Local Control Policies for Global Control and Navigation

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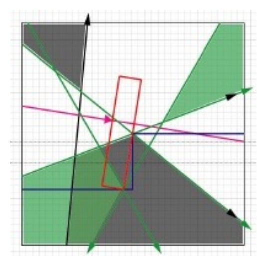


THE ROBOTICS INSTITUTE

Sensor Based Planning Lab

Microdynamic Systems Laboratory

Theme

Develop collections of local control policies

- Control policies respect local constraints
  - Obstacles
  - Dynamical constraints
- Convergence guaranteed over limited domain
- Global performance guaranteed by composition
- Deployment is automatic

Systems:

- Kinematic : $\dot{\mathbf{q}} = \mathbf{u}$
- Dynamical : $\ddot{\mathbf{q}} = \mathbf{u}$

Overview

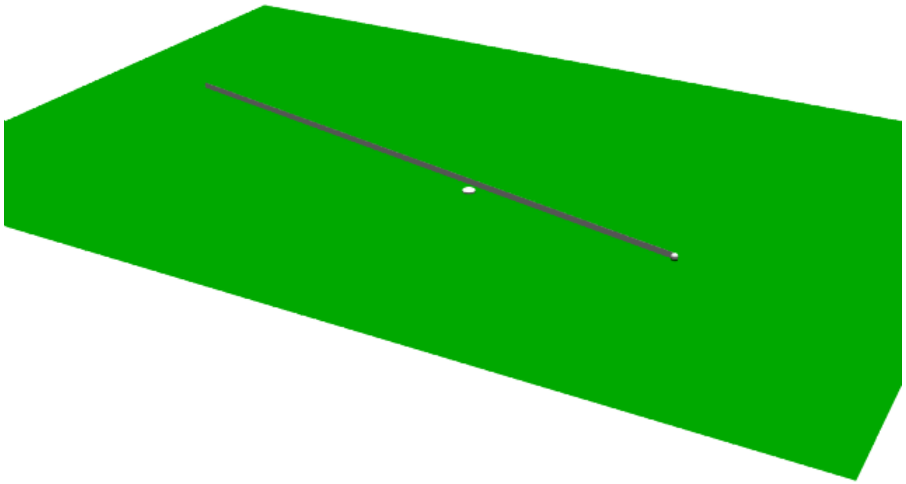
- GOLF: A Metaphor
  - Good OLd Fashioned Sequential Composition
  - Potential Fields
- Task and Spatial Decompositions
  - Convex Cellular Decompositions
  - Focus on $\mathbb{R}^2$ for clarity
- Local Control Policy Design
  - Kinematic Systems
  - Constrained Dynamical Systems
- Future Directions and Conclusion

GOLF : A Metaphor

- Open Loop
  - System:

    $$\dot{\mathbf{q}} = \mathsf{u} = \mathsf{u}_0$$
    $$\mathbf{q}, \mathsf{u} \in \mathbb{R}^2$$

  - Not robust to:
    - Modeling error
    - External disturbance



GOLF : A Metaphor

- Feedback

  System: $\dot{\mathbf{q}} = \mathsf{u} = f(\mathbf{q})$

  $$\mathbf{q}, \mathsf{u} \in \mathbb{R}^2$$

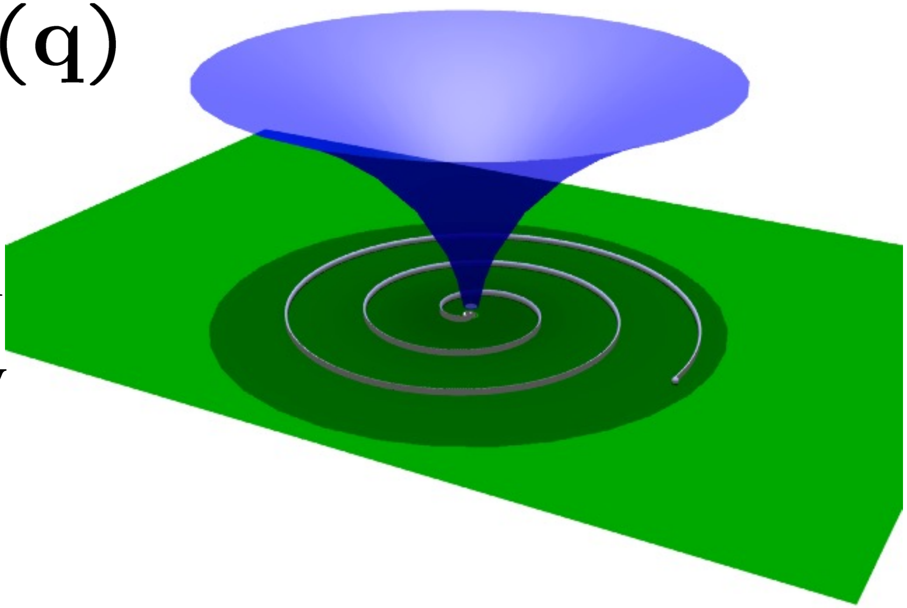
  – Disturbance rejection

  – Modeling uncertainty

- Lyapunov Methods

$$V = \frac{1}{2}\mathbf{q}^T\mathbf{q}$$

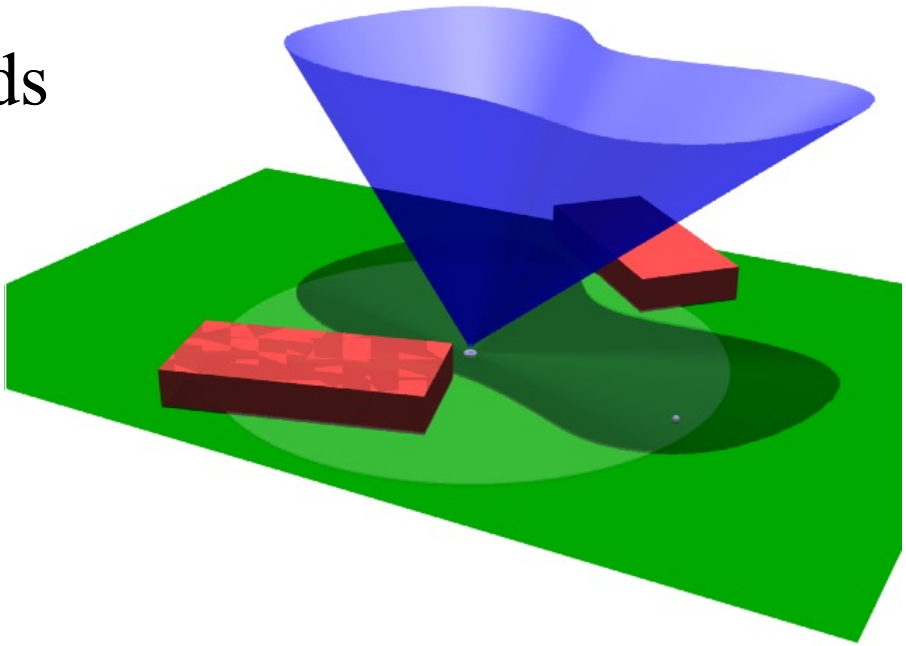
$$\dot{V} = \mathbf{q}^T\dot{\mathbf{q}} < 0 \quad \forall \|\mathbf{q}\| \neq 0$$



GOLF : A Metaphor

- Constraints
  - Potential Field Methods
  - Lyapunov Methods?

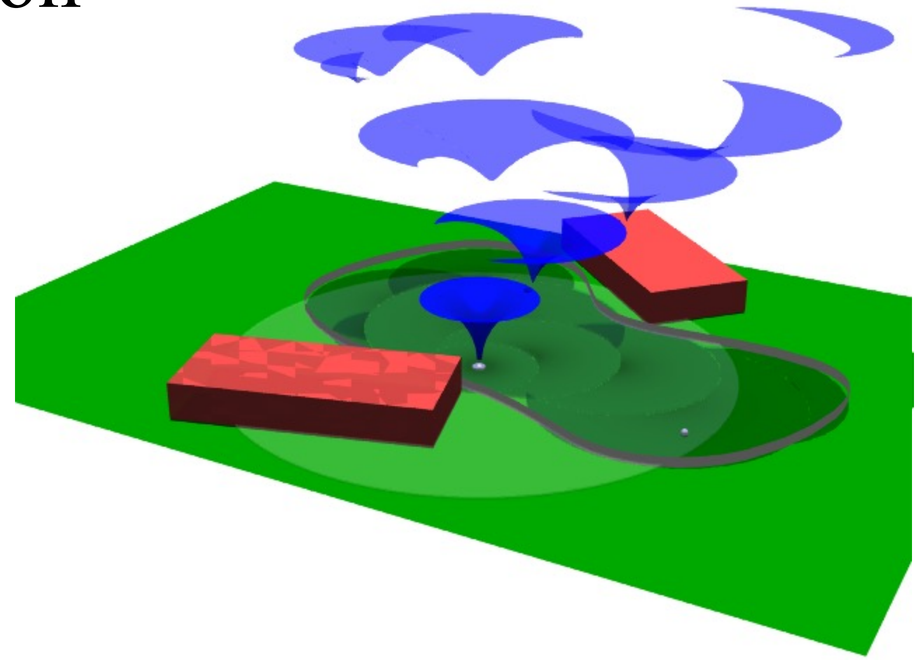
$$\forall \|\mathbf{q}\| \neq 0$$
$$V = g(\mathbf{q}) > 0$$



GOLF : A Metaphor

- Sequential Composition
  - Limited domain
  - Partial order
  - Prepares

$$\Phi_0 \succeq \Phi_1 \succeq \Phi_2$$

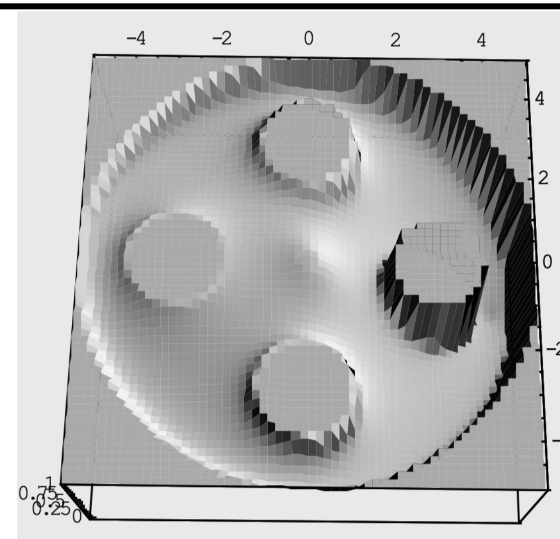


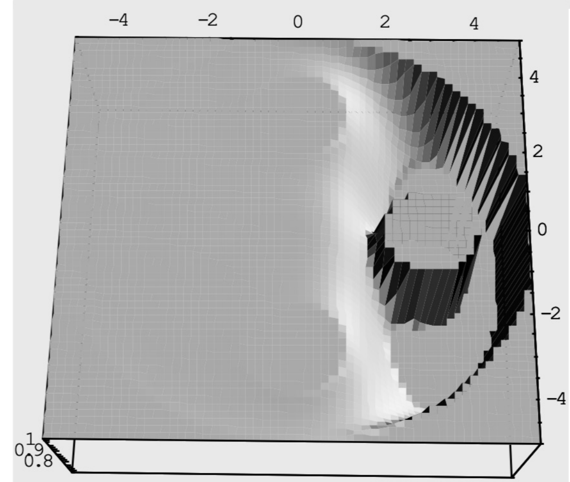
R. R. Burridge, A. A. Rizzi, and D. E. Koditschek. Sequential composition of dynamically dexterous robot behaviors. *International Journal of Robotics Research*, 18(6):534–555, 1999.

Potential Field Methods

- Potential Fields
  - Dynamical Performance
  - Topological Considerations
    - Saddle Points
    - Local minima

D. E. Koditschek. Exact robot navigation by means of potential functions: Some topological considerations. In *IEEE International Conference on Robotics and Automation*, pages 1–6, 1987.





- Navigation Functions

$$\varphi_k(\mathbf{q}) = \left(\rho_k \circ \sigma_1 \circ \frac{\gamma_k}{\beta}\right)(\mathbf{q}) = \frac{\|\mathbf{q}-\mathbf{q}_\mathrm{d}\|^2}{\left(\|\mathbf{q}-\mathbf{q}_\mathrm{d}\|^{2k} + \beta(\mathbf{q})\right)^{1/k}}$$

  - Free of local minima (large k)
  - Numerically difficult to implement

E. Rimon and D. E. Kodischek. Exact robot navigation using artificial potential functions. *IEEE Transactions on Robotics and Automation*, 8(5):501–518, October 1992.

Opportunities

- Control Policy Design
- Control Policy Deployment
- Global Performance
- Local Minima
- Planning Abstraction

Task Decomposition

Abstraction: Planning vs. Control

- Plan discrete goals
- Control policy generates continuous "behaviors"

<u>Example</u>: "Get me to CFR on time"

- Conventional version:
  - Plan : $\mathbf{q}(s) \subset \mathcal{FS}$
  - Time Scaling : $s(t) \in \mathbb{R}$
  - Control : $\mathsf{u} = \dot{\mathsf{q}} = \frac{d\mathbf{q}}{ds}\dot{s}$
- Our version:
  - Plan : Out of Smith - across road - NSH-…
  - Control : $\Phi_{\mathsf{EDSH}} \succeq \Phi_{\mathsf{road}} \succeq \Phi_{\mathsf{NSH}} \succeq \cdots$

Spatial Decomposition

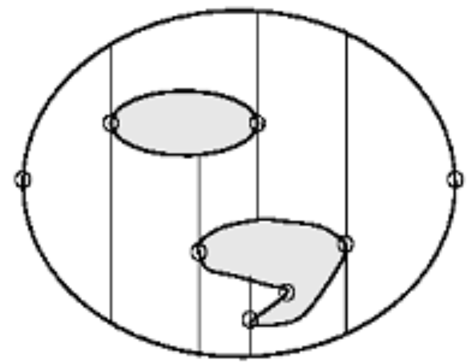
- Cellular decomposition
  - Collection of cells

$$\mathcal{K} = \{\mathcal{P}_i \subset \mathcal{FS} \;\mid\; i = 1 \ldots n,\; \mathcal{P}_i \cap \mathcal{P}_j = \emptyset \;\forall i \neq j\}$$

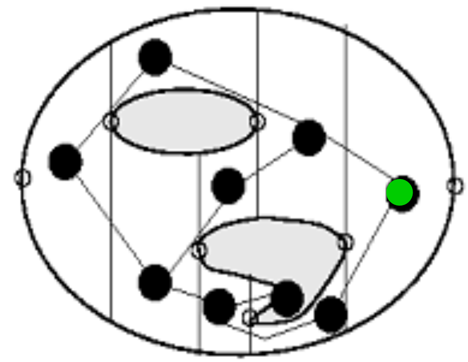
Exact $\bigcup_i \bar{\mathcal{P}}_i = \mathcal{FS}$

Approximate $\bigcup_i \bar{\mathcal{P}}_i \approx \mathcal{FS}$

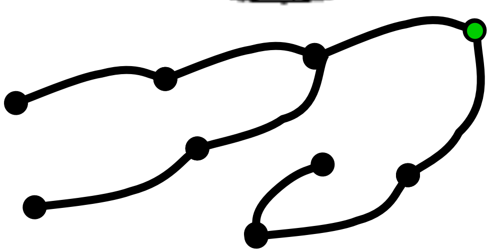
  - Computational geometry
    - NP-hard in general
    - Known algorithms for polygonal
    - Know maps a priori
- Adjacency graph
  - Dijkstra's Algorithm
  - Spanning Tree (Partial order)



Cell Decomp



Adjacency Graph



Spanning Tree

Control Policy Preconditions

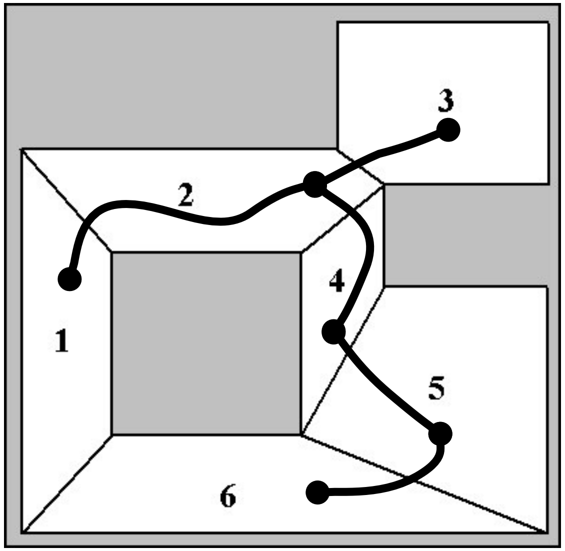
Must guarantee that the system:

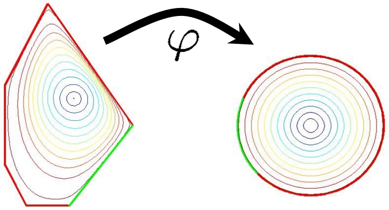
- respects
  - the boundaries of the cells
  - the dynamical constraints
- converges
  - to the goal located inside the cell
  - exits the cell within the desired *outlet zone*

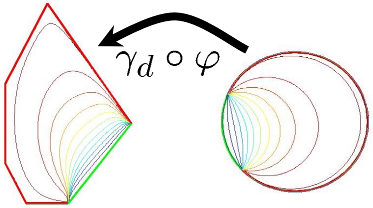
Each cell is ***conditionally*** *positive invariant*

Our Approach

- Decompose free space
  - Convex polytopes in $\mathbb{R}^n$
  - Open sets
  - Shared boundaries
- Partial order over cells
  - Adjacency graph
  - Spanning Tree
- Generic Control Policy
  - Map to unit ball
  - Potential field in ball
  - Pull back to cell
  - Control policy in cell

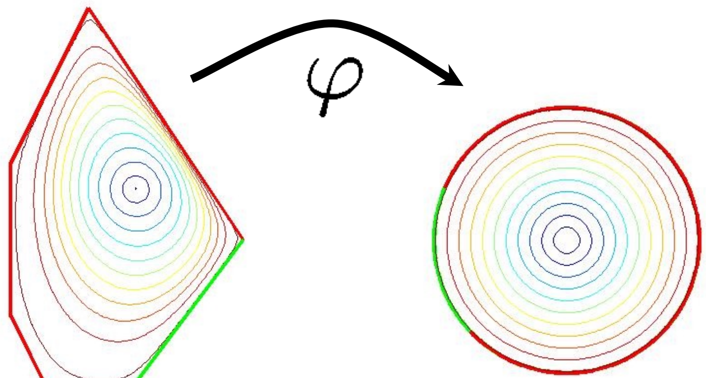




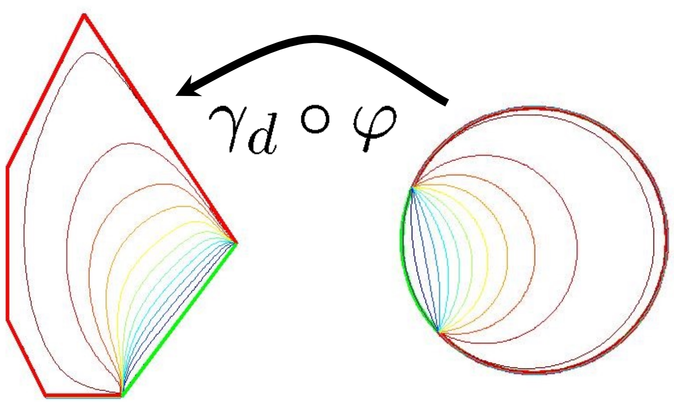


Mapping to Solution Space

Contours of Constant Disk Radius



Contours of Constant Potential



- Map cell to unit ball

$$\varphi : \mathcal{P} \to \mathcal{B}$$

  - C^k continuous on interior
  - Full rank on interior

- Solve Laplace's Equation on disk

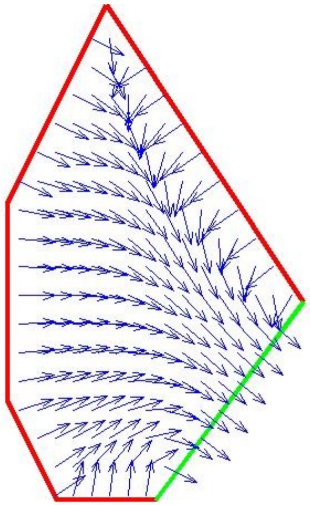
$$\nabla^2 \gamma_d = \frac{\partial^2 \gamma_d}{\partial {x_d}^2} + \frac{\partial^2 \gamma_d}{\partial {y_d}^2} = 0$$

- Pull potential solution back to cell

$$\gamma = \gamma_d \circ \varphi$$

C. I. Connolly, J. B. Burns, and R. Weiss. Path planning using laplace's equation. In *IEEE International Conference on Robotics and Automation*, volume 3, pages 2102–2106, May 1990.

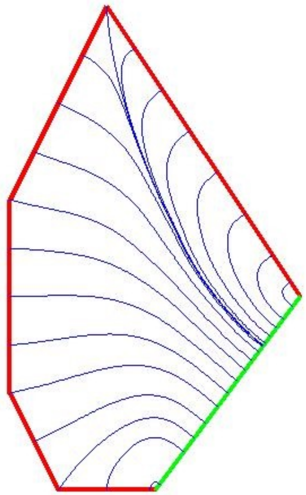
Vector Field Definition



Negative Normalized Gradient Vector Field

$$\hat{\mathsf{X}} = -\frac{D_{\mathbf{q}}\gamma}{\|D_{\mathbf{q}}\gamma\|} = -\frac{D_{\varphi(\mathbf{q})}\gamma_d \, D_{\mathbf{q}}\varphi}{\left\|D_{\varphi(\mathbf{q})}\gamma_d \, D_{\mathbf{q}}\varphi\right\|}$$

- Orthogonal to cell boundary (a.e.)
- Inward pointing along inlet boundary
- Outward pointing along outlet boundary



- Potential function γ is a C^k smooth function on the interior of the polygon
- No local minima or saddle points
- Flow along integral curves of $\hat{\mathsf{X}}$ induces desired behavior within a cell

Kinematic System

System

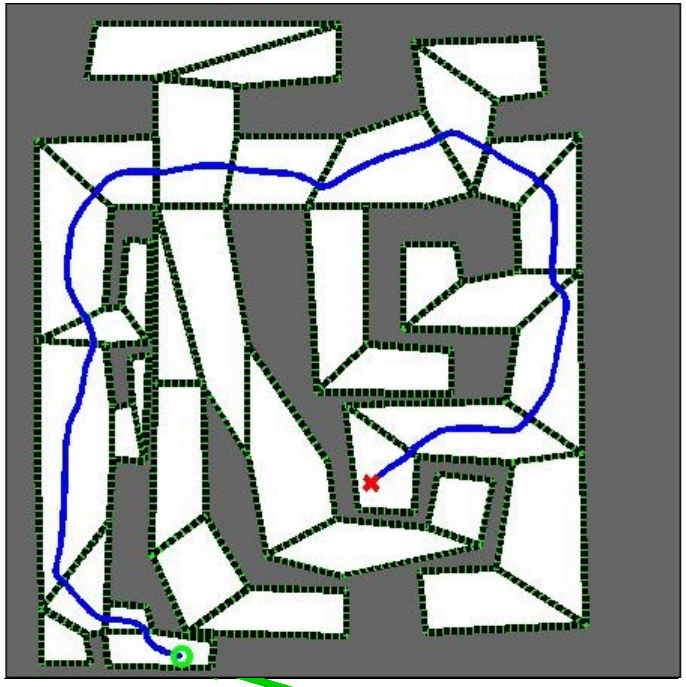
$$\dot{\mathbf{q}} = \mathbf{u}$$
$$\mathbf{q}\,,\ \mathbf{u} \in \mathbb{R}^2$$

Constraint

$$\|\dot{\mathbf{q}}\|_2 \leq \mathsf{V}_{\mathsf{max}}$$

Control

$$\mathbf{u} = s^* \hat{\mathsf{X}}$$
$$s^* \leq \mathsf{V}_{\mathsf{max}}$$



Automated generation of polygonal decomposition [Keil, '85]
Automated deployment of controllers based on adjacency graph
Controller switching is automatic based on region boundaries

Dynamical System

System

$$\ddot{\mathbf{q}} = \mathbf{u}$$

$$\mathbf{q}\,,\ \mathbf{u} \in \mathbb{R}^2$$

Constraints

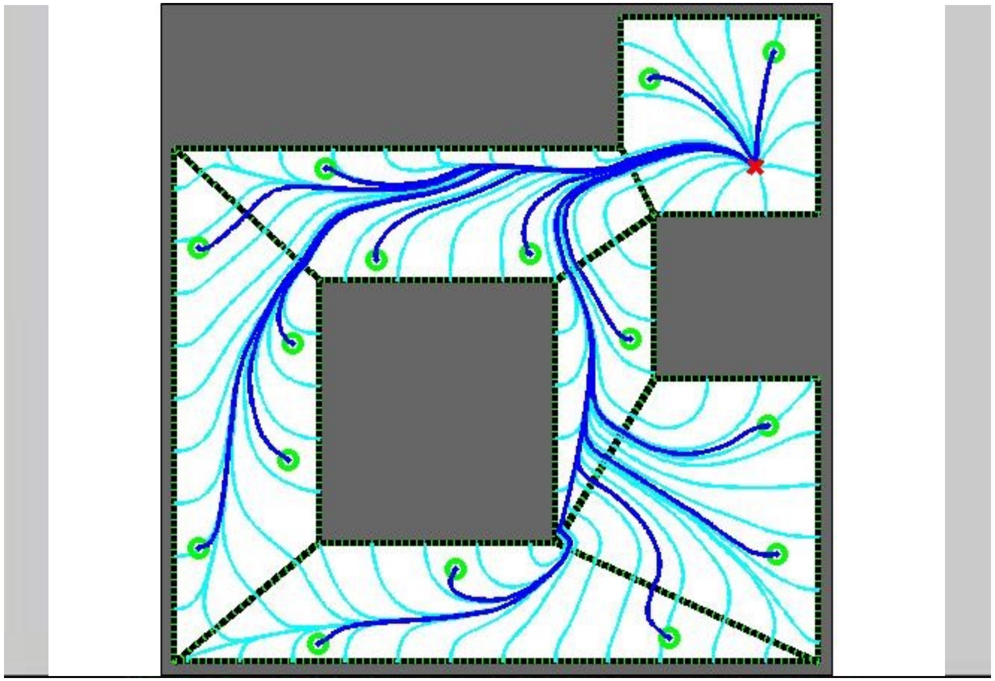
$$\cancel{\|\dot{\mathbf{q}}\|_2 \leq \mathsf{V}_{\mathsf{max}}}$$

$$\cancel{\|\mathbf{u}\|_2 \leq \mathsf{A}_{\mathsf{max}}}$$

Control

$$\mathbf{u} = K\ (\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}) + D_{\mathbf{q}}\mathsf{X}\,\dot{\mathbf{q}}$$

$$\mathsf{X}(\mathbf{q}) = s^*\hat{\mathsf{X}}$$



Integral Curves of $\mathsf{X}(\mathbf{q})$

Specification of the adjacency relationships induces a globally convergent controller for sufficiently high gain K .

Constrained Dynamical Systems

$\mathbf{u} = K\ (\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}) + D_{\mathbf{q}}\mathsf{X}\,\dot{\mathbf{q}}$ is not sufficient.

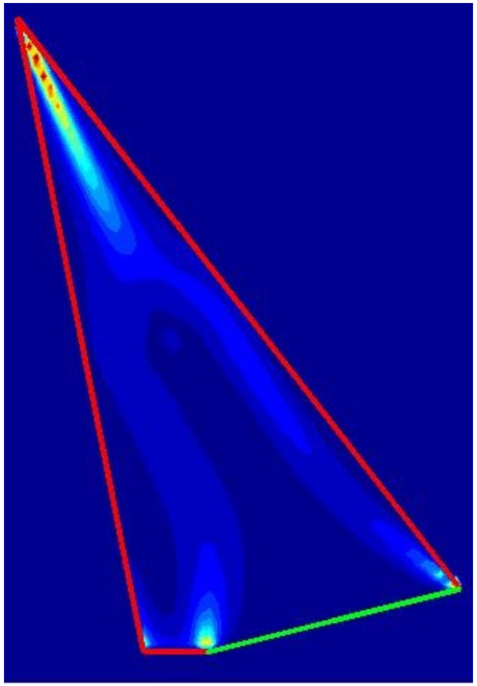
$$\dot{\mathbf{q}} = \mathsf{X}(\mathbf{q}) \;\Rightarrow\; \mathbf{u} = D_{\mathbf{q}}\mathsf{X}\,\dot{\mathbf{q}}$$

$$\|D_{\mathbf{q}}\mathsf{X}\,\dot{\mathbf{q}}\| = ?$$

Let $\mathsf{X}(\mathbf{q}) = s\,(\mathbf{q})\ \hat{\mathsf{X}}(\mathbf{q})$

$$s\,(\mathbf{q}) = \min\left(\frac{S^*}{\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|},\, \mathsf{V}_{\max}\right)$$

$$S^* \leq \min_{q} \sqrt{\frac{\mathsf{A}_{\max}}{\left\|D_{\mathbf{q}}\hat{\mathsf{X}} - \frac{\hat{\mathsf{X}}(\mathbf{q})D_q\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|}{\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|}\right\|}}\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|$$

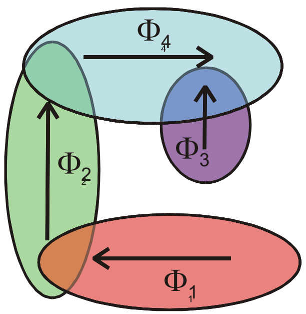


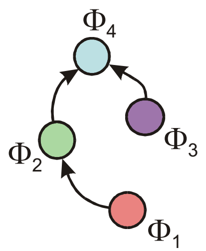
$\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|$

Spectral Norm $\|M\| = \max_{\|\mathbf{x}\|=1} \|M\,\mathbf{x}\|$

Constrained Dynamical Systems

- Hybrid control policies within cell
  - Save Φ_S
  - Align Φ_A
  - Join Φ_J
  - Flow Φ_F





$$\Phi_S \succeq \Phi_A \succeq \Phi_J \succeq \Phi_F$$

- Savable set $\mathcal{S} = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{P}, \, \Phi_S \Rightarrow \text{no collision}\}$

Alfred A. Rizzi. Hybrid control as a method for robot motion programming. In *IEEE International Conference on Robotics and Automation*, volume 1, pages 832 – 837, May 1998.

Arthur Quaid and Alfred A. Rizzi. Robust and efficient motion planning for a planar robot using hybrid control. In *IEEE International Conference on Robotics and Automation*, volume 4, pages 4021 – 4026, April 2000.

Constrained Dynamical Systems

- Save control policy Φ_S

$$d_c = \mathbf{n}_c^T(\mathbf{q} - \mathbf{q}_c)$$

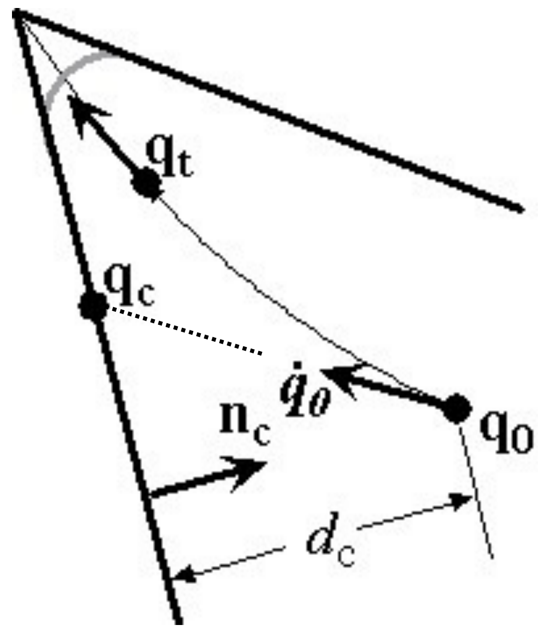
$$s_c = \mathbf{n}_c^T \dot{\mathbf{q}} \qquad t_c = \frac{d_c}{s_c}$$

$$d_b = \frac{s_c^2}{2\,\mathsf{A}_{\mathsf{max}}}$$

$$\zeta_c = \frac{d_b}{d_c} < 1$$

$$\Phi_S \,:\, \mathbf{u} = \mathsf{A}_{\mathsf{max}}\,\mathbf{n}_c$$

$$\dot{\zeta}_c < 0 \quad \text{if} \;\; d_b < d_c \;\; \text{and} \;\; \frac{d}{dt}\|\dot{\mathbf{q}}\| < 0$$



Domain : $\mathscr{D}(\Phi_S) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{P}\,,\, \zeta_c < 1\} = \mathcal{S}$

Goal Set : $\mathscr{G}(\Phi_S) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{\mathbf{q}} = 0\}$

Alfred A. Rizzi. Hybrid control as a method for robot motion programming. In *IEEE International Conference on Robotics and Automation*, volume 1, pages 832 – 837, May 1998.

Constrained Dynamical Systems

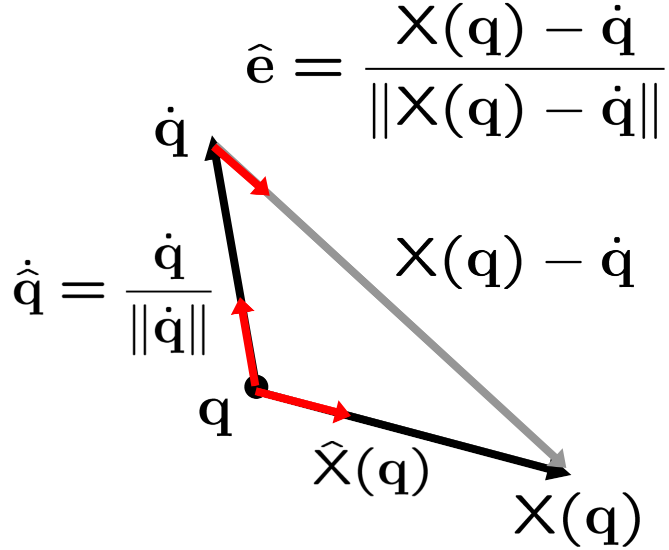
- Align control policy Φ_A

$$\mathscr{D}(\Phi_A) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{P}, \zeta_c < \mu\}$$

$$\mathscr{G}(\Phi_A) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{\mathbf{q}} = 0\}$$

$$\upsilon = \frac{\mu - \zeta_c}{\mu} \qquad \mu \in (0,1)$$

$$\sigma : \upsilon \to [0,1] \qquad \sigma(0) = 0, \quad \sigma(1) = 1, \quad \text{ex., } \sigma(\upsilon) = \sqrt{\upsilon}$$



$$\Phi_A : \mathbf{u} = \begin{cases} \mathsf{A}_{\max} \dfrac{(1-\sigma(\upsilon))\,\mathbf{n}_c + \sigma(\upsilon)\,\hat{\mathbf{e}}}{\|(1-\sigma(\upsilon))\,\mathbf{n}_c + \sigma(\upsilon)\,\hat{\mathbf{e}}\|} & \dot{\mathbf{q}}^T \mathsf{X} < \dot{\mathbf{q}}^T \dot{\mathbf{q}} \\ \mathsf{A}_{\max} \dfrac{(1-\sigma(\upsilon))\,\mathbf{n}_c - \sigma(\upsilon)\,\hat{\dot{\mathbf{q}}}}{\|(1-\sigma(\upsilon))\,\mathbf{n}_c - \sigma(\upsilon)\,\hat{\dot{\mathbf{q}}}\|} & \text{otherwise} \end{cases},$$

$\dfrac{d}{dt}\|\dot{\mathbf{q}}\| < 0$ and $\zeta_c < \mu$ Proof: Evaluate $\dot{\zeta}_c$ on boundary, $\zeta_c = \mu$

Constrained Dynamical Systems

- Join control policy Φ_J

$$\mathscr{D}(\Phi_J) = \left\{(\mathbf{q},\dot{\mathbf{q}}) \mid \dot{\mathbf{q}}^T \mathsf{X} \geq 0, \|\dot{\mathbf{q}}\| < \|\mathsf{X}(\mathbf{q})\|\right\}$$

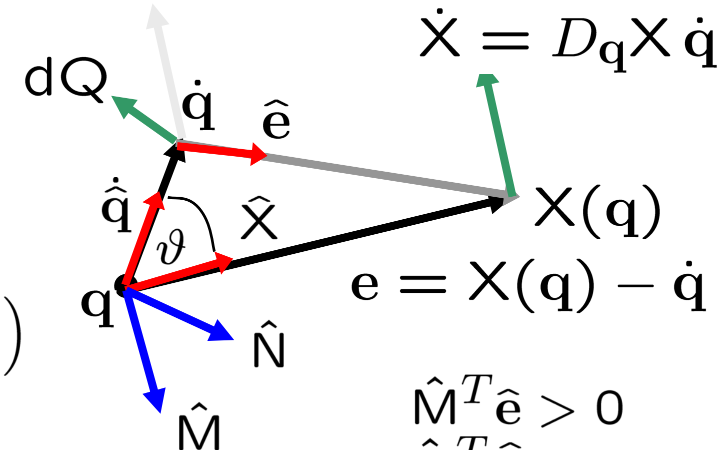
$$\mathscr{G}(\Phi_J) = \{(\mathbf{q},\dot{\mathbf{q}}) \mid \vartheta = 0 \text{ or } \mathbf{q} \in \partial\mathcal{P}_{\mathsf{outlet}},\ \|\dot{\mathbf{q}}\| \leq \|\mathsf{X}(\mathbf{q})\|\}$$

Want $\dot{\vartheta} \leq 0$

$$\mathsf{P} = \dot{\mathsf{X}} - \hat{\mathsf{M}}^T \dot{\mathsf{X}}\,\hat{\mathsf{M}} - \widehat{\mathsf{X}}^T \dot{\mathsf{X}}\,\widehat{\mathsf{X}}$$

$$\dot{\mathsf{X}} = \widehat{\mathsf{X}}^T \dot{\mathsf{X}}\,\widehat{\mathsf{X}} + \hat{\mathsf{M}}^T \dot{\mathsf{X}}\,\hat{\mathsf{M}} + \mathbf{P}$$

$$\mathbf{d}\mathsf{Q} = \frac{\|\dot{\mathbf{q}}\|}{\|\mathsf{X}(\mathbf{q})\|}\left(\widehat{\mathsf{X}}^T \dot{\mathsf{X}}\,\hat{\dot{\mathbf{q}}} + \hat{\mathsf{M}}^T \dot{\mathsf{X}}\,\hat{\mathsf{N}} + \mathbf{P}\right)$$



$$\hat{\mathsf{M}}^T \hat{\mathbf{e}} > 0$$
$$\hat{\mathsf{M}}^T \widehat{\mathsf{X}} = 0$$
$$\hat{\mathsf{N}}^T \hat{\mathbf{e}} > 0$$
$$\hat{\mathsf{N}}^T \dot{\mathbf{q}} = 0$$
$$\mathbf{P} \perp \hat{\mathsf{M}},\ \mathbf{P} \perp \hat{\mathsf{N}}$$
$$\mathbf{P} \perp \widehat{\mathsf{X}},\ \mathbf{P} \perp \hat{\dot{\mathbf{q}}}$$

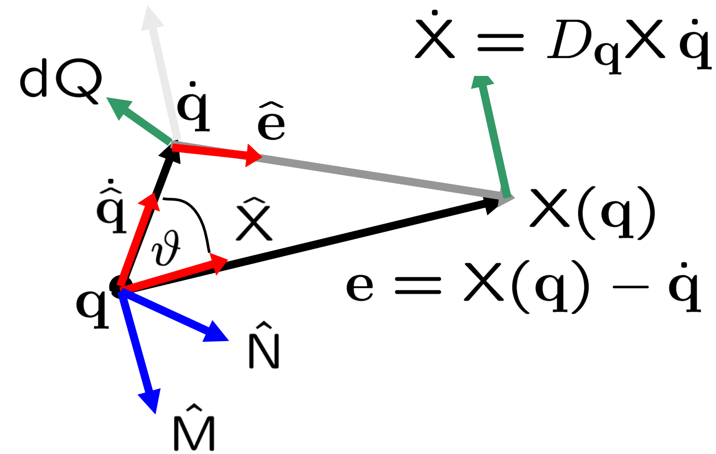
$$\left\|\dot{\mathsf{X}}\right\| \leq \mathsf{A}_{\mathsf{max}} \text{ and } \|\dot{\mathbf{q}}\| \leq \|\mathsf{X}(\mathbf{q})\| \Rightarrow \|\mathbf{d}\mathsf{Q}\| \leq \mathsf{A}_{\mathsf{max}}$$

$$\mathbf{u} = \mathbf{d}\mathsf{Q} + K^* \left(\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}\right)$$

Where $K^* > 0$ chosen such that $\|\mathbf{u}\| \leq \mathsf{A}_{\mathsf{max}}$

Constrained Dynamical Systems

- Join control policy Φ_J



Brake $\quad x = \min\left(0, \hat{\mathsf{X}}^T\dot{\mathsf{X}}\right)$

Turn $\quad m = \max\left(\hat{\mathsf{M}}^T\dot{\mathsf{X}}, 0\right)$

Bend $\quad \mathsf{P} = \dot{\mathsf{X}} - \hat{\mathsf{M}}^T\dot{\mathsf{X}}\,\hat{\mathsf{M}} - \hat{\mathsf{X}}^T\dot{\mathsf{X}}\,\hat{\mathsf{X}}$

$$\mathsf{dQ} = \frac{\|\dot{\mathbf{q}}\|}{\|\mathsf{X}(\mathbf{q})\|}\left(x\,\hat{\dot{\mathbf{q}}} + m\,\hat{\mathsf{N}} + \mathbf{P}\right)$$

Steer : $\quad s = \min\left(K\,\hat{\mathsf{N}}^T\mathbf{e}\,,\ \sqrt{\mathsf{A}_{\mathsf{max}}{}^2 - \frac{\dot{\mathbf{q}}^T\dot{\mathbf{q}}}{\mathsf{X}^T\mathsf{X}}\left(\mathsf{P}^T\mathsf{P} + x^2\right)} - \frac{\|\dot{\mathbf{q}}\|}{\|\mathsf{X}\|}m\right)$

Accelerate : $a = \min\left(K\hat{\dot{\mathbf{q}}}^T\mathbf{e},\ \sqrt{\mathsf{A}_{\mathsf{max}}{}^2 - \frac{\dot{\mathbf{q}}^T\dot{\mathbf{q}}}{\mathsf{X}^T\mathsf{X}}\mathsf{P}^T\mathsf{P} - \left(\frac{\|\dot{\mathbf{q}}\|}{\|\mathsf{X}\|}m + s\right)^2} - \frac{\|\dot{\mathbf{q}}\|}{\|\mathsf{X}\|}x\right)$

$$\Phi_J\ :\ \mathbf{u} = \mathsf{dQ} + s\,\hat{\mathsf{N}} + a\,\hat{\dot{\mathbf{q}}}$$

Constrained Dynamical Systems

- Flow control policy Φ_F

$$\mathscr{D}(\Phi_F) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{P}, \|\dot{\mathbf{q}}\| \leq \|\mathsf{X}(\mathbf{q})\|, \vartheta = 0\}$$

Set of zero measure, need to "fatten"

$$\mathscr{G}(\Phi_F) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \partial\mathcal{P}_{\mathsf{outlet}}, \|\dot{\mathbf{q}}\| \leq \|\mathsf{X}(\mathbf{q})\|, \vartheta = 0\}$$

$$\Phi_F : \mathbf{u} = K\,(\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}) + D_{\mathbf{q}}\mathsf{X}\,\dot{\mathbf{q}}$$

if $\|\mathbf{u}\| > \mathsf{A}_{\mathsf{max}}$, then

$$\mathbf{u} = \left((x+a)\,\hat{\dot{\mathbf{q}}} + \left(\hat{\mathsf{M}}^T\dot{\mathsf{X}}\right)\hat{\mathsf{N}} + \mathbf{P}\right)$$

Given $\mathsf{X}(\mathbf{q}) = \dfrac{s^*}{\left\|D_{\mathbf{q}}\hat{\mathsf{X}}\right\|}\hat{\mathsf{X}}(\mathbf{q})$, it is always true that

$\|\dot{\mathbf{q}}\| \leq \|\mathsf{X}\|$ (by Lemma 4.2 and IVT)

$\dot{\vartheta} = 0$ (by Lemma 4.3)

$\lim_{t\to\infty} \mathbf{q} \in \partial\mathcal{P}_{\mathsf{outlet}}$ (by Lemma 4.4)

Constrained Dynamical Systems

- Simulation Results

Hey!

I have to give you some reason to come to my proposal :-)

Future Work

- Velocity Constraint back-chaining
  - Flow leaving cell enters in savable set of neighbor
  - Look for less conservative velocity scalings
- Extend methods to
  - systems with non-holonomic constraints
  - underactuated systems
- Develop tools to allow behavior design
  - parallel parking
  - doorway navigation
- Interface with higher level AI reasoning

Conclusions

- Presented methods to automatically deploy local control policies
  - Local control policies respect local constraints
  - Composition guarantees global convergence
  - Fully actuated systems in $\mathbb{R}^n$
- Planning vs. control abstraction
- Future extensions
  - Non-holonomic constraints
  - Underactuated systems

References

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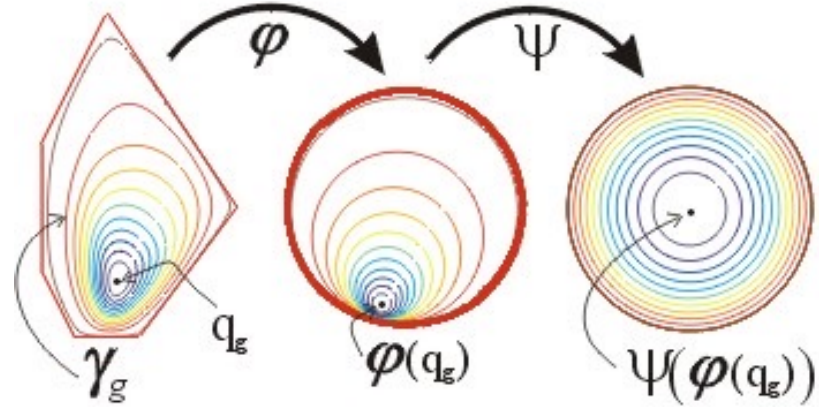
Convergent Control Policy

- Map cell to ball φ
- Map goal point in ball to origin of ball Ψ
- Potential

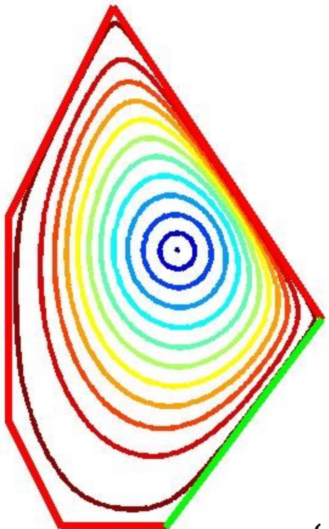


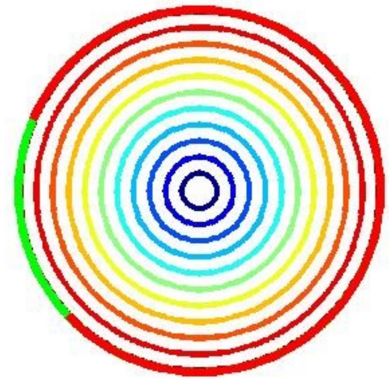
$$\gamma_g = \frac{1}{2}\|\psi \circ \varphi\|^2$$

$$\hat{\mathsf{X}}_g(\mathbf{q}) = -\frac{D_{\mathbf{q}}\gamma_g^T}{\|D_{\mathbf{q}}\gamma_g\|} \qquad \mathsf{X}_g(\mathbf{q}) = \underbrace{\frac{\|\mathbf{q}-\mathbf{q_g}\|^2}{\|\mathbf{q}-\mathbf{q_g}\|^2+\alpha}}_{s_g(\mathbf{q})}\hat{\mathsf{X}}_g \qquad \alpha \geq \frac{(\max_{\mathbf{q}} s_g(\mathbf{q}))^4}{16\,\mathsf{A_{max}}^2}$$

$$\mathsf{X}(\mathbf{q}) = \frac{s^*}{\|D_q\mathsf{X}_g(\mathbf{q})\|} s_g(\mathbf{q})\, \hat{\mathsf{X}}_g$$

Polygon to Disk Mapping

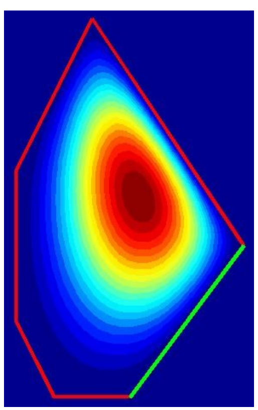




$$\varphi(\mathbf{q}) = \frac{\mathbf{q}}{\|\mathbf{q}\| + B\,\beta(\mathbf{q})}$$

$$\beta(\mathbf{q}) = \prod_{i=0}^{n} N_i \cdot (\mathbf{q} - P_i)$$

$$B = \frac{B_{max}^{1/N}}{B_{max}} \qquad B_{max} = \max_{\mathbf{q}}\ \beta(\mathbf{q})$$

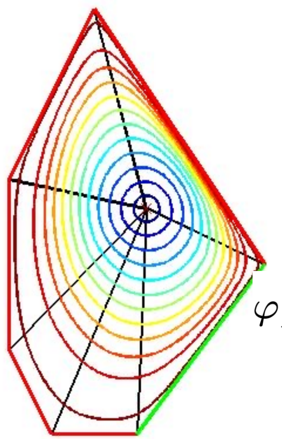


$\beta(\mathbf{q})$

Mapping Comparison

Our Mapping

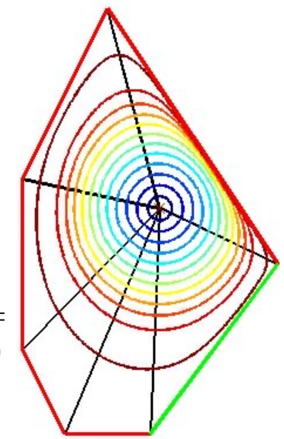
$$\varphi(\mathbf{q}) = \frac{\mathbf{q}}{\|\mathbf{q}\| + B\,\beta(\mathbf{q})}$$



Alternate Mapping

$$\varphi_{BR}(\mathbf{q}) = \frac{\mathbf{q}}{\sqrt[m]{\|\mathbf{q}\|^m + \beta(\mathbf{q})}}$$

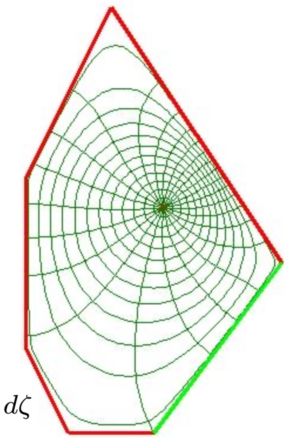
***m* is number of faces**



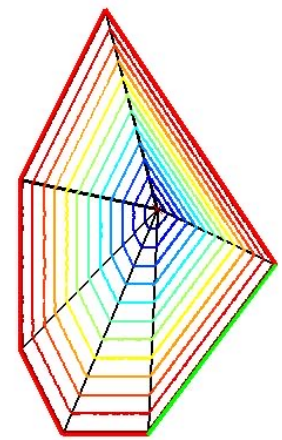
Schwarz-Christoffel Conformal Mapping

$$\varphi_{SC}^{-1}(\mathrm{q_d}) = A + C\int^{z} \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} d\zeta$$

$z_k = \varphi(w_k)$ is a prevertex



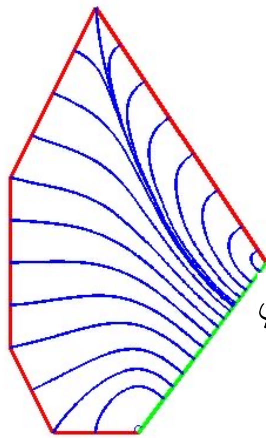
Linear Retraction Mapping



Mapping Comparison

Our Mapping

$$\varphi(\mathbf{q}) = \frac{\mathbf{q}}{\|\mathbf{q}\| + B\,\beta(\mathbf{q})}$$



Alternate Mapping

$$\varphi_{BR}(\mathbf{q}) = \frac{\mathbf{q}}{\sqrt[m]{\|\mathbf{q}\|^m + \beta(\mathbf{q})}}$$

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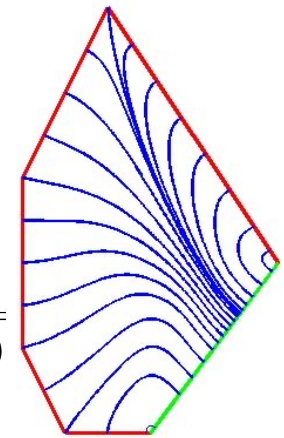
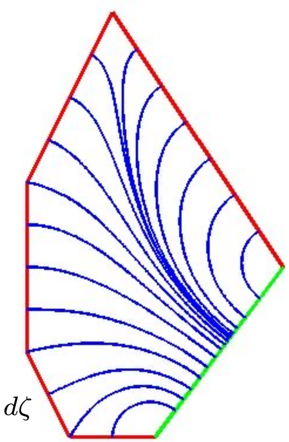


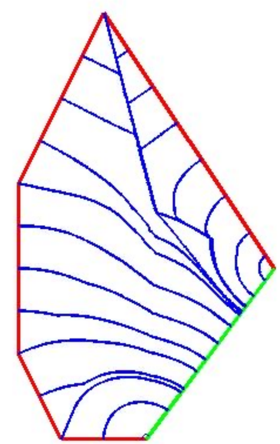
Schwarz-Christoffel Conformal Mapping

$$\varphi_{SC}^{-1}(\mathbf{q}_\mathrm{d}) = A + C \int^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} d\zeta$$

$z_k = \varphi(w_k)$ is a prevertex



Linear Retraction Mapping

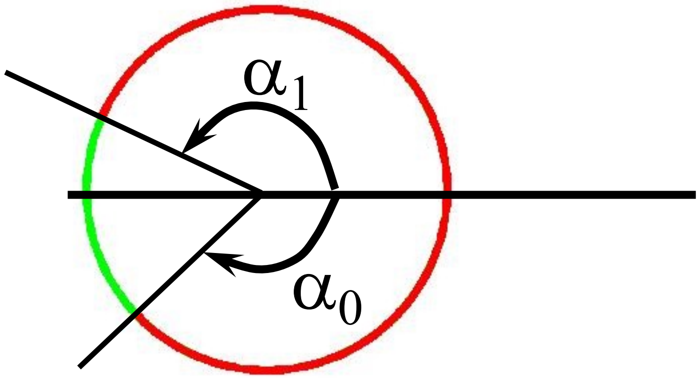


Laplace's Equation on Disk

Steady State Heat Equation

$$\Delta\gamma_d = \frac{\partial^2\gamma_d}{\partial x^2} + \frac{\partial^2\gamma_d}{\partial y^2} = 0$$

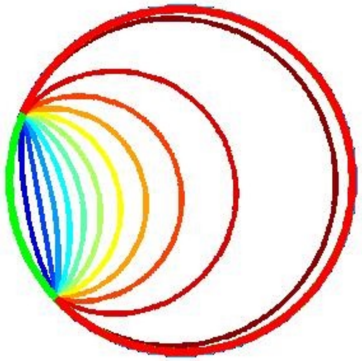
$$\Delta\gamma_d = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\gamma_d}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\gamma_d}{\partial\theta^2} = 0$$



Boundary Condition

$$\forall\theta \in (-\pi, \pi]\ ,\ r = 1$$

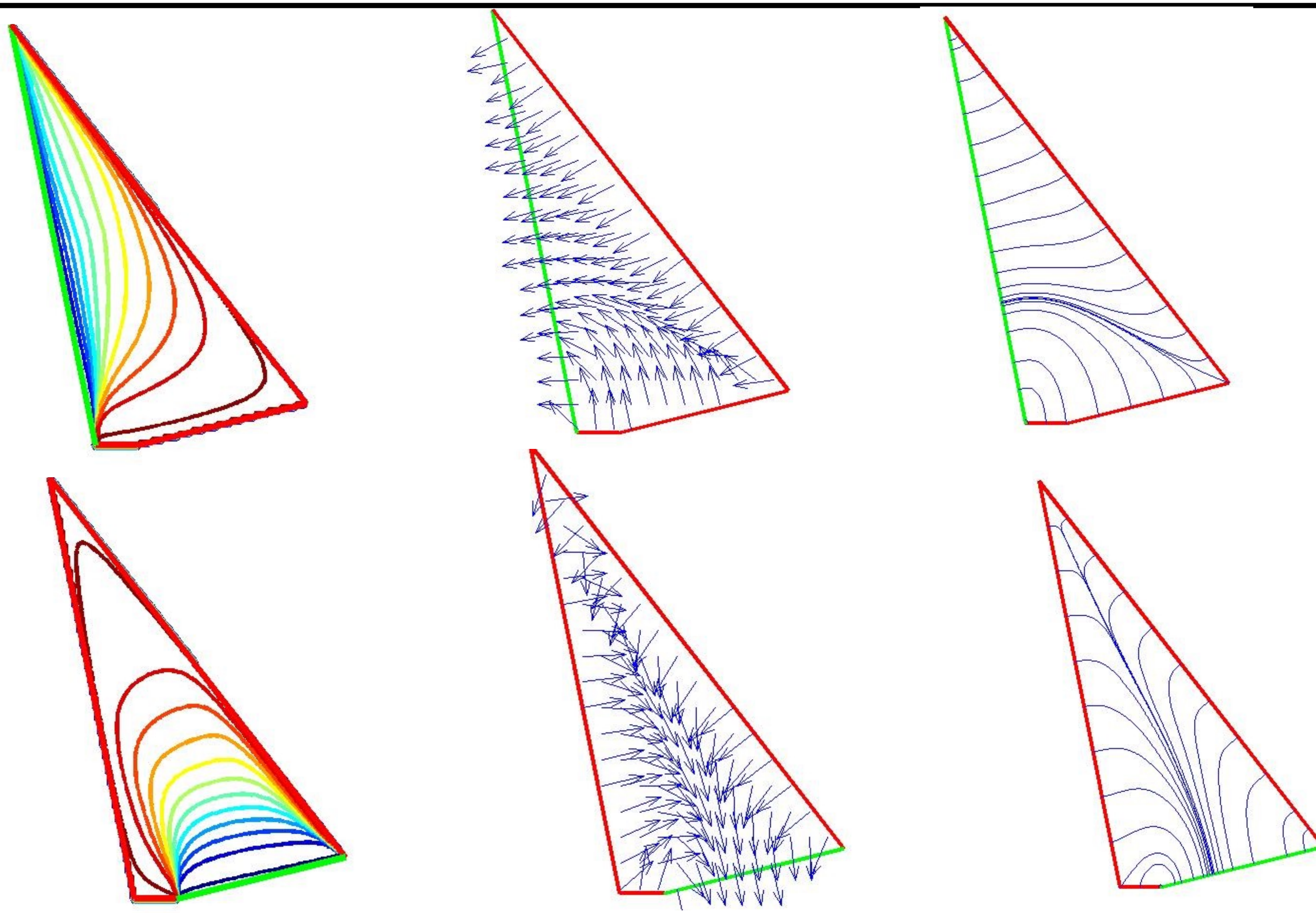
$$\gamma_d(\theta) = \begin{cases} 1 & \theta \in [\alpha_0, \alpha_1] \\ 0 & \text{otherwise} \end{cases}$$



Solution

$$\gamma_d(r,\theta) = \frac{\alpha_1-\alpha_0}{2\pi} + \frac{1}{\pi}\left(\tan^{-1}\left(\frac{r\sin(\alpha_1-\theta)}{1-r\cos(\alpha_1-\theta)}\right) - \tan^{-1}\left(\frac{r\sin(\alpha_0-\theta)}{1-r\cos(\alpha_0-\theta)}\right)\right)$$

Additional Examples



C^2 Fillet Curve Approximation

$$\mathbf{p}_i = (x_i\,,\,y_i) = \beta_R\left(\mathbf{n}_i^{\perp}\cdot\bar{\mathbf{n}}^{\perp}\,,\,\mathbf{n}_i^{\perp}\cdot\bar{\mathbf{n}}\right)\,,$$
$$\mathbf{p}_j = \left(x_j\,,\,y_j\right) = \beta_R\left(\mathbf{n}_j^{\perp}\cdot\bar{\mathbf{n}}^{\perp}\,,\,\mathbf{n}_j^{\perp}\cdot\bar{\mathbf{n}}\right)\,.$$

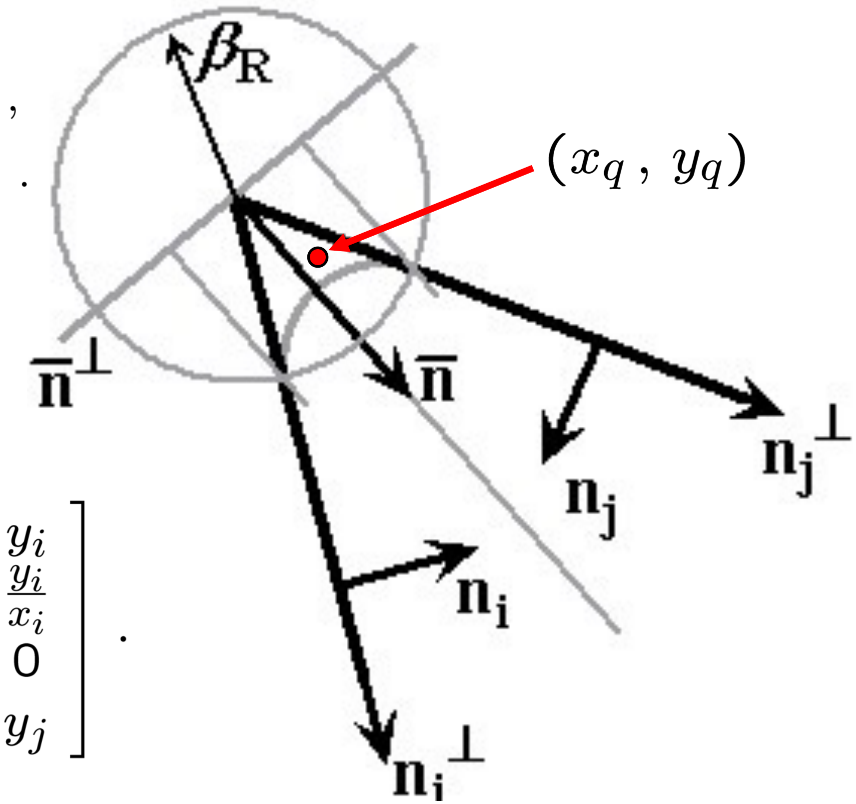
$$y\,(x) = a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$$

$$\frac{dy}{dx}(0) = 0 \;\Rightarrow d = 0$$

$$\begin{bmatrix} x_i^4 & x_i^3 & x_i^2 & 1 \\ 4\,x_i^3 & 3\,x_i^2 & 2\,x_i & 0 \\ 12\,x_i^2 & 6\,x_i & 2 & 0 \\ x_j^4 & x_j^3 & x_j^2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ e \end{bmatrix} = \begin{bmatrix} y_i \\ \frac{y_i}{x_i} \\ 0 \\ y_j \end{bmatrix}\,.$$

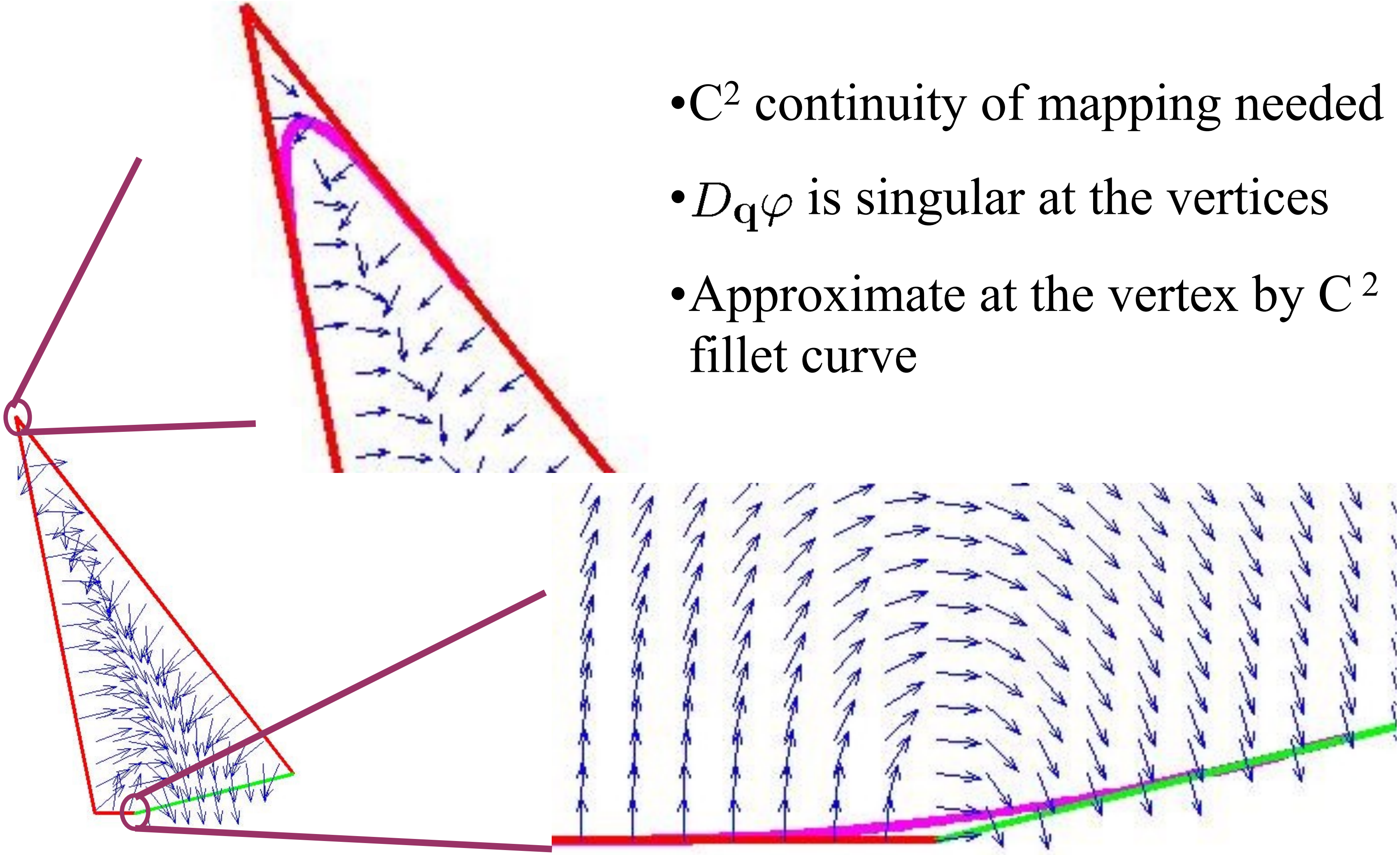
if $y\,(x_q) > y_q$ then $(x_q\,,\,y\,(x_q))$

otherwise $(x_q\,,\,y_q)$



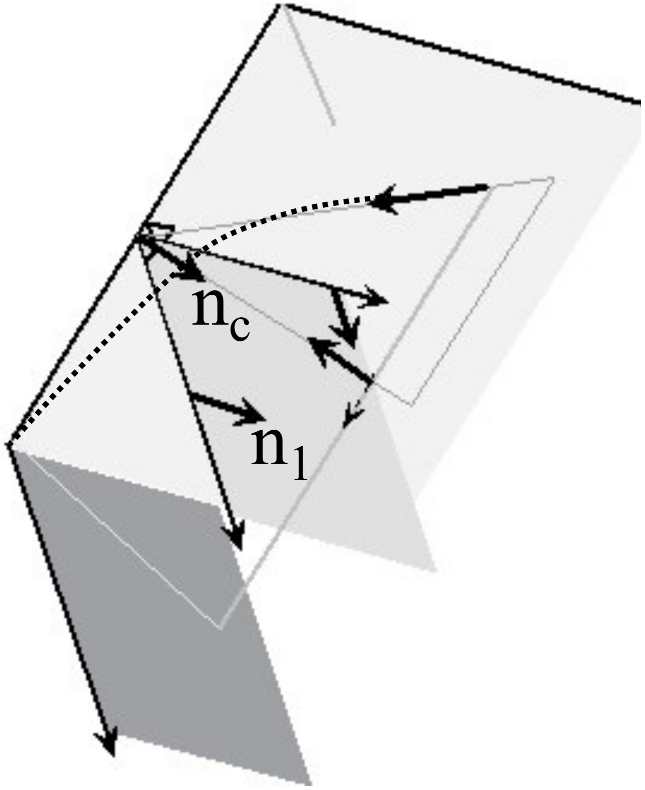
Polygonal Approximation

- C^2 continuity of mapping needed
- $D_{\mathbf{q}}\varphi$ is singular at the vertices
- Approximate at the vertex by C^2 fillet curve



Constrained Dynamical Systems

- Save control policy Φ_S



Constrained Dynamical Systems

- Align control policy Φ_A

$$\mathscr{D}(\Phi_A) = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \mathbf{q} \in \mathcal{P}, \zeta_c < \mu\}$$

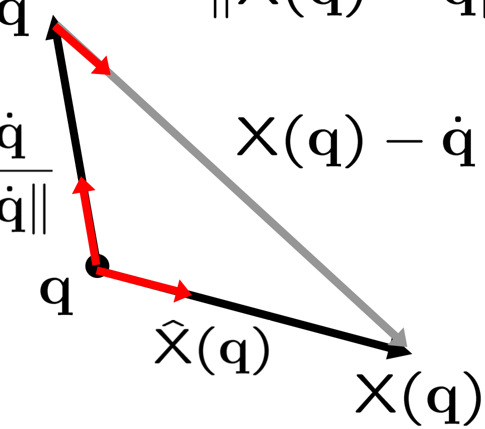
$$\mathscr{G}(\Phi_A) = \left\{(\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{\mathbf{q}}^T \mathsf{X} \geq 0, \|\dot{\mathbf{q}}\| < \|\mathsf{X}(\mathbf{q})\|\right\}$$

$$\upsilon = \max\left(\frac{\mu - \zeta_c}{\mu}, 0\right)$$

$$\Phi_A : \mathbf{u} = \begin{cases} \mathsf{A}_{\max} \dfrac{\left(1-\upsilon^{\frac{1}{2}}\right)\mathbf{n}_c + \upsilon^{\frac{1}{2}}\hat{\mathbf{e}}}{\left\|\left(1-\upsilon^{\frac{1}{2}}\right)\mathbf{n}_c + \upsilon^{\frac{1}{2}}\hat{\mathbf{e}}\right\|} & \dot{\mathbf{q}}^T \mathsf{X} < \dot{\mathbf{q}}^T \dot{\mathbf{q}} \\ \mathsf{A}_{\max} \dfrac{\left(1-\upsilon^{\frac{1}{2}}\right)\mathbf{n}_c - \upsilon^{\frac{1}{2}}\hat{\dot{\mathbf{q}}}}{\left\|\left(1-\upsilon^{\frac{1}{2}}\right)\mathbf{n}_c - \upsilon^{\frac{1}{2}}\hat{\dot{\mathbf{q}}}\right\|} & \text{otherwise} \end{cases},$$

$$\hat{\mathbf{e}} = \frac{\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}}{\|\mathsf{X}(\mathbf{q}) - \dot{\mathbf{q}}\|}$$

$$\hat{\dot{\mathbf{q}}} = \frac{\dot{\mathbf{q}}}{\|\dot{\mathbf{q}}\|}$$



$\frac{d}{dt}\|\dot{\mathbf{q}}\| < 0$ and $\zeta_c < \mu$ Proof: Evaluate $\dot{\zeta}_c$ on boundary, $\zeta_c = \mu$

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