BUNDESWETTBEWERB MATHEMATIK 2018
2. ROUND, PROBLEM 3

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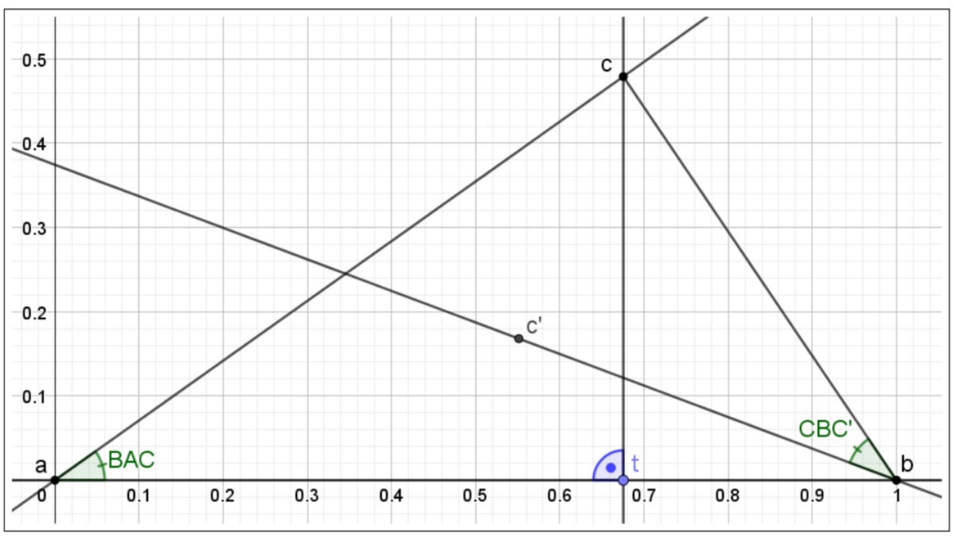
Problem. *Let AB be a line segment and let T be a point on it, that is closer to B than to A. Let C be a point on the line that is perpendicular to AB and that goes through T.*

a) *Show that there exists exactly one point D on the line segment AC, such that the angles $\angle CBD$ and $\angle BAC$ are the same.*
b) *Show that the line perpendicular to AC and going through D intersects the line segment AB in a point E that does not depend on the choice of C.*

Proof. $A = 0 + 0i$ and $B = 1 + 0i$ is assumed without loss of generality. We have $T = x + 0i$ with $x \in (0.5, 1]$, since T is a point on the line segment that is closer to B than to A. We also have $C = x + yi$ with $y \in \mathbb{R} \setminus \{0\}$, since C is a point on the line that is perpendicular to AB and that goes through T. $y > 0$ is assumed without loss of generality.



Let $C' := C(C - B) + B$. It is the result of rotating C counter-clockwise around B by the angle $\arg(C) = \angle BAC$ and scaling it by $|C|$. Therefore, we have $\angle BAC = \angle CBC'$.

$$
\begin{aligned}
C' &= C\left(C-B\right)+B \\
&= \left(x+yi\right)\left(x-1+yi\right)+1 \\
&= \left(\left(x-1\right)x-y^2\right)+\left(\left(x-1\right)y+xy\right)i+1 \\
&= \left(x^2-x-y^2\right)+\left(xy-y+xy\right)i+1 \\
&= \left(x^2-x-y^2+1\right)+\left(2xy-y\right)i
\end{aligned}
$$

An expression for the line AC is $A+\mu\left(C-A\right)=\mu C=\mu x+\mu yi$ with $\mu\in\mathbb{R}$. An expression for the line BC' is $B+\lambda\left(C'-B\right)$ with $\lambda\in\mathbb{R}$:

$$
\begin{aligned}
& B+\lambda\left(C'-B\right) \\
&= 1+\lambda\left(\left(x^2-x-y^2\right)+\left(2xy-y\right)i\right) \\
&= \left(\lambda x^2-\lambda x-\lambda y^2+1\right)+\left(2\lambda xy-\lambda y\right)i
\end{aligned}
$$

Equating both line expressions yields the following system of two linear equations. The solution of this system describes the intersection D of both lines with $\angle BAC=\angle CBC'=\angle CBD$.

$$
\begin{aligned}
\mu x &= \lambda x^2-\lambda x-\lambda y^2+1 \\
\mu y &= 2\lambda xy-\lambda y
\end{aligned}
$$

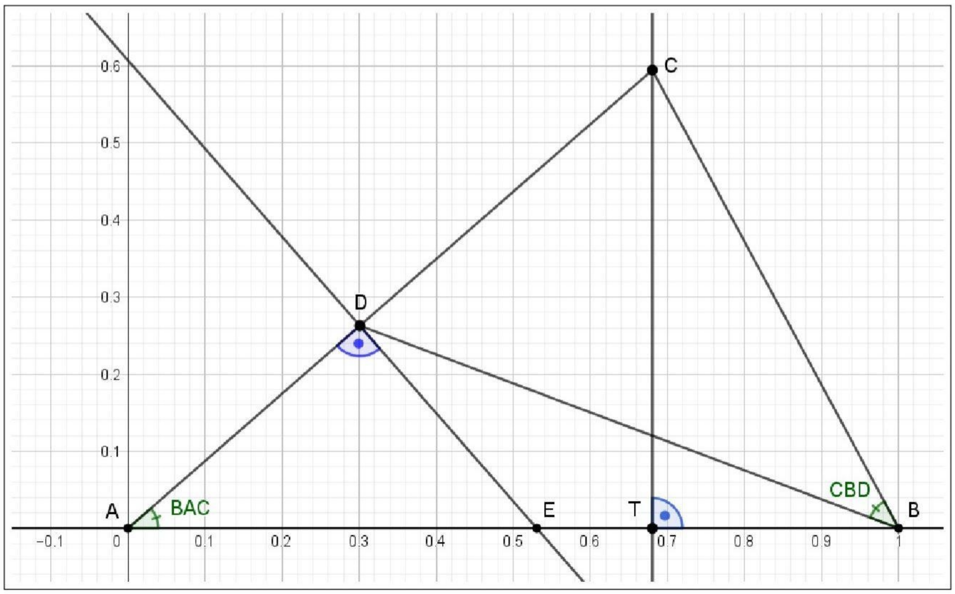
Dividing the second linear equation by $y\neq 0$ yields $\mu=2\lambda x-\lambda$. μ can now be eliminated from the first linear equation:

$$
\begin{aligned}
& \left(2\lambda x-\lambda\right)x=\lambda x^2-\lambda x-\lambda y^2+1 \\
\Leftrightarrow\ & 2\lambda x^2-\lambda x=\lambda x^2-\lambda x-\lambda y^2+1 \\
\Leftrightarrow\ & \lambda x^2+\lambda y^2=1 \\
\Leftrightarrow\ & \lambda=\frac{1}{x^2+y^2}
\end{aligned}
$$

Plugging $\lambda=\frac{1}{x^2+y^2}$ into $\mu=2\lambda x-\lambda$ yields $\mu=\frac{2x-1}{x^2+y^2}$. First, we have $x^2-2x+1=(x-1)^2\geq 0$ and therefore $x^2\geq 2x-1$. Because of $y^2>0$ we also have $x^2+y^2>2x-1$ and therefore $1>\frac{2x-1}{x^2+y^2}$. Next, we have $2x-1>0$, $x^2+y^2>0$ and $\frac{2x-1}{x^2+y^2}>0$ because of $x>0.5$. Therefore, $1>\frac{2x-1}{x^2+y^2}>0$.

The system of linear equations has exactly one solution $\lambda=\frac{1}{x^2+y^2}$ and $\mu=\frac{2x-1}{x^2+y^2}$. Also we have shown that $\mu=\frac{2x-1}{x^2+y^2}$ is always between 0 and 1. Therefore, there is exactly one point D between A and C, such that the angles $\angle CBD$ and $\angle BAC$ are the same.

$$
\begin{aligned}
D &= \left(\lambda x^2 - \lambda x - \lambda y^2 + 1\right) + (2\lambda xy - \lambda y)\, i \\
&= \left(\frac{x^2 - x - y^2}{x^2 + y^2} + 1\right) + \left(\frac{2xy - y}{x^2 + y^2}\right) i \\
&= \left(\frac{2x^2 - x}{x^2 + y^2}\right) + \left(\frac{2xy - y}{x^2 + y^2}\right) i
\end{aligned}
$$



Let $A' := i(A - D) + D$. It is the result of rotating A counter-clockwise around D by the angle $\arg(i) = 90°$ and scaling it by $|i| = 1$.

$$
\begin{aligned}
A' &= i(A - D) + D \\
&= D - Di \\
&= \left(\frac{2x^2 - x}{x^2 + y^2}\right) + \left(\frac{2xy - y}{x^2 + y^2}\right) i - \left(\frac{2x^2 - x}{x^2 + y^2}\right) i + \left(\frac{2xy - y}{x^2 + y^2}\right) \\
&= \left(\frac{2x^2 + 2xy - x - y}{x^2 + y^2}\right) + \left(\frac{-2x^2 + 2xy + x - y}{x^2 + y^2}\right) i
\end{aligned}
$$

An expression for the line DA' is $D + \zeta\,(A' - D)$ with $\zeta \in \mathbb{R}$:

$$
\begin{aligned}
& D+\zeta(A'-D) \\
&= D+\zeta(D-Di-D) \\
&= D-\zeta Di \\
&= \left(\frac{2x^2-x}{x^2+y^2}\right)+\left(\frac{2xy-y}{x^2+y^2}\right)i-\zeta\left(\frac{2x^2-x}{x^2+y^2}\right)i+\zeta\left(\frac{2xy-y}{x^2+y^2}\right) \\
&= \left(\frac{2x^2+2\zeta xy-x-\zeta y}{x^2+y^2}\right)+\left(\frac{-2\zeta x^2+2xy+\zeta x-y}{x^2+y^2}\right)i
\end{aligned}
$$

Equating the imaginary part with 0 and solving for ζ yields the point E where DA' meets AB.

$$
\begin{aligned}
& \frac{-2\zeta x^2+2xy+\zeta x-y}{x^2+y^2}=0 \\
\Leftrightarrow & -2\zeta x^2+2xy+\zeta x-y=0 \\
\Leftrightarrow & -2\zeta x^2+\zeta x=-2xy+y \\
\Leftrightarrow & \zeta=\frac{-2xy+y}{-2x^2+x}=\frac{y(-2x+1)}{x(-2x+1)}=\frac{y}{x}
\end{aligned}
$$

$$
\begin{aligned}
E= & \left(\frac{2x^2+2\zeta xy-x-\zeta y}{x^2+y^2}\right)+\left(\frac{-2\zeta x^2+2xy+\zeta x-y}{x^2+y^2}\right)i \\
= & \left(\frac{2x^2+\frac{2xy^2}{x}-x-\frac{y^2}{x}}{x^2+y^2}\right)+\left(\frac{-\frac{2x^2y}{x}+2xy+\frac{yx}{x}-y}{x^2+y^2}\right)i \\
= & \left(\frac{2x^3+2xy^2-x^2-xy^2}{x\left(x^2+y^2\right)}\right)+\left(\frac{-2xy+2xy+y-y}{x^2+y^2}\right)i \\
= & \frac{x^2(2x-1)+y^2(2x-1)}{x\left(x^2+y^2\right)} \\
= & \frac{\left(x^2+y^2\right)(2x-1)}{x\left(x^2+y^2\right)} \\
= & \frac{2x-1}{x}
\end{aligned}
$$

We have $1>x$. Therefore, $x>2x-1$ and $1>\frac{2x-1}{x}$. Also we have $x>0.5$. Therefore $2x-1>0$ and $\frac{2x-1}{x}>0$. We have shown that E is between A and B. Indeed, E is independent of the choice of C, since y doesn't appear in the expression of E. $\square$