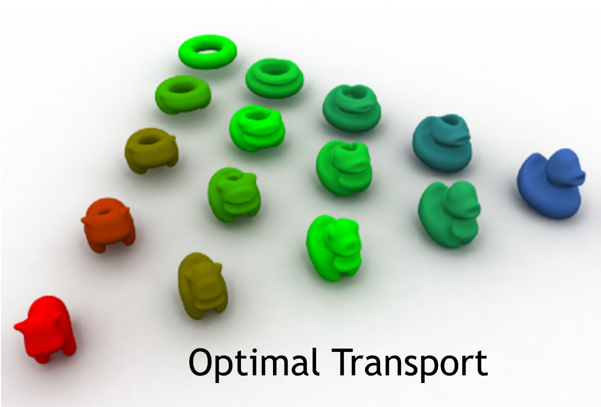
Mathematical Coffees

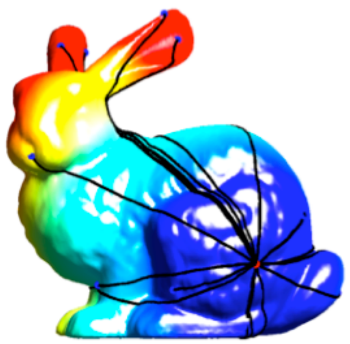
Huawei-FSMP joint seminars

https://mathematical-coffees.github.io

Organized by: Mérouane Debbah & Gabriel Peyré

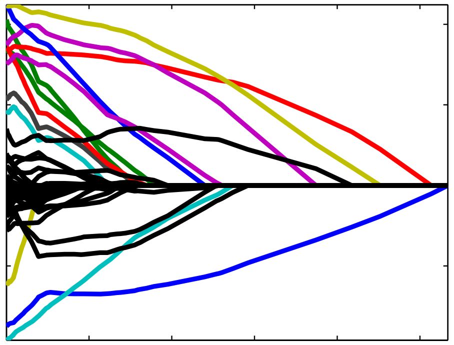


Optimal Transport

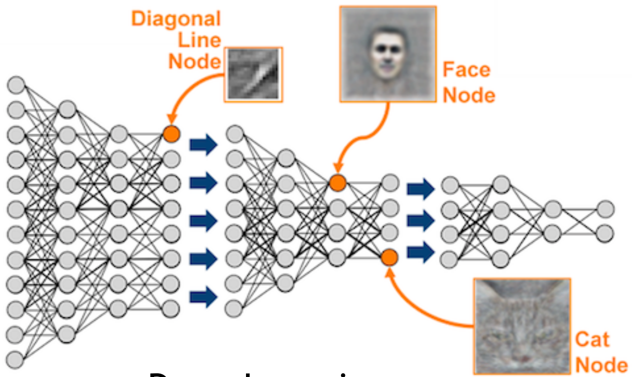


Geodesics

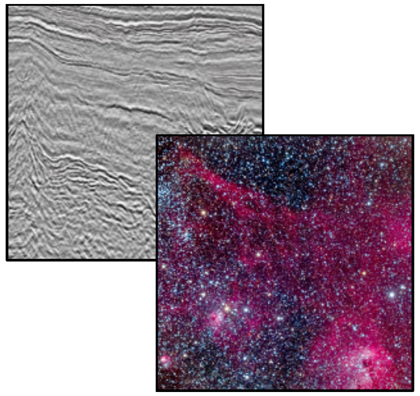
Meshes



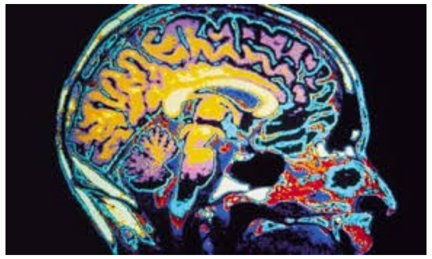
Optimization



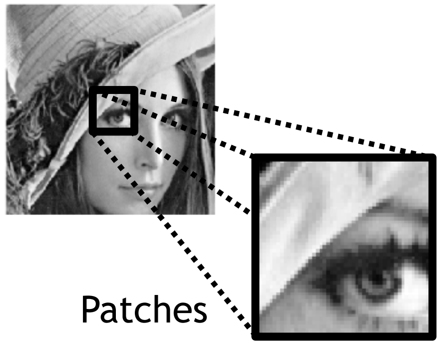
Deep Learning



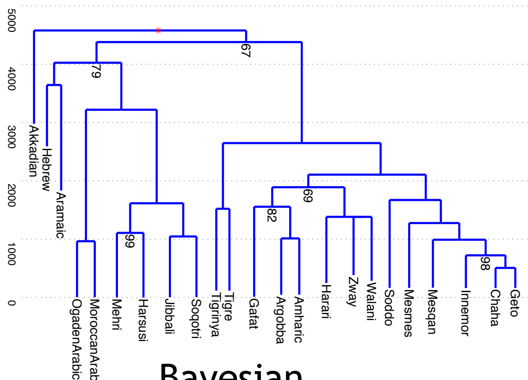
Sparsity



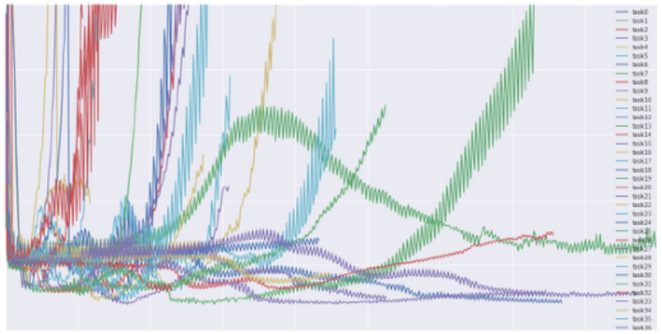
Neuro-imaging



Patches



Bayesian



Parallel/Stochastic

Alexandre Allauzen, Paris-Sud.
Pierre Alliez, INRIA.
Guillaume Charpiat, INRIA.
Emilie Chouzenoux, Paris-Est.

Nicolas Courty, IRISA.
Laurent Cohen, CNRS Dauphine.
Marco Cuturi, ENSAE.
Julie Delon, Paris 5.

Fabian Pedregosa, INRIA.
Guillaume Lecué, CNRS ENSAE
Julien Tierny, CNRS and P6.
Robin Ryder, Paris-Dauphine.
Gael Varoquaux, INRIA.

Jalal Fadili, ENSICaen.
Alexandre Gramfort, INRIA.
Matthieu Kowalski, Supelec.
Jean-Marie Mirebeau, CNRS,P-Sud.

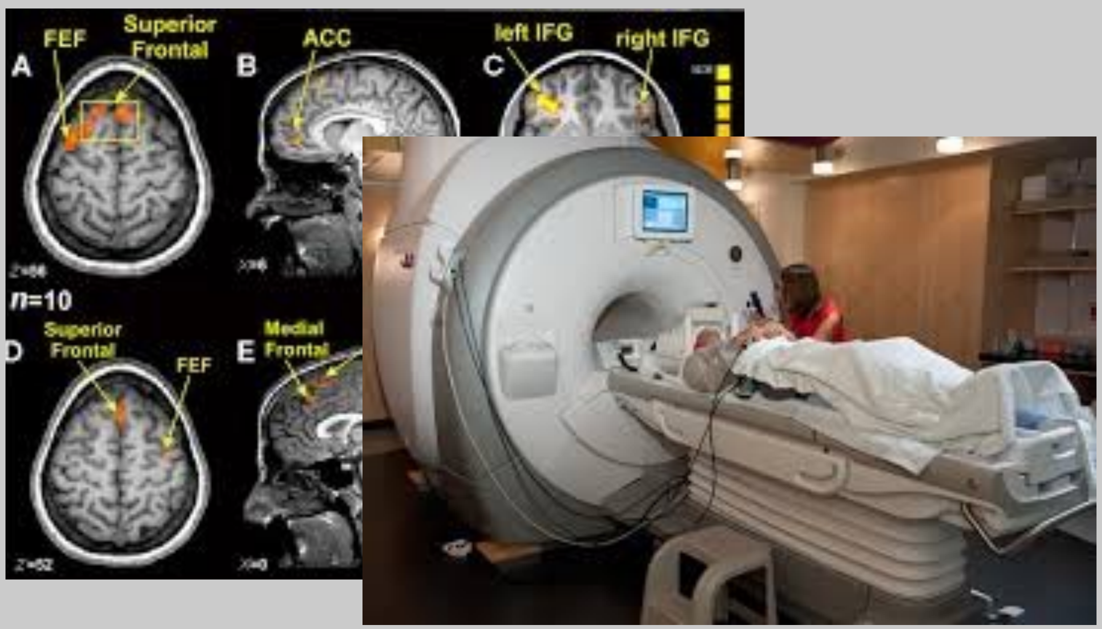
Curses and Blessings of High Dimension

Gabriel Peyré

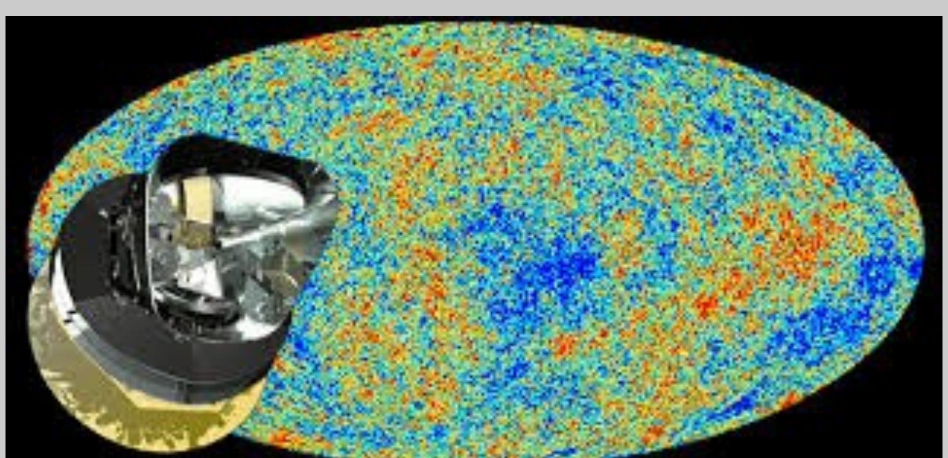
www.numerical-tours.com

Big Datasets (large *n*)

Giga pixels camera



fMRI imaging



Planck mission

Imaging sciences

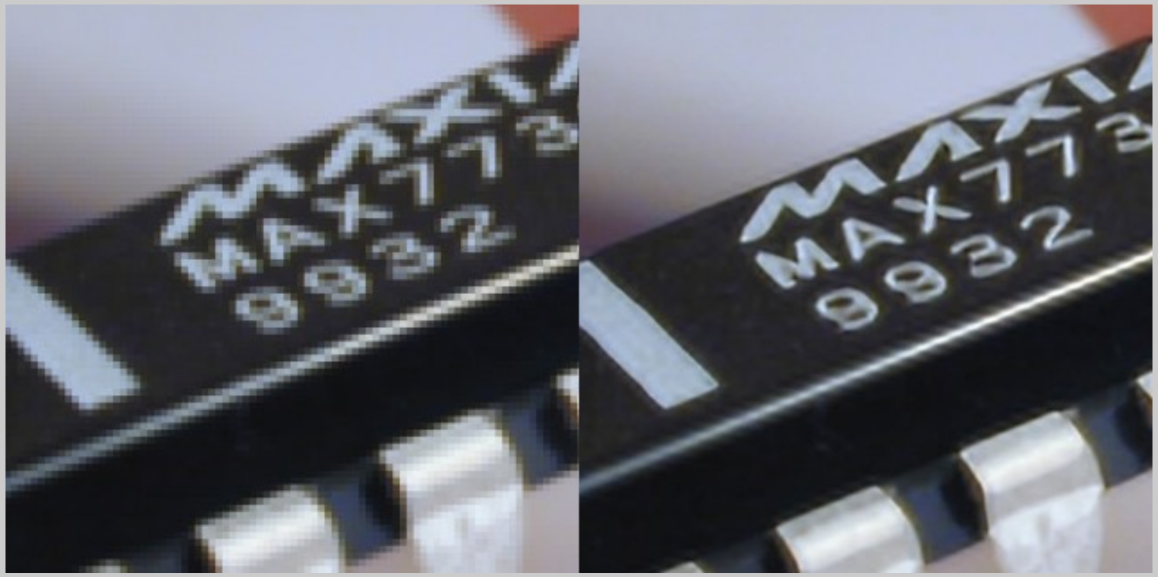
Machine learning

DNA microarrays

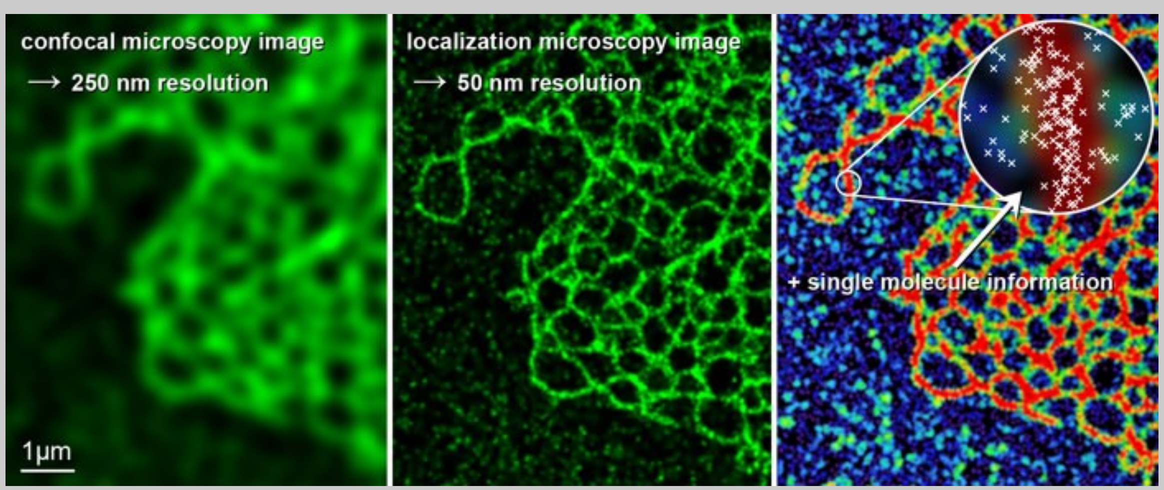
Network monitoring

Big Models (large p)

Super-resolution



Single molecule imaging



Imaging sciences

Machine learning

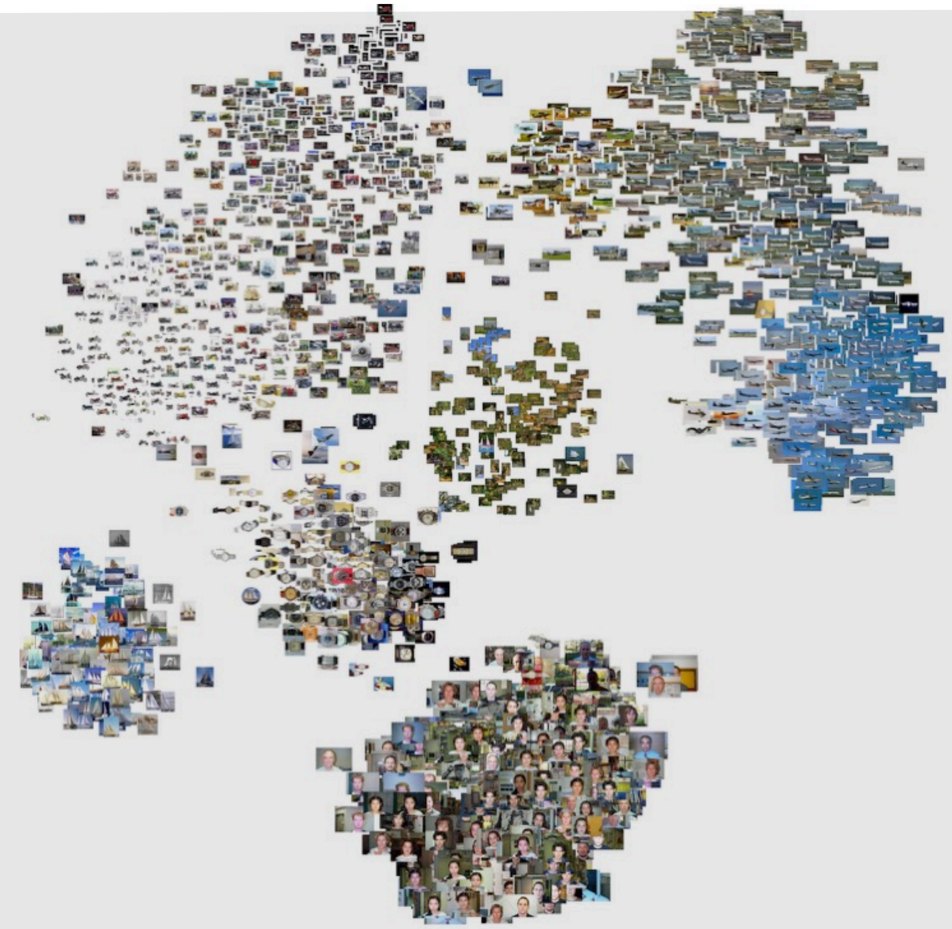
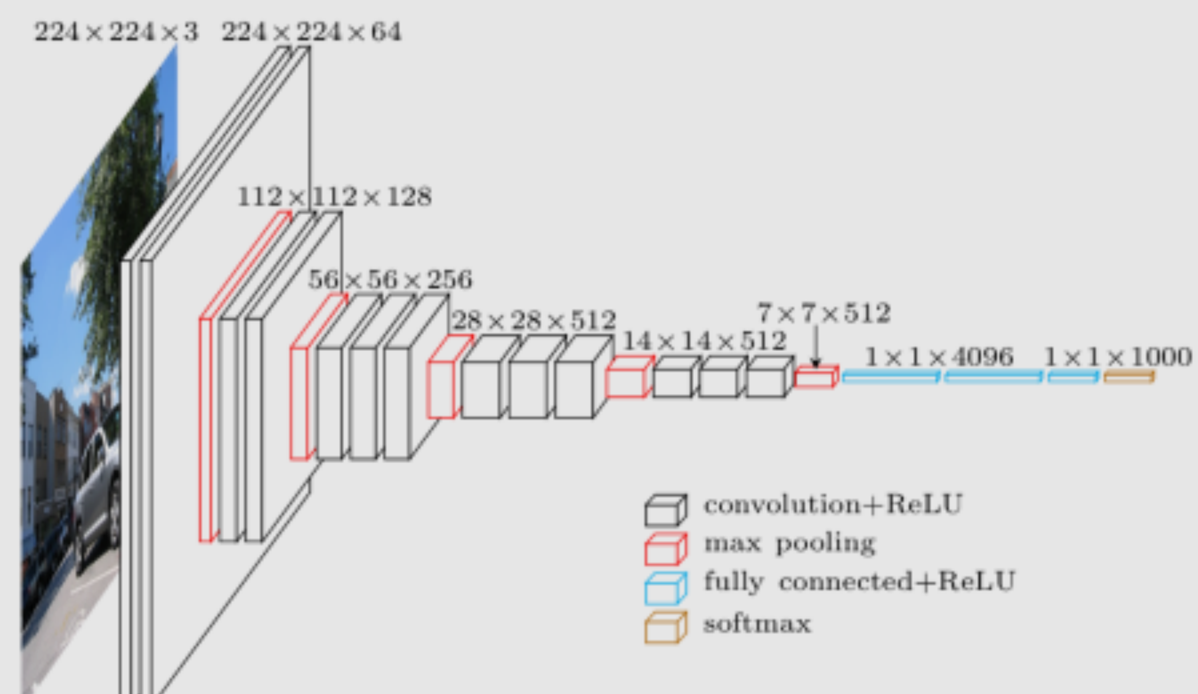


Image classification

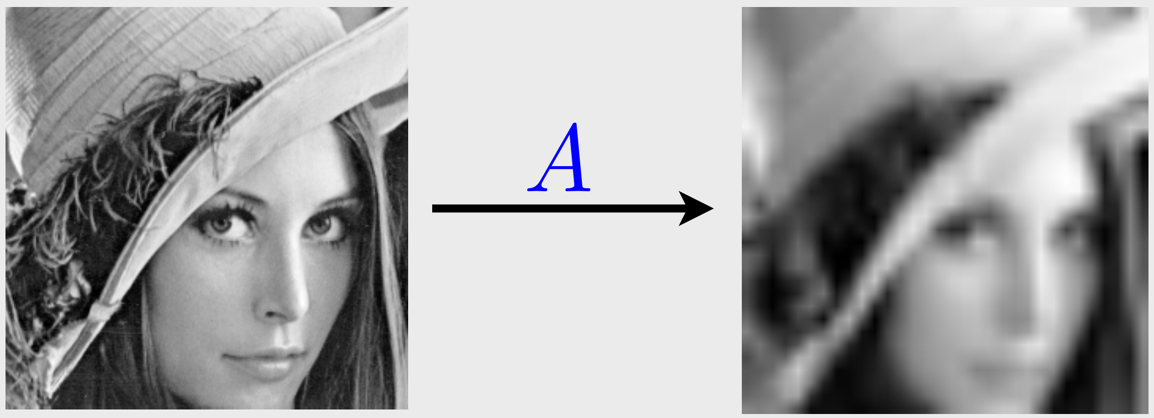


Very deep learning

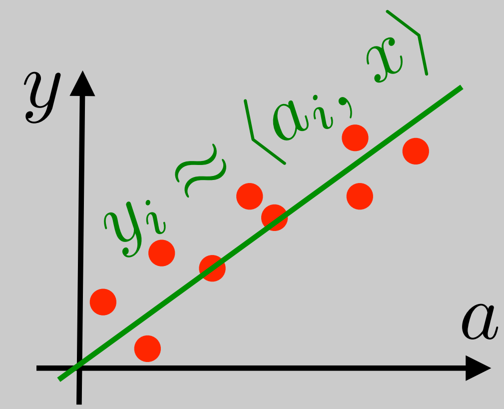
Linear Models

Solving $\quad y \underset{\approx}{=} A\, x \in \mathbb{R}^n \qquad A \in \mathbb{R}^{n \times p}$

Imaging sciences:



Machine learning:
Observations $(a_i, y_i)_{i=1}^p$,



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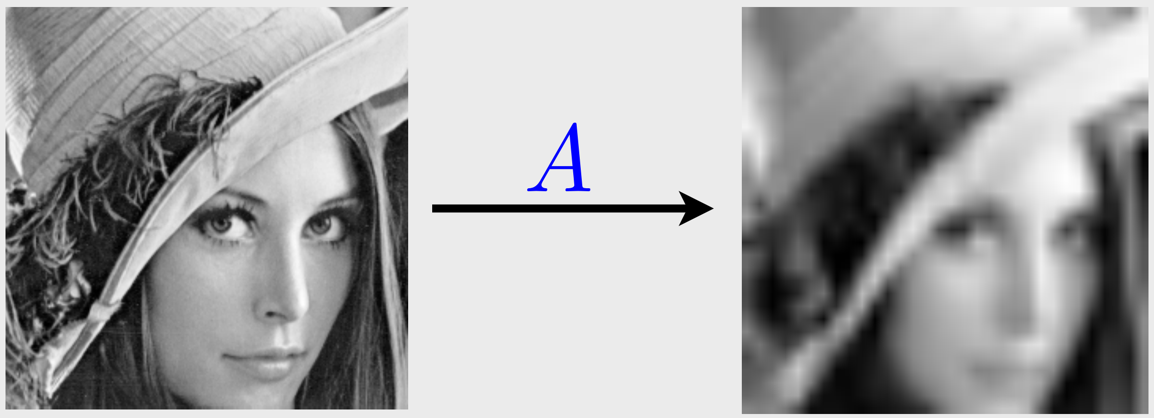

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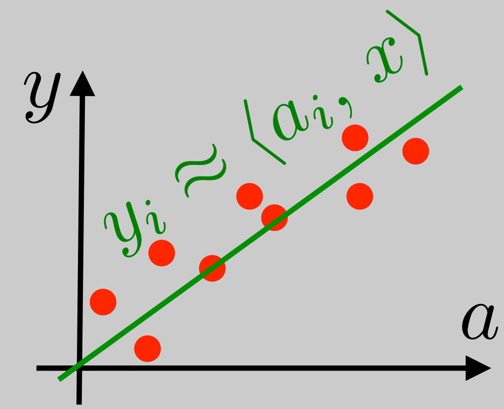
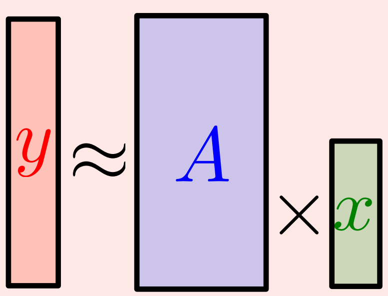


Over-determined $(n > p)$



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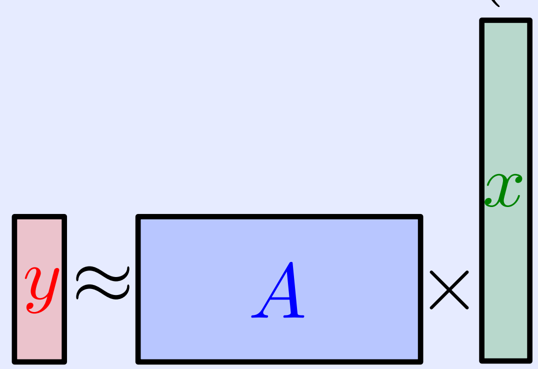
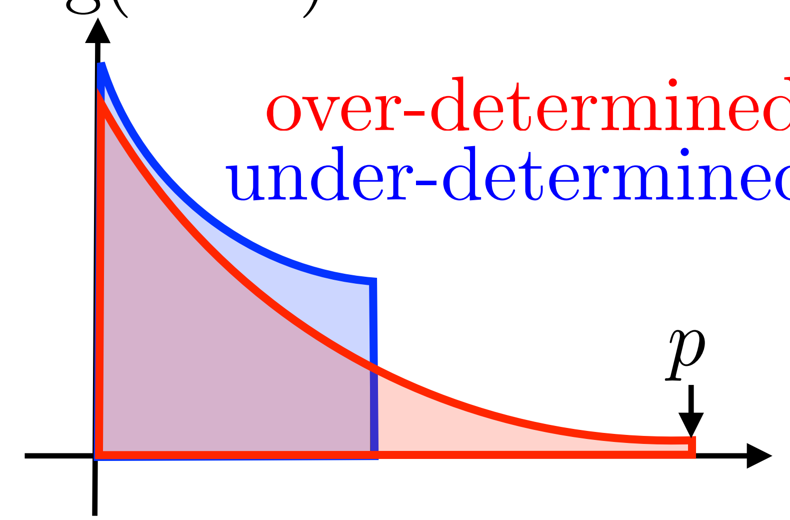


eig$(A^\top A)$



Linear Models

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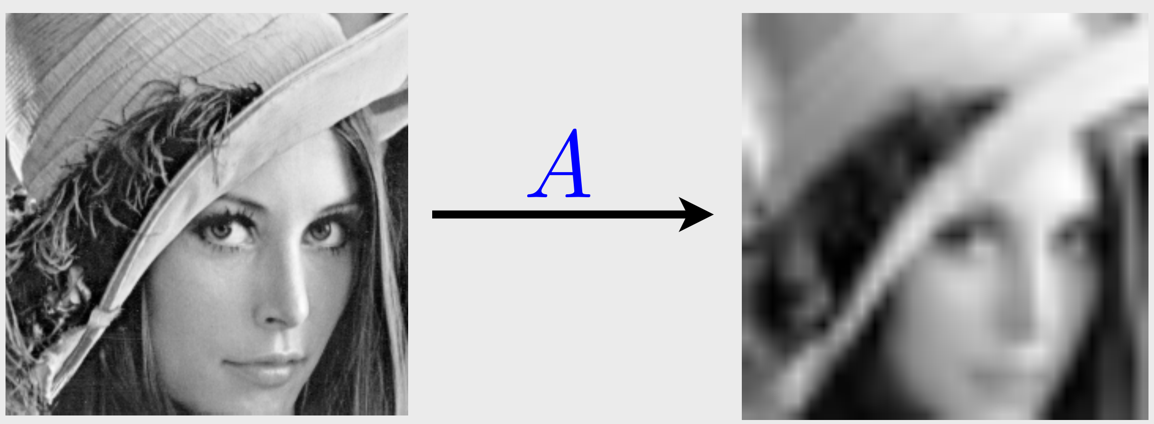
Imaging sciences:



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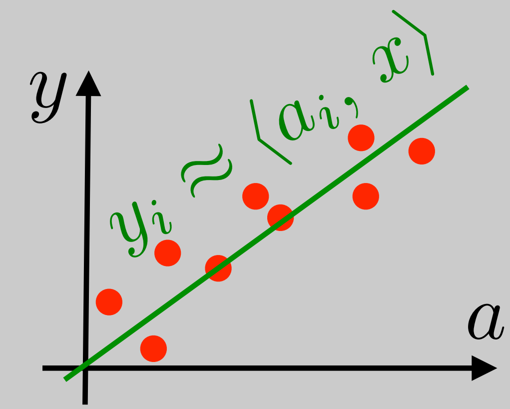

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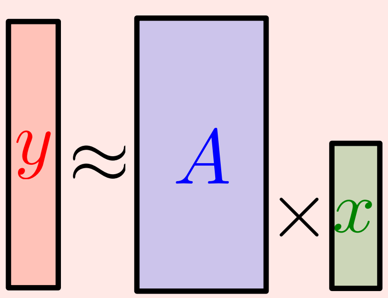
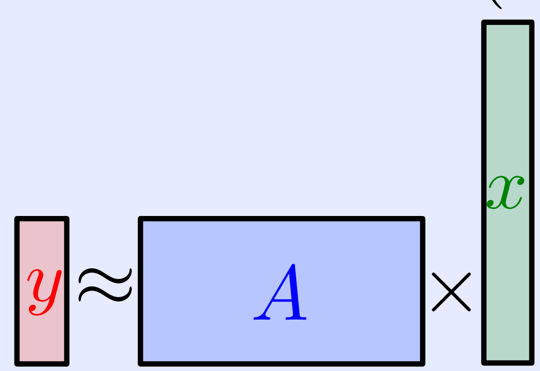
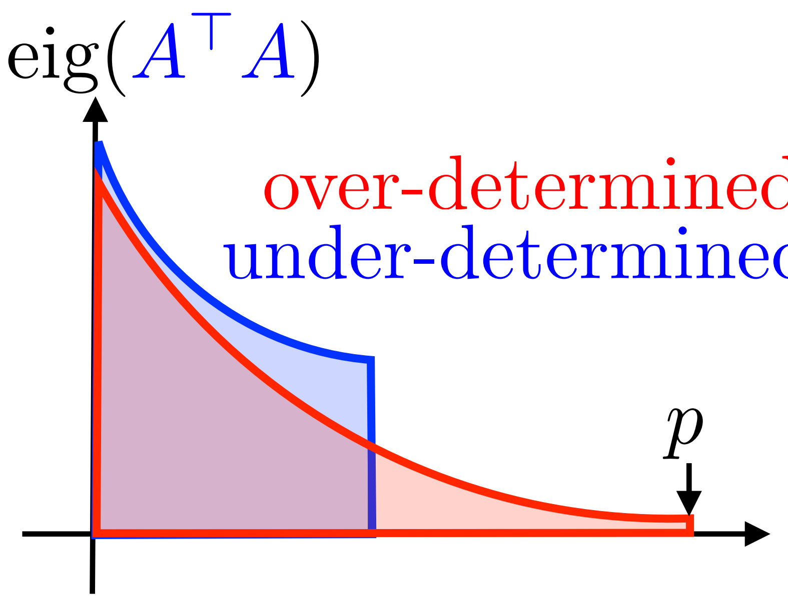


Under-determined $(n < p)$





Curse: Ill-posed, noisy, large size (n, p).

Blessing: unreasonable effectiveness of regularization in high dimension.

Algorithms for large (n,p)

Regularized least square / empirical risk minimization:

$$\min_f \|Ax - y\|^2 + \lambda \|x\|^2$$

$$x = \underbrace{(A^\top A + \lambda \mathrm{Id}_n)^{-1} A^\top y}_{\substack{\text{If } n < m \\ \text{(over-determined)}}} = \underbrace{A^\top (A A^\top + \lambda \mathrm{Id}_m)^{-1} y}_{\substack{\text{If } m < n \\ \text{(under-determined)}}}$$

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Large but finite (n, p): use first order methods.

Gradient descent, CG, BFGS, proximal splittings.

$\longrightarrow$ $O(np)$ or even $O(p)$ cost per iterate.

$\longrightarrow$ Extends to non-smooth regularization (e.g. ℓ^1).

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Very large or infinite p: use stochastic descent methods.

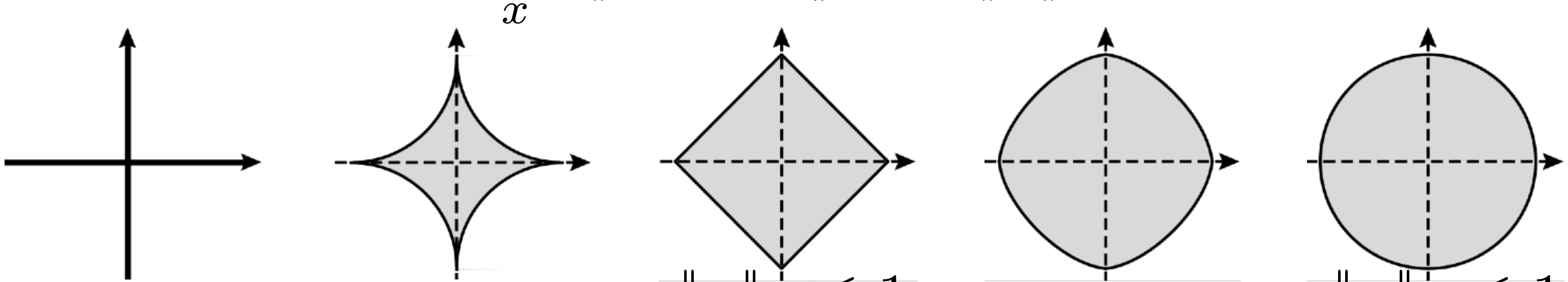
Draw (y_i, a_i) at random, then $x \leftarrow (1 - \tau_k \lambda) x - \tau_k (\langle a_i, x \rangle - y_i) a_i$

decays to 0

L1 and Dimensionality Reduction

Sparsity / model selection: replace $\|x\|^2$ by $\|x\|_1$.

$$\min_x \|Ax - y\|^2 + \lambda \|x\|_1$$



$\|x\|_1 \leqslant 1$ $\|x\|_2 \leqslant 1$

misleading
in high-dimension

$\longrightarrow$ Better model in imaging sciences. $\longrightarrow$ Support recovery with very large p.

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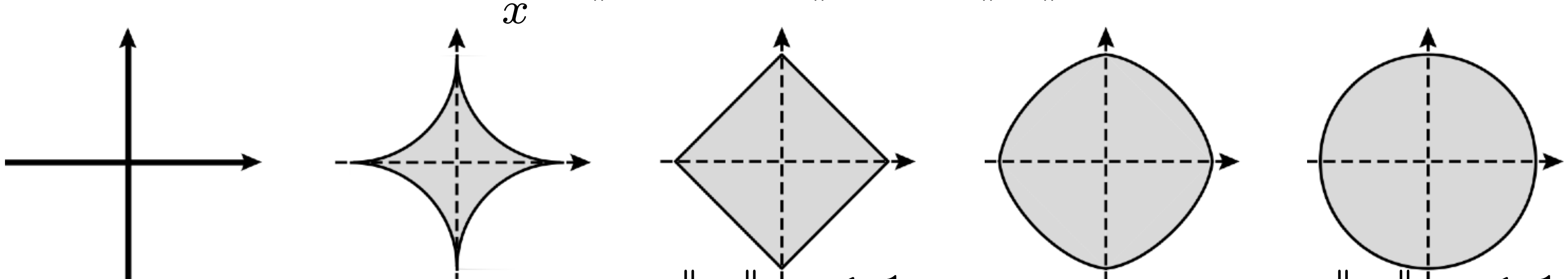

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misleading
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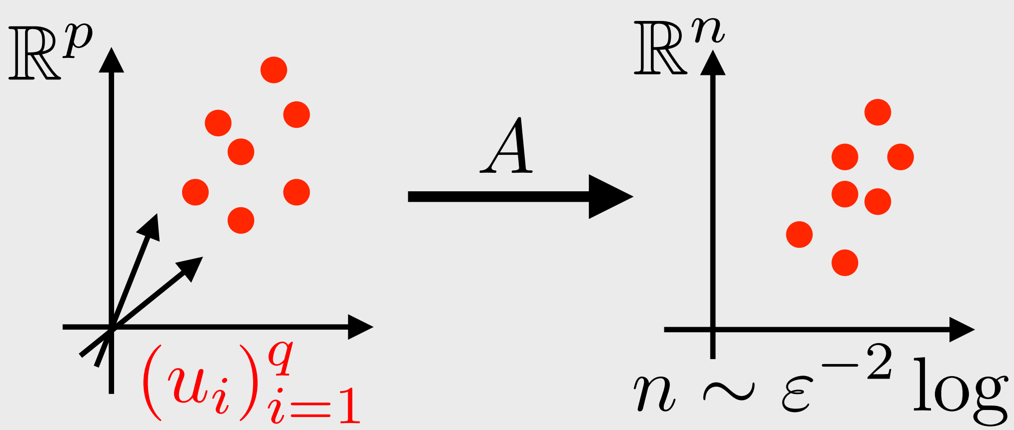
"Optimal" setting: choose $A \in \mathbb{R}^{n \times p}$ random.

Johnson-Lindenstrauss lemma



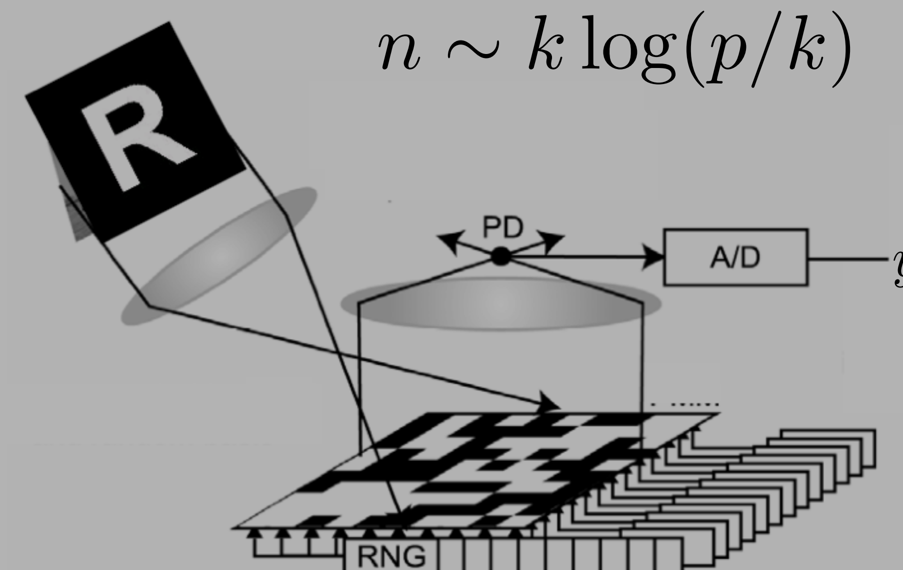
$n \sim \varepsilon^{-2} \log(q/\delta)$

$$\frac{\|u_i - u_j\|^2_{\mathbb{R}^p}}{\|A(u_i - u_j)\|^2_{\mathbb{R}^n}} \in [(1-\varepsilon), (1+\varepsilon)]$$

Compressed sensing

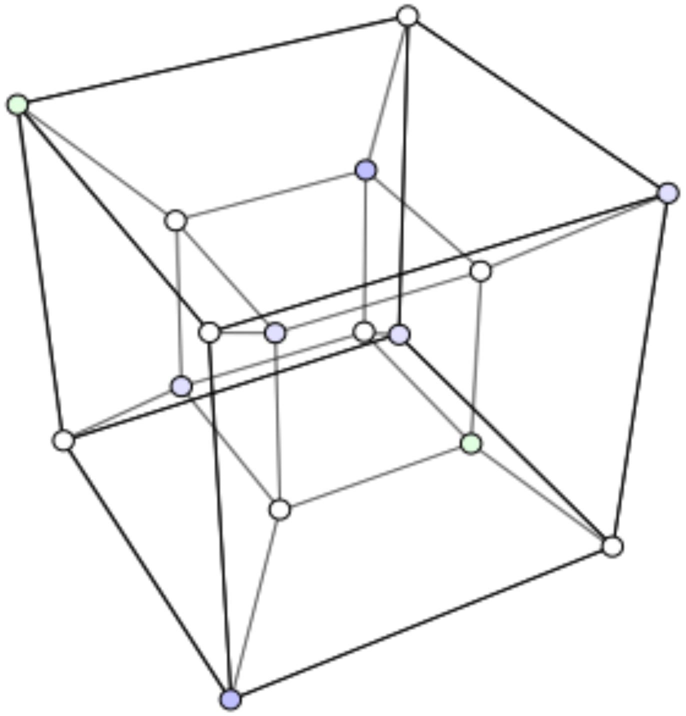
Perfect recovery of k-sparse input

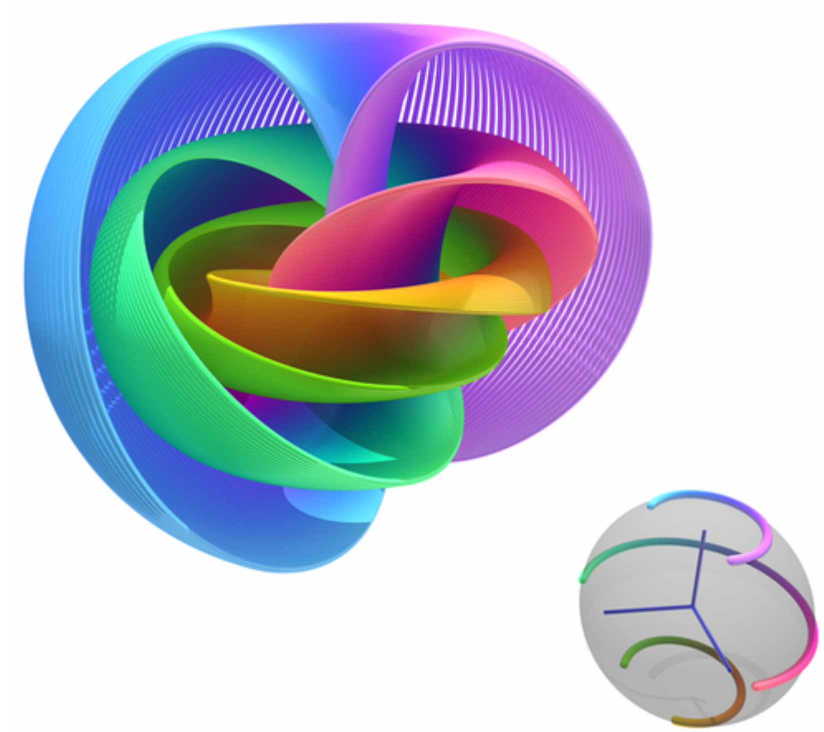
$n \sim k \log(p/k)$

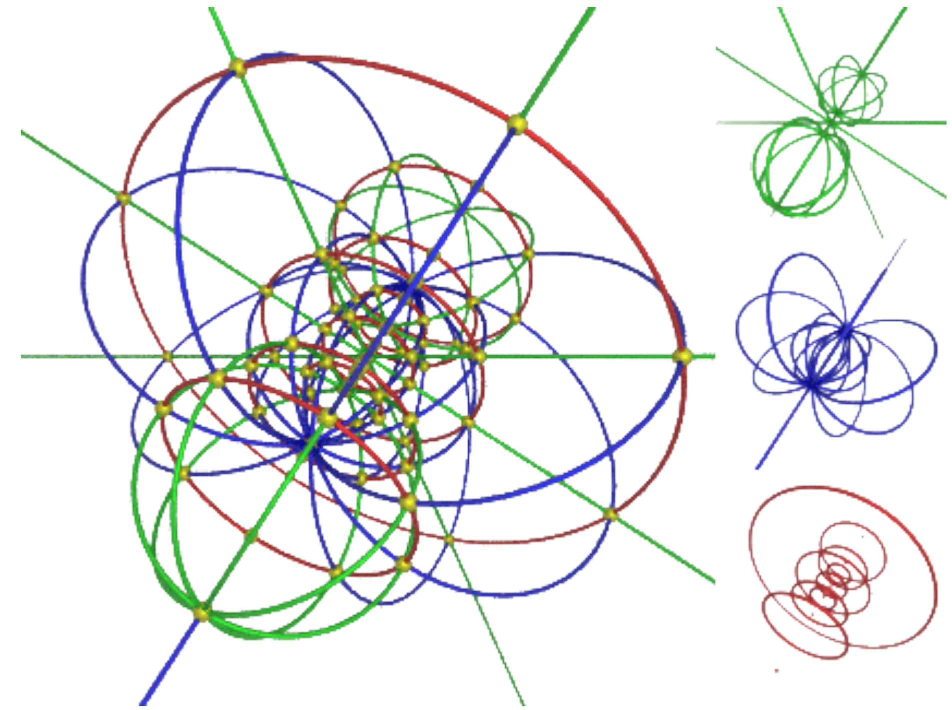


What’s Next

Julie Delon: not so intuitive phenomena in high dimension.







Jalal Fadili: model selection in high-dimension.