

Push-forward Measures for Parameter Identification under Uncertainty

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Introduction

Motivation

How do we update initial descriptions of uncertainty using model predictions and data?

Data-Consistent Inversion is a novel framework that uses push-forward and pull-back measures to ensure solutions are consistent with the observed distribution of data.

Question

How do we cast a Parameter Identification problem in the context of Data-Consistent Inversion?

Framework

| | |
|---|--|
| ▪ \mathbb{P}, π | Probability Measure, Density |
| ▪ $\Lambda \subset \mathbb{R}^P$ | Parameter Space |
| ▪ $\mathbf{o} : \Lambda \rightarrow \mathcal{O} \subset \mathbb{R}^D$ | Observables |
| ▪ $\Xi \subset \mathbb{R}^D$ | Noise Space |
| ▪ $\lambda^\dagger \in \Lambda$ | True Parameter |
| ▪ $\mathbf{d}(\xi) \subset \mathbb{R}^D$ | Possible Data, $\mathbf{d}_i(\xi) = \mathbf{o}_i(\lambda^\dagger) + \xi_i$ |
| ▪ $\xi^\dagger \in \Xi$ | Noise in Measurements |
| ▪ σ^2 | Variance of Noise |
| ▪ $\mathbf{d}^\dagger \in \mathbb{R}^D$ | Observed Data, $\mathbf{d}^\dagger = \mathbf{d}(\xi^\dagger)$ |
| ▪ $\mathbb{P}_{\text{in}}, \pi_{\text{in}}$ | Initial |
| ▪ $\mathbb{P}_{\text{obs}}, \pi_{\text{obs}}$ | Observed |
| ▪ $\mathbb{P}_{\text{pre}}, \pi_{\text{pre}}$ | Predicted (push-forward) |
| ▪ $\mathbb{P}_{\text{up}}, \pi_{\text{up}}$ | Updated (pull-back) |

Updating with Observations and Predictions

$$\mathbb{P}_{\text{up}} = \mathbb{P}_{\text{in}} \frac{\mathbb{P}_{\text{obs}}}{\mathbb{P}_{\text{pre}}} \quad \left| \quad \pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \frac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$$

References & Attribution

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Left to Right: Theory, Stability, BET, ConsistentBayes, Personal Website.
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Approach

Quantity of Interest Map

A Functional Relating Predictions and Data

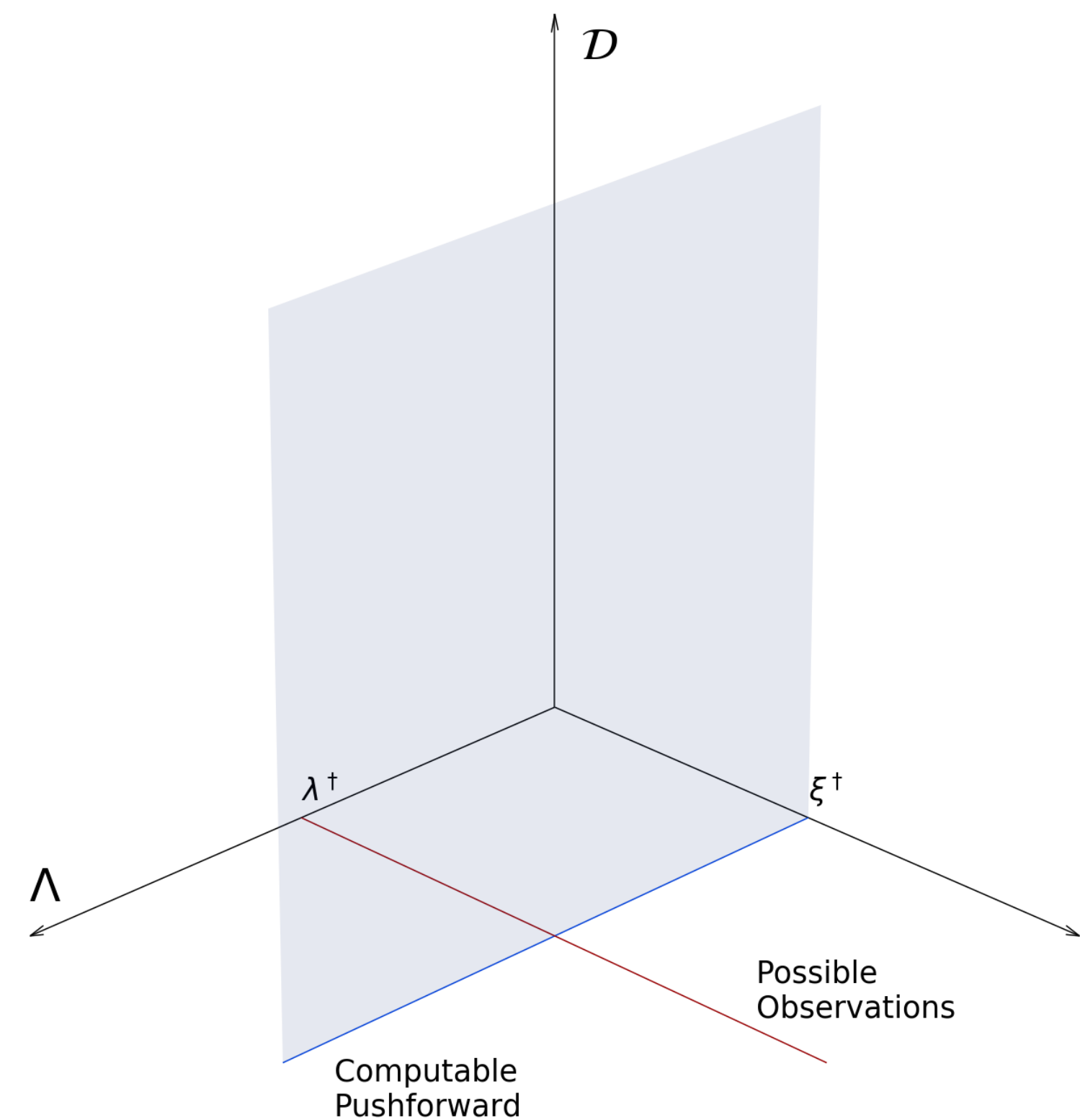
- Ideal
- Theoretical
- Practical
- Computable

$$Q(\lambda, \xi) = F(\mathbf{o}(\lambda), \mathbf{d}(\xi))$$

$$Q(\Lambda, \Xi) =: \mathcal{D}_{\mathcal{T}} \subset \mathbb{R}$$

$$Q(\lambda) = F(\mathbf{o}(\lambda), \mathbf{d}^\dagger)$$

$$Q(\Lambda) =: \mathcal{D}_{\mathcal{C}} \subset \mathcal{D}_{\mathcal{T}}$$



How do conditionals of Ξ compare to the joint density?

Observed Distribution

Given a functional, what measure do we invert?

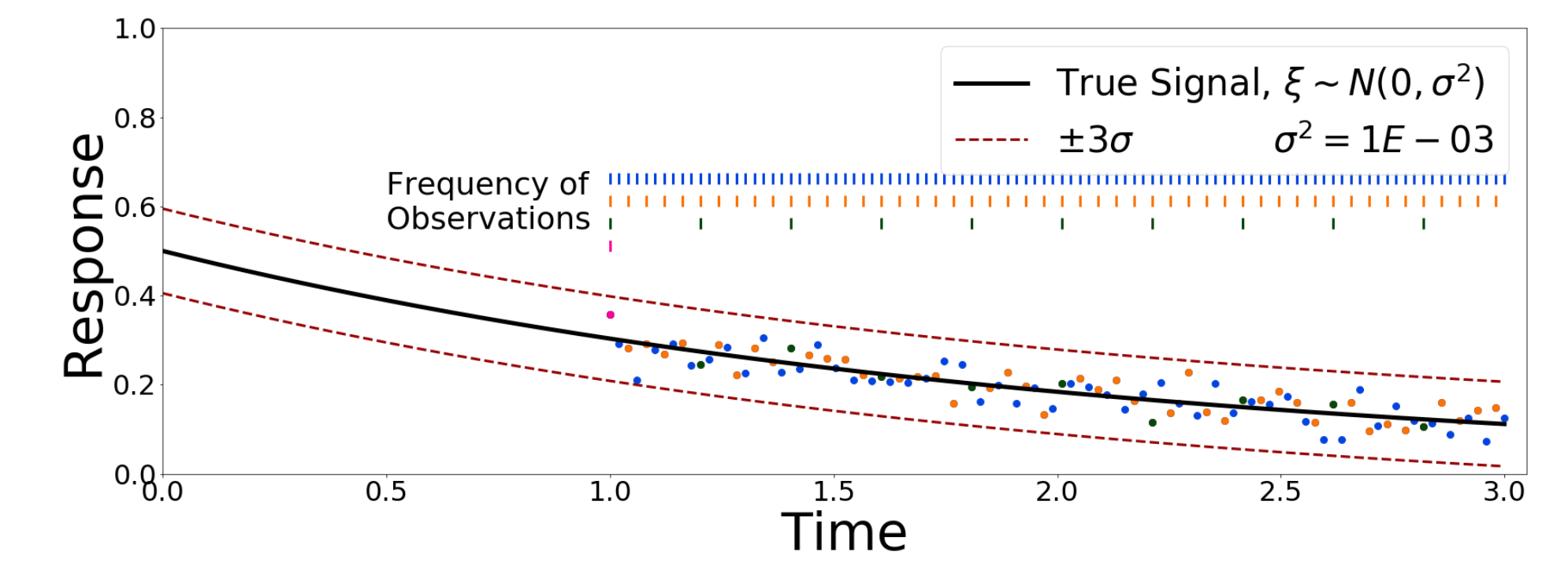
$$Q(\lambda^\dagger, \xi) \sim \pi_{\text{obs}} \text{ when we allow } \xi \text{ to vary over } \Xi$$

| $F(\mathbf{o}(\lambda), \mathbf{d}^\dagger)$ | ξ | π_{obs} |
|--|---------------------------|--------------------|
| $\frac{1}{\sigma\sqrt{D}} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)$ | $\xi \sim L^2$ | $N(0, 1)$ |
| $\frac{1}{\sigma^2} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)^2$ | $\xi \sim N(0, \sigma^2)$ | $\chi^2(D)$ |
| $\frac{1}{\sigma^2 D} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)^2$ | $\xi \sim N(0, \sigma^2)$ | $\Gamma(D/2, D/2)$ |
| \vdots | \vdots | \vdots |

Choices of F and associated π_{obs} for stochastic inverse problem with $\mathbf{d}^\dagger = \mathbf{o}_i(\lambda^\dagger) + \xi_i^\dagger$

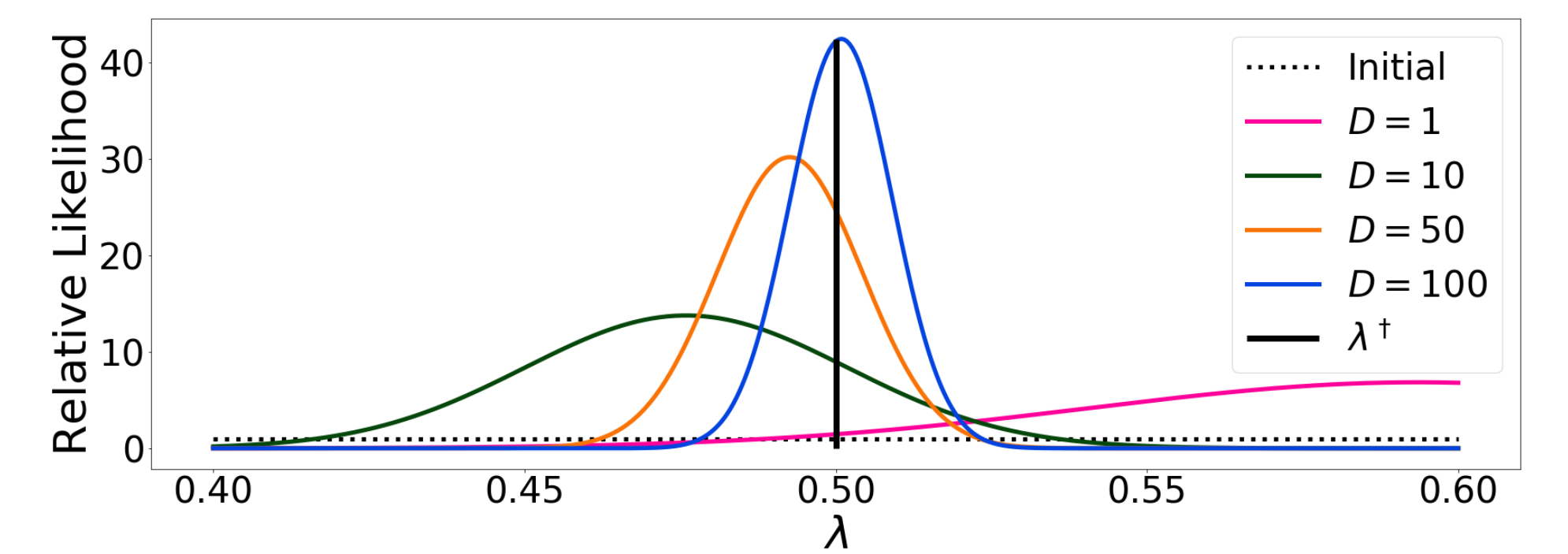
Example

Consider an exponential decay problem with uncertain initial condition:



Convergence

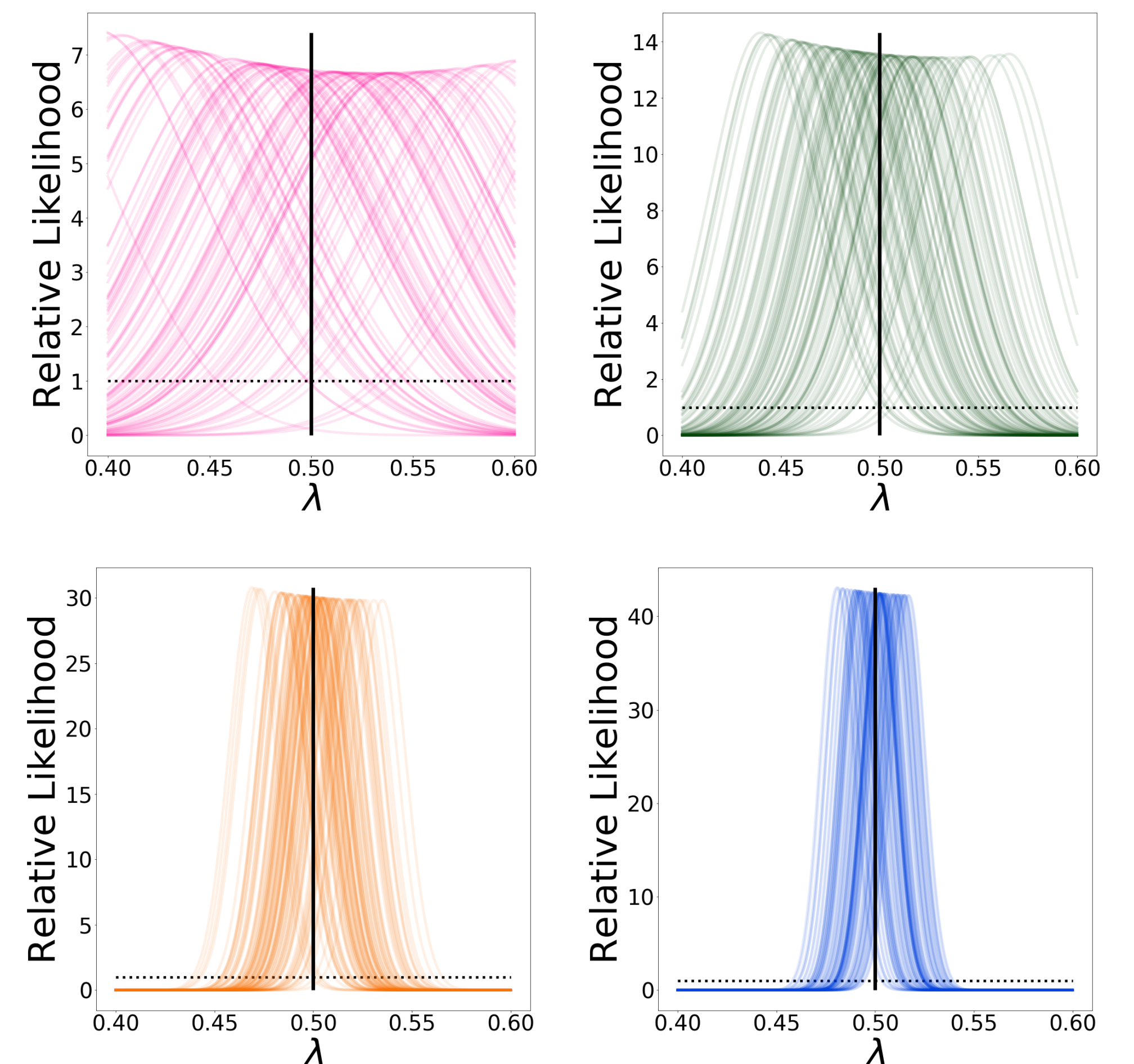
How do solutions change with more data?



λ^\dagger and π_{up} for $D = 1, 10, 50, 100$ for $N = 1000$

Stability

How do solutions on conditionals of Ξ compare?



λ^\dagger and π_{up} for one hundred realizations of ξ^\dagger for $D = 1, 10, 50, 100$