Determinantal point processes and random matrix theory in a nutshell
– part I –

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Contents

In these notes we review the main concepts about Determinantal Point Processes.

Determinantal point processes are of considerable current interest in Probability theory and Mathematical Physics. They were first introduced by Macchi ([8]) and they arise naturally in Random Matrix theory, non-intersecting paths, certain combinatorial and stochastic growth models and representation theory of large groups.

Classical references about determinantal processes are the papers by Hough *et al.* [2], Johansson [3], König [5] and Soshnikov [9].

1 Point Processes

Consider a random collection of points on the real line. A configuration $\mathcal{X}$ is a subset of $\mathbb{R}$ that locally contains a finite number of points, i.e. $\#(\mathcal{X} \cap [a,b]) < +\infty$ for every bounded interval $[a,b] \subset \mathbb{R}$.

Definition 1. A (locally finite) **point process** $\mathbb{P}$ on $\mathbb{R}$ is a probability measure on the space of all configurations of points $\{\mathcal{X}\}$.

Definition 2. $\mathbb{P}$ is an n-point process if $\mathbb{P}(\#\mathcal{X} = n) = 1$.

Loosely speaking, given a point process on $\mathbb{R}$, it is possible to evaluate the probability of any given configuration.

Example. If $P(x_1, \ldots, x_n)$ is a probability function on $\mathbb{R}^n$ (with respect to the Lebesgue measure) such that it is invariant under permutations

$$P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n) \qquad \forall \sigma \in S_n, \tag{1}$$

then P defines naturally a point process.

The mapping

$$A \mapsto \mathbb{E}[\#(\mathcal{X} \cap A)], \tag{2}$$

which assigns to a Borel set A the expected value of the number of points in A under the configuration $\mathcal{X}$, is a measure on $\mathbb{R}$. Assuming there exists a density ρ_1 with respect to the Lebesgue measure (**1-point correlation function**), we have

$$\mathbb{E}[\#(\mathcal{X} \cap A)] = \int_A \rho_1(x)\mathrm{d}x \tag{3}$$

and $\rho_1(x)dx$ represents the probability to have a point in the infinitesimal interval $[x, x + \mathrm{d}x]$.

More generally, a **k-point correlation function** ρ_k (if it exists) is a function of k variables such that for distinct points

$$\rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{4}$$

is the probability to have a point in each infinitesimal interval $[x_j, x_j + \mathrm{d}x_j]$, $j = 1, \ldots, k$. Thus, given disjoints sets $A_1, \ldots, A_k$, we have

$$\mathbb{E}\left[\prod_{j=1}^{k} \#(\mathcal{X} \cap A_j)\right] = \int_{A_1} \ldots \int_{A_k} \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{5}$$

i.e. the expected number of k-tuples $(x_1, \ldots, x_k) \in \mathcal{X}^k$ such that $x_j \in A_j$ for every j. In case the A_j's are not disjoint it is still possible to define the quantity above, with little modifications. For example, if $A_j = A$ for every j, then

$$\mathbb{E}\left[\left(\binom{\#\mathcal{X} \cap A}{k}\right)\right] = \frac{1}{k!}\int_A \ldots \int_A \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{6}$$

is the expected number of ordered k-tuples $(x_1, \ldots, x_k)$ such that $x_1 < x_2 < \ldots < x_k$ and $x_j \in A$ for every $j = 1, \ldots, k$.

Example (cont'd). If $P(x_1, \ldots, x_n)$ is a probability density function on $\mathbb{R}^n$, invariant under permutations of coordinates, then we can build an n-point process on $\mathbb{R}$ with correlation functions

$$\rho_k(x_1, \ldots, x_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} P(x_1, \ldots, x_n)\mathrm{d}x_{k+1} \ldots \mathrm{d}x_n. \tag{7}$$

The problem of existence and uniqueness of a random point field defined by its correlation functions was studied by Lenard in [6] and [7].

2 Determinantal Point Processes

Definition 3. A point process with correlation functions ρ_k is **determinantal** if there exists a kernel $K(x, y)$ such that for every k and every $x_1, \dots, x_k$ we have

$$\rho_k(x_1, \dots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^k. \tag{8}$$

The kernel K is called correlation kernel of the determinantal point process.

Remark 4. *The correlation kernel is not unique. If K is a correlation kernel, then the conjugation of K with any positive function $h(\cdot)$ gives an equivalent correlation kernel*

$$\widetilde{K}(x, y) := h(x)K(x, y)h(y)^{-1} \tag{9}$$

describing the same point process.

Determinantal processes became quite common as a model describing (random) points that tend to exclude one another. Indeed, it is easy to see that in a determinantal process there is a repulsion between nearby points and from this feature comes the denomination of DPP as Fermionic point process (such name is mostly found in physics literature).

Examples of determinantal processes can be constructed thanks to the following result. We refer to [9] for a thorough exposition.

Theorem 5. *Consider a kernel K with the following properties:*

- *trace-class:* $\operatorname{Tr} K = \int_{\mathbb{R}} K(x, x)\mathrm{d}x = n < +\infty$*;*
- *positivity:* $\det[K(x_i, x_j)]_{i,j=1}^n$ *is non-negative for every* $x_1, \dots, x_n \in \mathbb{R}$*;*
- *reproducing property:* $\forall\ x, y \in \mathbb{R}$

$$K(x, y) = \int_{\mathbb{R}} K(x, s)K(s, y)\mathrm{d}s. \tag{10}$$

Then,

$$P(x_1, \dots, x_n) := \frac{1}{n!} \det[K(x_i, x_j)]_{i,j=1}^n \tag{11}$$

is a probability measure on $\mathbb{R}^n$, invariant under coordinates permutations. The associated point process is a determinantal point process with K as correlation kernel.

3 Gap probability

In a determinantal process all information is contained in the correlation kernel and all quantities of interest can be expressed in terms of K. In particular, given a Borel set A, we are interested in the so called gap probability, i.e. the probability to find no points in A.

Consider a point process on $\mathbb{R}$ with correlation function ρ_k and let A be a Borel set such that, with probability 1, there are only finitely may points in A (for example, A is bounded). Denote

by $p_A(n)$ the probability that there are exactly n points in A. If there are n points in A, then the number of ordered k-tuples in A is $\binom{n}{k}$. Therefore, the following equality holds

$$\frac{1}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k=\sum_{n=k}^{\infty}\binom{n}{k}p_A(n). \tag{12}$$

Assume the following alternating series is absolutely convergent, then

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k&=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k\binom{n}{k}p_A(n)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\right)p_A(n).\end{aligned} \tag{13}$$

On the other hand,

$$\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases} \tag{14}$$

therefore,

$$p_A(0)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k, \tag{15}$$

where we call $p_A(0)$ **gap probability**, i.e. the probability to find no points in A. In particular, when a point process is determinantal, we have

$$p_A(0)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\det\left[K(x_i,x_j)\right]_{i,j=1}^{k}\mathrm{d}x_1\dots\mathrm{d}x_k, \tag{16}$$

which is clearly the Fredholm determinant

$$\det\left(\mathrm{Id}-\mathbf{K}\Big|_A\right) \tag{17}$$

of the (trace class) integral operator

$$\begin{aligned}\mathbf{K}\Big|_A : L^2(A)&\to L^2(A)\\ f(x)&\mapsto\mathbf{K}[f](x)=\int_{\mathbb{R}}K(x,y)f(y)\mathrm{d}y.\end{aligned} \tag{18}$$

It is actually possible to prove a more general result, which reduces to the one above when considering zero particles.

Theorem 6 (Theorem 2, [9]). *Consider a determinantal point process with kernel K. For any finite Borel sets $B_j\subseteq\mathbb{R}$, $j=1,\dots,n$, the generating function of the probability distribution of the occupation number $\#_{B_j}:=\#\{x_i\in B_j\}$ is given by*

$$\mathbb{E}\left(\prod_{i=1}^{n}z_j^{\#_{B_j}}\right)=\det\left(\mathrm{Id}-\sum_{j=1}^{n}(1-z_j)K\Big|_{B_j}\right). \tag{19}$$

In particular, the probability of finding any number of points k_j in the correspondent set B_j $\forall j$ is given by a suitable derivative of the generating function at the origin. We refer again to [9] for a detailed proof of the Theorem.

4 Examples of Determinantal Point Processes

4.1 Fermi gas

Consider the Schrödinger operator $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ with V a real-valued function, acting on the space $L^2(E)$, E is a separable Hausdorff space (for the sake of simplicity, E will be $\mathbb{R}$ or S^1), and let $\{\varphi_k\}_{k=0}^{\infty}$ be a set of orthonormal eigenfunctions for the operator H. The n^{th} exterior power of H is an operator $\bigwedge^n H := \sum_{i=1}^n \left[-\frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + V(x_i)\right]$ acting on $\bigwedge^n L^2(E)$ (the space of antisymmetric L^2-functions of n variables) and it describes the Fermi gas with n particle, i.e. an ensemble of n fermions.

The ground state of the Fermi gas is given by the so called Slater determinant

$$\psi(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^n \varphi_{i-1}(x_{\sigma(i)}) = \frac{1}{\sqrt{n!}} \det\left[\varphi_{i-1}(x_j)\right]_{i,j=1}^n . \tag{20}$$

It is known that the absolute value squared of the ground state defines the probability distribution of the particles. Therfore, we have

$$p_n(x_1, \ldots, x_n) = |\psi(x_1, \ldots, x_n)|^2 = \frac{1}{n!} \det\left[K_n(x_i, x_j)\right]_{i,j=1}^n \tag{21}$$

$$K_n(x, y) := \sum_{i=0}^{n-1} \varphi_{i-1}(x)\overline{\varphi_{i-1}(y)} \tag{22}$$

and $K_n(x, y)$ is the kernel of the orthogonal projection onto the subspace spanned by the first n eigenfunctions $\{\varphi_j\}$ of H.

The formula above defines a determinantal process with correlation functions

$$\rho_k^{(n)}(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int p_n(x_1, \ldots, x_n) \mathrm{d}x_{k+1} \ldots \mathrm{d}x_n = \det\left[K_n(x_i, x_j)\right]_{i,j=1}^k . \tag{23}$$

To give some practical examples, let's focus on two special cases of H. The first case is the harmonic oscillator on the real line $\mathbb{R}$

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2; \tag{24}$$

its eigenfunctions are the Weber-Hermite functions

$$\varphi_k(x) = \frac{(-1)^k}{\pi^{\frac{1}{4}}} (2^k k)^{\frac{1}{2}} e^{\frac{x^2}{2}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2} \tag{25}$$

and the correlation kernel is (using the Christoffel-Darboux formula)

$$K_n(x, y) = \left(\frac{n}{2}\right)^{\frac{1}{2}} \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_n(y)\varphi_{n-1}(x)}{x - y}. \tag{26}$$

The second case is the free particle on a circle S^1

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \tag{27}$$

and its correlation kernel is

$$K_n(\theta, \eta) = \frac{\sin\left(\frac{n}{2}(\theta - \eta)\right)}{2\pi \sin\left(\frac{\theta - \eta}{2}\right)}. \tag{28}$$

4.2 Dyson processes

Let $p_t(x;y)$ be the transition probability density from point x to point y at time t of a one-dimensional strong Markov process with continuous sample paths. A classical theorem by S. Karlin and J. McGregor [4] gives a determinantal formula for the probability that a number of paths with given starting and ending positions fall in certain sets at some later time without intersecting in the intermediate time interval (see Figure 1).

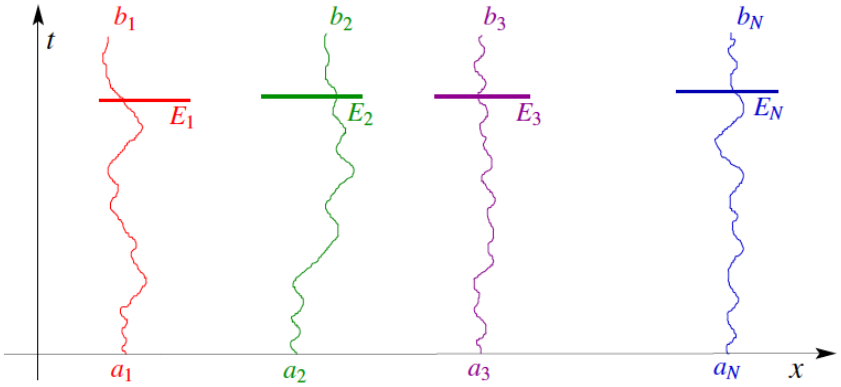

Figure 1: Non-intersecting Brownian paths with fixed starting points $\{a_j\}$.

Theorem 7 ([4]). *Consider n independent copies $X_1(t), \dots, X_n(t)$ of a one-dimensional strong Markov process with continuous sample paths, conditioned so that*

$$X_j(0) = a_j \tag{29}$$

for given values $a_1 < a_2 < \ldots < a_n \in \mathbb{R}$. Let $p_t(x,y)$ be the transition probability function of the Markov process and let $E_1, \dots, E_n \subseteq \mathbb{R}$ be disjoint Borel sets (more precisely, we assume $\sup E_j < \inf E_{j+1}$). Then,

$$\frac{1}{Z_n} \int_{E_1} \dots \int_{E_n} \det\left[p_t(a_i, x_j)\right]_{i,j=1}^{n} \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{30}$$

is equal to the probability that each path X_j belongs to the set E_j at time t, without any intersection between paths in the time interval $[0,t]$, for some normalizing constant Z_n.

Sketch of the proof (a heuristic argument). Let $n = 2$, then

$$\begin{aligned} &\frac{1}{Z_2} \int_{E_1} \int_{E_2} p_t(a_1,x_1)p_t(a_2,x_2) - p_t(a_1,x_2)p_t(a_2,x_1)\,\mathrm{d}x_1\mathrm{d}x_2 \\ &= P\left(X_1(t) \in E_1\right) P\left(X_2(t) \in E_2\right) - P\left(X_1(t) \in E_2\right) P\left(X_2(t) \in E_1\right) \\ &=: P\left(\mathcal{A}\right) - P\left(\mathcal{B}\right). \end{aligned} \tag{31}$$

On the other hand,

$$P\left(\mathcal{A}\right) - P\left(\mathcal{B}\right) = P\left(\mathcal{A}_1\right) + P\left(\mathcal{A}_2\right) - P\left(\mathcal{B}_1\right) - P\left(\mathcal{B}_2\right) \tag{32}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ represent the following events:

$$\begin{aligned}
\mathcal{A}_1 &= \{\, X_i(t) \in E_i \text{ respectively and the paths did not intersect} \,\} \\
\mathcal{A}_2 &= \{\, X_i(t) \in E_i \text{ respectively and the paths did intersect at least once} \,\} \\
\mathcal{B}_1 &= \{\, X_1(t) \in E_2,\, X_2(t) \in E_1 \text{ and the paths did not intersect} \,\} \\
\mathcal{B}_2 &= \{\, X_1(t) \in E_2,\, X_2(t) \in E_1 \text{ and the paths did intersect at least once} \,\}
\end{aligned}$$

Clearly $P(\mathcal{B}_1) = 0$. Moreover, consider the event $\mathcal{A}_2$: at the first time when the two path collide, we can interchange the labels. This is a bijection $\Psi : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{B}_2$. Since the process is Markovian and the two particles act independently, we have

$$P(\mathcal{A}_2) = P(\mathcal{B}_2)\,. \tag{33}$$

In conclusion,

$$\frac{1}{Z_2}\int_{E_1}\int_{E_2} \det\left[p_t(a_i, x_j)\right]_{i,j=1,2}\, \mathrm{d}x_1\mathrm{d}x_2 = P(\mathcal{A}_1)\,. \tag{34}$$

□

However, this is not a determinantal process, since the correlation functions are not expressible in terms of the determinant of a kernel. If we additionally condition the paths to end at time $T > 0$ at some given points $b_1 < \ldots < b_n$, without any intersection between the paths along the whole time interval $[0, T]$, then it can be shown (using an argument again based on the Markov property) that the random positions of the n paths at a given time $t \in [0, T]$ have the joint probability density function

$$\frac{1}{Z_n}\det\left[p_t(a_i, x_j)\right]_{i,j=1}^n \det\left[p_{T-t}(x_i, b_j)\right]_{i,j=1}^n = \frac{1}{Z_n}\det\left[K_n(x_i, x_j)\right]_{i,j=1}^n \tag{35}$$

with a suitable normalizing constant Z_n and kernel

$$K_n(x, y) := \sum_{j=1}^n \phi_j(x)\psi_j(y) \tag{36}$$

$$\phi_j \in \langle p_t(a_1, x), \ldots, p_t(a_n, x)\rangle, \quad \psi_k \in \langle p_{T-t}(x, b_1), \ldots, p_{T-t}(x, b_n)\rangle, \tag{37}$$

$$\int_{\mathbb{R}} \phi_j(x)\psi_k(x)\mathrm{d}x = \delta_{jk}. \tag{38}$$

Remark 8. *The model we just constructed is known in the literature as biorthogonal ensemble. We refer to [1] for a thorough exposition on the subject.*

Of interest is also the confluent case when two or more starting (or ending) points collapse together. For example, in the confluent limit as $a_j \to a$ and $b_j \to b$, for all j's (see Figure 2), applying l'Hôpital rule to (35) gives

$$\frac{1}{\widetilde{Z}_n}\det\left[\frac{\mathrm{d}^{i-1}}{\mathrm{d}a^{i-1}}p_t(a, x_j)\right]_{i,j=1}^n \det\left[\frac{\mathrm{d}^{j-1}}{\mathrm{d}b^{j-1}}p_{T-t}(x_i, b)\right]_{i,j=1}^n \tag{39}$$

which is still a determinantal point process with kernel derived along the same method as in (36)-(38).

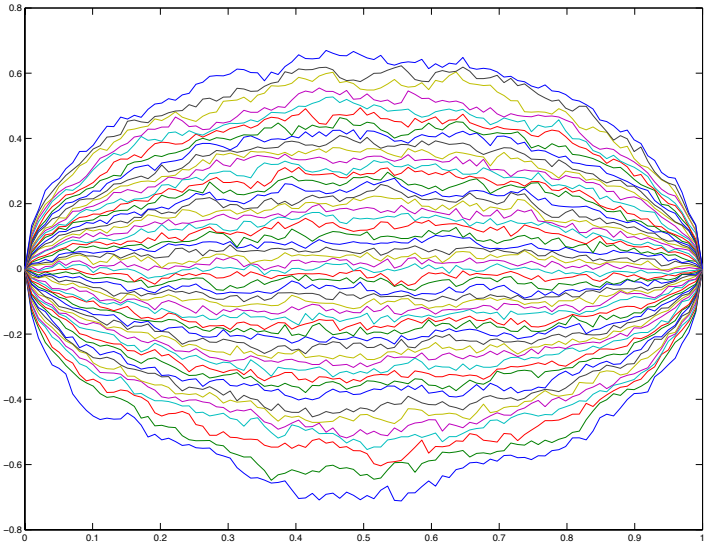

Figure 2: Numerical simulation of 50 non-intersecting Brownian paths in the confluent case with one starting and one ending point.

5 Transformations of Determinantal Point Processes

5.1 Scaling and translations

Let $f \in C^1(\mathbb{R})$ be a bijection, with continuously differentiable inverse $f^{-1} = g \in C^1(\mathbb{R})$. Then, f maps configurations on $\mathbb{R}$ to configurations on $\mathbb{R}$. The push-forward of a point process $\mathbb{P}$ under such mapping is $f(\mathbb{P})$ and it is again a point process.

Proposition 9. *If K is the kernels of a DPP $\mathbb{P}$, then $f(\mathbb{P})$ is also a DPP with kernel*

$$\widehat{K}(x,y) = \sqrt{g'(x)g'(y)}K(g(x),g(y)). \tag{40}$$

In particular, if f is a linear function $f(x) = \alpha\,(x - x^*)$, $\alpha > 0$ and $x^* \in \mathbb{R}$, then the transformed point process $f(\mathbb{P})$ is a scaled and translated version of the original point process $\mathbb{P}$. The correlation kernel K changes to

$$\widehat{K}(x,y) = \frac{1}{\alpha}K\left(x^* + \frac{x}{\alpha}, x^* + \frac{y}{\alpha}\right). \tag{41}$$

5.2 Thinning

Given a point process $\mathbb{P}$, fix $x^* \in R$ and condition on the event that $x^* \in \mathcal{X}$; then, remove this point from the configuration. The resulting process $\mathcal{X}\backslash\{x^*\}$ is a new point process.

Proposition 10. *Let K be the correlation kernel of a DPP and let x^* be a point such that $0 <$*

$K(x^*, x^*) < +\infty$. *Then, the point process obtained by removing* x^* *is determinantal with kernel*

$$\widehat{K}(x, y) = K(x, y) - \frac{K(x, x^*)K(x^*, y)}{K(x^*, x^*)}. \tag{42}$$

5.3 Limits (and universality)

Suppose that for each n we can construct a (finite) determinantal point process $\mathbb{P}_n$ with correlation kernel K_n. If the sequence of kernels $\{K_n\}$ converge in some sense to a limit kernel K as $n \to \infty$, one can expect that also the point processes $\mathbb{P}_n$ will converge to a new determinantal point process $\mathbb{P}$ with correlation kernel K.

This is indeed the case provided some mild assumptions.

Proposition 11. *Let* $\mathbb{P}$ *and* $\mathbb{P}_n$ *be determinantal point processes with kernels* K *and* K_n *respectively. Let* K_n *converge pointwise to* K

$$\lim_{n\to\infty} K_n(x, y) = K(x, y) \tag{43}$$

uniformly in $x,\ y$ *over compact subsets of* $\mathbb{R}$. *Then, the point processes* $\mathbb{P}_n$ *converge to* $\mathbb{P}$ *weakly.*

Remark 12. *The condition of uniform convergence on compact sets may be relaxed.*

Suppose we have a sequence of kernels K_n and a fixed reference point x_*. Before taking the limit, we first perform a centering and rescaling of the form

$$x \mapsto Cn^\gamma(x - x_*) \tag{44}$$

with suitable values of $C,\ \gamma > 0$. Then in many cases of interest the rescaled kernels have a limit

$$\lim_{n\to\infty} \frac{1}{Cn^\gamma} K_n\left(x_* + \frac{x}{Cn^\gamma}, x_* + \frac{y}{Cn^\gamma}\right) = K(x, y) \tag{45}$$

Therefore, the scaling limit K is a kernel that corresponds to a determinantal point process with an infinite number of points.

The physical meaning of this scaling and limiting procedure is the following: as the number of points tends to infinity, one is interested in the local (microscopic) behaviour of the system in specific points of the domain where the particles may lie, upon suitable rescaling: for example, in an infinitesimal neighbourhood entirely contained in the domain (the so-called "bulk") or in an infinitesimal neighbourhood only including the left-most or right-most particles on the line (the so-called "edge").

In many different situations the same scaling limit K may appear. The phenomenon is known as **universality** in Random Matrix Theory. Instances of limiting kernels are the sine kernel

$$K_{\text{sine}} = \frac{\sin \pi(x - y)}{\pi(x - y)} \tag{46}$$

and the Airy kernel

$$K_{\text{Ai}} = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}, \tag{47}$$

where Ai is the Airy function.

References

[1] A. Borodin. Biorthogonal ensembles. *Nucl. Phys. B*, 536:704–732, 1999.

[2] J.B. Hough, M. Krishnapur, Y. Peres, and B. Virag. Determinantal processes and independence. *Prob. Surveys*, 3:206–229, 2006.

[3] K. Johansson. Random matrices and determinantal processes. In *Mathematical Statistical Physics*, volume LXXXIII, chapter 1, pages 1–55. Elsevier, 1st edition, 2006.

[4] S. Karlin and J. McGregor. Coincidence probabilities. *Pacific J. Math.*, 9(4):1141 – 1164, 1959.

[5] W. König. Orthogonal polynomial ensembles in probability theory. *Prob. Surveys*, 2:385–447, 2005.

[6] A. Lenard. Correlation functions and the uniqueness of the state in classical statistical mechanics. *Comm. Math. Phys.*, 30:35–44, 1973.

[7] A. Lenard. States of classical statistical mechanical system of infinitely many particles, I-II. *Arch. Rational Mech. Anal.*, 59:219–256, 1975.

[8] O. Macchi. The coincidence approach to stochastic point processes. *Adv. Appl. Prob.*, 7:83–122, 1975.

[9] A. Soshnikov. Determinantal random point fields. *Russian Mathematical Surveys*, 55:923 – 975, 2000.