# Push-forward Measures for Parameter Identification under Uncertainty

Mathematical Michael

University of Colorado: Denver

## Introduction

### Motivation

How do we update initial descriptions of uncertainty using model predictions and data?

**Data-Consistent Inversion** is a novel framework that uses push-forward and pull-back measures to ensure solutions are consistent with the observed distribution of data.

### Question

*How do we cast a **Parameter Identification** problem in the context of Data-Consistent Inversion?*

## Framework

| | |
|---|---|
| $\mathbb{P},\ \pi$ | Probability Measure, Density |
| $\Lambda \subset \mathbb{R}^P$ | Parameter Space |
| $\boldsymbol{o} : \Lambda \to \mathcal{O} \subset \mathbb{R}^D$ | Observables |
| $\Xi \subset \mathbb{R}^D$ | Noise Space |
| $\lambda^\dagger \in \Lambda$ | True Parameter |
| $\boldsymbol{d}(\xi) \subset \mathbb{R}^D$ | Possible Data, $d_i(\xi) = \boldsymbol{o}_i(\lambda^\dagger) + \xi_i$ |
| $\xi^\dagger \in \Xi$ | Noise in Measurements |
| $\sigma^2$ | Variance of Noise |
| $\boldsymbol{d}^\dagger \in \mathbb{R}^D$ | Observed Data, $\boldsymbol{d}^\dagger = \boldsymbol{d}(\xi^\dagger)$ |
| $\mathbb{P}_{\text{in}},\ \pi_{\text{in}}$ | Initial |
| $\mathbb{P}_{\text{obs}},\ \pi_{\text{obs}}$ | Observed |
| $\mathbb{P}_{\text{pre}},\ \pi_{\text{pre}}$ | Predicted (**push-forward**) |
| $\mathbb{P}_{\text{up}},\ \pi_{\text{up}}$ | Updated (**pull-back**) |

### Updating with Observations and Predictions

$$\mathbb{P}_{\text{up}} = \mathbb{P}_{\text{in}} \frac{\mathbb{P}_{\text{obs}}}{\mathbb{P}_{\text{pre}}} \quad \Big| \quad \pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \frac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$$

## References & Attribution

Author: Michael Pilosov || Advisor: Dr. Troy Butler

Left to Right: Theory, Stability, BET, ConsistentBayes, Personal Website.

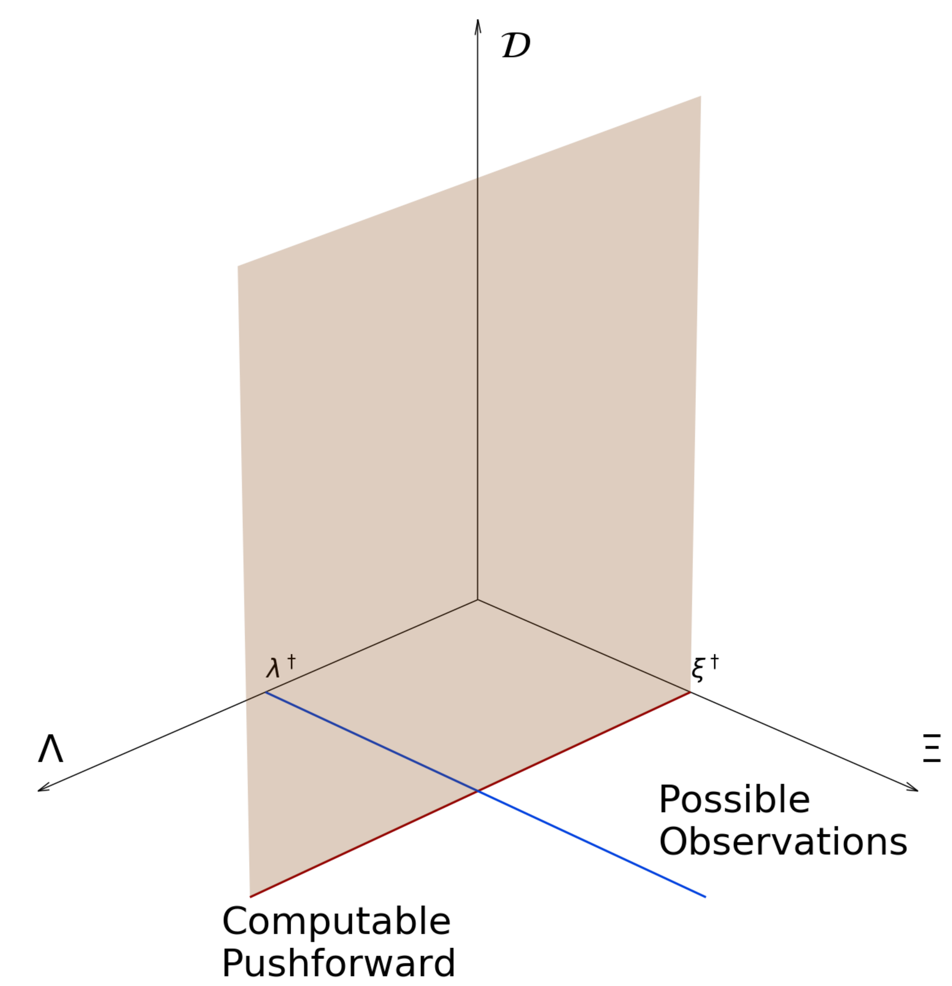
Funding provided by NSF DMS-1818941.

## Approach

### Quantity of Interest Map

*A Functional Relating **Predictions** and **Data***

- Ideal $\quad Q(\lambda, \xi) = F(\boldsymbol{o}(\lambda), \boldsymbol{d}(\xi))$
- Theoretical $\quad Q(\Lambda, \Xi) =: \mathcal{D}_{\mathcal{T}} \subset \mathbb{R}$
- Practical $\quad Q(\lambda) = F\left(\boldsymbol{o}(\lambda), \boldsymbol{d}^\dagger\right)$
- Computable $\quad Q(\Lambda) =: \mathcal{D}_{\mathcal{C}} \subset \mathcal{D}_{\mathcal{T}}$




How do conditionals of $\Xi$ compare to the joint density?

### Observed Distribution

Given a functional, what measure do we invert?

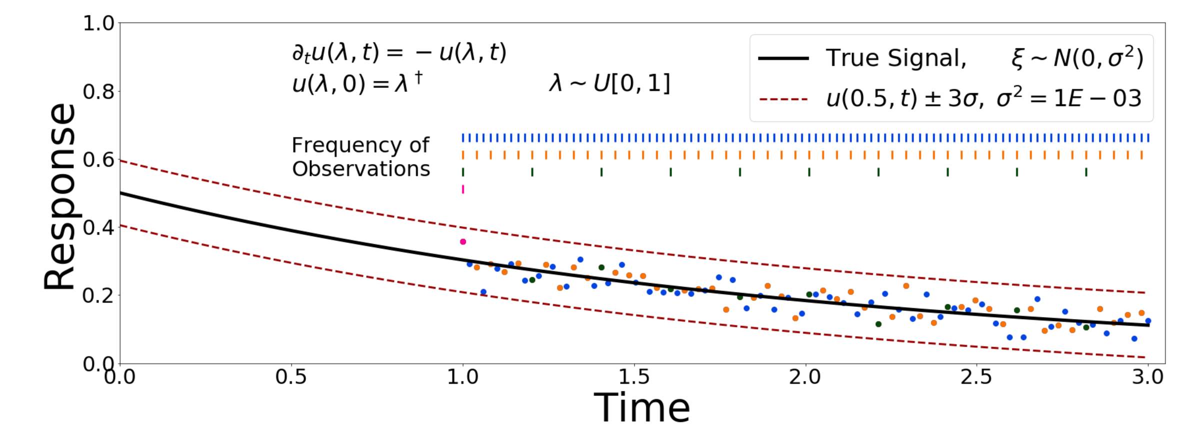
$Q(\lambda^\dagger, \xi) \sim \pi_{\text{obs}}$ when we allow $\xi$ to vary over $\Xi$

| $F(\boldsymbol{o}(\lambda), \boldsymbol{d}^\dagger)$ | $\xi$ | $\pi_{\text{obs}}$ |
|---|---|---|
| $\frac{1}{\sigma\sqrt{D}} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)$ | $\xi \sim L^2$ | $N(0,1)$ |
| $\frac{1}{\sigma^2} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)^2$ | $\xi \sim N(0,\sigma^2)$ | $\chi^2(D)$ |
| $\frac{1}{\sigma^2 D} \sum \left(\boldsymbol{o}_i(\lambda) - \boldsymbol{d}_i^\dagger\right)^2$ | $\xi \sim N(0,\sigma^2)$ | $\Gamma(D/2, D/2))$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Choices of $F$ and associated $\pi_{\text{obs}}$ for stochastic inverse problem with $\boldsymbol{d}^\dagger = \boldsymbol{o}_i(\lambda^\dagger) + \xi_i^\dagger$

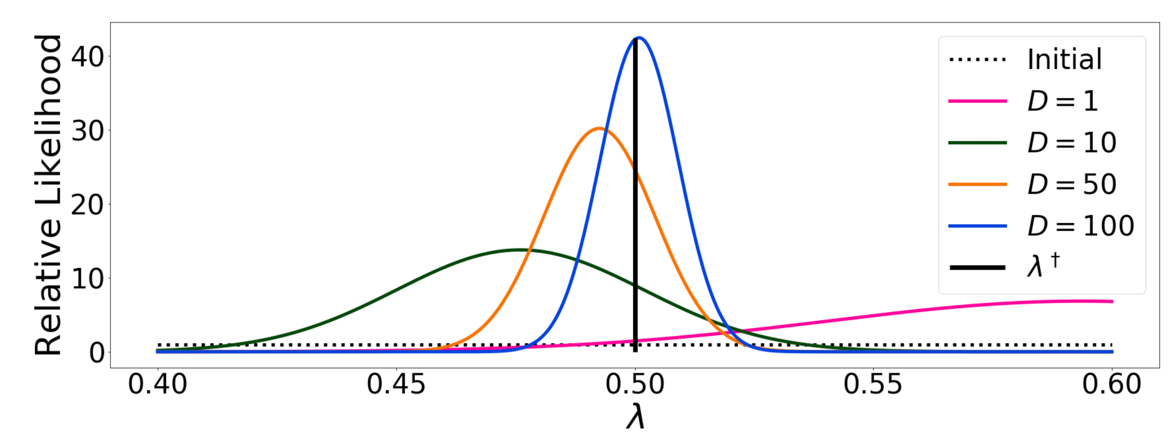
## Example

Consider an exponential decay problem with uncertain initial condition:




### Convergence

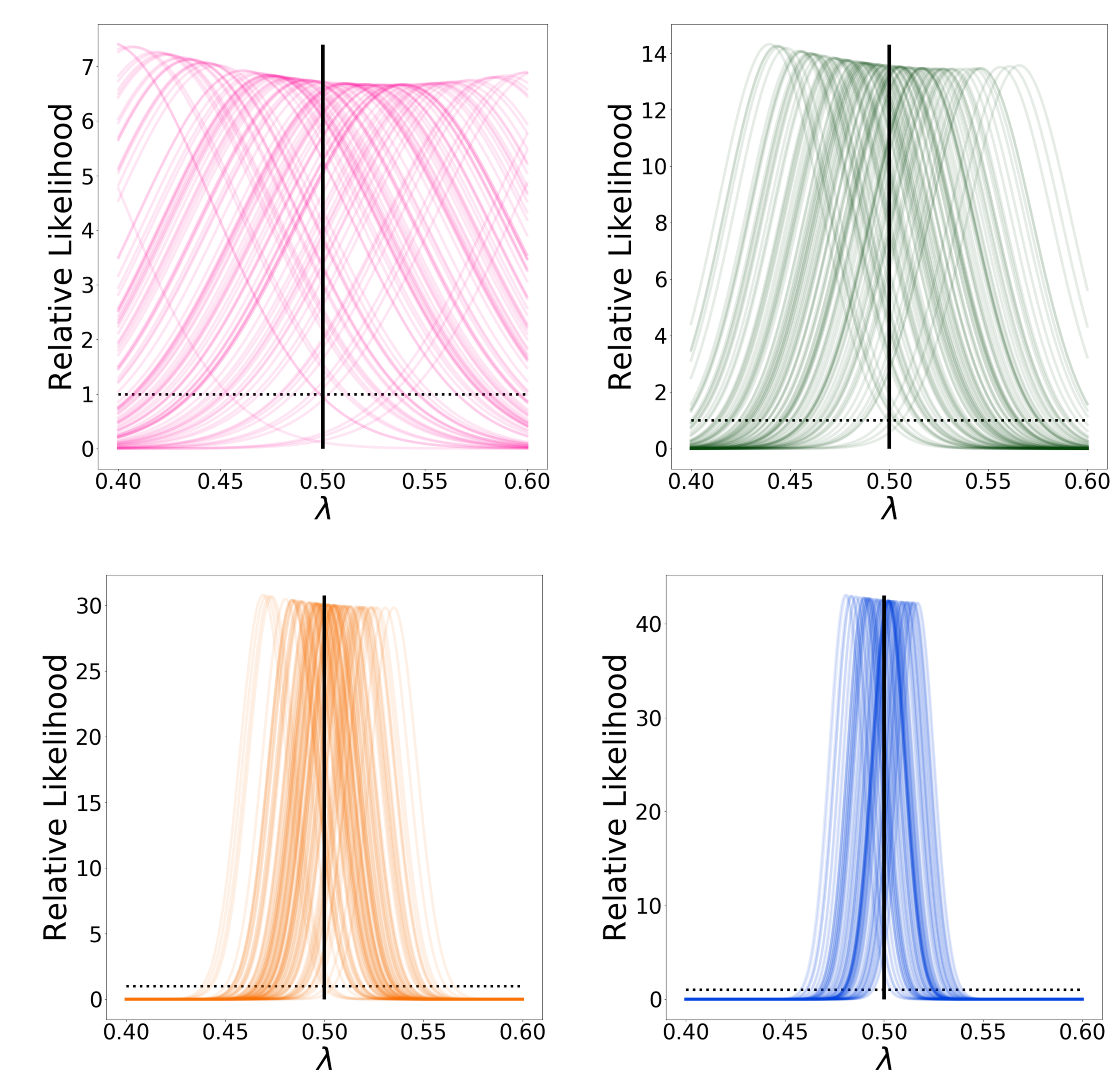
How do solutions change with more data?




$\lambda^\dagger$ and $\pi_{\text{up}}$ for $D = 1, 10, 50, 100$ for $N = 1000$

### Stability

How do solutions on conditionals of $\Xi$ compare?




$\lambda^\dagger$ and $\pi_{\text{up}}$ for one hundred realizations of $\xi^\dagger$ for $D = 1, 10, 50, 100$