Using Pushforward and Pullback Measures for Parameter Identification and Distribution Estimation

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Distribution Estimation

Goal: Obtain the Best Distribution of λ

- Bayesian Context: Regular Bayes insufficient, Hierarchical model required.
- Data Consistent: Use data to construct observed distribution.

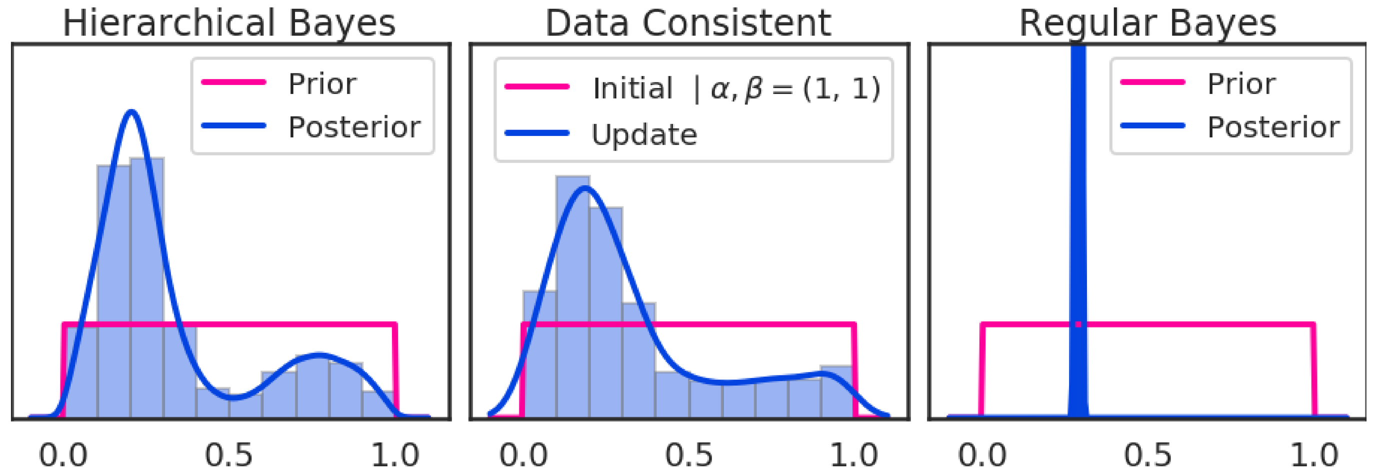
Example

Consider an exponential decay problem with uncertain decay rate:

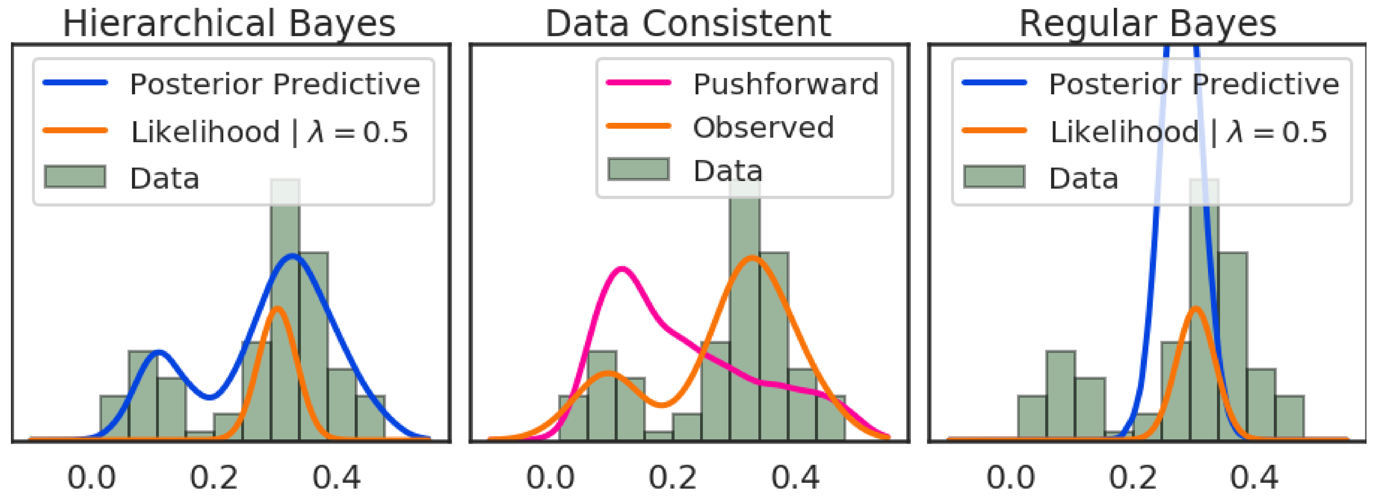
$$u(t) = u_0 \exp(-\lambda t),\ u_0 = 0.5,\ t = 2$$

Regular Bayes	$\pi_{\text{prior}} \sim U[0,1]\,,\quad \pi_{\text{L}}\left(\boldsymbol{d} \mid \lambda\right) \sim N\left(Q\left(\lambda\right), \sigma^2\right)$ $\pi_{\text{post}}\left(\lambda \mid \boldsymbol{d}\right) \propto \pi_{\text{prior}}\left(\lambda\right) \pi_{\text{L}}\left(\boldsymbol{d} \mid \lambda\right)$
Hierarchical Bayes	$\pi_{\text{prior}}\left(\alpha, \beta\right) \sim \chi_1^2\,,\quad \alpha, \beta \in \Omega := [0, \infty) \times [0, \infty)$ $\pi_{\text{prior}}\left(\lambda \mid \alpha, \beta\right) s \sim \text{Beta}\left(\alpha, \beta\right),\quad \pi_{\text{L}}\left(\boldsymbol{d} \mid \lambda\right) \sim N\left(Q\left(\lambda\right), \sigma^2\right)$ $\pi_{\text{post}}\left(\lambda \mid \boldsymbol{d}\right) \propto \int_\Omega \pi_{\text{prior}}\left(\lambda, \alpha, \beta\right) \pi_{\text{L}}\left(\boldsymbol{d} \mid \lambda, \alpha, \beta\right) \, d\Omega$
Data Consistent	$\pi_{\text{up}}\left(\lambda\right) = \pi_{\text{in}}\left(\lambda\right) \dfrac{\pi_{\text{obs}}\left(Q\left(\lambda\right)\right)}{\pi_{\text{pre}}\left(Q\left(\lambda\right)\right)}$

Plots of Concepts and Results



Λ Parameter Space

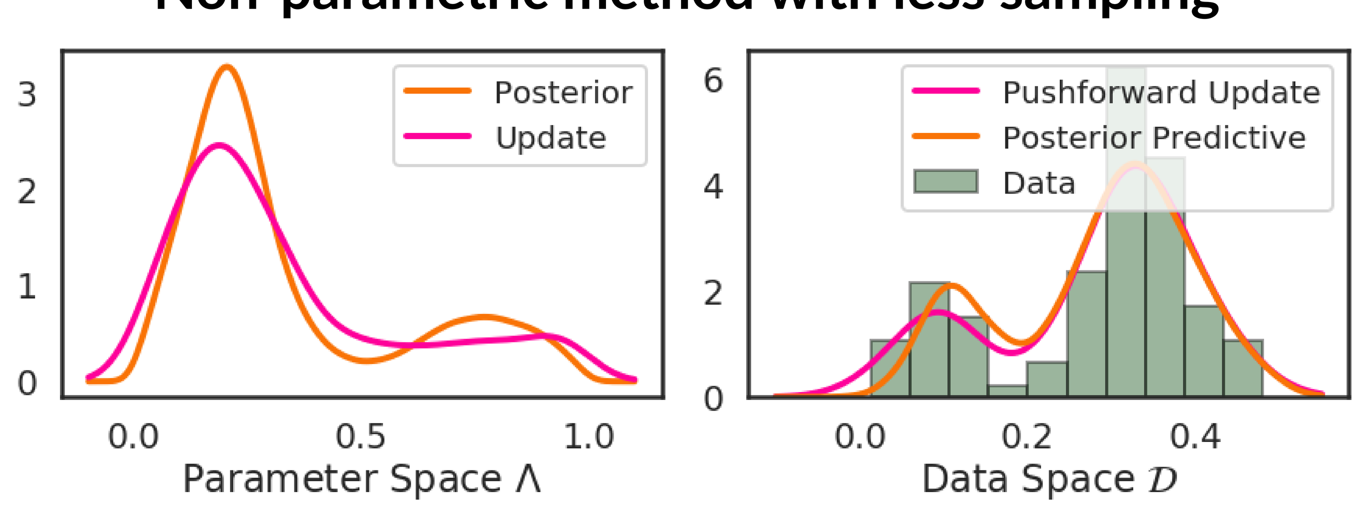


100 observations of data

$\mathcal{D}$ Data Space

Takeaways

Non-parametric method with less sampling



Data Consistent vs. Hierarchical Bayes

Intro to Data Consistent Inversion

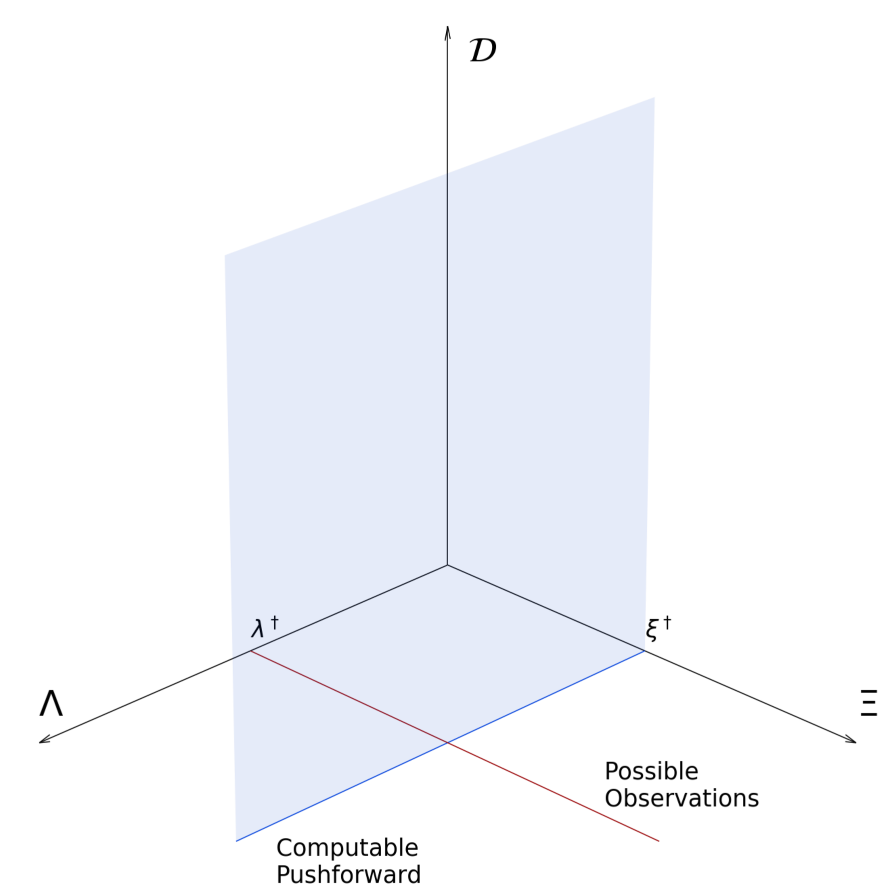
Solving Stochastic Inverse Problems

Data Consistent Inversion is a novel framework that uses pushforward and pullback measures to ensure solutions are consistent with the observed distribution of data.

The Data Consistent Approach

$$\pi_{\text{up}}\left(\lambda\right) = \pi_{\text{in}}\left(\lambda\right) \frac{\pi_{\text{obs}}\left(Q\left(\lambda\right)\right)}{\pi_{\text{pre}}\left(Q\left(\lambda\right)\right)}$$

Which Stochastic Inverse Problem?



Solve for a single parameter value, or for a parameter distribution?

Notation

$\lambda \in \Lambda,\ \xi \in \Xi$	Parameter Space, Noise Space
$\boldsymbol{d} \in \mathcal{D}$	Observables
$Q : \Lambda \to \mathcal{D}$	Quantity of Interest Map
$\pi_{\text{prior}},\ \pi_{\text{L}}$	Prior , Likelihood
$\pi_{\text{in}},\ \pi_{\text{obs}},\ \pi_{\text{pre}}$	Initial, Observed, Predicted (**pushforward**)
$\pi_{\text{post}},\ \pi_{\text{up}}$	Posterior, Update (**pullback**)

References & Attribution

Advisor: Dr. Troy Butler

University of Colorado: Denver

Left to Right: Theory, Stability, BET, Poster, Website. Funding: NSF DMS-1818941.

Parameter Identification

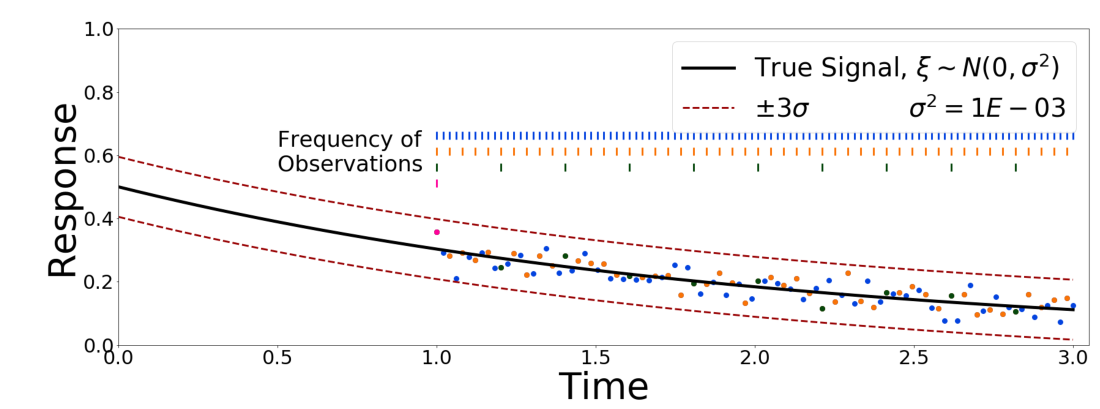
Goal: Obtain the Best Value of λ

- Bayesian Context: Model uses assumed likelihood function of data given λ.
- Data Consistent: Construct predicted distribution of residuals given λ.

Example

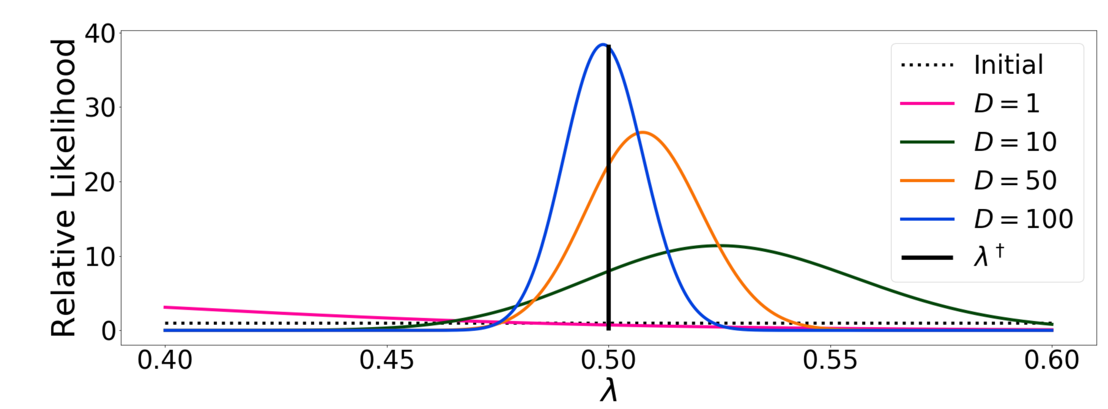
Consider an exponential decay problem with uncertain decay rate:

$$u(t) = u_0 \exp(-\lambda t),\ u_0 = 0.5,\ t \in [0, 3]$$



Convergence of Data Consistent Inversion

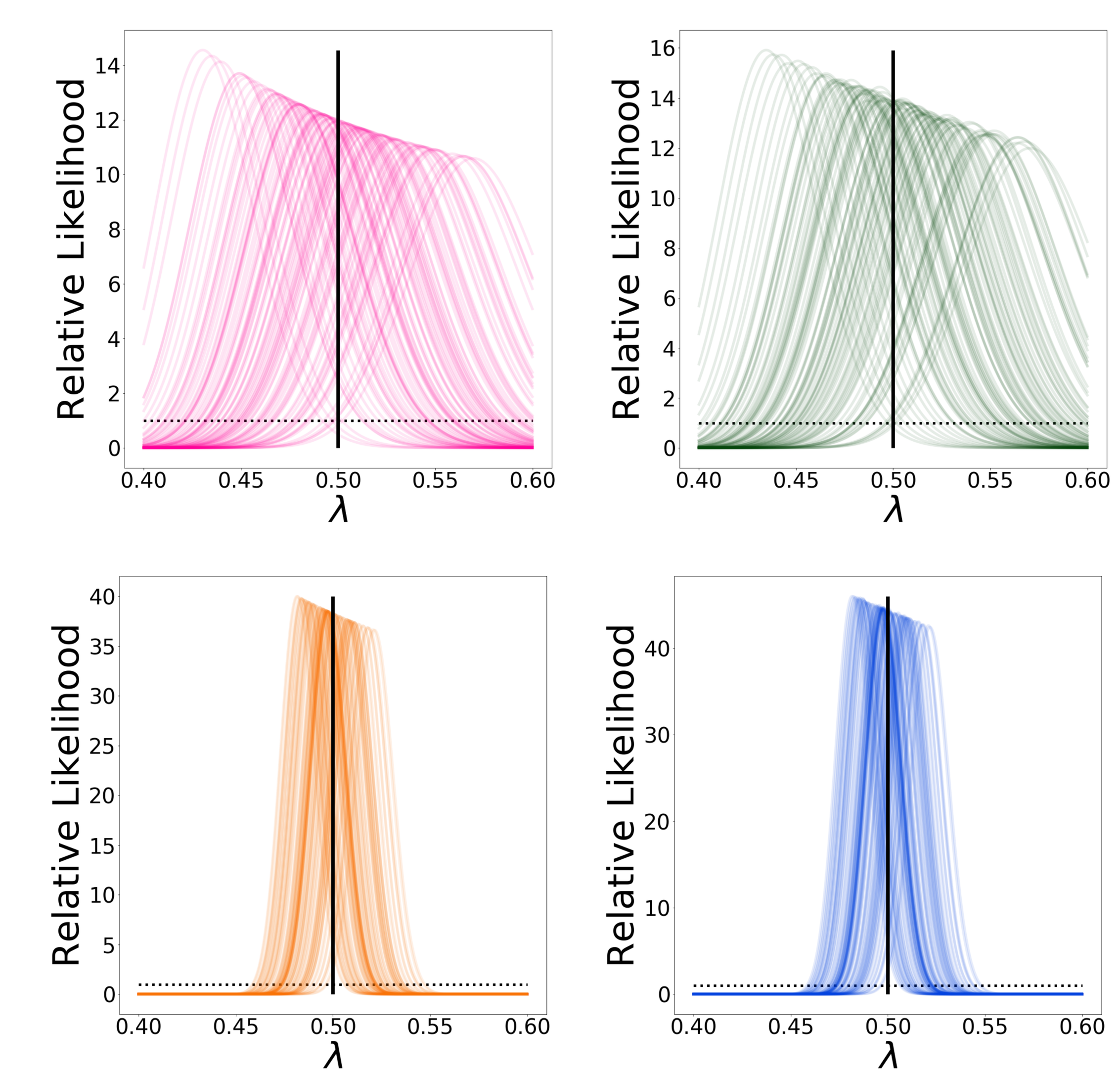
How do solutions change with more data?



$\lambda^\dagger$ and π_{up} for $D = 1, 10, 50, 100$ for $N = 1000$

Comparison to Regular Bayes

How do solutions on conditionals of Ξ compare?



100 realizations of data for $D = 10, 100$ for Data Consistent Inversion (left) and Regular Bayes (right).