Using Push-Forward and Pullback Measures for Parameter Identification and Distribution Estimation

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Distribution Estimation

Goal: Obtain the Best Distribution of λ

Bayesian Context: Simple Bayes model is insufficient, hierarchical Bayes model required.

Data Consistent Context: Use data to construct observed distribution.

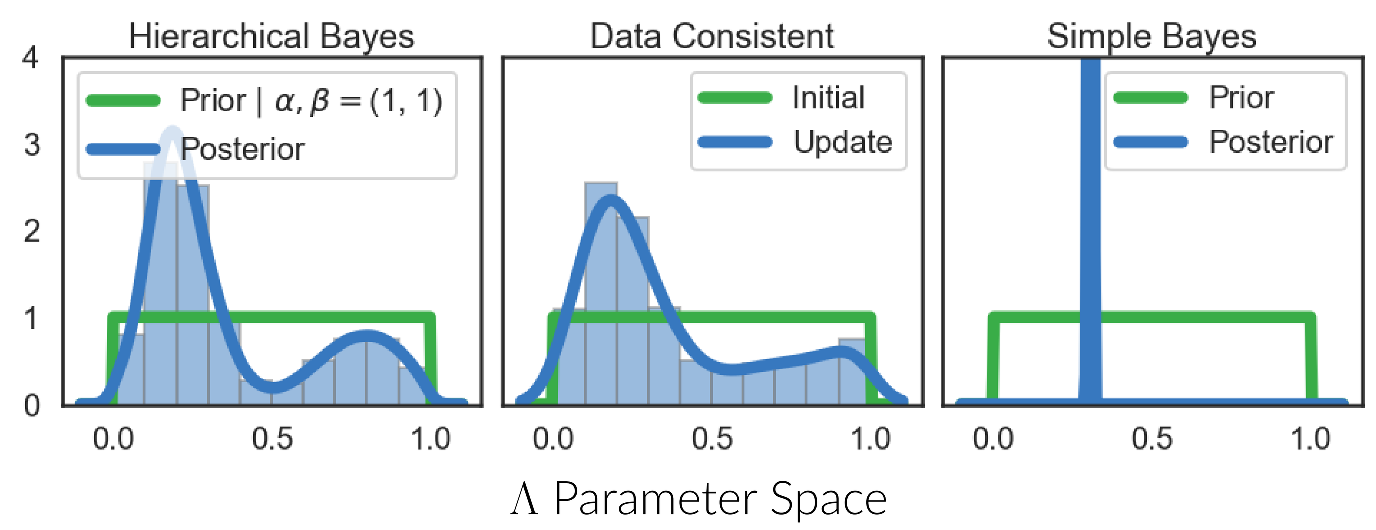
Example

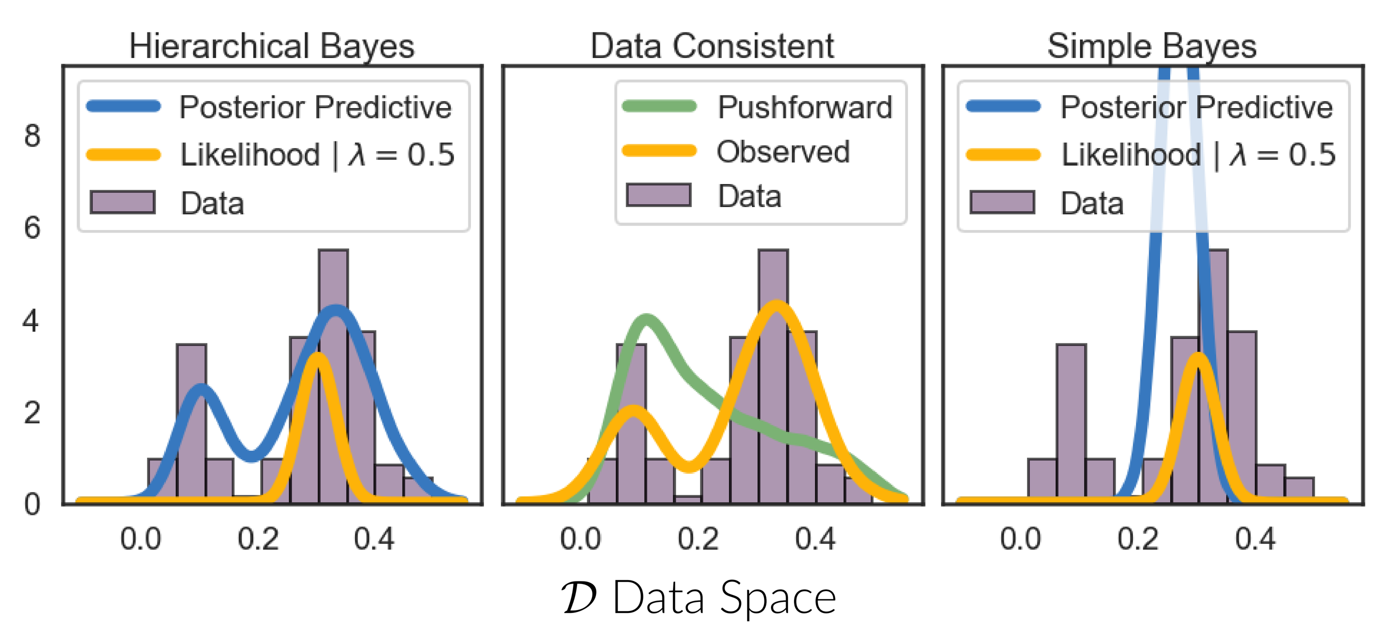
Consider an exponential decay problem with uncertain decay rate:

$$u(t) = u_0 \exp(-\lambda t),\ u_0 = 0.5,\ t = 2$$

Simple Bayes	$\pi_{\text{prior}} \sim U[0,1]\,, \quad \pi_{\text{L}}(\boldsymbol{d} \mid \lambda) \sim N\left(Q(\lambda), \sigma^2\right)$ $\pi_{\text{post}}(\lambda \mid \boldsymbol{d}) \propto \pi_{\text{prior}}(\lambda)\, \pi_{\text{L}}(\boldsymbol{d} \mid \lambda)$
Hierarchical Bayes*	$\pi_{\text{prior}}(\alpha,\beta) \sim \chi^2_1\,, \quad \alpha,\beta \in \Omega := [0,\infty) \times [0,\infty)$ $\pi_{\text{prior}}(\lambda \mid \alpha,\beta)\, s \sim \text{Beta}(\alpha,\beta), \quad \pi_{\text{L}}(\boldsymbol{d} \mid \lambda) \sim N\left(Q(\lambda), \sigma^2\right)$ $\pi_{\text{post}}(\lambda \mid \boldsymbol{d}) \propto \int_\Omega \pi_{\text{prior}}(\lambda,\alpha,\beta)\, \pi_{\text{L}}(\boldsymbol{d} \mid \lambda,\alpha,\beta)\, \mathrm{d}\Omega$
Data Consistent	$\pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \dfrac{\pi_{\text{obs}}(\text{Q}(\lambda))}{\pi_{\text{pre}}(\text{Q}(\lambda))}$

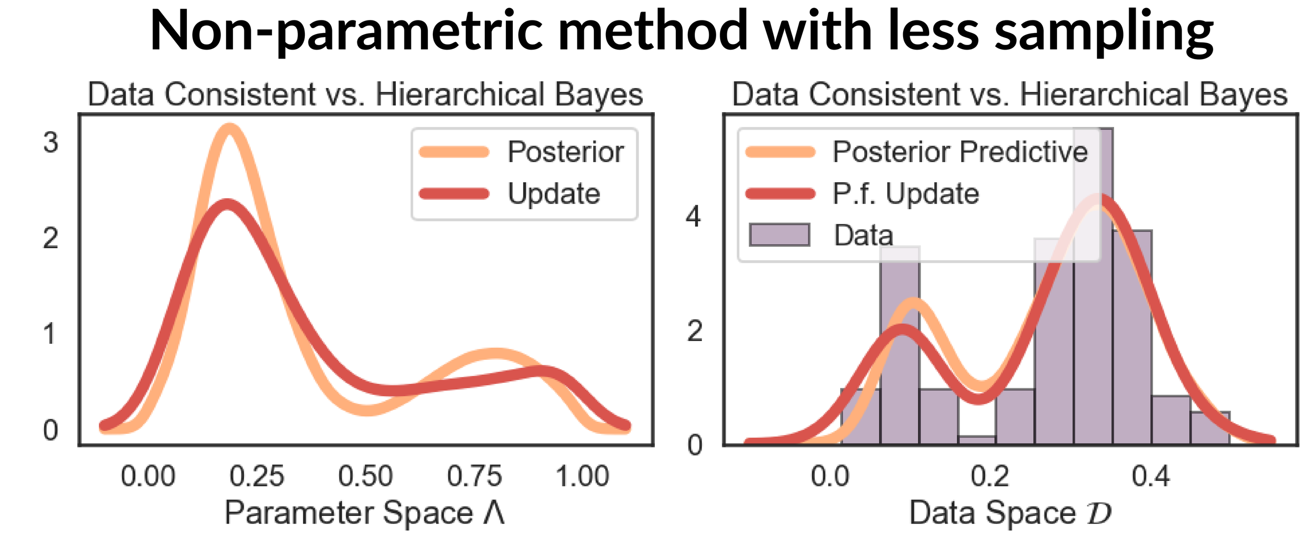
Plots of Concepts and Results





Takeaways

Non-parametric method with less sampling



Intro to Data Consistent Inversion

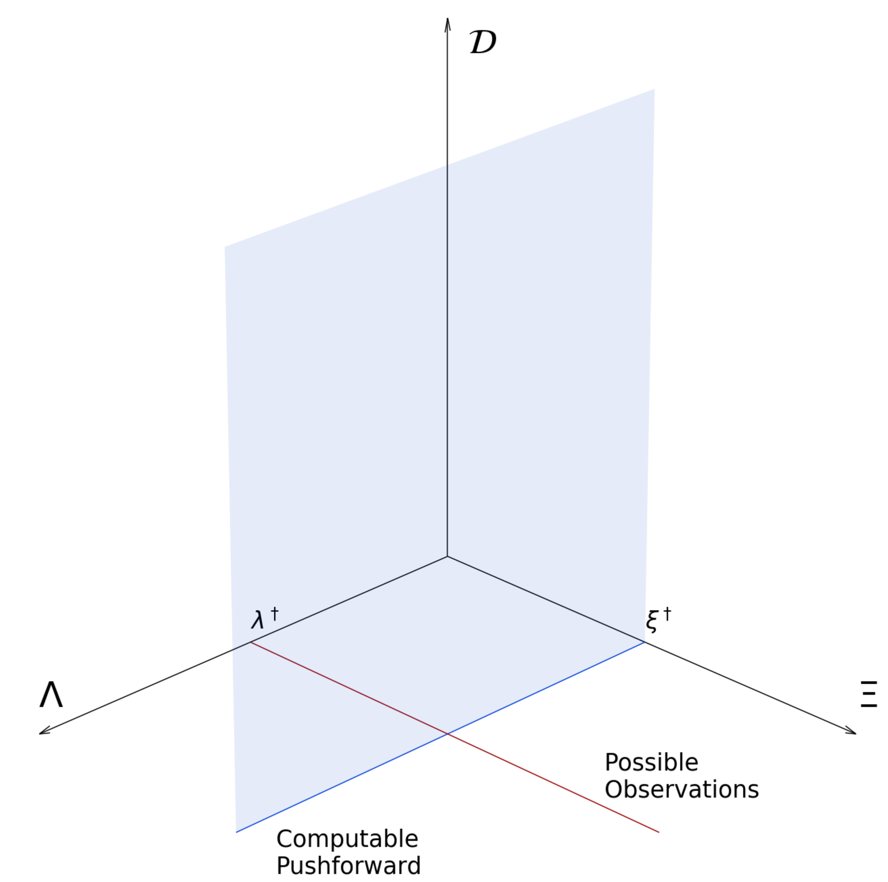
Solving Stochastic Inverse Problems

Data-Consistent Inversion is a novel framework that uses push-forward and pull-back measures to ensure solutions are consistent with the observed distribution of data.

The Data Consistent Approach

$$\pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \frac{\pi_{\text{obs}}(\text{Q}(\lambda))}{\pi_{\text{pre}}(\text{Q}(\lambda))}$$

Which Stochastic Inverse Problem?



Do you want to solve for a single parameter value or for a parameter distribution?

Notation

- $\lambda \in \Lambda,\ \xi \in \Xi$ — Parameter Space, Noise Space
- $\boldsymbol{d} \in \mathcal{D}$ — Observables
- $Q : \Lambda \to \mathcal{D}$ — Quantity of Interest Map
- $\pi_{\text{prior}},\ \pi_{\text{L}}$ — Prior , Likelihood
- $\pi_{\text{in}},\ \pi_{\text{obs}},\ \pi_{\text{pre}}$ — Initial, Observed, Predicted (**push-forward**)
- $\pi_{\text{post}},\ \pi_{\text{up}}$ — Posterior, Update (**pullback**)

References & Attribution

Author: Michael Pilosov || Advisor: Dr. Troy Butler

Left to Right: Theory, Stability, BET, ConsistentBayes, Personal Website.

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Parameter Identification

Goal: Obtain the Best Value of λ

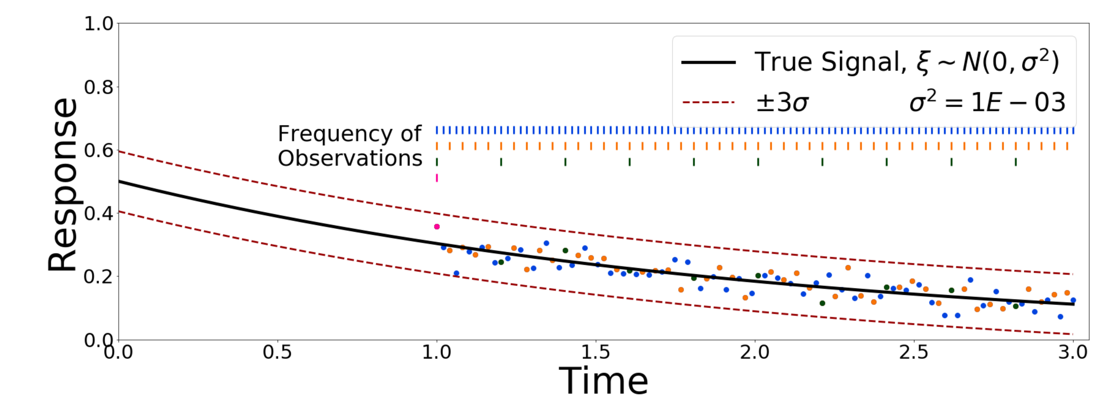
Bayesian Context: Simple Bayes model uses assumed likelihood function of data given λ.

Data Consistent Context: Uses data to construct a predicted distribution of the average residuals given λ.

Example

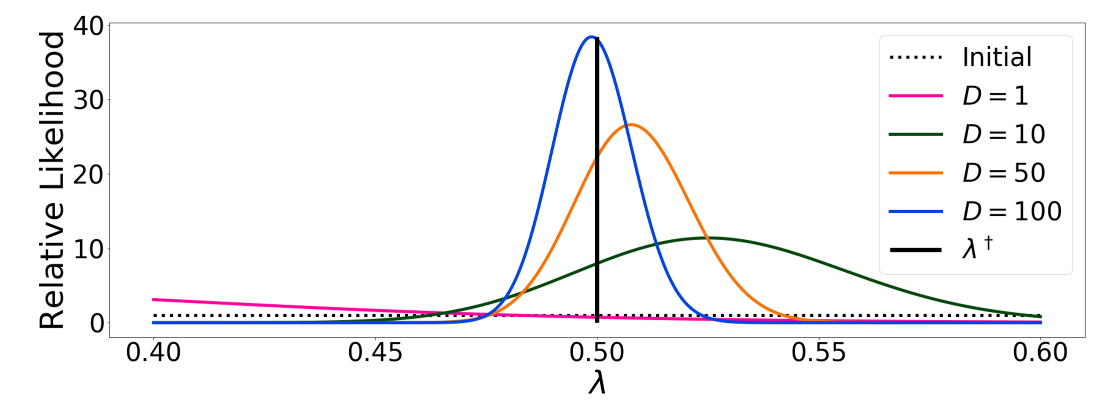
Consider an exponential decay problem with uncertain decay rate:

$$u(t) = u_0 \exp(-\lambda t),\ u_0 = 0.5,\ t \in [0,3]$$



Convergence of Data Consistent Approach

How do solutions change with more data?



$\lambda^\dagger$ and π_{up} for $D = 1, 10, 50, 100$ for $N = 1000$

Comparison to Bayes

How do solutions on conditionals of Ξ compare?