Using Pushforward and Pullback Measures for Parameter Identification and Distribution Estimation

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Distribution Estimation

Goal: Obtain the Best Distribution of λ

- Bayesian Context: Regular Bayes insufficient, Hierarchical model required.
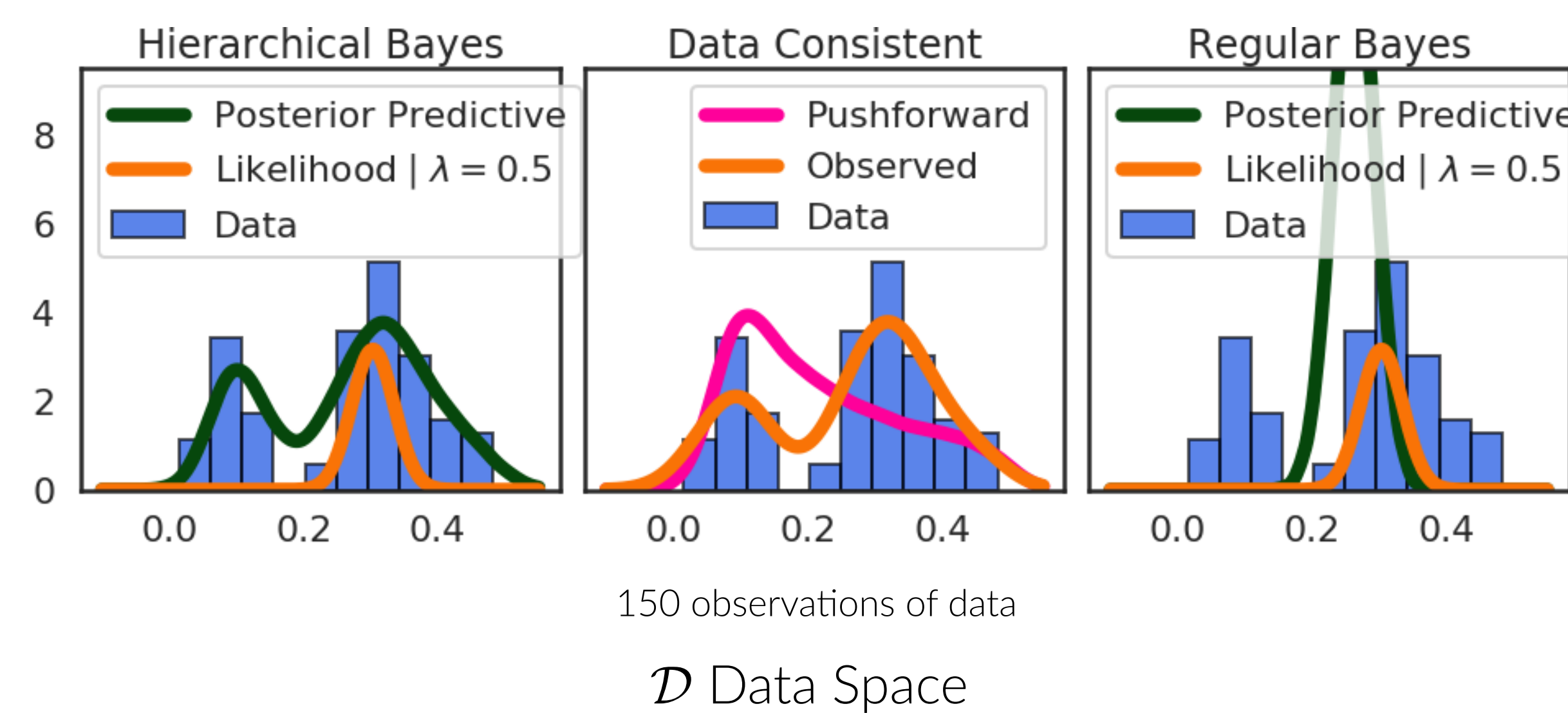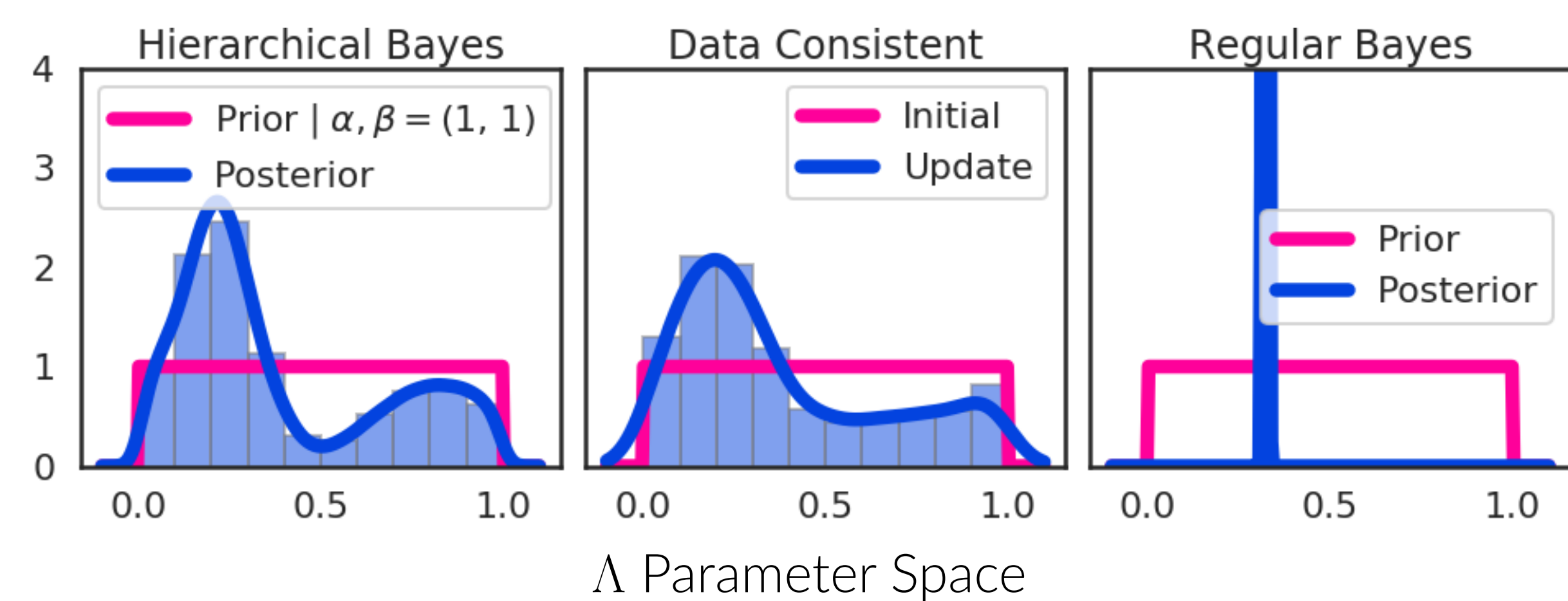- Data Consistent: Use data to construct observed distribution.

Example

Consider an exponential decay problem with uncertain decay rate:
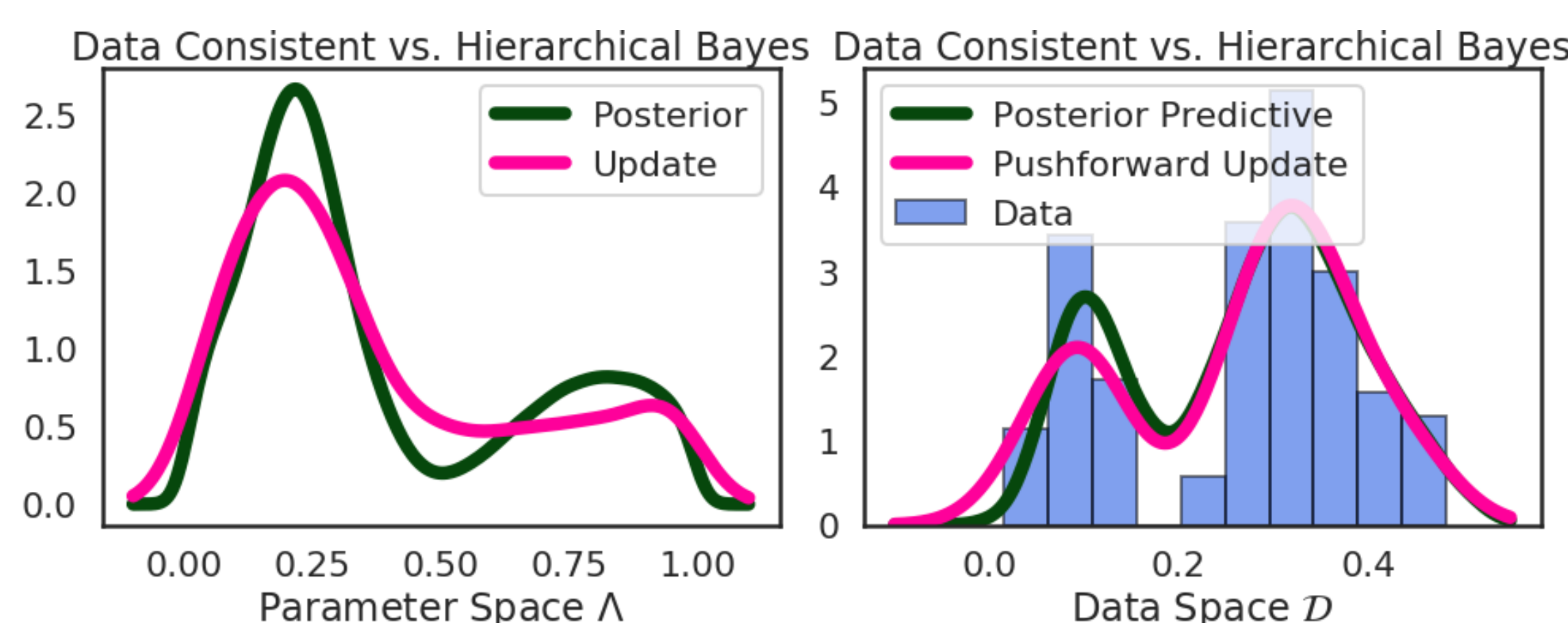
$$u(t) = u_0 \exp(-\lambda t), \ u_0 = 0.5, \ t = 2$$

Regular Bayes	$\pi_{\text{prior}} \sim U[0,1], \quad \pi_{\text{L}}(\boldsymbol{d} \mid \lambda) \sim N(Q(\lambda), \sigma^2)$ $\pi_{\text{post}}(\lambda \mid \boldsymbol{d}) \propto \pi_{\text{prior}}(\lambda)\,\pi_{\text{L}}(\boldsymbol{d} \mid \lambda)$
Hierarchical Bayes	$\pi_{\text{prior}}(\alpha,\beta) \sim \chi_1^2, \quad \alpha,\beta \in \Omega := [0,\infty) \times [0,\infty)$ $\pi_{\text{prior}}(\lambda \mid \alpha,\beta)\,s \sim \text{Beta}(\alpha,\beta), \quad \pi_{\text{L}}(\boldsymbol{d} \mid \lambda) \sim N(Q(\lambda), \sigma^2)$ $\pi_{\text{post}}(\lambda \mid \boldsymbol{d}) \propto \int_\Omega \pi_{\text{prior}}(\lambda,\alpha,\beta)\,\pi_{\text{L}}(\boldsymbol{d} \mid \lambda,\alpha,\beta)\,\mathrm{d}\Omega$
Data Consistent	$\pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda)\dfrac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$

Plots of Concepts and Results

Λ Parameter Space

150 observations of data

$\mathcal{D}$ Data Space

Takeaways

Non-parametric method with less sampling
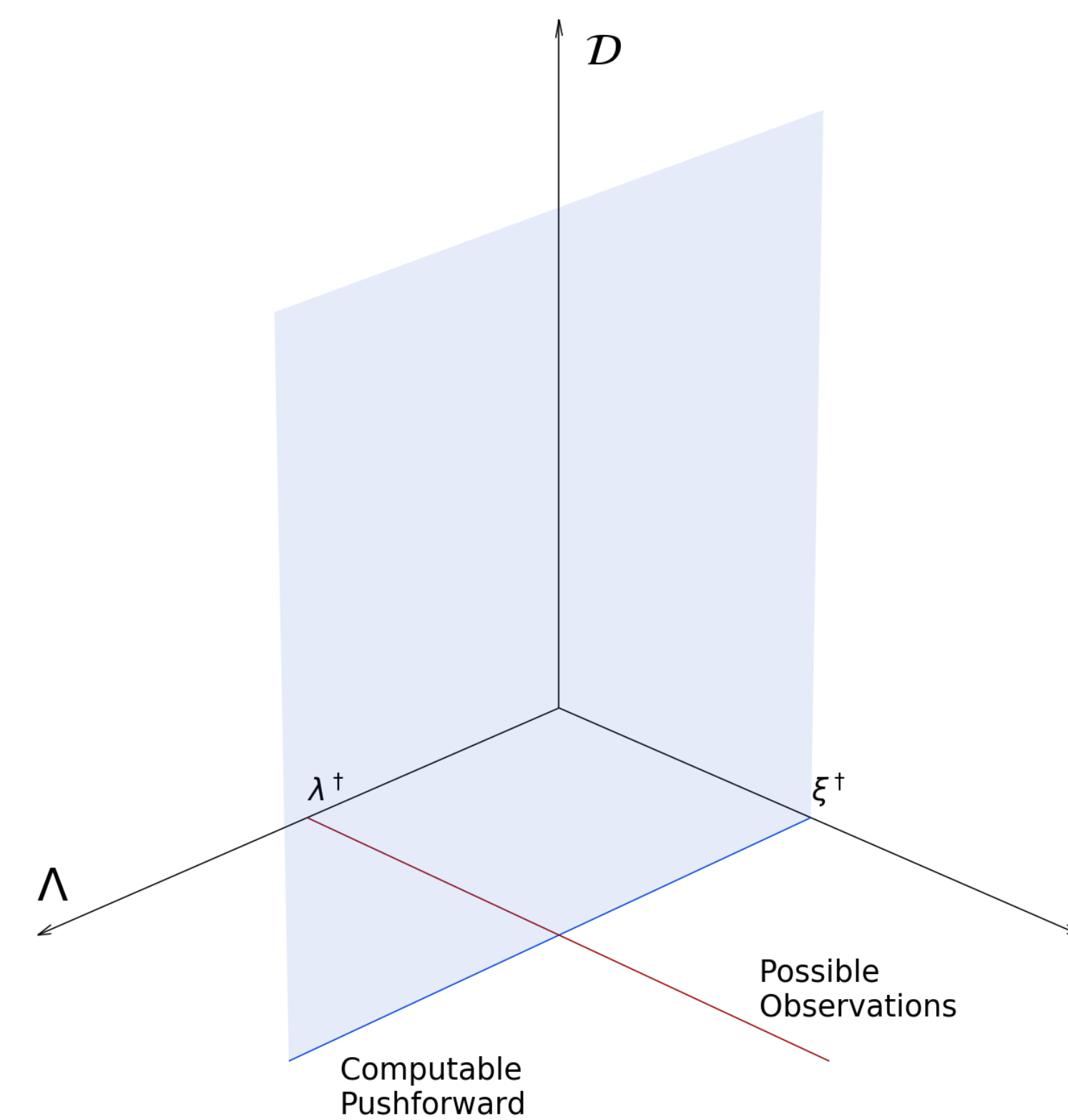

Intro to Data Consistent Inversion

Solving Stochastic Inverse Problems

Data Consistent Inversion is a novel framework that uses pushforward and pullback measures to ensure solutions are consistent with the observed distribution of data.

The Data Consistent Approach

$$\pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda)\frac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$$

Which Stochastic Inverse Problem?

Solve for a single parameter value, or for a parameter distribution?

Notation

- $\lambda \in \Lambda, \ \xi \in \Xi$ Parameter Space, Noise Space
- $\boldsymbol{d} \in \mathcal{D}$ Observables
- $Q : \Lambda \to \mathcal{D}$ Quantity of Interest Map
- $\pi_{\text{prior}}, \ \pi_{\text{L}}$ Prior , Likelihood
- $\pi_{\text{in}}, \ \pi_{\text{obs}}, \ \pi_{\text{pre}}$ Initial, Observed, Predicted (**pushforward**)
- $\pi_{\text{post}}, \ \pi_{\text{up}}$ Posterior, Update (**pullback**)

References & Attribution
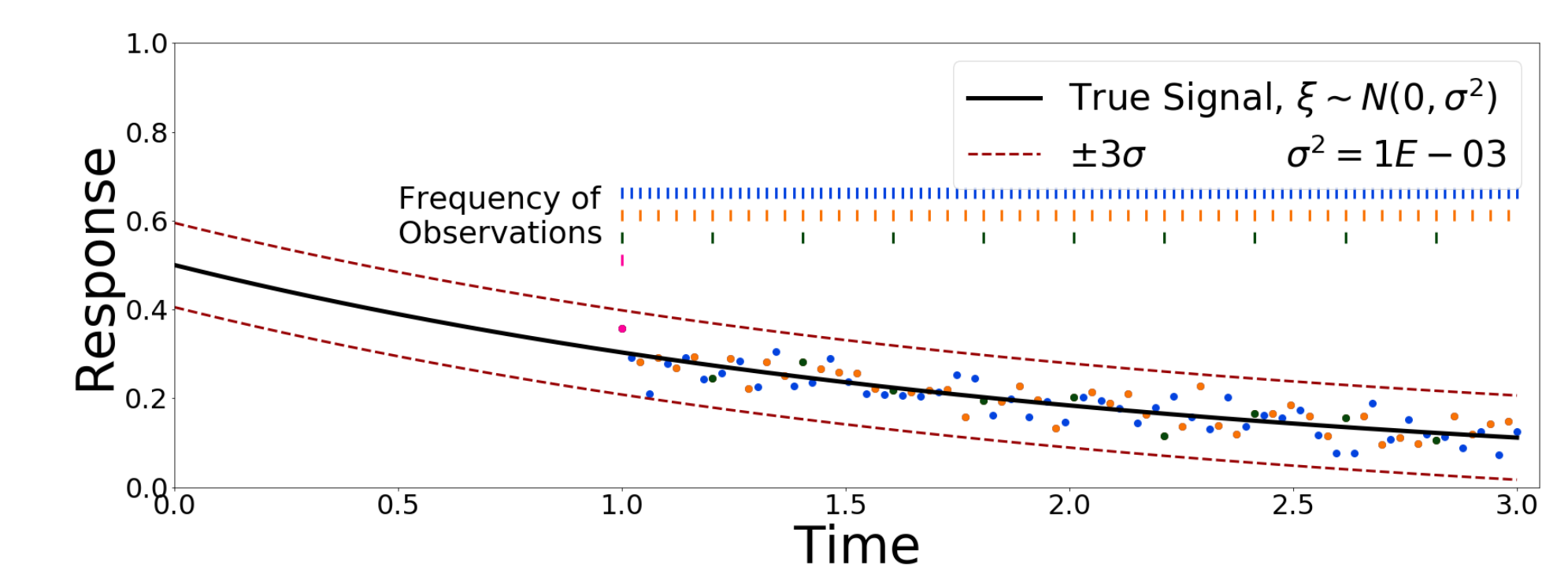
Parameter Identification

Goal: Obtain the Best Value of λ

- Bayesian Context: Model uses assumed likelihood function of data given λ.
- Data Consistent: Construct predicted distribution of residuals given λ.

Example
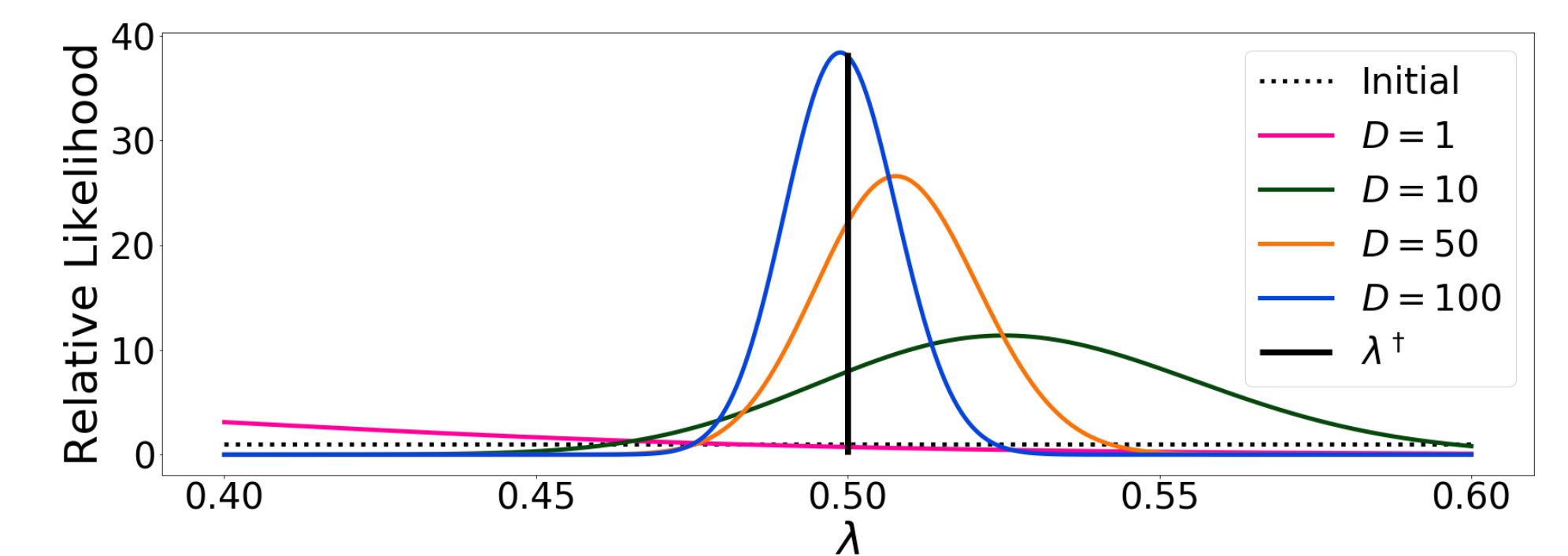
Consider an exponential decay problem with uncertain decay rate:

$$u(t) = u_0 \exp(-\lambda t), \ u_0 = 0.5, \ t \in [0,3]$$
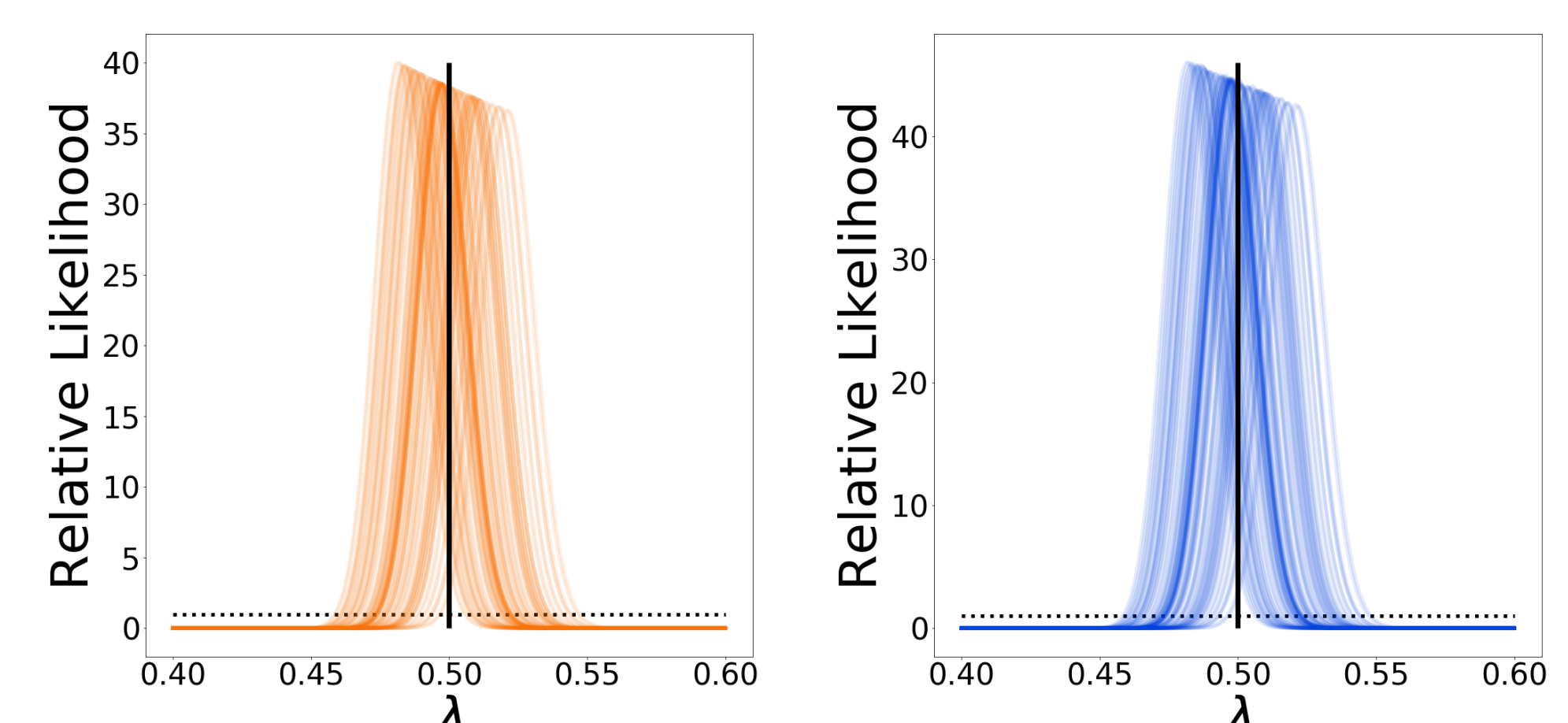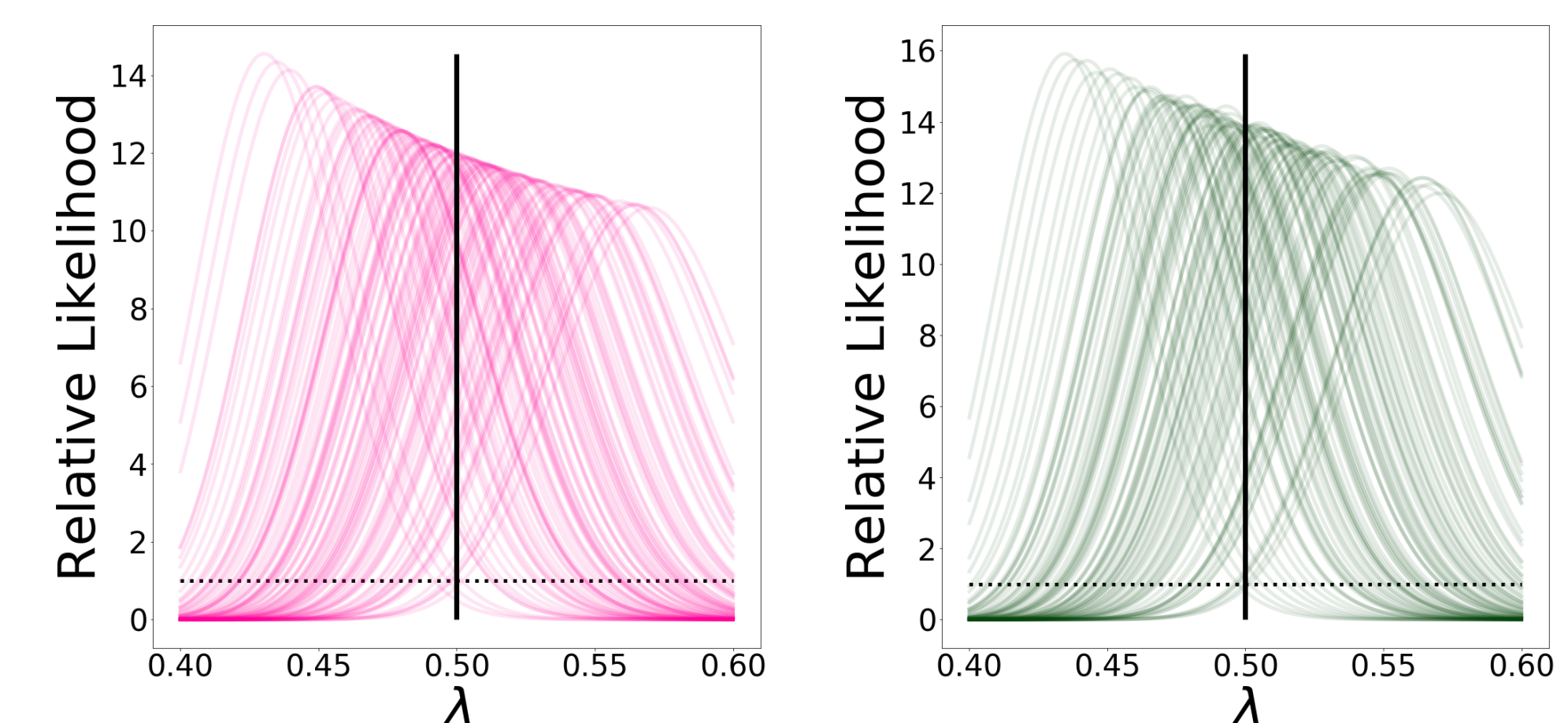

Convergence of Data Consistent Inversion

How do solutions change with more data?

$\lambda^\dagger$ and π_{up} for $D = 1, 10, 50, 100$ for $N = 1000$

Comparison to Regular Bayes

How do solutions on conditionals of Ξ compare?

100 realizations of data for $D = 10, 100$ for Data Consistent Inversion (left) and Regular Bayes (right).