Solving Olympiad Level Geometry Problems with Complex Numbers

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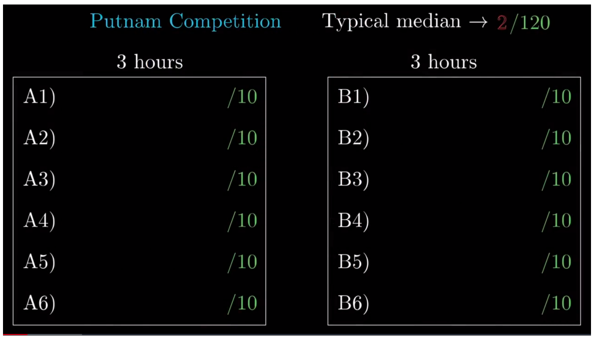
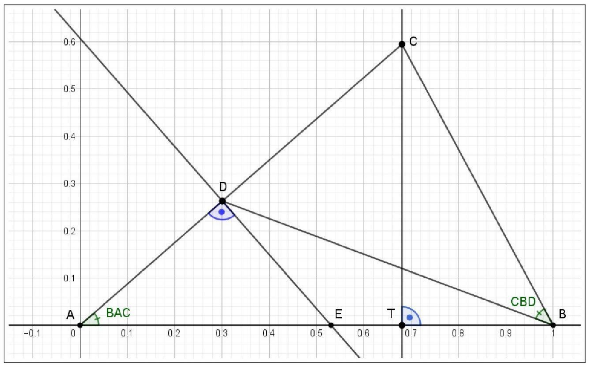
1 Introduction



Let me introduce you to the "Bundeswettbewerb Mathematik", one of the biggest German math competitions for school students. The first round is three months long and participants are asked to solve three out of four problems at home. Students managing to do so face four harder problems in the second round, but this time they only have two months and need to solve every problem in order to progress. They are not only allowed but actually encouraged to study math literature while problem solving. And yet, despite the abundance of time and resources available, a lot of students, unfortunately, fail to solve every problem. In the year 2018, only 22 percent of the competing students actually managed to pass the second round. This makes it apparent, that these problems are genuienly hard. In this video, I'm going to discuss a geometry problem from that year and show you a technique that turns it into a small exercise.

Intro-Animation



2 Problem

Here is the problem: Let AB be a line segment and let T be a point on it, that is closer to B than to A. Let C be a point on the line that is perpendicular to AB and that goes through T. First, show that there exists exactly one point D on the line segment AC, such that the angles $\angle CBD$ and $\angle BAC$ are the same. Second, show that the line perpendicular to AC and going through D intersects the line segment AB in a point E that does not depend on the choice of C.

I am going to reveal the solution to this geometry problem by the end of this video, but first I need to introduce you to our main problem solving tool: Complex numbers.

3 Basics

Let the number i be defined by the property that its square is (-1). Notice that it's not a real number, since the square of a real number cannot be negative. i is called *imaginary* for this reason. While performing any calculations, we are going to treat i like a constant while keeping in mind that we can

exchange i^2 for (-1).
A *complex number* is a number of the form $(a+bi)$, where a and b are real numbers. We refer to a as the real part and to b as the imaginary part. Let's have a look at two specific complex numbers: The first one has the real part 3 and the imaginary part (-1) while the second one has the real part (-1) and the imaginary part 1. Adding and subtracting these numbers is easy as long as you remember to treat i like a constant.

$$\begin{aligned}(3-i)+(-1+i) &= 3-i-1+i = 2\\ (3-i)-(-1+i) &= 3-i+1-i = 4-2i\end{aligned}$$

Multiplication is just as easy, but this time you also need to remember that you can exchange i^2 for -1.

$$\begin{aligned}(3-i)\cdot(-1+i) &= -3+3i+i-i^2\\ &= -3+3i+i-(-1)\\ &= -3+3i+i+1\\ &= -2+4i\end{aligned}$$

The trick when it comes to division, is to expand the fraction with the divisor, whose sign has been flipped in the imaginary part, in this case the term to expand with is $(-1-i)$. This operation might look the scariest, but it's actually just two parallel multiplications. Notice how i disappears in the divisor after multiplying. Please keep in mind that I chose this example specifically for this operation to work out nicely. In general, dividing two complex numbers yields a messy result.

$$\begin{aligned}\frac{(3-i)}{(-1+i)} &= \frac{(3-i)(-1-i)}{(-1+i)(-1-i)}\\ &= \frac{-3-3i+i+i^2}{1+i-i-i^2}\\ &= \frac{-3-3i+i-1}{1+i-i-(-1)}\\ &= \frac{-3-3i+i-1}{1+i-i+1}\\ &= \frac{-4-2i}{2}\\ &= -2-i\end{aligned}$$

Regarding the general case, it's useful to express these operations as formulas. Let's consider two general complex numbers $(a+bi)$ and $(c+di)$, where a, b, c, d are real numbers. Pause and try to derive the following equations. (In case of division, c and d may not be both 0, since division by 0 is undefined.)

$$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\ (a+bi)-(c+di) &= (a-c)+(b-d)i\\ (a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i\\ \frac{(a+bi)}{(c+di)} &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

Here are the results:

$$\begin{aligned}(a+bi)+(c+di) &= a+bi+c+di\\ &= (a+c)+(b+d)i\end{aligned}$$

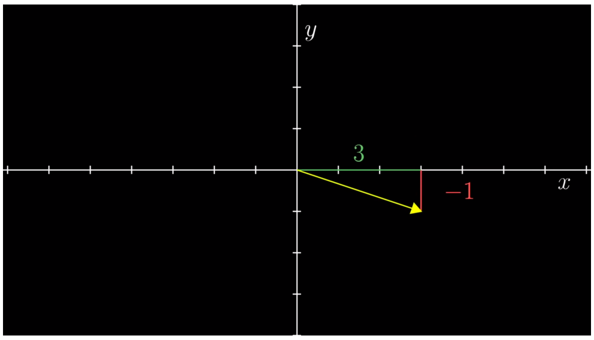
$$\begin{aligned}(a+bi)-(c+di) &= a+bi-c-di\\ &= (a-c)+(b-d)i\end{aligned}$$

$$\begin{aligned}(a+bi)\cdot(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

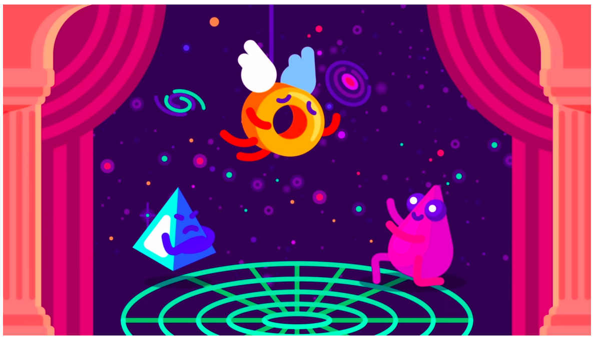
$$\begin{aligned}\frac{(a+bi)}{(c+di)} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)}\\ &= \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2}\\ &= \frac{ac-adi+bci+bd}{c^2-cdi+cdi+d^2}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i\end{aligned}$$

4 Complex Plane

Real numbers can be thought of as points on the number line, but i is not a real number. It lives on a line perpendicular to the number line called the *imaginary axis* and every multiple of i can be found on it, too.
You can think of a complex number, e.g. $(3-i)$, as a point in this 2D-space. The real part - in this case the number 3 - tells you how far to walk along the real axis and its sign tells you in which direction. A positive real part means walking to the right, while a negative real part means walking to the left. The imaginary part - in this case the number (-1) - tells



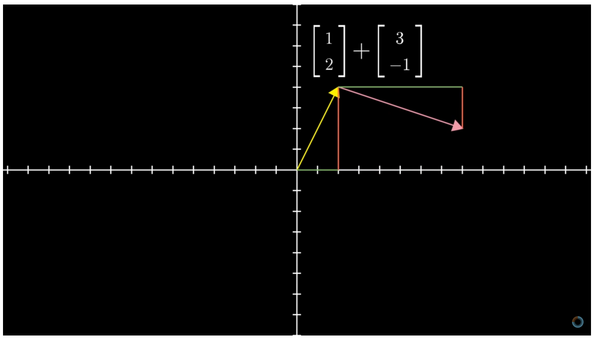
you how far to walk parallely to the imaginary axis and its sign tells you in which direction. A positive imaginary part means walking upward, while a negative imaginary part means walking downward. Notice how every point in this 2D-space is associated with one and only one complex number. Likewise, every complex number is associated with one and only one point. For this reason, we call this 2D-space the *complex plane*.



If you think of our geometry problem as a theatrical performance, the complex plane is its stage and the complex numbers are its actors and actresses. You, however, are the director writing the script: You need to tell everyone on stage where they have to be and how they have to move. Let's have a look at some stage directions you can give.

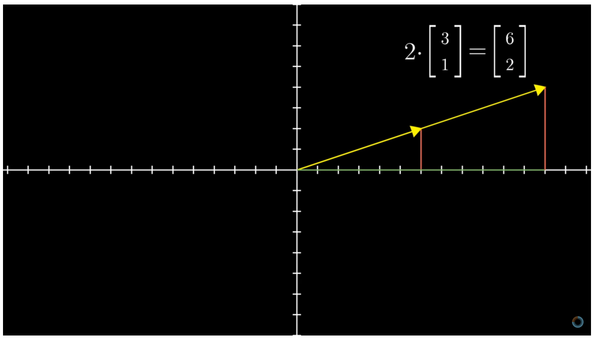
5 Addition and Subtraction

Consider the complex numbers $(3 - i)$ and $(-1 + i)$ again. The point associated with their sum can be found with the following four-step instruction: From the origin, move three units to the right, one unit down, one unit to the left and one unit up. As we already calculated, the result is indeed the



complex number 2. Notice how the first two steps just lead to the point $(3 - i)$. We could have taken a shortcut by drawing an arrow from the origin to this point. The last two steps can be simplified by drawing a second arrow from the origin to $(-1 + i)$ and sliding its tail to the tip of the first arrow.
Subtraction is just like addition, but the direction of the last two steps is inverted: From the origin, move three units to the right, one unit down, - but this time - one unit to the right and one unit down. We can keep the first arrow. Only the second arrow needs to be flipped to resemble the change of direction.

6 Scaling



Thinking of a complex number as a walking instruction or as an arrow makes it easier to visualize. Take $(2 + i)$ as an example. Multiplying it with 2 corresponds with following the walking instruction twice or - alternatively - with stretching the arrow to twice its length. Conversely, multiplying with $\frac{1}{2}$ squishes the arrow to half its length. A negative factor does the same, but also flips the arrow after

stretching or squishing it. In general, we refer to this action as *scaling*.
Notice how every point on the line going through the origin and the complex number can be reached by scaling the complex number accordingly. Choosing a parameter λ as the factor allows us to describe the line as a whole.

7 Lines

We are now able to describe the unique line going through the origin and a given complex number. We can tweak this method in order to describe the line going through two general complex numbers. Take $(-1+i)$ and $(1+2i)$ as an example and notice, that there is a unique line going through them. Choose one of the two complex numbers - in this case I chose $(-1+i)$ - and subtract the chosen complex number from the two original complex numbers. Here we get 0 and $(2+i)$ as a result. Notice how this shifts our two original complex numbers in a way that one of them becomes the origin. We can now describe the line going through them, in this case $\lambda(2+i)$, but we need to shift this line to where it was by adding the subtracted value from before. This yields $(-1+i)+\lambda(2+i)$. Of course, this expression can be tidied up a little bit:

$$\begin{aligned} &(-1+i)+\lambda(2+i) \\ =& -1+i+2\lambda+\lambda i \\ =& (2\lambda-1)+(\lambda+1)i \end{aligned}$$

This process can be done more quickly by condensing it into a single formula: Let z_1 and z_2 be two general complex numbers. The unique line going through them is described by the expression $z_1+\lambda(z_2-z_1)$. Consider the complex numbers $(4+i)$ and $(5-i)$ as an example. Since we already used λ, we are going to use a different parameter like μ, in order to distinguish both lines.

$$\begin{aligned} &(4+i)+\mu((5-i)-(4+i)) \\ =& 4+i+\mu(5-i-4-i) \\ =& 4+i+\mu(1-2i) \\ =& 4+i+\mu-2\mu i \\ =& (\mu+4)+(-2\mu+1)i \end{aligned}$$

8 Intersection of Lines

We now gathered two expressions, with each of them describing a line in the complex plane:

$$\begin{aligned} &(2\lambda-1)+(\lambda+1)i \\ &(\mu+4)+(-2\mu+1)i \end{aligned}$$

One interesting thing we can do with two different lines is to determine where they intersect. We do this by equating both expressions and solving the resulting equation.

$$(2\lambda-1)+(\lambda+1)i=(\mu+4)+(-2\mu+1)i$$

As of right now, we have one equation with two variables. However, notice that two complex numbers are equal if, and only if, their real parts and their imaginary parts are equal. The result is an easy to solve system of two linear equations:

$$\begin{aligned} &2\lambda-1=\mu+4 \\ &\lambda+1=-2\mu+1 \quad \Leftrightarrow \; \lambda=-2\mu \end{aligned}$$

$$\Rightarrow -4\mu-1=\mu+4 \quad \Leftrightarrow \; -5\mu=5 \Leftrightarrow \; \mu=-1$$

Plugging $\mu=-1$ into the original line expression yields the point where the two lines intersect:

$$(-1+4)+(-2\cdot(-1)+1)i=3+3i$$

Again, the result turned out nicely just because I chose the example accordingly. Please don't be scared if you encounter massive fractions while working out the intersection of two lines and keep in mind that there might not even be an intersection since both lines could be parallel. In this case, the system of linear equations doesn't admit a solution.

9 Absolute Value and Argument

We oftentimes want to refer to the distance between a complex number $(a+bi)$ and the origin. We call this distance the *absolute value* of $(a+bi)$ and we use this notation with two vertical lines: $|a+bi|$. If we are only talking about real numbers, the absolute value can be understood as dropping any existant minus sign. When it comes to complex numbers, it can be calculated by using the Pythagorean theorem:

$$|a+bi| = \sqrt{a^2+b^2}$$

Now think of the angle between the real axis and the arrow described by a complex number $(a+bi)$. We refer to this angle as the *argument* of $(a+bi)$ and use the notation $\arg(a+bi)$, where "arg" is short for "argument". It can be calculated through the means of trigonometry, but this won't be necessary for this video.

10 Multiplication and Division

The astute scholars amongst you may have noticed, that we only looked at linear algebra up to this point. This is certainly correct, but now comes the part where complex numbers outshine 2D-vectors completely.
Consider the complex numbers $(3-i)$ and $(-1+i)$ again. Their product can be derived as follows: First, scale $(3-i)$ by the absolute value of $(-1+i)$. Second, rotate the result counter-clockwise around the origin by the argument of $(-1+i)$.
The result of $\frac{3-i}{-1+i}$ can be derived in a similiar way: First, scale $(3-i)$ by one over the absolute value of $(-1+i)$. Second, rotate the result clockwise around the origin by the argument of $(-1+i)$.
Of course, we already have two convenient formulas for actually calculating the product or the quotient of two complex numbers:

$$(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$$
$$\frac{(a+bi)}{(c+di)} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

The main takeaway here is that multiplication describes scaling and <u>counter-clockwise</u> rotation around the origin, while division describes scaling and <u>clockwise</u> rotation around the origin.

11 Rotation

Let's consider the complex number $(1+i)$. We want to rotate it counter-clockwise around the origin by $90°$. Notice that the complex number i has $90°$ as its argument. Additionally, the absolute value of i is 1, so no scaling is going to occur while multiplying with i.

$$(1+i)\cdot i = -1+i$$

We could have also rotated $(1+i)$ clockwise around the origin by dividing by i.

$$\frac{1+i}{i} = \frac{(1+i)\cdot(-i)}{i\cdot(-i)} = \frac{1-i}{1} = 1-i$$

But what if we wanted to rotate $(1+i)$ around another point, like $(2i)$ for example? First, shift both points by subtracting $2i$ from them. Notice how $2i$ was shifted to the origin.

$$(1+i)-2i = (1-i)$$

Second, multiply the result with i in order to rotate it counter-clockwise around the origin by $90°$.

$$(1-i)\cdot i = (1+i)$$

Third, shift both points back by adding $2i$.

$$(1+i)+2i = (1+3i)$$

This might seem surprising, but we now have everything we need to know in order to solve the geometry problem I presented at the start of this video.

12 Without Loss of Generality

The problem starts out with the following proposition: "Let AB be a line segment." We want the problem to take place in the complex plane, but where do we put A and B exactly? Some might suggest that we have to treat A and B as two general complex numbers like $(a+bi)$ and $(c+di)$. This would work, but there is a way to make future calculations easier.
Think of two ancient Greek mathematicians sitting at the beach and studying geometry by drawing figures in the sand. They are asked to imagine a line segment and so they do by drawing one wherever they happen to sit. From there, they follow the same unambiguous steps to produce some geometric construction. Both resulting figures are geometrically similiar meaning that we can scale, rotate, mirror and shift one figure in order to perfectly

match the other. Consequently, it's sufficient to study just one figure, because any result is applicable to the other as well.

Thanks to geometric similiarity, we are free to choose A and B however we like, for example $A = 0$ and $B = 1$. The idea of focusing on a special case that is geometrically similiar to the general case can be reffered to with the expression "*without loss of generality*". For example, mathematicans would distil everything we talked about in this section down to the following sentence: "$A = 0$ and $B = 1$ is assumed without loss of generality."

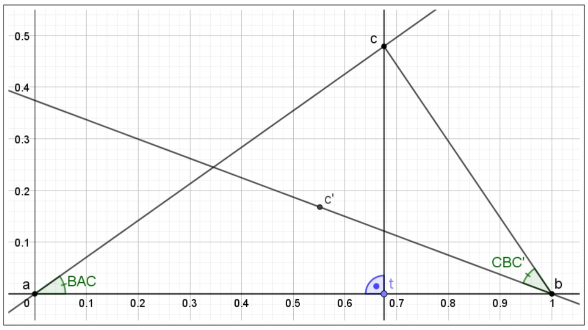
The geometry problem now asks for us to imagine a point T on the line segment, that is closer to B than to A. Now you might be thinking that we are free choose T, just like we were free to choose A and B. Unfortunately, it does not work this way, because the instruction "closer to B than to A" is ambiguous. Imagine the ancient greek mathematicians again. If one of them chose T to be near the middle of the line segment and the other one chose T to be near to the point B, they would end up with figures, which are not geometrically similiar. There is no way to match them by scaling, rotating, mirroring or shifting them. For this reason, we need to treat T as a general point. We do this by making use of a parameter $T = x$, where x is a real number in the semi-open intervall $]0.5, 1]$.

Now the problem asks us to imagine a point C on the line that is perpendicular to AB and that goes through T. Just like with T, we need to treat C as a general point, but notice that we can assume C to be above the line segment without loss of generality. If C were to be below it, we could just look at the drawing mirrored along the real axis and end up with a geometrically similar drawing where C happens to be above the line segment. We can therefore assume $C = x + yi$ without loss of generality, where y is a real number bigger than 0.

Abusing geometric similiarity in order to make simplifying assumptions acts as the concrete foundation for our mathematical proof, because it fixes many moving parts. We can now start to dig into the main calculations.

13 Solution - Part 1

We first have to show that there exists exactly one point D on the line segment AC, such that



the angles $\angle CBD$ and $\angle BAC$ are the same. Here is how we are going to do that: First, rotate C counter-clockwise around B by the angle $\angle BAC$ and call the result C'. Second, find the intersection of the line going through B and C' with the line going through A and C. This intersection is the point D the problem was talking about. At the end, check if D lies on the line segment AC.

Notice that C itself has the argument $\angle BAC$, so let's start with the rotation: Subtract B from the points B and C. This shifts B to the origin and C to the point $(x-1)+yi$. Now, multiply this result with C in order to rotate it counter-clockwise around the origin by $\angle BAC$. Some scaling occurs as well, but this effect doesn't matter for reasons that are going to become apparent.

$$\begin{aligned}
(a+bi)\cdot(c+di) &= (ac-bd)+(ad+bc)i \\
((x-1)+yi)\cdot(x+yi) &= ((x-1)x-y^2)+((x-1)y+xy)i \\
&= (x^2-x-y^2)+(xy-y+xy)i \\
&= (x^2-x-y^2)+(2xy-y)i
\end{aligned}$$

Finally, we need to shift the result back by adding B to it.

$$C' = (x^2 - x - y^2 + 1) + (2xy - y)i$$

We have finally found an expression for C', but we only care about the line going through B and C'. Any scaling happening during multiplication didn't matter to us, because the result is the same line. Now let's calculate its expression:

First, subtract B from the points B and C'. This again shifts B to the origin and C' to the following point:

$$(x^2 - x - y^2) + (2xy - y)i$$

Second, multiply it by the paramter λ to describe the line:

$$(\lambda x^2 - \lambda x - \lambda y^2) + (2\lambda xy - \lambda y)i$$

And finally, shift the result back by adding B to it:

$$BC' :\ (\lambda x^2 - \lambda x - \lambda y^2 + 1) + (2\lambda xy - \lambda y)i$$

The line going through A and C is easier to describe, since A is already at the origin. We just need to scale C by a paramter like μ:

$$AB :\ (\mu x) + (\mu y)i$$

Equating both expressions leads to the following system of linear equations:

$$\begin{aligned} \mu x &= \lambda x^2 - \lambda x - \lambda y^2 + 1 \\ \mu y &= 2\lambda xy - \lambda y \end{aligned}$$

We need to figure out the values of the parameters μ and λ in order to get the point where both lines intersect. Since y is bigger than 0, we can divide the second equation by it:

$$\mu = 2\lambda x - \lambda$$

μ can now be eliminated in the first equation by replacing it with $(2\lambda x - \lambda)$:

$$\begin{aligned} &(2\lambda x - \lambda)x &&= \lambda x^2 - \lambda x - \lambda y^2 + 1 && \\ \Leftrightarrow\ &2\lambda x^2 - \lambda x &&= \lambda x^2 - \lambda x - \lambda y^2 + 1 && \mid -\lambda x^2 + \lambda x \\ \Leftrightarrow\ &\lambda x^2 &&= -\lambda y^2 + 1 && \mid +\lambda y^2 \\ \Leftrightarrow\ &\lambda x^2 + \lambda y^2 &&= 1 && \\ \Leftrightarrow\ &\lambda(x^2 + y^2) &&= 1 && \mid \cdot \frac{1}{x^2 + y^2} \\ \Leftrightarrow\ &\lambda &&= \frac{1}{x^2 + y^2} && \end{aligned}$$

We have solved for λ and can use this result to solve for μ as well:

$$\begin{aligned} \mu &= 2\lambda x - \lambda \\ &= \frac{2x}{x^2 + y^2} - \frac{1}{x^2 + y^2} \\ &= \frac{2x - 1}{x^2 + y^2} \end{aligned}$$

You could go ahead and calculate the point D, but we can already tell that D lies on the line segment AC by just looking at the calculated value for the paramter μ. If we can prove that this value is between 0 and 1, then D has to be between A and C. Since $x > 0.5$, the numerator $(2x-1)$ is bigger than 0:

$$\begin{aligned} &x && > 0.5 && \mid \cdot 2 \\ \Leftrightarrow\ &2x && > 1 && \mid -1 \\ \Leftrightarrow\ &2x - 1 && > 0 && \end{aligned}$$

The denominator $(x^2 + y^2)$ is the sum of two squared real non-zero numbers, making it bigger than 0 as well. So we know that the fraction $\frac{2x-1}{x^2+y^2}$ is bigger than 0. We are left to show that it's not bigger than 1 and there is a pretty cool trick to do that: Consider $(x-1)^2$. It is the square of some real number, therefore, it's not negative. From here it's just simplification:

$$\begin{aligned} &(x-1)^2 && \geq 0 && \\ \Leftrightarrow\ &x^2 - 2x + 1 && \geq 0 && \mid +2x - 1 \\ \Leftrightarrow\ &x^2 && \geq 2x - 1 && \mid \cdot \frac{1}{x^2} \\ \Leftrightarrow\ &1 && \geq \frac{2x-1}{x^2} && \end{aligned}$$

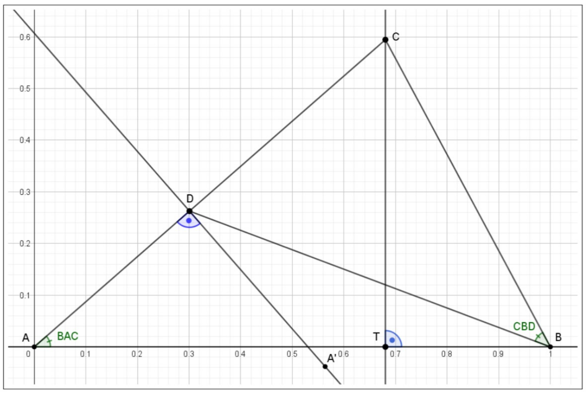
The fraction $\frac{2x-1}{x^2}$ is not bigger than 1, but it is even smaller if y^2 is added to the denominator.

$$1 > \frac{2x - 1}{x^2 + y^2}$$

The system of linear equations has exactly one solution for the values of λ and μ. Also we have shown that the value of μ is always between 0 and 1. Therefore, there is exactly one point D between A and C, such that the angles $\angle CBD$ and $\angle BAC$ are the same.
We have solved the first part of the problem! Let's move on the second one.

14 Solution - Part 2

Show that the line perpendicular to AC and going through D intersects the line segment AB in a point E that does not depend on the choice of C. This part of the problem is going to be solved in the exact same way. As of right now, we just know that D is somewhere on the line segment AB, but we need to calculate its exact position. We do this by



simply plugging in the calculated value for a parameter into the corresponding line expression. Take λ for example:

$$\begin{aligned}
BC' : \quad & (\lambda x^2 - \lambda x - \lambda y^2 + 1) + (2\lambda xy - \lambda y)i \\
\lambda \quad & = \frac{1}{x^2+y^2} \\
\Rightarrow D \quad & = \left(\frac{x^2 - x - y^2}{x^2+y^2} + 1\right) + \left(\frac{2xy - y}{x^2+y^2}\right)i \\
& = \left(\frac{2x^2 - x}{x^2+y^2}\right) + \left(\frac{2xy - y}{x^2+y^2}\right)i
\end{aligned}$$

Next, we need to describe the line perpendicular to AC and going through D. We can get this line's expression by rotating A around D by $90°$ and calling the result A'. Subtract D from the points A and D. This shifts D to the origin and A to the following point:

$$\left(\frac{-2x^2 + x}{x^2+y^2}\right) + \left(\frac{-2xy + y}{x^2+y^2}\right)i$$

Multiply the result by the imaginary number i with an argument of exactly $90°$. This corresponds with a counter-clockwise rotation around the origin by $90°$.

$$\begin{aligned}
& \left(\frac{-2x^2 + x}{x^2+y^2}\right)i + \left(\frac{-2xy + y}{x^2+y^2}\right)i^2 \\
= & \left(\frac{-2x^2 + x}{x^2+y^2}\right)i + \left(\frac{2xy - y}{x^2+y^2}\right) \\
= & \left(\frac{2xy - y}{x^2+y^2}\right) + \left(\frac{-2x^2 + x}{x^2+y^2}\right)i
\end{aligned}$$

Shift the result back by adding D to it. We have now obtained A'.

$$\begin{aligned}
A' = & \left(\frac{2xy - y}{x^2+y^2}\right) + \left(\frac{-2x^2 + x}{x^2+y^2}\right)i \\
& + \left(\frac{2x^2 - x}{x^2+y^2}\right) + \left(\frac{2xy - y}{x^2+y^2}\right)i \\
= & \left(\frac{2x^2 + 2xy - x - y}{x^2+y^2}\right) + \left(\frac{-2x^2 + 2xy + x - y}{x^2+y^2}\right)i
\end{aligned}$$

Again, any scaling during multiplication doesn't affect the line, but notice that the imaginary number i has the absolute value 1. In this particular case, there was in fact no scaling.
Now comes the part there we calculate the line expression for DA'. Subtract D from D and A'. This shifts D to the origin and A' to the following point:

$$\left(\frac{2xy - y}{x^2+y^2}\right) + \left(\frac{-2x^2 + x}{x^2+y^2}\right)i$$

Multiply this result with a parameter like ζ:

$$\left(\frac{2\zeta xy - \zeta y}{x^2+y^2}\right) + \left(\frac{-2\zeta x^2 + \zeta x}{x^2+y^2}\right)i$$

Add D back to this expression to shift the line accordingly:

$$\begin{aligned}
DA' : & \left(\frac{2\zeta xy - \zeta y}{x^2+y^2}\right) + \left(\frac{-2\zeta x^2 + \zeta x}{x^2+y^2}\right)i \\
& + \left(\frac{2x^2 - x}{x^2+y^2}\right) + \left(\frac{2xy - y}{x^2+y^2}\right)i \\
= & \left(\frac{2\zeta xy - \zeta y + 2x^2 - x}{x^2+y^2}\right) + \left(\frac{-2\zeta x^2 + \zeta x + 2xy - y}{x^2+y^2}\right)i
\end{aligned}$$

Please don't get intimidated by this line expression! We just need to know where this line intersects the real axis in order to obtain the point E. We do this by equating the imaginary part of the line expression with 0 and solving for ζ.

$$\begin{aligned}
\frac{-2\zeta x^2 + \zeta x + 2xy - y}{x^2+y^2} & = 0 & & \mid \cdot (x^2+y^2) \\
\Leftrightarrow -2\zeta x^2 + \zeta x + 2xy - y & = 0 & & \mid -2xy + y \\
\Leftrightarrow -2\zeta x^2 + \zeta x & = -2xy + y & & \mid \cdot \frac{1}{-2x^2 + x} \\
\Leftrightarrow \zeta & = \frac{-2xy + y}{-2x^2 + x}
\end{aligned}$$

The line DA' meets the real axis where ζ equals the fraction we just obtained, but it can be simplified further:

$$\zeta = \frac{-2xy + y}{-2x^2 + x} = \frac{y(-2x+1)}{x(-2x+1)} = \frac{y}{x}$$

What a nice result! Now let's plug this value into the original line expression of DA' in order to obtain E. Try to appreciate how everything works out perfectly in this calculation:

$$\begin{aligned}
DA': \quad & \left(\frac{2\xi xy - \xi y + 2x^2 - x}{x^2+y^2}\right) \\
& + \left(\frac{-2\xi x^2 + \xi x + 2xy - y}{x^2+y^2}\right) i \\
\xi \quad & = \frac{y}{x} \\
\Rightarrow E \quad & = \left(\frac{2y^2 - \frac{y^2}{x} + 2x^2 - x}{x^2+y^2}\right) \\
& + \left(\frac{-2xy + y + 2xy - y}{x^2+y^2}\right) i \\
& = \frac{2y^2 - \frac{y^2}{x} + 2x^2 - x}{x^2+y^2} \\
& = \frac{2xy^2 - y^2 + 2x^3 - x^2}{x(x^2+y^2)} \\
& = \frac{y^2(2x-1) + x^2(2x-1)}{x(x^2+y^2)} \\
& = \frac{(x^2+y^2)(2x-1)}{x(x^2+y^2)} \\
& = \frac{2x-1}{x}
\end{aligned}$$

Look at the expression for the point E and notice the absence of the symbol y. This proves that E is indeed independent of the choice of y. No matter how we choose C, the point E stays at this exact spot we just calculated. And this is what the problem wanted us to prove.

15 Further Reading

This video is an effort to make higher level mathematics more accessible to school students. You can find the proof I just presented in the video description. Feel free to use it as a template for your own proofs.

If you are interested in preparing for a math olympiad and if you happen to speak German, French or Italian, I highly encourage you to study the free online ressources provided by the Swiss Math Olympiad. The link can be found in the video description.

If you are interested in an elementary solution to the discussed geometry problem, and if you happen to speak German, you can check out the official homepage of the "Bundeswettbewerb Mathematik". There you can find further information about this math competition as well as past problems with solutions. Again, the link can be found in the video description.

Thanks for watching.