Push-forward Measures for Parameter Identification under Uncertainty

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Introduction

Motivation

How do we update initial descriptions of uncertainty using model predictions and data?

Data-Consistent Inversion is a novel framework that uses push-forward and pull-back measures to ensure solutions are consistent with the observed distribution of data.

Question

*How do we cast a **Parameter Identification** problem in the context of Data-Consistent Inversion?*

Framework

- $\mathbb{P},\ \pi$ — Probability Measure, Density
- $\Lambda \subset \mathbb{R}^P$ — Parameter Space
- $\boldsymbol{o} : \Lambda \to \mathcal{O} \subset \mathbb{R}^D$ — Observables
- $\Xi \subset \mathbb{R}^D$ — Noise Space
- $\lambda^\dagger \in \Lambda$ — True Parameter
- $\boldsymbol{d}(\xi) \subset \mathbb{R}^D$ — Possible Data, $d_i(\xi) = \boldsymbol{o}_i(\lambda^\dagger) + \xi_i$
- $\xi^\dagger \in \Xi$ — Noise in Measurements
- σ^2 — Variance of Noise
- $\boldsymbol{d}^\dagger \in \mathbb{R}^D$ — Observed Data, $\boldsymbol{d}^\dagger = \boldsymbol{d}(\xi^\dagger)$
- $\mathbb{P}_{\text{in}},\ \pi_{\text{in}}$ — Initial
- $\mathbb{P}_{\text{obs}},\ \pi_{\text{obs}}$ — Observed
- $\mathbb{P}_{\text{pre}},\ \pi_{\text{pre}}$ — Predicted (**push-forward**)
- $\mathbb{P}_{\text{up}},\ \pi_{\text{up}}$ — Updated (**pull-back**)

Updating with Observations and Predictions

$$\mathbb{P}_{\text{up}} = \mathbb{P}_{\text{in}} \frac{\mathbb{P}_{\text{obs}}}{\mathbb{P}_{\text{pre}}} \qquad \pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \frac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$$

References & Attribution

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Left to Right: Theory, Stability, BET, ConsistentBayes, Personal Website.

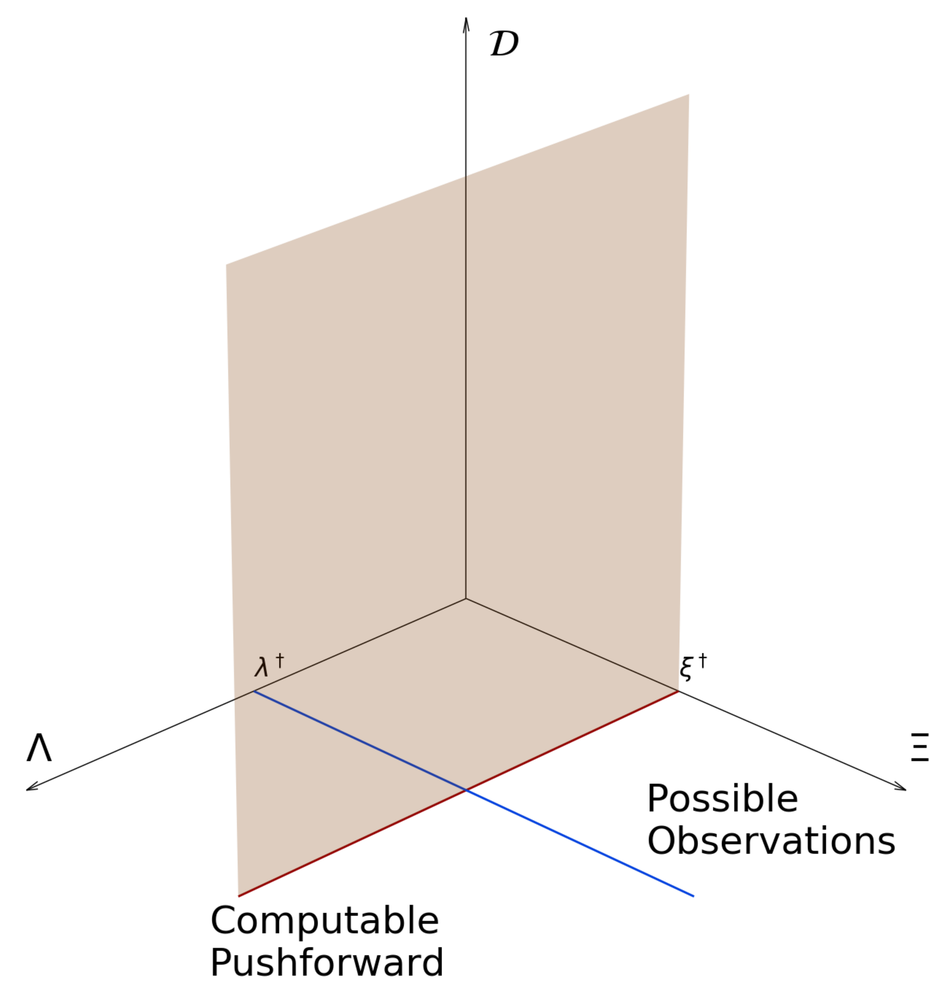
Funding provided by NSF DMS-1818941.

Approach

Quantity of Interest Map

*A Functional Relating **Predictions** and **Data***

- Ideal: $Q(\lambda, \xi) = F(\boldsymbol{o}(\lambda), \boldsymbol{d}(\xi))$
- Theoretical: $Q(\Lambda, \Xi) =: \mathcal{D}_{\mathcal{T}} \subset \mathbb{R}$
- Practical: $Q(\lambda) = F\left(\boldsymbol{o}(\lambda), \boldsymbol{d}^\dagger\right)$
- Computable: $Q(\Lambda) =: \mathcal{D}_{\mathcal{C}} \subset \mathcal{D}_{\mathcal{T}}$



How do conditionals of Ξ compare to the joint density?

Observed Distribution

Given a functional, what measure do we invert?

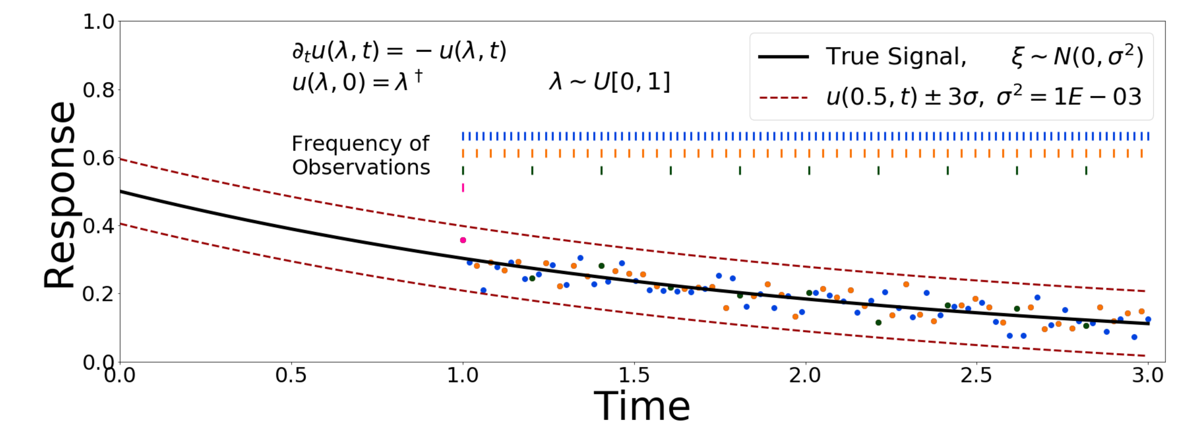
$Q(\lambda^\dagger, \xi) \sim \pi_{\text{obs}}$ when we allow ξ to vary over Ξ

$F(\boldsymbol{o}(\lambda), \boldsymbol{d}^\dagger)$	ξ	π_{obs}
$\frac{1}{\sigma\sqrt{D}} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)$	$\xi \sim L^2$	$N(0,1)$
$\frac{1}{\sigma^2} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)^2$	$\xi \sim N(0,\sigma^2)$	$\chi^2(D)$
$\frac{1}{\sigma^2 D} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)^2$	$\xi \sim N(0,\sigma^2)$	$\Gamma(D/2, D/2)$
$\vdots$	$\vdots$	$\vdots$

Choices of F and associated π_{obs} for stochastic inverse problem with $\boldsymbol{d}^\dagger = \boldsymbol{o}_i(\lambda^\dagger) + \xi_i^\dagger$

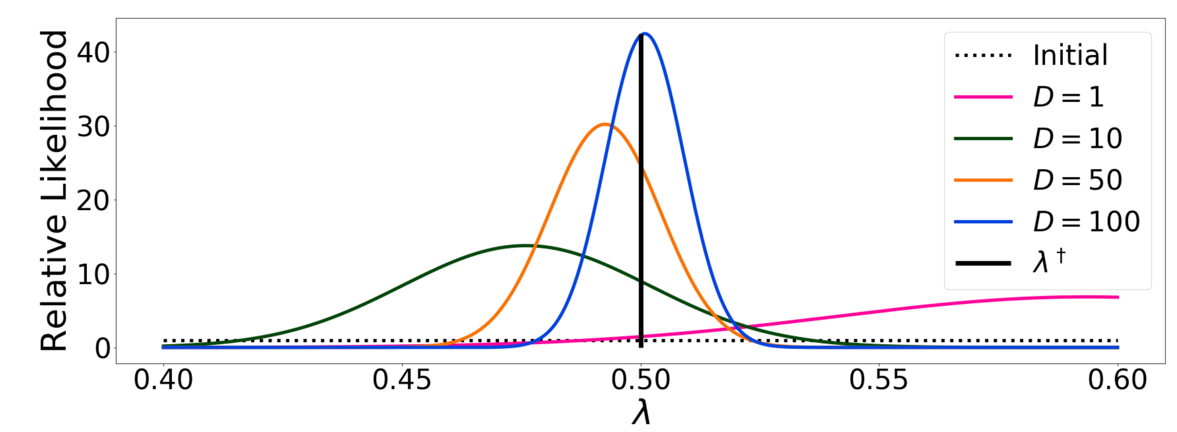
Example

Consider an exponential decay problem with uncertain initial condition:



Convergence

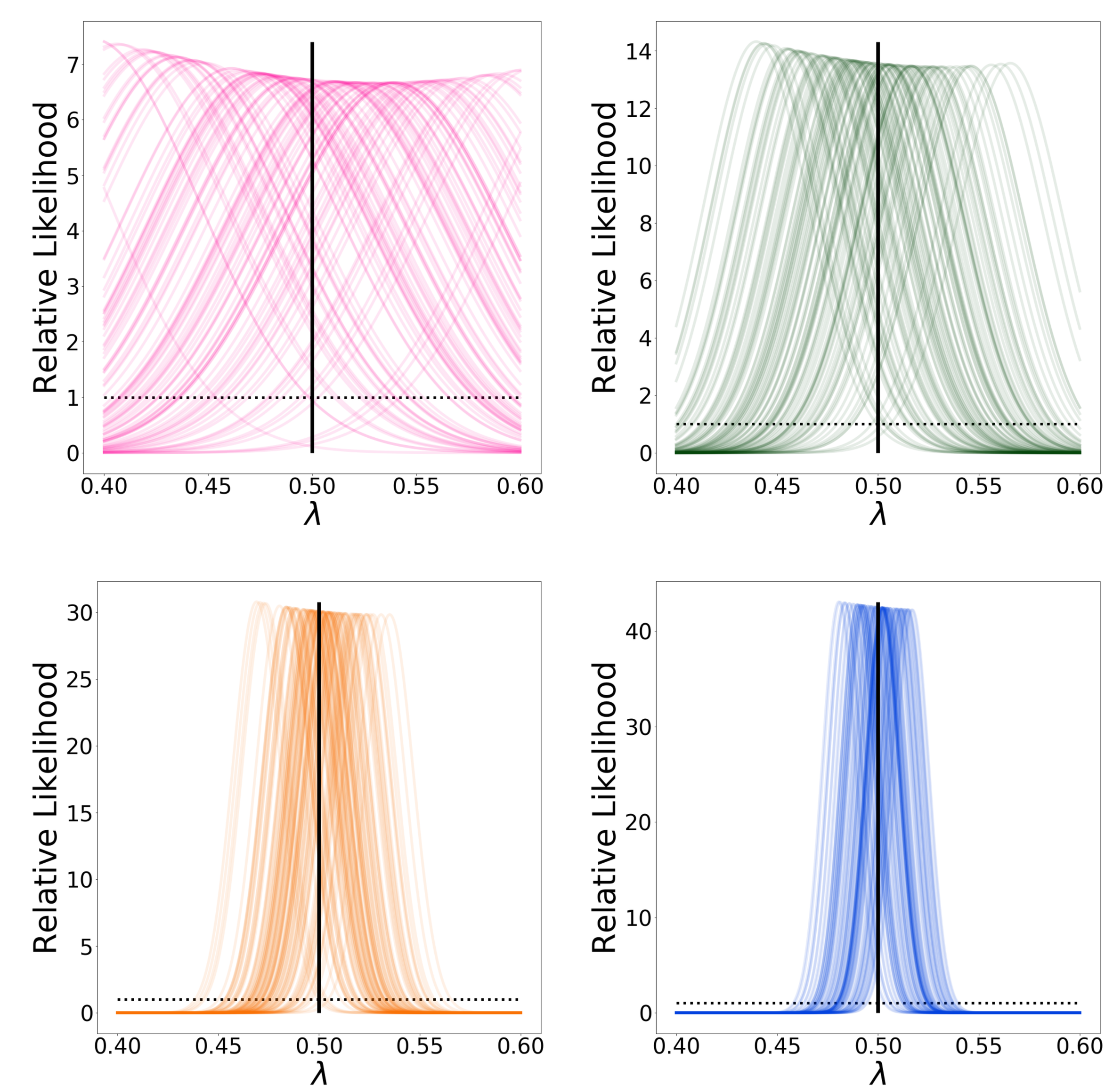
How do solutions change with more data?



$\lambda^\dagger$ and π_{up} for $D = 1, 10, 50, 100$ for $N = 1000$

Stability

How do solutions on conditionals of Ξ compare?



$\lambda^\dagger$ and π_{up} for one hundred realizations of $\xi^\dagger$ for $D = 1, 10, 50, 100$