ■

We call the reader's attention to the points below:

- When applied to driftless control systems (7.15) such that the vector fields g_j are linearly independent at x_e (as in the previous example), Brockett's theorem implies $m = n$ as a *necessary and sufficient* condition for C^1-stabilizability. However, if the dimension of the distribution $\Delta = \text{span}\{g_1, \ldots, g_m\}$ drops at x_e, such condition is no more necessary.

- If system (7.15) cannot be stabilized by C^1 feedback, the same negative result holds for its dynamic extension (7.16)–(7.17). In other words, the topological obstruction to C^1-stabilizability expressed by Theorem 4 is preserved under dynamic extension [7].

- Brockett's theorem does not apply to *time-varying* feedback laws $u = u(x, t)$.

As a conclusion, underactuated ($m < n$) systems without drift that satisfy the independence assumption on the g_j's cannot be stabilized via continuously differentiable static feedback laws. This has consequences on the design of feedback controllers for nonholonomic systems, as we shall see in Section 7.8.

7.4 Classification of Nonholonomic Systems

On the basis of the controllability results recalled in the last section, we shall now give conditions for the integrability of the set of kinematic constraints (7.2), which is repeated below for convenience

$$a_i^T(q)\dot{q} = 0, \qquad i = 1, \ldots, k < n, \tag{7.20}$$

together with the associated kinematic model

$$\dot{q} = \sum_{j=1}^{m} g_j(q) u_j = G(q)u, \qquad m = n - k. \tag{7.21}$$

The accessibility distribution Δ_C of the kinematic model is the involutive closure of the nonsingular distribution $\Delta = \text{span}\{g_1, \ldots, g_m\}$. We have now

Proposition 1. *The set of k Pfaffian constraints (7.20) is holonomic if and only if its associated kinematic model (7.21) is such that*

$$\dim \Delta_{\mathcal{C}} = \dim \Delta = m, \tag{7.22}$$

i.e., the distribution Δ is involutive.

Proof (sketch of). We make use of the condition given in the first remark at the end of Section 7.3.2. Assume that $\dim \Delta_{\mathcal{C}} = m$. Then, the set of reachable states from any point of the configuration space is contained in an m-dimensional integral manifold of $\Delta_{\mathcal{C}}$. This implies that the set of kinematic constraints is holonomic. Conversely, if constraint (7.20) is holonomic, the system motion is confined to an m-dimensional manifold. Hence, the rank of the accessibility algebra must equal m in an open and dense subset of $\mathcal{Q}$. ∎

Two remarks are in order, namely:

- The reader may verify that, in the case of a single differential constraint (7.6), condition (7.22) coincides with the integrability conditions (7.8).
- In the special case of $k = n - 1$ kinematic constraints, the associated kinematic model consists of a single vector field ($m = 1$). As pointed out in Section 7.3.1, the corresponding distribution is always involutive. Hence, the mechanical system is holonomic. In particular, this happens for a two-dimensional system subject to a scalar differential constraint, as we shall see through an example in Section 7.5.2.

Proposition 1 shows that $\dim \Delta_{\mathcal{C}} > m$ is a necessary and sufficient condition for the set of kinematic constraints (7.20) to be nonholonomic. However, we may be more precise, and distinguish between partial or complete nonholonomy.

Proposition 2. *The set of k Pfaffian constraints (7.20) contains a subset of p integrable constraints if and only if the associated kinematic model (7.21) is such that*

$$\dim \Delta_{\mathcal{C}} = n - p.$$

If $p = 0$, or

$$\dim \Delta_{\mathcal{C}} = n,$$

the system is completely nonholonomic.

Again, the proof of this result follows easily from the accessibility conditions given in Section 7.3.2. In particular, note that, if $p \geq 1$, by Frobenius' theorem there exists an $(n-p)$-dimensional integral manifold of $\Delta_{\mathcal{C}}$ on which the system motion is confined (a leaf of the corresponding foliation). In the special case $p = 0$, Chow's theorem applies.

As indicated by Proposition 2, a system subject to k kinematic constraints is completely nonholonomic if the associated accessibility distribution $\Delta_{\mathcal{C}}$ spans $\mathbb{R}^n$. To verify this condition, one must perform the iterative procedure of Section 7.3.2, which amounts to computing repeated Lie brackets of the input vector fields $g_1, \dots, g_m$ of system (7.21). The level of bracketing needed to span $I\!R^n$ is related to the complexity of the motion planning problem [12]. For this reason, we give below a classification of nonholonomic systems based on the sequence and order of Lie brackets in the corresponding accessibility algebra [13].

Let $\Delta = \text{span}\,\{g_1, \dots, g_m\}$. Define the *filtration* generated by the distribution Δ as the sequence $\{\Delta_i\}$, with

$$\begin{aligned} \Delta_1 &= \Delta \\ \Delta_i &= \Delta_{i-1} + [\Delta_1, \Delta_{i-1}], \quad i \geq 2, \end{aligned}$$

where

$$[\Delta_1, \Delta_{i-1}] = \text{span}\,\{[g_j, v] | g_j \in \Delta_1, v \in \Delta_{i-1}\}.$$

Note that, by construction, $\Delta_i \subseteq \Delta_{i+1}$. Also, from the Jacobi identity follows that $[\Delta_i, \Delta_j] \subseteq [\Delta_1, \Delta_{i+j-1}] \subseteq \Delta_{i+j}$.

A filtration is *regular* in a neighborhood V of q_0 if

$$\dim \Delta_i(q) = \dim \Delta_i(q_0), \qquad \forall q \in V.$$

For a regular filtration, if $\dim \Delta_{i+1} = \dim \Delta_i$, then Δ_i is involutive and $\Delta_{i+j} = \Delta_i$, for all $j \geq 0$. Since $\dim \Delta_1 = m$ and $\dim \Delta_i \leq n$ always, the termination condition is met after at most $n - m = k$ steps, i.e., the number of original kinematic constraints.

If the filtration generated by a distribution Δ is regular, it is possible to define the *degree of nonholonomy* of Δ as the smallest integer κ such that

$$\dim \Delta_{\kappa+1} = \dim \Delta_\kappa.$$

The foregoing reasoning implies that $\kappa \leq k + 1$. The *growth vector* $r \in \mathcal{Z}^\kappa$ is defined as

$$r_i = \dim \Delta_i, \qquad i = 1, \dots, \kappa,$$

and the *relative growth vector* $\sigma \in \mathcal{Z}^\kappa$ is

$$\sigma_i = r_i - r_{i-1}, \qquad i = 1, \dots, \kappa, \quad r_0 = 0.$$

The previous conditions for holonomy, partial nonholonomy and complete nonholonomy may be restated in terms of the above defined concepts. The set of k kinematic constraints (7.20) is:

1. Holonomic, if $\kappa = 1$ (or $\dim \Delta_\kappa = m$).
2. Nonholonomic, if $2 \leq \kappa \leq k$. In particular, the constraint set (7.20) is:
   - 2a. Partially nonholonomic, if $m + 1 \leq \dim \Delta_\kappa < n$.
   - 2b. Completely nonholonomic, if $\dim \Delta_\kappa = n$.

We conclude this section by pointing out that a similar analysis can be used for kinematic constraints in a more general form than eq. (7.20), namely

$$a_i^T(q)\dot{q} = \gamma_i, \quad i = 1, \ldots, k < n, \qquad \text{or} \qquad A^T(q)\dot{q} = \gamma, \tag{7.23}$$

with constant γ_i's. For example, this form arises from conservation of a *nonzero* angular momentum in space robots (see Section 7.5.2).

The kinematic model associated with differential constraints of the form (7.23) describes the admissible generalized velocities as

$$\dot{q} = f(q) + \sum_{j=1}^{m} g_j(q)u_j = f(q) + G(q)u, \qquad m = n - k, \tag{7.24}$$

i.e., a nonlinear control system with drift. The columns of G are again a basis for the m-dimensional nullspace of matrix $A^T(q)$, while the drift vector field can be obtained, e.g., via pseudoinversion as

$$f(q) = A^{\#}(q)\gamma = A(q)\left(A^T(q)A(q)\right)^{-1}\gamma.$$

To decide if the *generalized Pfaffian* constraints (7.23) are nonholonomic, one may use the same tools given in this section, namely analyze the dimension of the accessibility distribution Δ_C of the kinematic model (7.24). However, when dealing with the *control* of systems with generalized Pfaffian constraints, the presence of a drift term $f \notin \text{span}\{g_1, \ldots, g_m\}$ implies that accessibility is not equivalent to controllability[2]. In this case, one may use Sussmann's sufficient condition (Theorem 3, Section 7.3.2) to verify small-time local controllability.

7.5 Kinematic Modeling Examples

In this section we shall examine several kinematic models of nonholonomic mechanical systems. In particular, three different sources of nonholonomy are considered: rolling contacts without slipping, conservation of angular momentum in multibody systems, and robotic devices under special control operation.

In the first class, typical applications are:

- Wheeled mobile robots and vehicles, where the rolling contact takes place between the wheels and the ground [3, 14–18].
- Dextrous manipulation with multifingered robot hands, with the constraint arising from the rolling contact of fingertips with the objects [19–21].

[2] It is readily verified that $A^TG = 0$ implies $g_j^TA = 0$ and then $g_j^Tf = 0$, for $j = 1, \ldots, m$, so that $f(q)$ is orthogonal to any vector in $\text{span}\{g_1(q), \ldots, g_m(q)\}$.

A second situation in which nonholonomic constraints come into play is when multibody systems are allowed to float freely, i.e., without having a fixed base. The conservation of angular momentum yields then a *differential* constraint that is not integrable in general. Systems that fall into this class are

- Robotic manipulators mounted on space structures [22–28].

- Dynamically balanced hopping robots in the flying phase, mimicking the maneuvers of gymnasts or divers [29–31].

- satellites with reaction (or momentum) wheels for attitude stabilization [32, 33].

In these cases, we have expressly used the term 'differential' in place of 'kinematic' for the constraints, because conservation laws depend on the generalized inertia matrix of the system, and thus contain also dynamical parameters.

Finally, another source of nonholonomic behavior is the particular control operation adopted in some robotic structures. As illustrative examples we cite:

- Redundant robots under a particular inverse kinematics control [34].

- Underwater robotic systems where forward propulsion is allowed only in the pointing direction [35, 36].

- Robotic manipulators with one or more passive joints [37–39].

We emphasize that in this class, the nonholonomic behavior is a consequence of the available control capability or chosen actuation strategy. In fact, all these examples fall into the category of *underactuated* systems, with less control inputs than generalized coordinates. Note also that, in the last kind of system the nonholonomic constraint is always expressed at the acceleration level.

Next, we present examples of wheeled mobile robots, space robots with planar structure, and redundant robots under kinematic inversion. For each case, we derive the kinematic model and proceed with the analysis by computing their degree of nonholonomy and the associated quantities.

7.5.1 Wheeled Mobile Robots

The basic element of a wheeled mobile robot is the rolling wheel. Indeed, the rolling disk of Section 7.1 provides a model for this component. We will start analyzing it and then deal with vehicles having unicycle or car-like kinematics, possibly towing trailers.

Rolling Disk

For this system, the configuration space has dimension $n = 4$ (see Fig. 7.1). By letting $q = (x, y, \theta, \phi)$, the input vector fields of the kinematic model corresponding to eqs. (7.4)–(7.5) are computed as

$$g_1 = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

To build the filtration associated with $\Delta = \text{span}\,\{g_1, g_2\}$, compute

$$g_3 = [g_1, g_2] = \begin{bmatrix} \rho\sin\theta \\ -\rho\cos\theta \\ 0 \\ 0 \end{bmatrix}, \qquad g_4 = [g_2, g_3] = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 0 \end{bmatrix}.$$

It is apparent that $\dim\Delta_1 = 2$, $\dim\Delta_2 = 3$ and $\dim\Delta_3 = \dim\Delta_{\mathcal{C}} = 4$. Thus, the rolling disk is completely nonholonomic with degree of nonholonomy $\kappa = 3$, growth vector $r = (2, 3, 4)$, and relative growth vector $\sigma = (2, 1, 1)$.

Unicycle

Many types of wheeled mobile robots with multiple wheels have a kinematic model *equivalent* to that of a unicycle, whose configuration is described by $q = (x, y, \theta)$, where (x, y) are the Cartesian coordinates of the ideal contact point and θ is the orientation of the vehicle with respect to the x axis (see Fig. 7.3). Real-world examples include the commercial robots Nomad 200 of Nomadic Technologies and TRC Labmate, as well as the research prototype Hilare developed at LAAS [16].

The kinematic rolling constraint is expressed as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \dot{q} = 0, \tag{7.25}$$

which imposes a zero lateral velocity for the vehicle. The nullspace of the constraint matrix is spanned by the columns of

$$G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}.$$

All the admissible generalized velocities are obtained as linear combinations of the two columns g_1 and g_2 of G. In particular, denoting by u_1 the *driving* velocity and by u_2 the *steering* velocity input, the following kinematic control system is obtained

$$\dot{q} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2. \tag{7.26}$$

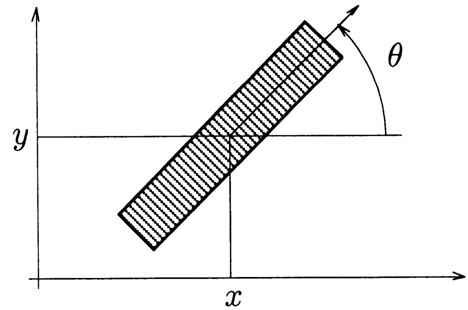

Figure 7.3: Unicycle

To show that the unicycle is subject to a nonholonomic constraint, we compute the Lie bracket

$$[g_1, g_2] = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix},$$

which does not belong to span $\{g_1, g_2\}$. As a consequence, the accessibility distribution Δ_C has dimension 3, and the system is nonholonomic. In particular, one has $\dim \Delta_1 = 2$ and $\dim \Delta_2 = 3$. Hence, the unicycle has degree of nonholonomy $\kappa = 2$, growth vector $r = (2,3)$ and relative growth vector $\sigma = (2,1)$.

Car-Like Robot

Consider a robot having the same kinematic model of an automobile, as shown in Fig. 7.4. For simplicity, we assume that the two wheels on each axis (front and rear) collapse into a single wheel located at the midpoint of the axis (*bicycle* model). The front wheel can be steered while the rear wheel orientation is fixed.

The generalized coordinates are $q = (x, y, \theta, \phi)$, where (x, y) are the Cartesian coordinates of the rear-axle midpoint, θ measures the orientation of the car body with respect to the x axis, and ϕ is the steering angle.

The system is subject to two nonholonomic constraints, one for each wheel:

$$\begin{aligned} \dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) &= 0 \\ \dot{x} \sin\theta - \dot{y} \cos\theta &= 0, \end{aligned}$$

(x_f, y_f) being the position of the front-axle midpoint. By using the rigid-body constraint

$$\begin{aligned} x_f &= x + \ell \cos\theta \\ y_f &= y + \ell \sin\theta, \end{aligned}$$

where ℓ is the distance between the axles, the first kinematic constraint becomes

$$\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \dot{\theta}\, \ell \cos\phi = 0.$$

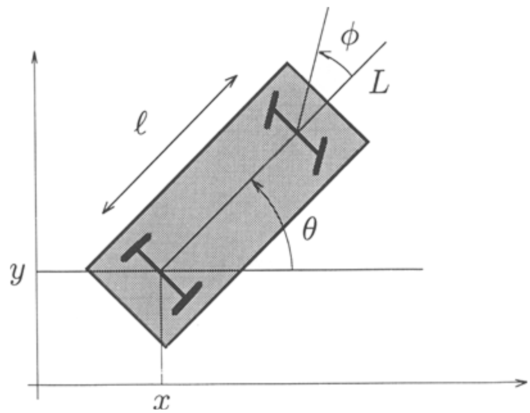

Figure 7.4: Car-like robot

The constraint matrix is

$$A^T(q) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 \end{bmatrix},$$

and has constant rank 2. Its nullspace is two-dimensional, and all the admissible generalized velocities are obtained as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{1}{\ell}\sin\phi \\ 0 \end{bmatrix} \alpha_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \alpha_2.$$

Since the front wheel can be steered, we set $\alpha_2 = u_2$, where u_2 is the *steering* velocity input. The expression of α_1 depends on the location of the driving input u_1. If the car has *front-wheel driving*, we have directly $\alpha_1 = u_1$. The corresponding control system will be

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \frac{1}{\ell}\sin\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$

Simple calculations yield

$$g_3 = [g_1, g_2] = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ -\frac{1}{\ell}\cos\phi \\ 0 \end{bmatrix}, \qquad g_4 = [g_1, g_3] = \begin{bmatrix} -\frac{\sin\theta}{\ell} \\ \frac{\cos\theta}{\ell} \\ 0 \\ 0 \end{bmatrix}.$$

These two vector fields are sometimes called *wriggle* and *slide*, respectively, in view of their physical meaning. The dimension of the accessibility distribution $\Delta_{\mathcal{C}}$ is 4. In particular, the front-wheel drive car has degree of nonholonomy $\kappa = 3$, growth vector $r = (2, 3, 4)$, relative growth vector $\sigma = (2, 1, 1)$.

The model for *rear-wheel driving* can be derived by letting $\alpha_1 = u_1/\cos\phi$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2. \tag{7.27}$$

As expected, the first two equations are those of a unicycle (without steering input). Also, there is a control singularity at $\phi = \pm\pi/2$, where the first vector field blows out. This corresponds to the rear-wheel drive car becoming jammed when the front wheel is normal to the longitudinal axis L of the car body. Instead, this singularity does not occur for the front-wheel drive car, that in the same situation can still pivot about its rear-axle midpoint.

We have

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\ell\cos^2\phi} \\ 0 \end{bmatrix}, \qquad g_4 = [g_1, g_3] = \begin{bmatrix} -\frac{\sin\theta}{\ell\cos^2\phi} \\ \frac{\cos\theta}{\ell\cos^2\phi} \\ 0 \\ 0 \end{bmatrix}.$$

Again, the degree of nonholonomy is $\kappa = 3$, the growth vector $r = (2, 3, 4)$, the relative growth vector $\sigma = (2, 1, 1)$. However, $\dim\Delta_{\mathcal{C}} = 4$ only away from the singularity $\phi = \pm\pi/2$, that corresponds to a loss of controllability for the vehicle. The relevance of this singularity is limited, due to the restricted range of the steering angle ϕ in many practical cases.

N-Trailer Robot

A more complex wheeled vehicle is obtained by attaching N one-axle trailers to a car-like robot with rear-wheel drive. For simplicity, each trailer is assumed to be connected to the axle midpoint of the previous one (*zero hooking*), as shown in Fig. 7.5. The car length is ℓ, and the hinge-to-hinge length of the i-th trailer is ℓ_i. One possible generalized coordinate vector that uniquely describes the configuration of this system is $q = (x, y, \phi, \theta_0, \theta_1, \ldots, \theta_N) \in I\!R^{N+4}$, obtained by setting $\theta_0 = \theta$ and extending the configuration of the car-like robot with the orientation θ_i, $i = 1, \ldots, N$, of each trailer. As a consequence, $n = N + 4$.

The $N + 2$ nonholonomic constraints are

$$\begin{aligned} \dot{x}_f \sin(\theta_0 + \phi) - \dot{y}_f \cos(\theta_0 + \phi) &= 0 \\ \dot{x}\sin\theta_0 - \dot{y}\cos\theta_0 &= 0 \\ \dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i &= 0, \qquad i = 1, \ldots, N, \end{aligned}$$

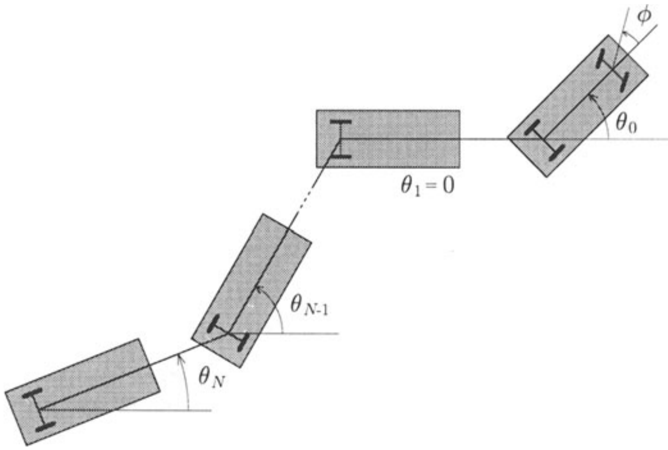

Figure 7.5: N-trailer robot

where (x_f, y_f) is the position of the midpoint of the front axle of the car. Since

$$\begin{aligned} x_i &= x - \sum_{j=1}^{i} \ell_j \cos\theta_j \\ y_i &= y - \sum_{j=1}^{i} \ell_j \sin\theta_j \end{aligned}$$

for $i = 1, \ldots, N$, the kinematic constraints become

$$\begin{aligned} \dot{x}\sin(\theta_0 + \phi) - \dot{y}\cos(\theta_0 + \phi) - \dot{\theta}_0\,\ell\cos\phi &= 0 \\ \dot{x}\sin(\theta_0) - \dot{y}\cos(\theta_0) &= 0 \\ \dot{x}\sin\theta_i - \dot{y}\cos\theta_i + \sum_{j=1}^{i} \dot{\theta}_j\,\ell_j\cos(\theta_i - \theta_j) &= 0, \quad i = 1, \ldots, N. \end{aligned}$$

The nullspace of the constraint matrix is spanned by the two columns of

$$G(q) = \begin{bmatrix} \cos\theta_0 & 0 \\ \sin\theta_0 & 0 \\ 0 & 1 \\ \frac{1}{\ell}\tan\phi & 0 \\ -\frac{1}{\ell_1}\sin(\theta_1 - \theta_0) & 0 \\ -\frac{1}{\ell_2}\cos(\theta_1 - \theta_0)\sin(\theta_2 - \theta_1) & 0 \\ \vdots & \vdots \\ -\frac{1}{\ell_i}\left(\prod_{j=1}^{i-1}\cos(\theta_j - \theta_{j-1})\right)\sin(\theta_i - \theta_{i-1}) & 0 \\ \vdots & \vdots \\ -\frac{1}{\ell_N}\left(\prod_{j=1}^{N-1}\cos(\theta_j - \theta_{j-1})\right)\sin(\theta_N - \theta_{N-1}) & 0 \end{bmatrix} = \begin{bmatrix} g_1(q) & g_2(q) \end{bmatrix}. \quad (7.28)$$

Thus, the kinematic control system is

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2,$$

where u_1 is the rear-wheel driving velocity and u_2 the front-wheel steering velocity of the towing car.

For this mobile robot, Laumond [40] has devised a cleverly organized proof of controllability—and hence, of complete nonholonomy. However, the degree of nonholonomy of the system is not strictly defined, since the relative filtration is not regular. Recently, Sørdalen [41] has introduced a slightly modified model in which N trailers are attached to a *unicycle* with driving and steering inputs. In this case, $n = N + 3$. Moreover, he has pointed out that a more convenient model format is obtained by choosing (x, y) as the position coordinates of the last trailer rather than of the towing vehicle. It has been shown [42] that the *maximum* degree of nonholonomy (i.e., its maximum value for $q \in I\!R^n$) for this modified robot is F_{N+3}, where F_k indicates the k-th Fibonacci number. By analogy, we conjecture that the maximum degree of nonholonomy for our model (7.28) should be F_{N+4}.

7.5.2 Space Robots with Planar Structure

Consider an n-body planar open kinematic chain which floats freely, as shown in Fig. 7.6. One of the bodies (say, the first) may represent the bulk of the space structure (a satellite), and the other $n - 1$ bodies are the manipulator links. An interesting control problem arises when no gas jets are used for controlling the satellite attitude, while the only available control inputs for reconfiguring the space structure are the manipulator joint torques, which are *internal* generalized forces. In fact, it may be convenient to refrain from using the satellite actuators, so to minimize fuel consumption. As we shall see, it is generally possible to change the configuration of the whole structure by moving only the manipulator joints.

For the i-th body, let ℓ_i be the hinge-to-hinge length and d_i the distance from joint $i - 1$ to its center of mass. Further, denote by r_i and v_i, respectively, the position and the linear velocity of the center of mass, and by ω_i the angular velocity of the body (all vectors are embedded in $I\!R^3$ and expressed in an inertial frame). Finally, m_i indicates the mass of the i-th body and I_i its inertia matrix with respect to the center of mass.

When no external force is applied, and in the absence of gravity and dissipation forces, the linear and angular momenta of the multibody system are conserved. Assume that initially they are all zero. The law of conservation of linear momentum is written, in general, as

$$\sum_{i=1}^{n} m_i v_i = \sum_{i=1}^{n} m_i \dot{r}_i = 0,$$

that can be obviously integrated to

$$\sum_{i=1}^{n} m_i r_i = m_t r_{c0} = c,$$

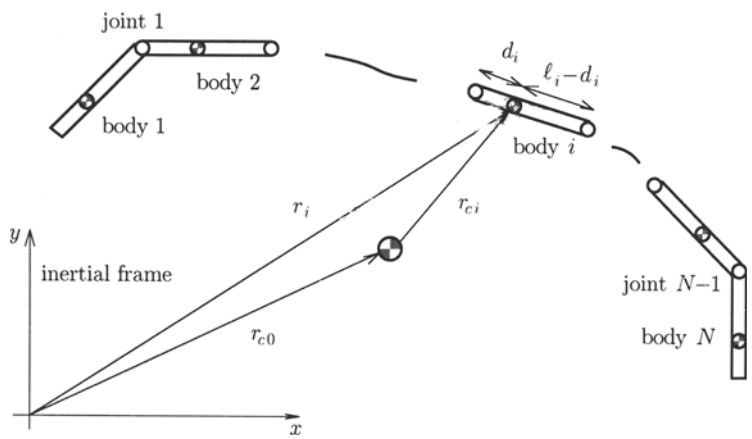

Figure 7.6: An n-body planar space structure (satellite+manipulator)

where m_t is the total mass of the system, r_{c0} is the position vector of the system center of mass and c is a vector constant. As a consequence, the conservation of linear momentum gives rise to three holonomic constraints, indicating that the system center of mass does not move.

The conservation of angular momentum is expressed as

$$\sum_{i=1}^{n} [I_i\omega_i + m_i(r_i \times v_i)] = 0, \tag{7.29}$$

i.e., as three differential constraints that are, in general, nonholonomic in the three-dimensional case. Since in the present case, motion is constrained to the xy-plane, there is only one nontrivial differential constraint in eqs. (7.29), namely the one relative to the z direction. We shall prove below the latter is a nonholonomic constraint provided that $n > 2$. However, to perform the analysis, it will be first necessary to convert this constraint into a Pfaffian form in terms of the system generalized coordinates q.

Recall that the kinetic energy of an n-body system can be put in the form

$$T = \frac{1}{2}\dot{q}^T B(q)\dot{q},$$

where $B(q)$ is the symmetric and positive-definite *inertia matrix*. For the actual computation of T it is convenient to choose the origin of the inertial frame at the center of mass of the whole system, so that $r_{c0} = 0$ and $r_i = r_{ci}$. The *generalized momenta* are defined as

$$p_i = \frac{\partial(T-U)}{\partial \dot{q}_i} = b_i^T(q)\dot{q}, \quad i = 1, \ldots, n,$$

where b_i is the i-th column of B. Here, we have exploited the assumption that the system is freely floating in space, so that U is constant. The angular momentum of a

xy-planar system along the z axis is then computed as $\sum_{i=1}^{n} p_i$ (see, for example, [43]). Therefore, the conservation of *zero* angular momentum along the z axis can be written in the form of a single ($k = 1$) Pfaffian constraint as follows

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} b_i^T(q)\dot{q} = \mathbf{1}^T B(q)\dot{q} = a^T(q)\dot{q} = 0, \tag{7.30}$$

where $\mathbf{1}^T = (1, 1, \ldots, 1)$. Indeed, conservation of a *nonzero* value for this angular momentum leads to a single differential constraint in the form (7.23).

Two-Body Robot

The Pfaffian constraint (7.30) is in general nonholonomic, but it is integrable in the particular case of $n = 2$. In fact, in this case we have $n - k = 1$, and, therefore, the accessibility distribution is always involutive, as pointed out in Section 7.4. We shall now give a detailed derivation of this fact.

Consider the structure shown in Fig. 7.7. The orientation of the i-th body with respect to the x axis of the inertial frame is denoted by θ_i ($i = 1, 2$).

The two vector equations

$$\begin{aligned}
\begin{bmatrix} r_{c1,x} \\ r_{c1,y} \end{bmatrix} + (\ell_1 - d_1)\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + d_2\begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} &= \begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} \\
m_1\begin{bmatrix} r_{c1,x} \\ r_{c1,y} \end{bmatrix} + m_2\begin{bmatrix} r_{c2,x} \\ r_{c2,y} \end{bmatrix} &= 0
\end{aligned}$$

may be solved for the two position vectors as

$$\begin{bmatrix} r_{c1} \\ r_{c2} \end{bmatrix} = \begin{bmatrix} r_{c1,x} \\ r_{c1,y} \\ r_{c2,x} \\ r_{c2,y} \end{bmatrix} = \begin{bmatrix} k_{11}\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{12}\begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} \\ k_{21}\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + k_{22}\begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \end{bmatrix} \end{bmatrix},$$

where

$$\begin{aligned}
k_{11} &= -m_2(\ell_1 - d_1)/m_t \\
k_{12} &= -m_2 d_2/m_t \\
k_{21} &= m_1(\ell_1 - d_1)/m_t \\
k_{22} &= m_1 d_2/m_t,
\end{aligned}$$

with $m_t = m_1 + m_2$.

As a consequence, setting $J_i = I_{i,zz}$, the kinetic energy of the i-th body is

$$T_i = \frac{1}{2}m_i\dot{r}_{ci}^T\dot{r}_{ci} + \frac{1}{2}J_i\dot{\theta}_i^2, \qquad i = 1, 2,$$

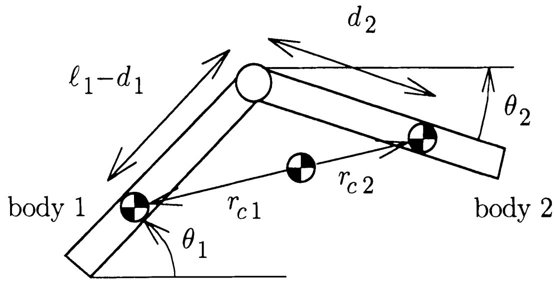

Figure 7.7: Two-body planar space robot

for $i = 1, 2$. The kinetic energy of the system becomes

$$T = T_1 + T_2 = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} \bar{J}_1 & b_{12}(\theta_1, \theta_2) \\ b_{12}(\theta_1, \theta_2) & \bar{J}_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix},$$

where

$$\bar{J}_i = J_i + m_1 k_{1i}^2 + m_2 k_{2i}^2, \qquad i = 1, 2,$$

and

$$b_{12}(\theta_1, \theta_2) = (m_1 k_{11} k_{12} + m_2 k_{21} k_{22}) \cos(\theta_2 - \theta_1).$$

Since T is a function of the *shape angle* $\phi_1 = \theta_2 - \theta_1$ only, the Pfaffian constraint (7.30) can be written as

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{J}_1 & b_{12}(\phi_1) \\ b_{12}(\phi_1) & \bar{J}_2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{\phi}_1 \right) = 0,$$

from which

$$\dot{\theta}_1 = -\frac{\bar{J}_2 + b_{12}(\phi_1)}{\bar{J}_1 + \bar{J}_2 + 2b_{12}(\phi_1)} \dot{\phi}_1,$$

where the denominator is strictly positive due to the positive-definiteness of the inertia matrix. Taking the single joint velocity as input ($u = \dot{\phi}_1$), and defining the generalized coordinate vector as

$$q = \begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix},$$

all the admissible motions are described by the driftless control system

$$\dot{q} = g(q)u = \begin{bmatrix} -\dfrac{\bar{J}_2 + b_{12}(\phi_1)}{\bar{J}_1 + \bar{J}_2 + 2b_{12}(\phi_1)} \\ 1 \end{bmatrix} u.$$

Since there is a single input vector field, this system is not controllable. Equivalently, conservation of angular momentum gives an integrable differential constraint in this case. In particular, if $\bar{J}_1 = \bar{J}_2$ one has

$$\dot{\theta}_1 = -\frac{1}{2}\dot{\phi}_1 \qquad \Longrightarrow \qquad \theta_1 = -\frac{1}{2}\phi_1 + c,$$

where c is a constant depending on the initial conditions.

In summary, the conservation of angular momentum is a holonomic constraint for a planar space robot with $n = 2$ bodies. Hence, for this mechanical system it is not possible to steer u so as to achieve *any pair* of absolute orientation θ_1 and internal shape ϕ_1.

n-Body Robot

Let us turn to the general case of $n \geq 3$ bodies. As before, we assume that the inertial reference frame is located at the system center of mass, and denote by θ_i the absolute orientation of the i-th body. Generalizing the expression derived for the case $n = 2$, the position of the center of mass of the i-th body is

$$\begin{bmatrix} r_{ci,x} \\ r_{ci,y} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} k_{ij} \cos\theta_j \\ \sum_{j=1}^{n} k_{ij} \sin\theta_j \end{bmatrix},$$

where

$$k_{ij} = \begin{cases} \frac{1}{m_t}\left[\ell_j \sum_{h=1}^{j-1} m_h + (\ell_j - d_j) m_j\right] & \text{for } j < i, \\ \frac{1}{m_t}\left[d_i \sum_{h=1}^{i-1} m_h - (\ell_i - d_i) \sum_{k=i+1}^{n} m_k\right] & \text{for } j = i, \\ \frac{1}{m_t}\left[-\ell_j \sum_{h=j+1}^{n} m_h - d_j m_j\right] & \text{for } j > i. \end{cases}$$

In turn, the kinetic energy of i-th body is expressed as

$$T_i = \frac{1}{2} m_i \dot{r}_{ci}^T \dot{r}_{ci} + \frac{1}{2} J_i \dot{\theta}_i^2 = \frac{1}{2} m_i \left[\sum_{h=1}^{n} \sum_{j=1}^{n} k_{ij} k_{ih} \cos(\theta_h - \theta_j) \dot{\theta}_h \dot{\theta}_j\right] + \frac{1}{2} J_i \dot{\theta}_i^2$$

Therefore, the kinetic energy of the system is

$$T = \sum_{i=1}^{n} T_i = \frac{1}{2} \dot{\theta}^T B(\theta) \dot{\theta},$$

with the generic element of the inertia matrix B taking the form

$$b_{ij}(\theta_i, \theta_j) = \begin{cases} \sum_{h=1}^{n} m_h k_{hi} k_{hj} \cos(\theta_i - \theta_j) & \text{for } j \neq i, \\ J_i + \sum_{h=1}^{n} m_h k_{hh}^2 & \text{for } j = i. \end{cases} \tag{7.31}$$

Note that b_{ij} depends only on the relative angle between the i-th and the j-th bodies. Define

$$\phi_i = \theta_{i+1} - \theta_i, \qquad i = 1, \ldots, n-1.$$

The vector $\phi = (\phi_1, \ldots, \phi_{n-1})$ can be obtained as

$$\phi = P\theta,$$

where P is an $(n-1) \times n$ matrix whose generic element is defined as

$$p_{ij} = \begin{cases} -1 & \text{for } j = i \\ +1 & \text{for } j = i+1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.32}$$

Choose the generalized coordinates vector as

$$q = \begin{bmatrix} \theta_1 \\ \phi_1 \\ \vdots \\ \phi_{n-1} \end{bmatrix} = \left[\frac{1 \quad \mathbf{0}^T}{P}\right] \theta = \left[\begin{array}{ccccc} 1 & 0 & 0 & \dots & \dots \\ \hline -1 & 1 & 0 & \dots & \dots \\ 0 & -1 & 1 & 0 & \dots \\ & & \dots & & \\ \dots & \dots & 0 & -1 & 1 \end{array}\right] \theta,$$

with the inverse mapping

$$\theta = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots \\ 1 & 1 & 0 & \dots & \dots \\ 1 & 1 & 1 & 0 & \dots \\ & & \dots & & \\ 1 & \dots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \phi \end{bmatrix} = [\mathbf{1} \quad S] \begin{bmatrix} \theta_1 \\ \phi \end{bmatrix},$$

where S is an $n \times (n-1)$ matrix. Correspondingly, the conservation of angular momentum becomes

$$\mathbf{1}^T B(\phi) \left(\mathbf{1}\dot{\theta}_1 + S\dot{\phi}\right) = 0,$$

from which

$$\dot{\theta}_1 = -\frac{\mathbf{1}^T B(\phi) S}{\mathbf{1}^T B(\phi) \mathbf{1}} u,$$

where $\dot{\phi} = u$ are the joint velocities, taken as inputs.

In summary, the *kinematic model* of the n-body space robot is

$$\dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} s_1(\phi) & s_2(\phi) & \dots & s_{n-1}(\phi) \\ & I_{n-1} & & \end{bmatrix} u, \tag{7.33}$$

in which

$$s_i(\phi) = -\frac{s_i'(\phi)}{\mathbf{1}^T B(\phi) \mathbf{1}}, \qquad i = 1, \dots, n-1,$$

where the numerator is

$$s_i'(\phi) = \sum_{j=i+1}^{n} \left[\bar{J}_j + \sum_{h=1}^{n} \sum_{l=1}^{n} m_l k_{lj} k_{lh} \cos\left(\sum_{r=h}^{j-1} \phi_r\right)\right],$$

with

$$\bar{J}_j = J_j + \sum_{h=1}^{n} m_h k_{hj}^2,$$

and the strictly positive denominator is given by

$$\mathbf{1}^T B(\phi) \mathbf{1} = \sum_{j=1}^{n} \bar{J}_j + \sum_{j=1}^{n} \sum_{\substack{h=1 \\ h \neq j}}^{n} \sum_{l=1}^{n} m_l k_{lj} k_{lh} \cos\left(\sum_{r=h}^{j-1} \phi_r\right).$$

To get more insight into these expressions and into the control properties of system (7.33), we consider the case of a planar space structure with $n = 3$ bodies. Simple calculations yield in this case

$$\begin{aligned} s_1'(\phi) &= \bar{J}_2 + \bar{J}_3 + h_{12}\cos\phi_1 + 2h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2) \\ s_2'(\phi) &= \bar{J}_3 + h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2) \end{aligned}$$

and

$$\mathbf{1}^T B(\phi)\mathbf{1} = \bar{J}_1 + \bar{J}_2 + \bar{J}_3 + 2\left[h_{12}\cos\phi_1 + h_{23}\cos\phi_2 + h_{13}\cos(\phi_1+\phi_2)\right],$$

with

$$\begin{aligned} \bar{J}_1 &= J_1 + m_1 k_{11}^2 + m_2 k_{21}^2 + m_3 k_{31}^2 \\ \bar{J}_2 &= J_2 + m_1 k_{12}^2 + m_2 k_{22}^2 + m_3 k_{32}^2 \\ \bar{J}_3 &= J_3 + m_1 k_{13}^2 + m_2 k_{23}^2 + m_3 k_{33}^2 \end{aligned}$$

and

$$\begin{aligned} h_{12} &= m_1 k_{11} k_{12} + m_2 k_{21} k_{22} + m_3 k_{31} k_{32} \\ h_{13} &= m_1 k_{11} k_{13} + m_2 k_{21} k_{23} + m_3 k_{31} k_{33} \\ h_{23} &= m_1 k_{12} k_{13} + m_2 k_{22} k_{23} + m_3 k_{32} k_{33} \end{aligned}$$

,

where

$$\begin{array}{lll} k_{11} = -\frac{(\ell_1-d_1)(m_2+m_3)}{m_t} & k_{12} = -\frac{\ell_2 m_3 + d_2 m_2}{m_t} & k_{13} = -\frac{d_3 m_3}{m_t} \\ k_{21} = \frac{(\ell_1-d_1)m_1}{m_t} & k_{22} = \frac{d_2 m_1 - (\ell_2-d_2)m_3}{m_t} & k_{23} = k_{13} \\ k_{31} = k_{21} & k_{32} = \frac{\ell_2 m_1 + (\ell_2-d_2)m_2}{m_t} & k_{33} = \frac{d_3(m_1+m_2)}{m_t} \end{array}$$

and $m_t = m_1 + m_2 + m_3$.

Thus, the kinematic model (7.33) is characterized for the three-body space robot by the two vector fields

$$g_1 = \begin{bmatrix} s_1(\phi) \\ 1 \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} s_2(\phi) \\ 0 \\ 1 \end{bmatrix}, \tag{7.34}$$

whose Lie bracket is

$$g_3 = [g_1, g_2] = \begin{bmatrix} \frac{\partial s_2(\phi)}{\partial \phi_1} - \frac{\partial s_1(\phi)}{\partial \phi_2} \\ 0 \\ 0 \end{bmatrix}.$$

One may verify that, in the case of equal bodies with uniform mass distribution, it is $g_3 = 0$ on the one-dimensional manifold $\phi_1 + \phi_2 = 0$. Hence, the filtration is not regular, and the degree of nonholonomy, the growth vector, and the relative growth vector are not strictly defined. However, by using higher order brackets, it can be shown that the accessibility distribution has dimension $3 = n$ and, hence, the system is completely nonholonomic and controllable. An equivalent result may be established for any number of bodies $n \geq 3$ (see [26]).

7.5.3 Redundant Robots under Kinematic Inversion

Recall that a manipulator with n joints performing an m-dimensional task is said to be *redundant* if $m < n$. Typically, the primary task is to position the end-effector in the workspace with a given orientation. Redundant robots are characterized by the one-to-many nature of the inverse kinematic map: a given end-effector pose can be realized by an infinity of joint configurations. Hence, with respect to conventional manipulators, they possess a higher dexterity.

Denoting by $q \in I\!R^n$ the joint-coordinate vector and by $p \in I\!R^m$ the task vector, we have at the differential level

$$\dot{p} = \frac{\partial k}{\partial q}(q)\dot{q} = J_k(q)\dot{q}, \tag{7.35}$$

where $k(\cdot)$ is the direct kinematic map for the chosen task and $J_k(q)$ is the analytic Jacobian of this map. Note that $J_k(q)$ is *not* the standard robot Jacobian $J(q)$ relating the linear velocity v of the tip of the end-effector and the angular velocity ω of the end-effector to the generalized joint velocities, i.e.,

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = J(q)\dot{q},$$

as discussed in [44]. In particular, if $m = 6$ and $p = (r, o)$, where r is the position of the end-effector and o is a minimal representation of its orientation (e.g., via Euler angles), then,

$$J_k(q) = \frac{\partial k}{\partial q} = \begin{bmatrix} I & 0 \\ 0 & T_k(q) \end{bmatrix} J(q),$$

with $T_k(q)$ invertible almost everywhere.

A typical way to generate joint motion at the velocity level is by inverting eq. (7.35), i.e.,

$$\dot{q} = G(q)u, \tag{7.36}$$

in which G is any *generalized inverse* of the Jacobian J_k and $u \in I\!R^m$ is a task command. A generalized inverse [45] satisfies $J_k G J_k = J_k$, implying that any feasible vector u is exactly realized. A usual choice is $G = J_k^{\#}$, the unique *pseudoinverse* of J_k.

One drawback of the inverse kinematics scheme (7.36) is that, for most choices of G, the joint motion corresponding to a closed end-effector path is not closed. This problem, referred to as *non-cyclicity* or *non-repeatability*, has been intensively studied in the last years [46–49]. However, only recently has such phenomenon been recognized as a manifestation of the nonholonomic behavior of redundant robots under inverse kinematic control [34]. In fact, a peculiar feature of nonholonomic systems is the nonzero drift (also called *holonomy* or *phase angle*) that occurs in $n - m$ generalized coordinates when the remaining m perform a cyclic motion [50].

The kinematic control system (7.36) is underactuated. Depending on the choice of the generalized inverse G, i.e., on the chosen inversion strategy, system (7.36) may or

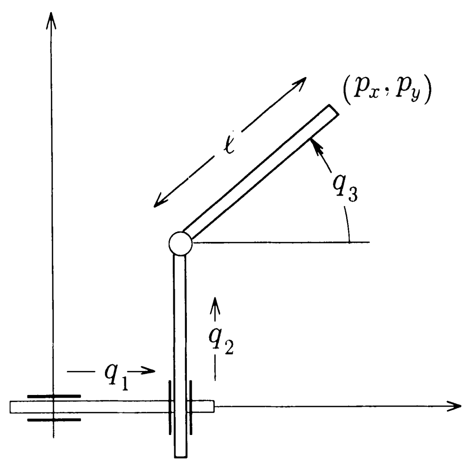

Figure 7.8: PPR robot

may not be controllable. Correspondingly, the robot motion is subject to a kinematic constraint that is nonholonomic or holonomic. Below, we shall study in detail one simple redundant manipulator.

PPR Robot

Let us consider the planar PPR robot shown in Fig. 7.8, having two prismatic and one revolute joints. This robot is redundant for the task of positioning the tip of the end-effector in the plane with unspecified orientation of the end-effector ($n = 3$, $m = 2$). Denoting by ℓ the length of the third link, the direct kinematic equations are

$$\begin{aligned} p_x &= q_1 + \ell c_3 \\ p_y &= q_2 + \ell s_3, \end{aligned}$$

where $s_3 = \sin q_3$ and $c_3 = \cos q_3$. The differential kinematics is expressed as eq. (7.35), where the analytic Jacobian matrix

$$J(q) = \begin{bmatrix} 1 & 0 & -\ell s_3 \\ 0 & 1 & \ell c_3 \end{bmatrix}$$

is always of full rank. The k subscript in the Jacobian has been dropped for compactness.

Assume that a *weighted* pseudoinverse [45] is chosen for the inversion scheme (7.36), i.e., $G = J_W^{\#}$ with weighting matrix

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w^2 \end{bmatrix} > 0,$$

where w is a constant with the same units of ℓ. For the PPR robot, we have

$$J_W^{\#}(q) = W^{-1}J^T(q)\left(J(q)W^{-1}J^T(q)\right)^{-1} = \frac{1}{1+(\ell/w)^2}\begin{bmatrix} 1+(\ell/w)^2c_3^2 & (\ell/w)^2 s_3 c_3 \\ (\ell/w)^2 s_3 c_3 & 1+(\ell/w)^2 s_3^2 \\ -(\ell/w^2)s_3 & (\ell/w^2)c_3 \end{bmatrix}.$$

Note that the weighting factor w^2 is introduced for achieving dimensional homogeneity.

All the admissible velocities are then obtained as

$$\dot q = \frac{1}{1+(\ell/w)^2}\begin{bmatrix} 1+(\ell/w)^2c_3^2 \\ (\ell/w)^2 s_3 c_3 \\ -(\ell/w^2)s_3 \end{bmatrix} u_1 + \frac{1}{1+(\ell/w)^2}\begin{bmatrix} (\ell/w)^2 s_3 c_3 \\ 1+(\ell/w)^2 s_3^2 \\ (\ell/w^2)c_3 \end{bmatrix} u_2. \tag{7.37}$$

The accessibility rank condition for system (7.37) is satisfied if the Lie bracket of the two columns j_1 and j_2 of $J_W^{\#}$ does not belong to span $\{j_1, j_2\}$, with

$$[j_1, j_2] = \beta \begin{bmatrix} \ell s_3 \\ -\ell c_3 \\ 1/w \end{bmatrix}, \qquad \beta = \frac{1}{w}\frac{(\ell/w)^2}{\left(1+(\ell/w)^2\right)^2}.$$

It can be readily verified that this is the case, and thus, the PPR robot under pseudoinversion is controllable. Hence, the kinematic constraint associated with system (7.37), i.e.,

$$\begin{bmatrix} \ell s_3 & -\ell c_3 & w^2 \end{bmatrix} \dot q = 0,$$

is nonholonomic.

It is noteworthy that there exist other choices of G yielding a holonomic system, such as

$$G(q) = \begin{bmatrix} 1+(\ell/w)s_3 & (\ell/w)s_3 \\ -(\ell/w)c_3 & 1-(\ell/w)c_3 \\ 1/w & 1/w \end{bmatrix}. \tag{7.38}$$

In fact, it can be verified that $[g_1, g_2] = 0$, so that the accessibility rank condition is violated and the system is not controllable. As expected, the kinematic constraint

$$\begin{bmatrix} 1 & 1 & -w+\ell(c_3 - s_3) \end{bmatrix} \dot q = 0$$

associated with the generalized inverse (7.38) can be integrated as

$$h(q) = q_1 + q_2 - wq_3 + \ell(s_3 + c_3) = c,$$

where c is a constant. In this case, the motion of the PPR robot is constrained to a leaf of $h(q)$, i.e., a two-dimensional manifold. As a consequence, the corresponding inversion scheme is cyclic.

7.6 Dynamics of Nonholonomic Systems

In this section, we introduce the dynamical model of an n-dimensional mechanical system with k kinematic constraints (7.2) using Lagrange formulation, and show how it can be partially linearized via feedback. For the derivation of the dynamical model, the reader may consult [51, 52]. The partial linearization procedure is taken from [53]; a similar approach can be found in [27].

Define the *Lagrangian* $\mathcal{L}$ as the difference between the system kinetic and potential energy:

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - U(q) = \frac{1}{2}\dot{q}^T B(q)\dot{q} - U(q),$$

where $B(q)$ is the positive-definite inertia matrix of the mechanical system. The *Euler-Lagrange* equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial q}\right)^T = A(q)\lambda + S(q)\tau, \tag{7.39}$$

in which $S(q)$ is an $n \times m$ matrix mapping the $m = n - k$ external inputs τ into forces/torques performing work on q, and $\lambda \in I\!R^m$ is the vector of the unknown *Lagrange multipliers*. The term $A(q)\lambda$ represents the vector of constraint forces. Note that we have assumed that the number of available inputs equals the number of degrees of freedom of the system.

The *dynamical model* of the constrained mechanical system is

$$B(q)\ddot{q} + n(q,\dot{q}) = A(q)\lambda + S(q)\tau \tag{7.40}$$

$$A^T(q)\dot{q} = 0, \tag{7.41}$$

with

$$n(q,\dot{q}) = \dot{B}(q)\dot{q} - \frac{1}{2}\left(\frac{\partial}{\partial q}\left(\dot{q}^T B(q)\dot{q}\right)\right)^T + \left(\frac{\partial U(q)}{\partial q}\right)^T.$$

Consider a matrix $G(q)$ whose columns are a basis for the nullspace of $A^T(q)$, as in eq. (7.10), so that $A^T(q)G(q) = 0$. The Lagrange multipliers may be eliminated by left-multiplying both sides of eq. (7.40) by $G^T(q)$, thereby obtaining the *reduced dynamical model*

$$G^T(q)\left[(B(q)\ddot{q} + n(q,\dot{q})\right] = G^T(q)S(q)\tau$$

as a set of m differential equations.

Assume now that

$$\det\left[G^T(q)S(q)\right] \neq 0,$$

which is a realistic hypothesis in most practical cases. It is convenient to merge the kinematic and dynamical models into an $(n+m)$-dimensional *reduced state-space model* of the form

$$\begin{aligned} \dot{q} &= G(q)v \\ \dot{v} &= M^{-1}(q)m(q,v) + M^{-1}(q)G^T(q)S(q)\tau, \end{aligned}$$

where $v \in I\!R^m$ is a reduced vector of velocities and

$$\begin{aligned} M(q) &= G^T(q)B(q)G(q) > 0 \\ m(q,v) &= G^T(q)B(q)\dot{G}(q)v + G^T(q)n(q, G(q)v), \end{aligned}$$

with

$$\dot{G}(q)v = \sum_{i=1}^{m} \left(v_i \frac{\partial g_i}{\partial q}(q) \right) G(q)v.$$

It is possible to perform a *partial linearization* of the reduced state-space model via a *computed torque* nonlinear feedback

$$\tau = \left[G^T(q)S(q)\right]^{-1} (M(q)a + m(q,v)), \tag{7.42}$$

where $a \in I\!R^m$ is a reduced vector of accelerations. The resulting system is

$$\begin{aligned} \dot{q} &= G(q)v \\ \dot{v} &= a, \end{aligned}$$

where the first n equations are the kinematic model and the second m equations act as a dynamic extension (see eqs. (7.16)–(7.17) in Section 7.3.2). Note that the computed torque input (7.42) requires the measure of v, which may not be directly accessible. However, v can be computed by pseudoinverting the kinematic model as

$$v = \left(G^T(q)G(q)\right)^{-1} G^T(q)\dot{q},$$

provided that q and $\dot{q}$ are measurable.

Defining the state vector $x = (q, v) \in I\!R^{n+m}$ and the input vector $u = a \in I\!R^m$, the *state-space model* of the closed-loop system may be rewritten in compact form as

$$\dot{x} = f(x) + g(x)u = \begin{bmatrix} G(q)v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} u, \tag{7.43}$$

i.e., a nonlinear control system with drift, also known as the *second-order kinematic model* of the constrained mechanical system.

In summary, for nonholonomic systems it is possible to cancel dynamic parameters via nonlinear feedback, assuming that *(i)* the dynamical model is exactly known, and *(ii)* the complete system state is measurable. If this is the case, the control problem can be directly solved at the velocity level, namely, by synthesizing v in such a way that the kinematic model

$$\dot{q} = G(q)v$$

behaves as desired. When v is available, eq. (7.42) can be used to obtain the control input τ at the generalized force level, provided that v is sufficiently smooth (at least differentiable, since $a = \dot{v}$ appears). This motivates further the search for smooth stabilizers for nonholonomic systems. We shall come back on this issue in Section 7.8.

7.7 Dynamic Modeling Examples

In this section we introduce the dynamical model of two nonholonomic systems that have already been considered in Section 7.5.

Unicycle

Let m be the mass of the unicycle, J its moment of inertia about the vertical axis, τ_1 the driving force and τ_2 the steering torque. With the nonholonomic constraint expressed as eq. (7.25), the dynamical model (7.40)–(7.41) takes the form

$$\begin{aligned} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} &= \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \lambda + \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \\ \dot{x}\sin\theta - \dot{y}\cos\theta &= 0. \end{aligned}$$

Next, we perform the reduction procedure introduced in the previous section. Since

$$\begin{aligned} G(q) &= S(q) \\ G^T(q)S(q) &= I_2 \\ G^T(q)B\dot{G}(q) &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} \dot{q} &= G(q)v \\ \dot{v} &= M^{-1}(q)\tau, \end{aligned}$$

i.e., the five dynamical equations

$$\begin{aligned} \dot{x} &= v_1\cos\theta \\ \dot{y} &= v_1\sin\theta \\ \dot{\theta} &= v_2 \\ \dot{v}_1 &= \tau_1/m \\ \dot{v}_2 &= \tau_2/J. \end{aligned}$$

By letting

$$\tau = Mu = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix},$$

the partially linearized dynamical model is obtained as

$$\dot{\xi} = \begin{bmatrix} v_1\cos\theta \\ v_1\sin\theta \\ v_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 = f(\xi) + g_1u_1 + g_2u_2,$$

with the state vector $\xi = (x, y, \theta, v_1, v_2) \in I\!R^5$. By computing the Lie brackets

$$[g_1, f] = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [g_2, f] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [g_2, [f, [g_1, f]]] = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

it is apparent that the dynamical model of the unicycle satisfies both the accessibility and the small-time controllability conditions. In fact, the vector fields

$$g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]]$$

span a space of dimension five at each ξ, and satisfy the conditions of Theorem 3.

n-Body Robot

Denote by $\tau \in I\!R^{n-1}$ the vector of torques at the $n-1$ joints. As shown in [27], the dynamical model (7.40)–(7.41) for the n-body space robot of Fig. 7.6 takes the form

$$\begin{aligned} B(\theta)\ddot{\theta} + n(\theta, \dot{\theta}) &= P^T \tau \\ \mathbf{1}^T B(\theta)\dot{\theta} &= 0, \end{aligned}$$

with the elements of $B(\theta)$ and P respectively given by eqs. (7.31) and (7.32), and $n(\theta, \dot{\theta})$ computed as

$$n(\theta, \dot{\theta}) = \dot{B}(\theta)\dot{\theta} - \frac{1}{2}\left(\frac{\partial}{\partial\theta}\left(\dot{\theta}^T B(\theta)\dot{\theta}\right)\right)^T.$$

The reduced state-space model consists of $2n-1$ first-order differential equations, namely,

$$\begin{aligned} \dot{\theta}_1 &= -\frac{\mathbf{1}^T B(\phi) S}{\mathbf{1}^T B(\phi)\mathbf{1}} v \\ \dot{\phi} &= v \\ \dot{v} &= M^{-1}(\phi)\left[-m(\phi, v) + \tau\right], \end{aligned}$$

where

$$\begin{aligned} M(\phi) &= P B(\phi) P^T \\ m(\phi, v) &= \dot{M}(\phi) v - \frac{1}{2}\frac{\partial}{\partial\phi}\left(v^T M(\phi) v\right). \end{aligned}$$

It can be proven that this system satisfies the conditions for accessibility and small-time local controllability (see [27]). Note that the right-hand side of the state-space model is independent of θ_1. This is so because the system Lagrangian is independent of θ_1, i.e., because θ_1 is a *cyclic* coordinate for the system. Mechanical systems with this special structure are referred to as nonholonomic *Čaplygin* systems [52].

7.8 Control of Nonholonomic Systems

In this section, we shall review some control techniques for nonholonomic systems. In particular, the problem will be addressed at the first-order kinematic level, i.e., on the driftless control system

$$\dot{q} = \sum_{j=1}^{m} g_j(q)u_j = G(q)u, \qquad m = n - k, \tag{7.44}$$

which is directly associated with the presence of k nonholonomic constraints in a mechanical system with n configuration variables.

Focusing on the control of first-order kinematic models of nonholonomic systems is not really a limitation. In fact, it has been shown in Section 7.6 that, provided that the dynamical model is exactly known and the whole state is measurable, then it is possible to transform through feedback the system into the second-order kinematic model (7.43). Once the control problem has been solved for system (7.44), the extension to the second order and then to the dynamical level is straightforward (see the remark at the end of Section 7.6).

Assume that the initial state of system (7.44) is q_0, and note that any q is actually an equilibrium configuration (denoted by q_e) of the unforced system. The objective is to transfer the system from q_0 to a desired q_f by using a proper sequence of input commands. From our discussion on the integrability of constraints, and, more specifically, Proposition 2 of Section 7.4, the kinematic model of a (completely) nonholonomic system is controllable, so that at least one suitable input sequence certainly exists. Typically, to achieve robustness with respect to perturbations, we would like to solve this control problem by using a feedback strategy, i.e., by driving the input u through the current *configuration error* $e = (q_f - q)$. However, there is one major difficulty to be faced.

Proposition 3. *For the kinematic model (7.44) of a nonholonomic system, there exists no C^1 static state feedback law $u = \alpha(q, e)$ which can make a point q_e asymptotically stable. Moreover, exponential stability cannot be achieved in any case.*

Proof. The first part of the proposition is readily established on the basis of Brockett's theorem. In fact, system (7.44) is underactuated and the vector fields g_j's are linearly independent by construction. Hence, the necessary condition for static C^1-stabilizability is violated. As for exponential stability, it suffices to note that the approximate linearization of system (7.44) at any q_e has $n - m$ zero uncontrollable eigenvalues. ∎

Because of the limitation expressed by this proposition, there are two possible approaches for solving our control problem:

1. Steer the system state from q_0 to q_f through an *open-loop* or *feedforward* control, in which the input u does not depend on the system state q nor on the error e. As a byproduct, one obtains a trajectory connecting q_0 to q_f that is feasible, i.e., complies with the nonholonomic constraints. Therefore, this approach is often referred to as *motion planning* for nonholonomic systems. However, such a solution is not robust with respect to disturbances, errors on the initial conditions, or modeling inaccuracies.

2. Use a feedback control that is not ruled out by Proposition 3, namely, a *nonsmooth* law $u = \alpha(q, e)$ and/or a *time-varying* law $u = \alpha(q, e, t)$. The design of this type of controllers is more difficult, but on the other hand they are preferable for real-time motion control under uncertain or perturbed conditions.

Some representative methods of both classes are presented below. However, we shall focus on open-loop control, because this allows to introduce concepts (e.g., the chained form) that are of great relevance also for the synthesis of feedback stabilizing laws.

7.8.1 Open-Loop Control

The most intuitive approach to open-loop control for a nonholonomic system is to find *canonical* paths, i.e., feasible paths that allow us to achieve any desired reconfiguration. The corresponding feedforward control $u(t)$ is derived by using the kinematic model. In Section 7.1, we have already seen an example of such procedure for the rolling disk.

Example. The unicycle of Fig. 7.3 may be steered from any initial configuration $q_0 = (x_0, y_0, \theta_0)$ to any final configuration $q_f = (x_f, y_f, \theta_f)$ through the motion sequence below:

1. Rotate the unicycle so as to align the projection of its midplane with the straight line joining (x_0, y_0) and (x_f, y_f).

2. Roll forward until the contact point with the ground is (x_f, y_f).

3. Rotate the unicycle to achieve the final orientation θ_f.

From the kinematic model (7.26) it is apparent that the above motion prescribes the sequential activation of u_2, u_1 and u_2 again. The value of each control input depends on the desired velocity profile for the generalized coordinates. ■

The applicability of this approach is limited to very simple structures, such as the unicycle, in which our geometric intuition is sufficient. A more systematic way to solve the problem is to note that there exist *canonical* model structures for which the steering problem can be solved efficiently, and to transform the considered nonholonomic systems into one of these forms. The most common one is the so-called chained form.

Chained Forms

Consider a driftless two-input control system of the form

$$\begin{aligned}\dot{q}_1 &= u_1\\ \dot{q}_2 &= u_2\\ \dot{q}_3 &= q_2u_1\\ \dot{q}_4 &= q_3u_1\\ &\vdots\\ \dot{q}_n &= q_{n-1}u_1,\end{aligned} \qquad \text{or} \qquad \dot{q} = g_1(q)u_1 + g_2(q)u_2, \tag{7.45}$$

with

$$g_1 = \begin{bmatrix} 1\\ 0\\ q_2\\ q_3\\ \vdots\\ q_{n-1}\end{bmatrix}, \quad g_2 = \begin{bmatrix} 0\\ 1\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}.$$

System (7.45) is a special case of *chained form* called *one-chain system.* If we set $u_1 = 1$, the system becomes linear and behaves like a chain of integrators from q_2 to q_n, driven by the input u_2. Note that the two-input case is sufficiently broad to cover most of the kinematic models of practical wheeled mobile robots. For the case $m \geq 3$ and for the corresponding *multi-chain* forms, the reader is referred to [54], where these canonical forms were originally introduced.

The structure of system (7.45) is particularly interesting, in that controllability is achieved via g_1, g_2 and (repeated) Lie brackets of the form $\mathrm{ad}^k_{g_1} g_2$, where

$$\mathrm{ad}_{g_1} g_2 = \mathrm{ad}^1_{g_1} g_2 = [g_1, g_2], \qquad \mathrm{ad}^k_{g_1} g_2 = [g_1, \mathrm{ad}^{k-1}_{g_1} g_2].$$

In fact, a simple computation shows that

$$\mathrm{ad}^k_{g_1} g_2 = \begin{bmatrix} 0\\ \vdots\\ (-1)^k\\ \vdots\\ 0\end{bmatrix}, \qquad 1 \leq k \leq n-2,$$

where the nonzero element is the $(k+2)$-nd entry. As a consequence, the n vector fields

$$\{g_1, g_2, \mathrm{ad}^1_{g_1} g_2, \ldots, \mathrm{ad}^{n-2}_{g_1} g_2\}$$

are linearly independent everywhere, and the controllability rank condition is satisfied. Equivalently, we may say that system (7.45) is completely nonholonomic, with degree of nonholonomy $\kappa = n-1$, growth vector $r = (2, 3, 4, \ldots, n)$, and relative growth vector $\sigma = (2, 1, 1, \ldots, 1)$. In connection with the foregoing calculation, u_1 is called the *generating* input, while q_1 and q_2 are often referred to as *base variables.* Note also that, for $k > n-2$, the repeated Lie brackets are identically zero; this property of the system is called *nilpotency.*

It is natural to ask how general the chained form (7.45) is. In particular, we are interested in conditions for converting the driftless control system (7.44) into chained form by means of the invertible input transformation $v = \beta(x)u$ and the change of coordinates $z = \phi(q)$. If these conditions hold, then we can design the controller for the chained form and apply a precompensator to the system, that performs the input and state transformations.

Recently, Murray [55] established a set of necessary and sufficient conditions for the conversion of a two-input system into chained form. Let $\Delta = \mathrm{span}\,\{g_1, g_2\}$ and define the two filtrations:

$$\begin{aligned} E_1 &= \Delta & F_1 &= \Delta \\ E_2 &= E_1 + [E_1, E_1] & F_2 &= F_1 + [F_1, F_1] \\ &\vdots & &\vdots \\ E_{i+1} &= E_i + [E_i, E_i], & F_{i+1} &= F_i + [F_i, F_1]. \end{aligned}$$

Then, system (7.44), with $m = 2$, can be cast in chained form if and only if

$$\dim E_i = \dim F_i = i + 1, \qquad i = 1, \ldots, n-1.$$

By applying this condition, one can show that completely nonholonomic systems with two inputs and relative growth vector $(2, 1)$ (obtained for $n = 3$) or $(2, 1, 1)$ (obtained for $n = 4$) can always be put in chained form.

The proof of this result relies on the theory of exterior differential systems, and in particular on the *Goursat normal form,* which is the dual of the chained form. The constructive procedure given in the proof has general validity, and has been used for example to find local transformations which convert the N-trailer system into chained form (see [41, 56]).

However, there is a simpler constructive algorithm based on the following *sufficient* condition for local conversion of a two-input system to chained form. Define the distributions

$$\begin{aligned} \Delta_0 &= \mathrm{span}\,\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-2} g_2\} \\ \Delta_1 &= \mathrm{span}\,\{g_2, \mathrm{ad}_{g_1} g_2, \ldots \mathrm{ad}_{g_1}^{n-2} g_2\} \\ \Delta_2 &= \mathrm{span}\,\{g_2, \mathrm{ad}_{g_1} g_2, \ldots, \mathrm{ad}_{g_1}^{n-3} g_2\}. \end{aligned}$$

If for some open set U, $\dim \Delta_0 = n$ and Δ_1, Δ_2 are involutive, and there exists a smooth function $h_1 : U \mapsto I\!R^n$ such that

$$dh_1 \cdot \Delta_1 = 0 \qquad \text{and} \qquad dh_1 \cdot g_1 = 1, \tag{7.46}$$

then there exists a local feedback transformation and change of coordinates that transform the system into chained form.

In particular, the change of coordinates $z = \phi(q)$ is given by

$$\begin{aligned} z_1 &= h_1 \\ z_2 &= L_{g_1}^{n-2} h_2 \\ &\vdots \\ z_{n-1} &= L_{g_1} h_2 \\ z_n &= h_2, \end{aligned}$$

with h_2 independent from h_1 and such that

$$dh_2 \cdot \Delta_2 = 0. \tag{7.47}$$

The existence of independent h_1 and h_2 with the above properties is guaranteed by Frobenius' theorem, since Δ_1 and Δ_2 are involutive. Using the invertible input transformation

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= \left(L_{g_1}^{n-1} h_2\right) u_1 + \left(L_{g_2} L_{g_1}^{n-2} h_2\right) u_2 \end{aligned}$$

results in the transformed system

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1. \end{aligned}$$

To apply this procedure, we must solve two sets of partial differential equations, namely, eqs. (7.46)–(7.47). If g_1 and g_2 have the special form

$$g_1 = \begin{bmatrix} 1 \\ g_{12}(q) \\ \vdots \\ g_{1n}(q) \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 \\ g_{22}(q) \\ \vdots \\ g_{2n}(q) \end{bmatrix},$$

with arbitrary g_{ij}'s, then it is easy to verify that Δ_1 is always involutive and we can choose $h_1 = q_1$. In this case, we only have to verify that Δ_2 is involutive, and solve the associated partial differential equation (7.47). To this end, one may in general use the constructive procedure given in the proof of Frobenius' theorem (see [4, p. 26]). Note that it is always possible to cast g_1 and g_2 in the above special form, by reordering variables and by virtue of the independence assumption on the input vector fields.

Example. Consider the kinematic model of the rear-wheel drive car-like robot, given by eq. (7.27). First, scale the input u_1 so that it enters directly into $\dot{x}$. Denoting by $\tilde{u}_1$ the new input and setting $\tilde{u}_2 = u_2$, we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\theta \\ \frac{1}{\ell}\sec\theta\tan\phi \\ 0 \end{bmatrix} \tilde{u}_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u}_2.$$

Since g_1 and g_2 have the special structure, we may set $h_1 = x$. Also, it is immediate to take the y coordinate as h_2, since it annihilates the distribution Δ_2. The resulting change of coordinates is

$$\begin{aligned} z_1 &= x \\ z_2 &= \frac{1}{\ell}\sec^3\theta\tan\phi \\ z_3 &= \tan\theta \\ z_4 &= y, \end{aligned}$$

while the input transformation $\tilde{u} = \beta^{-1}(q)v$ is

$$\begin{aligned} \tilde{u}_1 &= v_1 \\ \tilde{u}_2 &= -\frac{3}{\ell}\frac{\sin\theta}{\cos^2\theta}\sin^2\phi\, v_1 + \ell\cos^3\theta\cos^2\phi\, v_2. \end{aligned}$$

The transformed system becomes

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ \dot{z}_4 &= z_3 v_1. \end{aligned}$$

Note that the above change of coordinates (and thus, the obtained chained form) is only locally defined. ■

As another example, the conversion of the unicycle model into chained form will be given in Sect 7.8.2. Note that the aforementioned sufficient condition is not satisfied for the N-trailer robot, which nevertheless can be put in chained form by resorting to the general constructive procedure, as mentioned above.

We point out that there are other canonical forms that can be successfully used for steering nonholonomic systems, namely the *Čaplygin form* (see the end of Sect 7.7) and the *power form*. It is noteworthy that, for $m = 2$ inputs, the three canonical forms are equivalent, since there exist global coordinate transformations that allow to convert one into the others [57].

Some techniques for the open-loop control of chained forms are introduced below. In particular, we shall briefly review the use of

1. sinusoidal inputs

2. piecewise-constant inputs

3. polynomial inputs

for steering the two-input chained form (7.45), and report some simulation results for the unicycle and the PPR robot.

Sinusoidal Inputs

An effective approach to the open-loop control of system (7.45) is provided by *optimal control.* In fact, the problem of identifying trajectories minimizing a given cost functional in the presence of a differential constraint may be solved with the powerful tools of the Calculus of Variation, namely, the Euler-Lagrange equations and Pontryagin's Maximum Principle. Reasonable cost functionals are the path length, the input energy or the traveling time.

The sinusoidal steering method of [54] for chained-form systems was inspired by Brockett's work on optimal control for a particular class of systems [58]. As a matter of fact, chained forms were introduced as a generalization of special first- and second-order controllable systems for which sinusoidal steering minimizes the integral norm of the input.

The control algorithm consists of two phases.

1. Steer the base variables q_1 and q_2 to their desired values q_{f1} and q_{f2} in finite time t_1, by using suitable inputs u_1, u_2. Depending on the particular input choice, the remaining variables $q_3, \ldots, q_n$ will move to $q_3(t_1), \ldots, q_n(t_1)$.

2. For each k, $1 \leq k \leq n-2$, steer q_{k+2} to its final value $q_{f,k+2}$ using

$$u_1 = \alpha \sin \omega t \qquad u_2 = \beta \cos k\omega t$$

over one period $T = 2\pi/\omega$, where α, β are such that

$$q_{f,k+2} - q_{k+2}(t_1 + (k-1)T) = \frac{\alpha^k \beta}{k!(2\omega)^k} = \frac{\alpha^k \beta}{k!(4\pi)^k} \cdot T^k.$$

Note that:

- The total time needed to steer the system from q_0 to q_f is $t_f = t_1 + (n-2)T$. The choice of a small value of T leads to a faster overall motion but, accordingly, to larger applied inputs.

- The first phase may also be performed in feedback, while the second specifies $u(t)$ only in a feedforward mode. In particular, one may even use a linear (smooth) controller for the first phase, by allowing an arbitrarily small error tolerance in the final positioning of q_1 and q_2.

- The second phase is organized so to adjust one variable q_{k+2} at a time. At the end of each period, one has $q_i(t_1+kT) = q_i(t_1+(k-1)T)$ for $i < k+2$. However, phase 2 can be executed also all at once, by using simultaneous sinusoids at different frequencies [59].

- The inputs prescribed by the sinusoidal steering method are optimal only in the second phase. As a consequence, the trajectories thus produced are suboptimal for general reconfiguration tasks.

A related method, which applies to more general driftless systems, was proposed in [60].

Piecewise-Constant Inputs

Another effective method to steer chained-form systems comes from multirate digital control [61] and is based on the use of piecewise-constant inputs. As already noted, under such kind of control, system (7.45) behaves as a piecewise-linear system. Hence, forward integration of the motion equations is very simple.

Divide the total traveling time t_f from q_0 to q_f in $n-1$ subintervals of duration δ over which constant inputs are applied:

$$\begin{aligned} u_1(\tau) &= u_1 \\ u_2(\tau) &= u_{2k}, \end{aligned} \qquad \tau \in [(k-1)\delta, k\delta), \quad k = 1, \dots, n-1,$$

Note that u_1 is kept always constant with value

$$u_1 = \frac{q_{f1} - q_{01}}{t_f},$$

while the $n-1$ constant values of input u_2

$$u_{21}, u_{22}, \dots, u_{2,n-1}$$

are found from the forward integration of the model equations, that leads to a triangular system linear in the unknown u_{2k}'s, with the initial error components $q_{fi} - q_{0i}$, for $i = 2, \dots, n$, appearing in the constant term.

If $q_{f1} = q_{01}$, an intermediate point q_{int} must be added. Also, in principle, one may choose small values of δ to obtain fast convergence, but large inputs will be required as a consequence, similarly to the choice of the period T in the sinusoidal steering method.

A more general approach to design piecewise-constant controllers for system (7.44) has been proposed in [62]. This algorithm achieves exact steering for the case of nilpotent systems—such as chained-form systems—and approximate steering in general cases.

Polynomial Inputs

The idea of using polynomial inputs for steering chained systems is similar to the approach based on piecewise-constant inputs, but with added differentiability properties. Provided that $q_{f1} \neq q_{01}$, let

$$\begin{aligned} u_1 &= \operatorname{sign}(q_{f1} - q_{01}) \\ u_2 &= c_0 + c_1 t + \ldots + c_{n-2} t^{n-2}. \end{aligned}$$

The total traveling time from q_0 to q_f is $t_f = q_{f1} - q_{01}$, while the $n-1$ unknown c_i's are found as before by imposing the desired overall reconfiguration to the symbolically integrated equations. This requires the solution of a system of the form

$$M(t_f) \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \end{bmatrix} + m(q_0, t_f) = \begin{bmatrix} q_{f2} \\ q_{f3} \\ \vdots \\ q_{fn} \end{bmatrix},$$

where $M(t_f)$ is a nonsingular matrix if $t_f \neq 0$ [56].

As a final remark, pertinent to all the above steering methods, it should be stressed that motion is planned for the chained form (7.45), typically obtained only after a change of coordinates $z = \phi(q)$. Once the motion is mapped back to the original coordinates q, the obtained trajectories may be rather unnatural.

7.8.2 Open-Loop Control: Case studies

We report below simulation results of open-loop controllers for two nonholonomic systems, namely, the unicycle and the PPR robot under pseudoinversion. For the first system, the application of sinusoidal, piecewise-constant and polynomial controls will be considered. The second system will be used instead to introduce another similar technique for steering nonholonomic systems, based on the concept of holonomy angle.

Unicycle

The kinematic model (7.26) of the unicycle can be easily converted into chained form by letting

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta \end{aligned}$$

and

$$u = \begin{bmatrix} z_3 & 1 \\ 1 & 0 \end{bmatrix} v,$$

thereby obtaining

$$\begin{aligned}\dot{z}_1 &= v_1\\ \dot{z}_2 &= v_2\\ \dot{z}_3 &= z_2 v_1.\end{aligned}$$

The above input and state transformation are globally defined.

The three steering methods introduced before have been simulated for two different initial conditions. In both cases, the unicycle must reach the final desired configuration $q_f = (0,0,0)$. In case I, the system starts from $q_0 = (-1, 3, \pi/3)$, while, in case II, from $q_0 = (1, 3, \pi/3)$. The simulation results in Figs. 7.9–7.11 show the planned Cartesian paths. Note that the sinusoidal steering method requires larger motions, due to its two-phase structure, while the other two controllers behave similarly. Moreover, the presence of cusps in the path corresponds to motion inversions with zero velocity, similarly to what occurs in the parking maneuvers of a car.

PPR robot

Consider again the kinematic model of the PPR robot of Fig. 7.8 under weighted pseudoinversion, as expressed by eq. (7.37). A particular class of reconfiguration tasks is the one in which the initial and final configurations provide the same end-effector position, that is

$$k(q_f) = k(q_0). \tag{7.48}$$

Correspondingly, as the robot moves under the task control input u, the end-effector will trace a closed path. Let T be the desired duration of a reconfiguration cycle. One way to find suitable open-loop commands is to transform system (7.37) into chained form, which is certainly possible for a nonholonomic system with $m = 2$ and $n = 3$, and use one of the previous methods. Below, we show a different approach.

System (7.37) is conveniently rewritten in terms of the new set of coordinates (p_x, p_y, q_3) as

$$\begin{aligned}\dot{p}_x &= u_1\\ \dot{p}_y &= u_2\\ \dot{q}_3 &= \alpha(-s_3 u_1 + c_3 u_2),\end{aligned}$$

with $\alpha = \ell/(w^2 + \ell^2)$. These equations show that, in order to have a cyclic end-effector motion, the command $u(t)$ should be cyclic with the same period T. Typically, a parameterized class of inputs (say, piecewise-constant) is chosen for ease of computation.

For this particular robot, reconfiguring the arm amounts to driving q_3 to its final value q_{f3}. Accordingly, q_1 and q_2 will move to the unique values q_{f1} and q_{f2} obtained from the direct kinematic constraint (7.48). The third model equation can be used to compute the so-called *holonomy angle* [50]

$$\gamma = \oint_\Gamma dq_3(u), \tag{7.49}$$

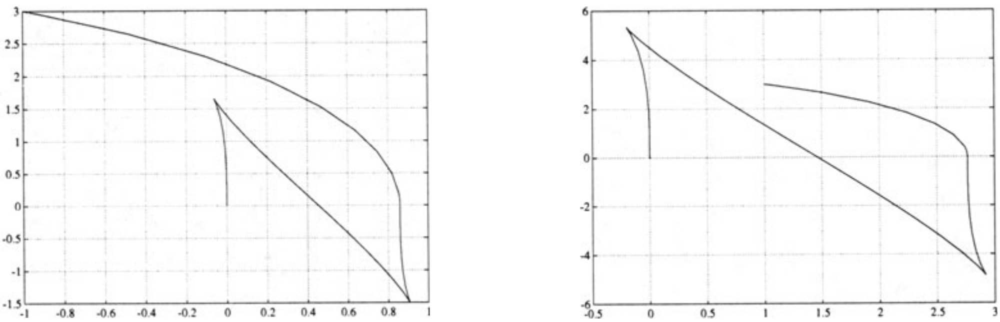

Figure 7.9: Sinusoidal open-loop control of the unicycle: case I (left) and II (right)

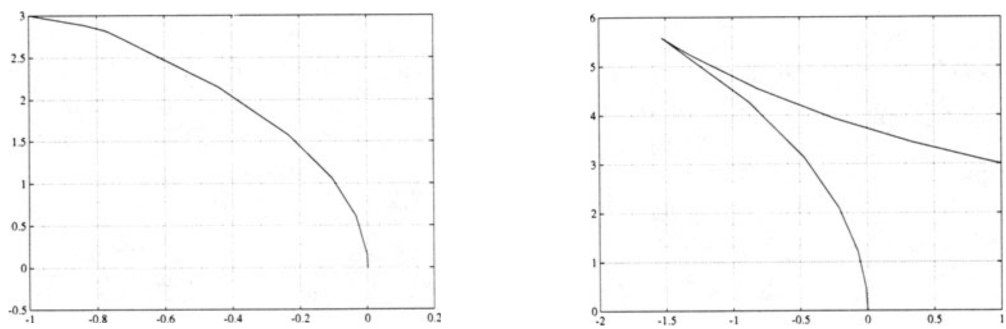

Figure 7.10: Piecewise-constant open-loop control of the unicycle: case I (left) and II (right)

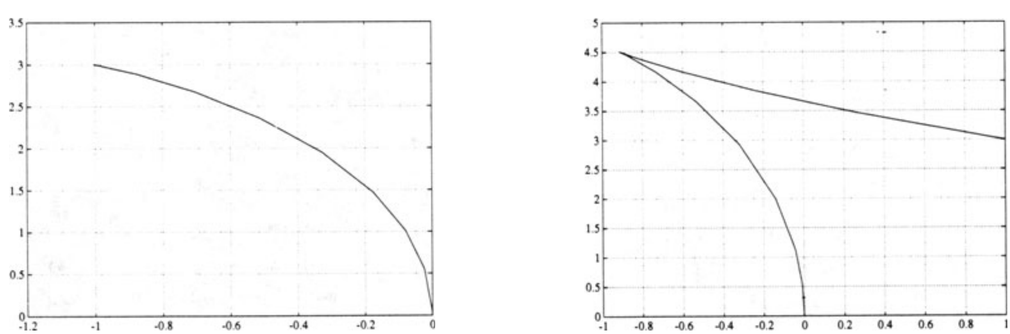

Figure 7.11: Polynomial open-loop control of the unicycle: case I (left) and II (right)

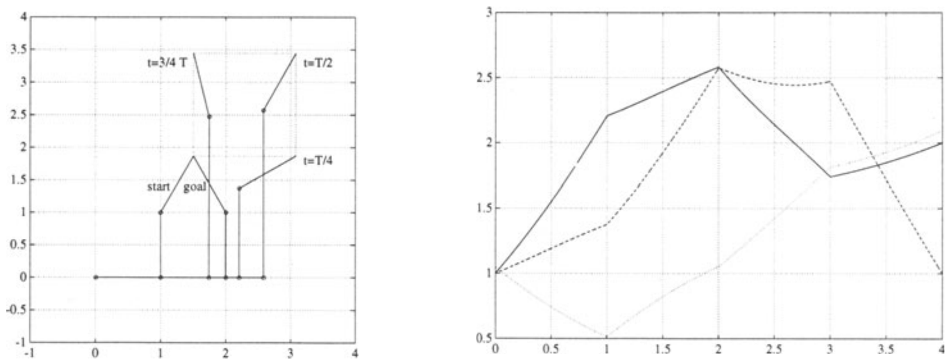

Figure 7.12: Open-loop control of the PPR robot: Cartesian motion (left) and joint motion (right)

which measures the third joint drift after one cycle Γ of u, due to the nonholonomic behavior under pseudoinversion. We would like to solve the inverse problem; i.e., given $\gamma = q_{f3} - q_{03}$, determine the parameter values that identify Γ in the chosen class of inputs such that eq. (7.49) is satisfied.

Consider as input class a sequence of piecewise-constant velocity commands, in which only one input is active at each instant:

$$u(t) = \begin{cases} u_1(t) = 4\Delta/T,\ u_2(t) = 0, & t \in [0, T/4), \\ u_1(t) = 0,\ u_2(t) = 4\Delta/T, & t \in [T/4, T/2), \\ u_1(t) = -4\Delta/T,\ u_2(t) = 0, & t \in [T/2, 3T/4), \\ u_1(t) = 0,\ u_2(t) = -4\Delta/T, & t \in [3T/4, T). \end{cases}$$

For a given T, this class of paths is parameterized by Δ. The robot end-effector will trace a square of side Δ in the counterclockwise direction, located right and above $p_0 = k(q_0)$ for $\Delta > 0$, or left and below p_0 for $\Delta < 0$. For the special case $\ell = w = 1$ ($\alpha = 1/2$), integration of the third model equation gives

$$\tan\left(\frac{q_3(T)}{2} + \frac{\pi}{4}\right) = \frac{(e^{\Delta/2} + e^{-\Delta/2}) + (1 + 2e^{-\Delta/2} - e^{\Delta})\tan(q_{03}/2)}{(1 + 2e^{\Delta/2} - e^{\Delta}) - (e^{\Delta/2} + e^{-\Delta/2})\tan(q_{03}/2)}. \tag{7.50}$$

Imposing the desired reconfiguration γ from q_{03} and using the transformation $x = e^{\Delta/2}$ leads to a 4-th order polynomial equation to be solved. Only the real positive roots of the polynomial equation are relevant: for a root $x > 1$, we have $\Delta > 0$ and an upper-right square; for a root $0 < x < 1$, $\Delta < 0$ and a lower-left square. Indeed, $x = 1$ ($\Delta = 0$) is a root only if $\gamma = 0$.

This method has been simulated for $q_0 = (1, 1, \pi/3)$, $q_f = (2, 1, 2\pi/3)$, and $T = 4$ s. The corresponding reconfiguration is $\gamma = \pi/3$, while the initial and final Cartesian position of the end-effector is $k(q_f) = k(q_0) = (1.5, \sqrt{3}/2)$. The desired reconfiguration was obtained along a square path of side $\Delta = 1.578$, the solution of eq. (7.50) with the smallest magnitude. Figure 7.12 shows the motion of the arm along the square path and the evolution of the joint variables. The peaks correspond to the sudden changes of the end-effector velocity at the square corners.

7.8.3 Feedback Control

We shall now give a brief review of existing methods for the feedback stabilization of nonholonomic systems. As already said, only nonsmooth or time-varying feedback laws are of interest for the stabilization to an equilibrium point—the only problem we are considering here. Interestingly enough, the case of nonholonomic systems is one in which it is more difficult to drive the system via feedback to a fixed point (*regulation problem*) than towards a reference trajectory (*tracking problem*), contrary to the usual belief.

Nonsmooth Feedback

The simpler approach to designing feedback controllers for nonholonomic systems is probably the one proposed in [63]. It consists of two stages: first, find an open-loop strategy that can achieve any desired reconfiguration for the particular system under consideration. Second, transform the motion sequence into a succession of equilibrium manifolds, which are then stabilized by feedback. The overall resulting feedback is necessarily discontinuous, because of the switching of the target manifolds. Each stabilization problem in the succession should be completed in finite time (that is, not just asymptotically), so as to have a well-defined procedure. In order to achieve such convergence behavior, discontinuous feedback is used within each stabilizing phase.

As an example, consider the unicycle reconfiguration strategy given at the beginning of Section 7.8.1. The three equilibrium manifolds are in this case:

$$\begin{array}{lrcl} 1. & S_1 & = & \{(x, y, \theta) \,|\, \theta = \text{atan}\,((y_f - y_0)/(x_f - x_0))\} \\ 2. & S_2 & = & \{(x, y, \theta) \,|\, x = x_f, y = y_f\} \\ 3. & S_3 & = & \{(x, y, \theta) \,|\, \theta = \theta_f\}. \end{array}$$

The design of feedback laws that stabilize these manifolds is straightforward:

$$\begin{array}{lrcl} 1. & u_1 & = & 0,\ u_2 = k_\theta\,(\text{atan}\,((y_f - y_0)/(x_f - x_0)) - \theta) \\ 2. & u_1 & = & -k_p e_p,\ u_2 = 0 \\ 3. & u_1 & = & 0,\ u_2 = k_\theta\,(\theta_f - \theta)\,, \end{array}$$

where $k_\theta > 0$, $k_p > 0$, and $e_p = \sqrt{(x_f - x)^2 + (y_f - y)^2}$. Note that only u_2 is needed for the first and the third manifolds, while only u_1 is needed for the second. A similar approach achieving stability with exponential rate of convergence for the unicycle is presented in [64].

The weakness of this approach is that it requires the ability to devise an open-loop strategy for the system. Moreover, any perturbation occurring on a variable that is not directly controlled during the current phase will result in a final error. As a result, feedback robustness is achieved only with respect to perturbation of the initial conditions. In some systems, a reasonable alternative solution is to perform in feedback mode a cyclic motion yielding the desired reconfiguration, like in the holonomy angle method for the PPR robot (Section 7.8.2). We shall see an example of this method in Section 7.8.4.

Time-Varying Feedback

The idea of allowing the feedback law to depend explicitly on time is due to Samson [65, 66], who presented smooth stabilization schemes for the unicycle and the car-like kinematic models. The use of time-varying feedback is suggested by the argument below.

Consider again the unicycle model (7.26), and denote by $q_r(t) = (x_r(t), y_r(t), \theta_r(t))$ an admissible reference trajectory obtained under the input $u_r(t) = (u_{r1}(t), u_{r2}(t))$, i.e.,

$$\begin{aligned}
\dot{x}_r &= \cos\theta_r \, u_{r1} \\
\dot{y}_r &= \sin\theta_r \, u_{r1} \\
\dot{\theta}_r &= u_{r2}.
\end{aligned}$$

As already noted in general for nonholonomic systems, the approximate linearization of system (7.26) around a *fixed point* leads to an uncontrollable driftless linear system. Instead, consider the linearization of system (7.26) around the *reference trajectory* $q_r(t)$. Let

$$\delta x = x - x_r \quad \delta y = y - y_r \quad \delta\theta = \theta - \theta_r \quad \delta u_i = u - u_{ri}, \; i = 1, 2.$$

We have, then, the approximate linearization

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta y} \\ \dot{\delta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin\theta_r \, u_{r1} \\ 0 & 0 & \cos\theta_r \, u_{r1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta\theta \end{bmatrix} + \begin{bmatrix} \cos\theta_r & 0 \\ \sin\theta_r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}, \tag{7.51}$$

or

$$\dot{\delta q} = A(t)\delta q + B(t)\delta u,$$

that is, a linear time-varying system. It is easy to see that the controllability rank condition (7.18) is satisfied provided that $u_{r1} \neq 0$, i.e., if there is nominal linear motion. In fact

$$\text{rank} \left[B(t) \; A(t)B(t) \; A^2(t)B(t) \right] = \text{rank} \begin{bmatrix} \cos\theta_r & 0 & 0 & -\sin\theta_r \, u_{r1} & 0 & 0 \\ \sin\theta_r & 0 & 0 & \cos\theta_r \, u_{r1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} = 3 = n.$$

It can be shown [67] that system (7.51) is easily stabilizable to zero by using linear or nonlinear time-invariant smooth feedback. This result is mainly due to the presence of a time-varying drift term in the *linearized model.*

Conversely, when considering the point-stabilization of a time-invariant nonholonomic system, the introduction of a time-varying component in the *control law* may lead to a smoothly stabilizable system. Indeed, this intuition should be worked out and supported by rigorous arguments, as done in [65, 66]. In particular, stability proofs rely on identifying a suitable time-varying Lyapunov function, and make use of LaSalle's techniques. Coron [68] and, particularly, Pomet [69] have extended these results to more general classes of controllable systems without drift. When applied to nonholonomic systems, the typical drawbacks of these methods are an erratic behavior, slow convergence towards the equilibrium point and a difficult tuning of the various parameters of the controller.

A hybrid strategy for the stabilization of the unicycle has been proposed in [70], namely, combining the advantages of smooth static feedback far from the target and time-varying feedback close to the target. The application of time-varying feedback for stabilizing chained-form and power-form systems has been investigated in [71] and [72], respectively.

Another steering method that falls into this class is a generalization of the sinusoidal steering techniques that we have seen in detail for the open-loop case [73]. The basic idea is to weight the amplitude of the sinusoidal inputs with the error in the corresponding variable. A nonsmooth time-varying controller for stabilization of chained-form systems with exponential rate of convergence has been proposed in [74]. Finally, we mention that feedback laws based on *dynamic compensation* [53, 75] may also be viewed as time-varying stabilizers.

7.8.4 Feedback Control: Case Studies

We present below numerical simulations of feedback controllers for two nonholonomic systems, viz. the unicycle and the three-body space robot. For the unicycle, results obtained with time-varying feedback will be shown and compared with those obtained with the open-loop controllers given in Sect. 7.8.2. For the three-body space robot, we shall devise a nonsmooth control strategy that is a feedback version of the holonomy angle method, already introduced for the PPR robot.

Unicycle

The time-varying controller of [65] and the hybrid controller of [70] have been simulated for the same cases I and II considered in Section 7.8.2. The obtained results are shown in Figs. 7.13–7.14, respectively. Note the poor convergence of the purely time-varying controller.

By comparing these results with those obtained with any open-loop controller (Figs. 7.9–7.11), it appears that the latter produce more reasonable trajectories. On

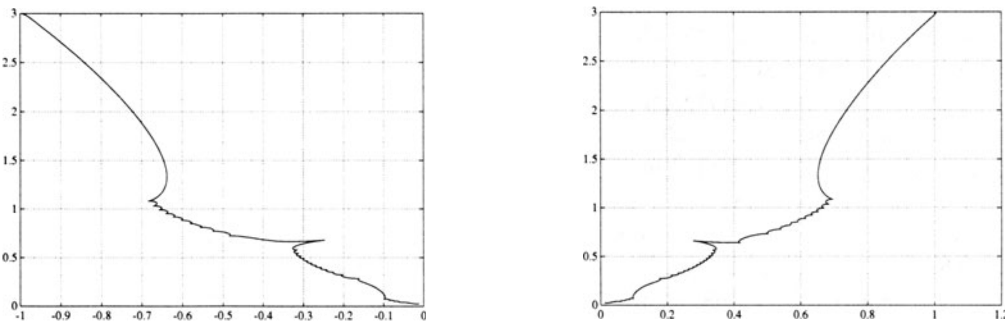

Figure 7.13: Time-varying feedback control of the unicycle: case I (left) and II (right)

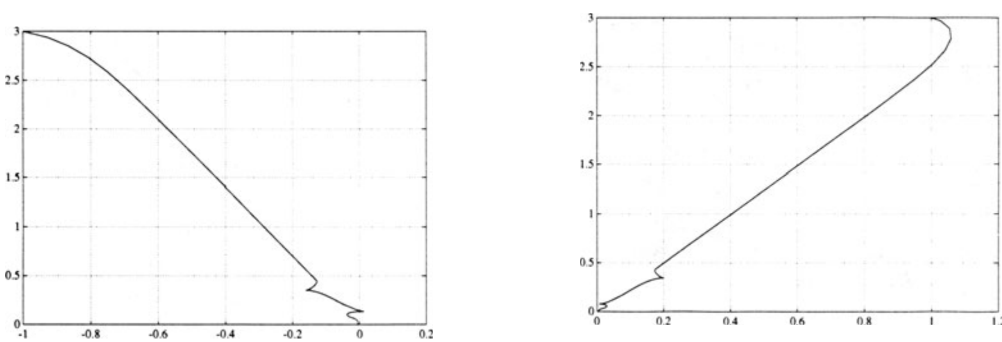

Figure 7.14: Hybrid feedback control of the unicycle: case I (left) and II (right)

the other hand, a feedback law could achieve the desired configuration even in the presence of disturbances, or starting from uncertain initial conditions. This suggests that a practical approach to controlling nonholonomic mobile robots requires a planning phase, in which a feasible trajectory for point-to-point motion is produced, along with a feedback control phase to guarantee robust execution of the planned trajectory.

Three-Body Space Robot

Consider the three-body space robot, whose kinematic model is obtained from the two input vector fields (7.34), or

$$\dot{\theta}_1 = s_1(\phi_1, \phi_2)u_1 + s_2(\phi_1, \phi_2)u_2 \tag{7.52}$$
$$\dot{\phi}_1 = u_1 \tag{7.53}$$
$$\dot{\phi}_2 = u_2. \tag{7.54}$$

A simple open-loop strategy for achieving any reconfiguration is to drive first ϕ_1 and ϕ_2 to their desired values ϕ_{f1} and ϕ_{f2}, and then to perform a cyclic (say, square) motion on ϕ_1, ϕ_2 so that also θ_1 reaches its desired value θ_{f1}. Note that, since system (7.52)–(7.54) is in Čaplygin form, the forward integration for computing the side of the needed square is simplified, with respect to the PPR robot case of Section 7.8.2.

However, it is easy to implement the above motion sequence also in a feedback mode, with the two phases below:

1. Reach the desired shape ϕ_f in finite time t_1, by letting

$$\begin{aligned} u_1 &= k_\phi \, \mathrm{sign}\,(\phi_{f1} - \phi_1) \\ u_2 &= k_\phi \, \mathrm{sign}\,(\phi_{f2} - \phi_2), \end{aligned}$$

   with $k_\phi > 0$. The first body orientation θ_1 will go from θ_{01} to a value $\theta_1(t_1)$ that we can measure.

2. Compute, by forward integration of eq. (7.52), the side Δ of the square cycle on ϕ_1, ϕ_2 needed to achieve the desired holonomy angle $\gamma = \theta_{f1} - \theta_1(t_1)$, and apply the input sequence:

$$u(t) = \begin{cases} u_1(t) = k_\theta \, \mathrm{sign}\,(\phi_{f1} + \Delta - \phi_1), \ u_2(t) = 0, & t \in [t_1, t_2), \\ u_1(t) = 0, \ u_2(t) = k_\theta \, \mathrm{sign}\,(\phi_{f2} + \Delta - \phi_2), & t \in [t_2, t_3), \\ u_1(t) = k_\theta \, \mathrm{sign}\,(\phi_{f1} - \phi_1), \ u_2(t) = 0, & t \in [t_3, t_4), \\ u_1(t) = 0, \ u_2(t) = k_\theta \, \mathrm{sign}\,(\phi_{f2} - \phi_2), & t \in [t_4, t_5). \end{cases}$$

   with $k_\theta > 0$. Here, t_{i+1} $(i = 1, \dots, 4)$ is the finite time at which the i-th corner of the square cycle is reached.

The above feedback stabilization scheme has been simulated for a reconfiguration task from $q_0 = (30°, -30°, 30°)$ to $q_f = (90°, 30°, -30°)$. The robot parameters have been chosen as $m_i = 10$ kg, $\ell_i = 1$ m, and $d_i = 0.5$ m, for $i = 1, 2, 3$. It has been assumed that each body of the chain can be modeled as a uniform thin rod. The simulation results obtained with $k_\phi = k_\theta = 1$ are shown in Figs. 7.15–7.18.

The evolution of θ_1, ϕ_1 and ϕ_2 during the first phase is given in Fig. 7.15. At time $t_1 = 1.04$ s, the system reaches the desired shape $(\phi_{f1}, \phi_{f2}) = (30°, -30°)$, while $\theta_1(t_1) = 58.5°$. The holonomy angle that can be obtained by cycling *from this configuration* on a square of side Δ in the shape space is shown in Fig. 7.16, as a function of Δ. In this case, Δ has been chosen so as to yield a reconfiguration of 31.5° for θ_1. The corresponding evolution of θ_1, ϕ_1 and ϕ_2 during the second phase is shown in Fig. 7.17. Finally, Fig. 7.18 shows a stroboscopic view of the overall motion of the three-body system. Frames 1 and 14 correspond, respectively, to q_0 and q_f. Frames 1–6 refer to the first phase, while the cycling phase over ϕ_1, ϕ_2 is shown in frames 6–14. The number identifying each frame is always attached to the first body of the kinematic chain.

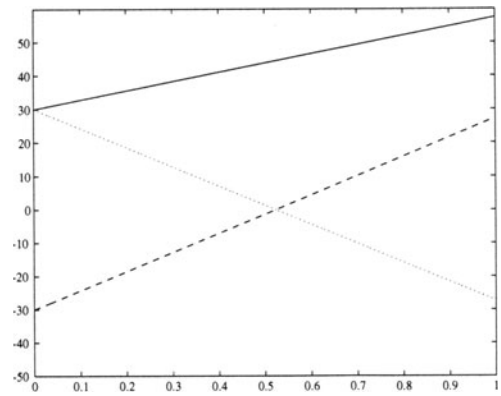

Figure 7.15: Evolution of $q = (\theta_1, \phi_1, \phi_2)$ during the first phase (in deg)

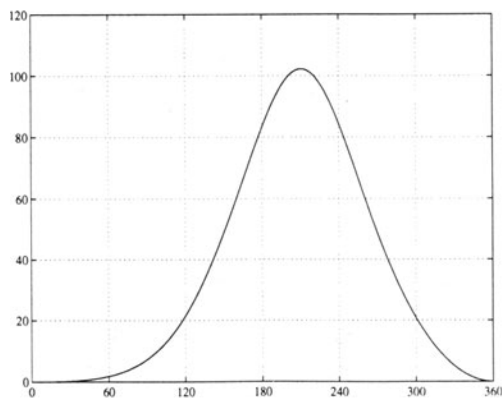

Figure 7.16: Holonomy angle for θ_1 vs. Δ (both in deg)

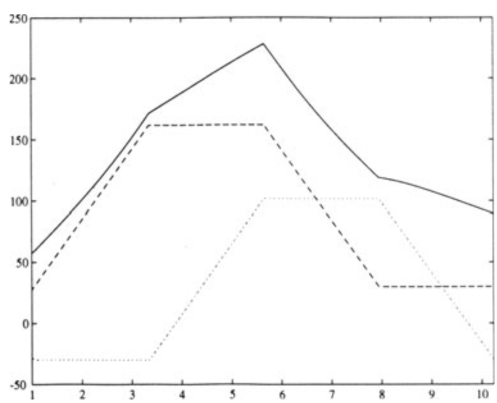

Figure 7.17: Evolution of $q = (\theta_1, \phi_1, \phi_2)$ during the second phase (in deg)

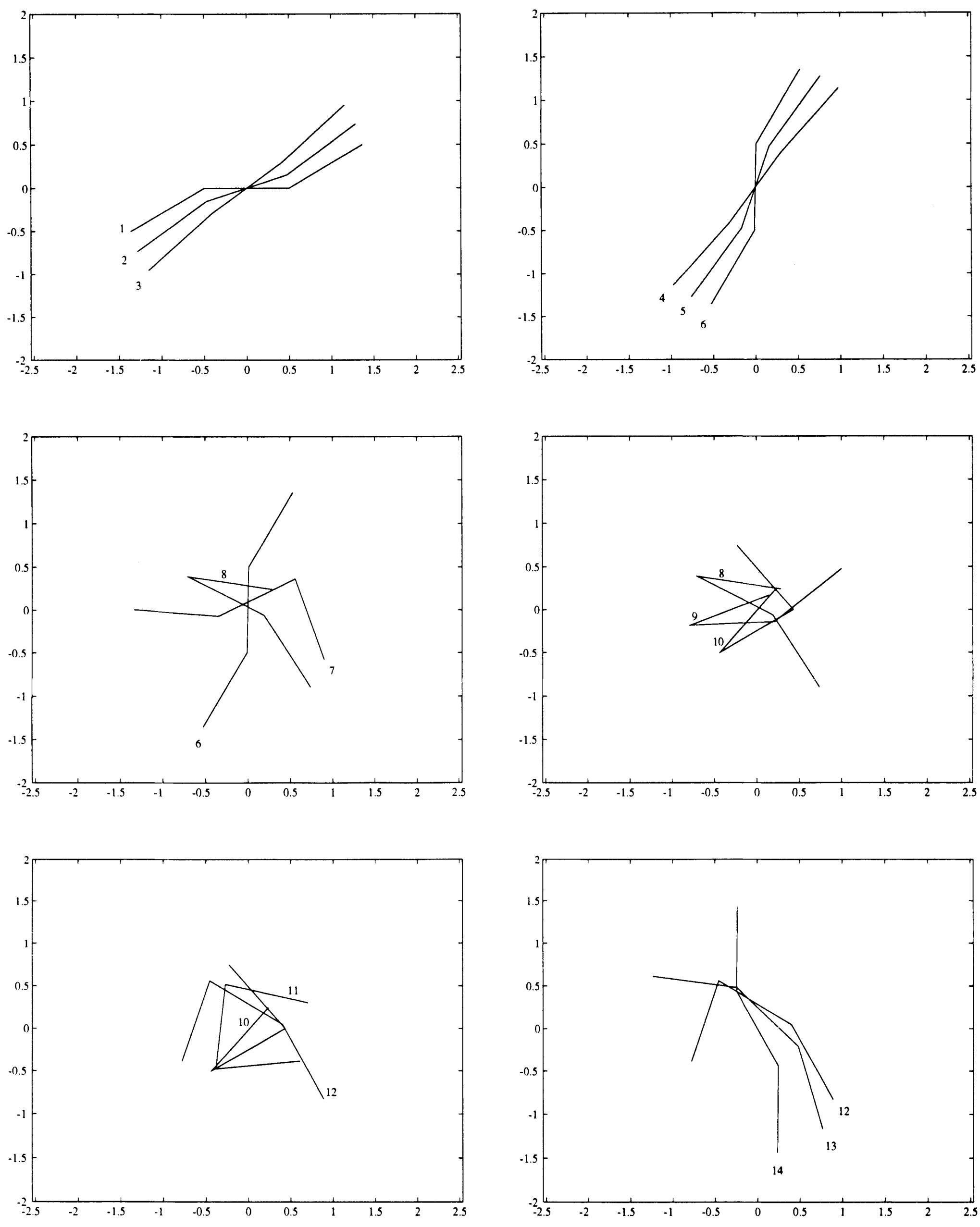

Figure 7.18: Feedback reconfiguration of the three-body space robot via holonomy angle method: stroboscopic view

7.8.5 Related Problems

The design of open-loop controllers for nonholonomic systems is often referred to as *motion planning*. However, to be consistent with the robotics literature, this term should indicate the search of feasible trajectories *in the presence of obstacles*. The existing strategies [60, 76] for solving this problem—indeed, a very difficult one—are based on the same approach, namely, first computing a collision-free holonomic path, and then approximating this path by an admissible nonholonomic one. In this context, it has been conjectured [12] that, for a free space of size ε (i.e., such that the minimum clearance in the configuration space $\mathcal{Q}$ is ε), the time complexity, in terms of number of maneuvers, of the motion planning problem is $O(\varepsilon^{-\kappa})$, where κ is the degree of nonholonomy of the system.

Finally, we point out that the absence of smooth stabilizability for nonholonomic systems is limited to the case of stabilization of an equilibrium point. Indeed, both stabilization to equilibrium manifolds [27, 38] and non-degenerate trajectories [77–79] are possible using smooth feedback.

Acknowledgments

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