

# Push-forward Measures for Parameter Identification under Uncertainty

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## Introduction

### Motivation

How do we update initial descriptions of uncertainty using data and model predictions?

### Background

*Data Consistent Inversion* is a framework that ensures solutions are consistent with the distribution of data.

### Question

How do we perform *Parameter Identification* in the context of *Data-Consistent Inversion*?

## Framework

▪ $\mathbb{P}, \pi$	Probability Measure, Density
▪ $\Lambda \subset \mathbb{R}^P$	Parameter Space
▪ $\mathbf{o} : \Lambda \rightarrow \mathcal{O} \subset \mathbb{R}^D$	Observables
▪ $\Xi \subset \mathbb{R}^D$	Noise Space
▪ $\lambda^\dagger \in \Lambda$	True Parameter
▪ $\mathbf{d}(\xi) \in \mathbb{R}^D$	Possible Data, $\mathbf{d}_i(\xi) = \mathbf{o}_i(\lambda^\dagger) + \xi_i$
▪ $\xi^\dagger \in \Xi$	Noise in Measurements
▪ $\sigma$	Variance of Noise
▪ $\mathbf{d}^\dagger \in \mathbb{R}^D$	Observed Data, $\mathbf{d}^\dagger = \mathbf{d}(\xi^\dagger)$
▪ $\mathbb{P}_{\text{in}}, \pi_{\text{in}}$	Initial
▪ $\mathbb{P}_{\text{obs}}, \pi_{\text{obs}}$	Observed
▪ $\mathbb{P}_{\text{pre}}, \pi_{\text{pre}}$	Predicted ( <b>Push-forward</b> )
▪ $\mathbb{P}_{\text{up}}, \pi_{\text{up}}$	Updated

### Updating with Observations and Predictions

$$\mathbb{P}_{\text{up}} = \mathbb{P}_{\text{in}} \frac{\mathbb{P}_{\text{obs}}}{\mathbb{P}_{\text{pre}}} \quad \left| \quad \pi_{\text{up}}(\lambda) = \pi_{\text{in}}(\lambda) \frac{\pi_{\text{obs}}(Q(\lambda))}{\pi_{\text{pre}}(Q(\lambda))}$$

## References & Attribution

Advisor: Dr. Troy Butler



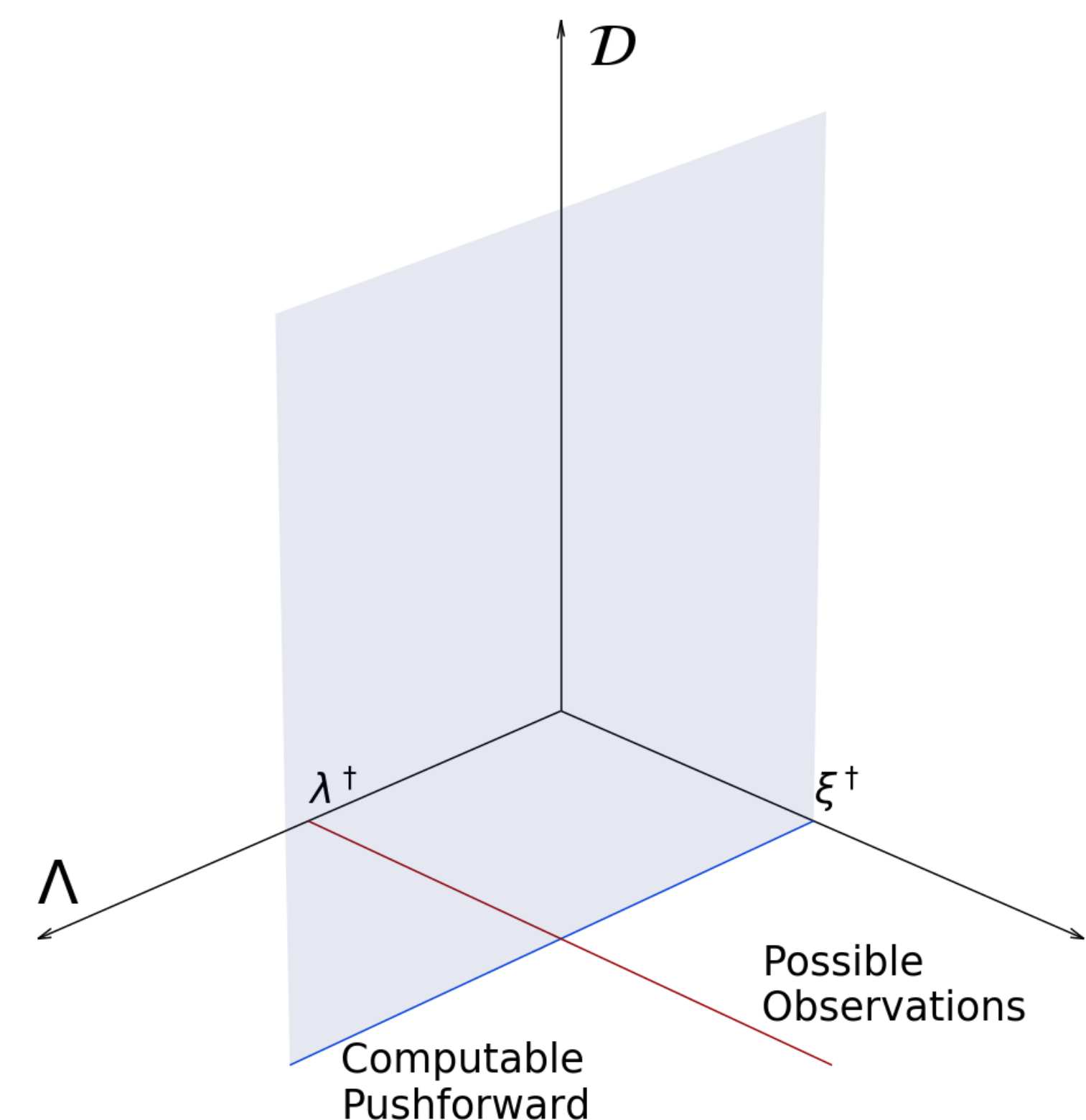
(Left to Right): Theory, Stability, BET, ConsistentBayes, Personal Website.  
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## Approach

### Quantity of Interest Map

A Functional Relating *Predictions* and *Data*

- Ideal  $Q(\lambda, \xi) = F(\mathbf{o}(\lambda), \mathbf{d}(\xi))$
- Theoretical  $Q(\Lambda, \Xi) =: \mathcal{D}_{\mathcal{T}} \subset \mathbb{R}$
- Practical  $\hat{Q}(\lambda) = F(\mathbf{o}(\lambda), \mathbf{d}^\dagger)$
- Computable  $\hat{Q}(\Lambda) =: \mathcal{D}_{\mathcal{C}} \subset \mathcal{D}_{\mathcal{T}}$



How do conditionals of  $\Xi$  compare to the joint density?

### Observed Distribution

Given a functional, what measure do we invert?

$Q(\lambda^\dagger, \xi) \sim \pi_{\text{obs}}$  when we allow  $\xi$  to vary over  $\Xi$

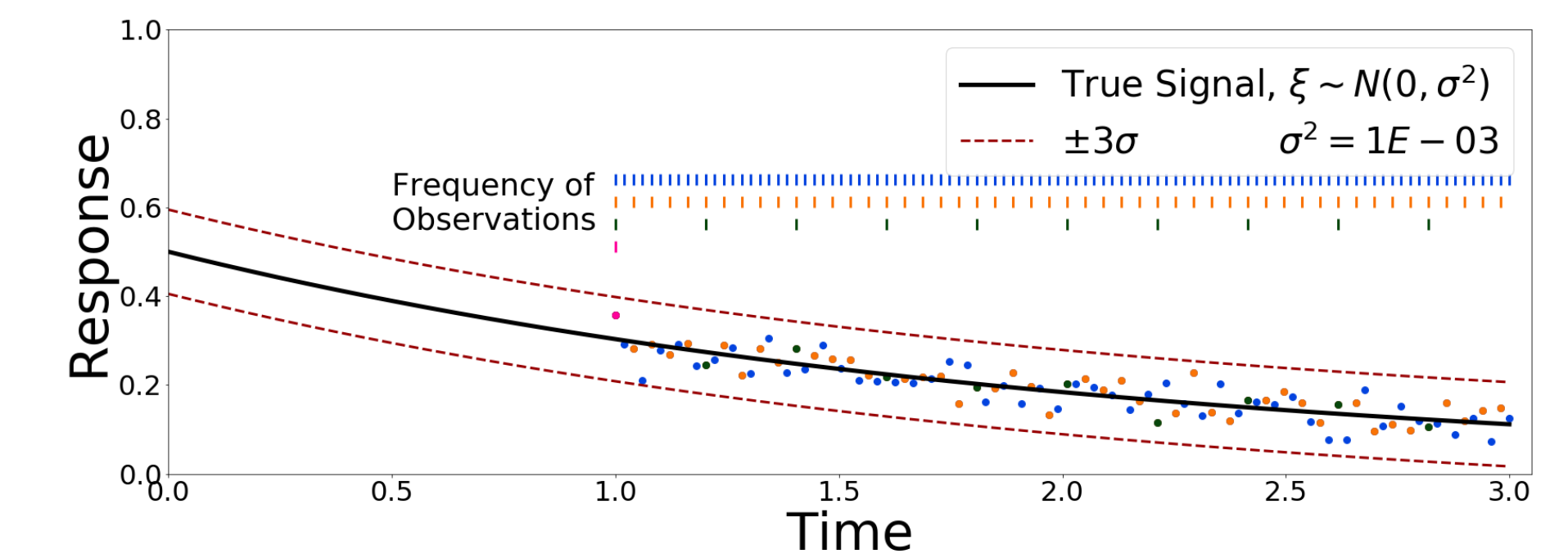
$F(\mathbf{o}, \mathbf{d}^\dagger)$	$\xi$	$\pi_{\text{obs}}$
$\frac{1}{\sigma\sqrt{D}} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)$	$\xi \sim L^2$	$N(0, 1)$
$\frac{1}{\sigma^2} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)^2$	$\xi \sim N(0, \sigma^2)$	$\chi^2(D)$
$\frac{1}{\sigma^2 D} \sum (\mathbf{o}_i(\lambda) - \mathbf{d}_i^\dagger)^2$	$\xi \sim N(0, \sigma^2)$	$\Gamma(D/2, D/2)$
$\vdots$	$\vdots$	$\vdots$

Choices of  $F$  and associated  $\pi_{\text{obs}}$  for stochastic inverse problem

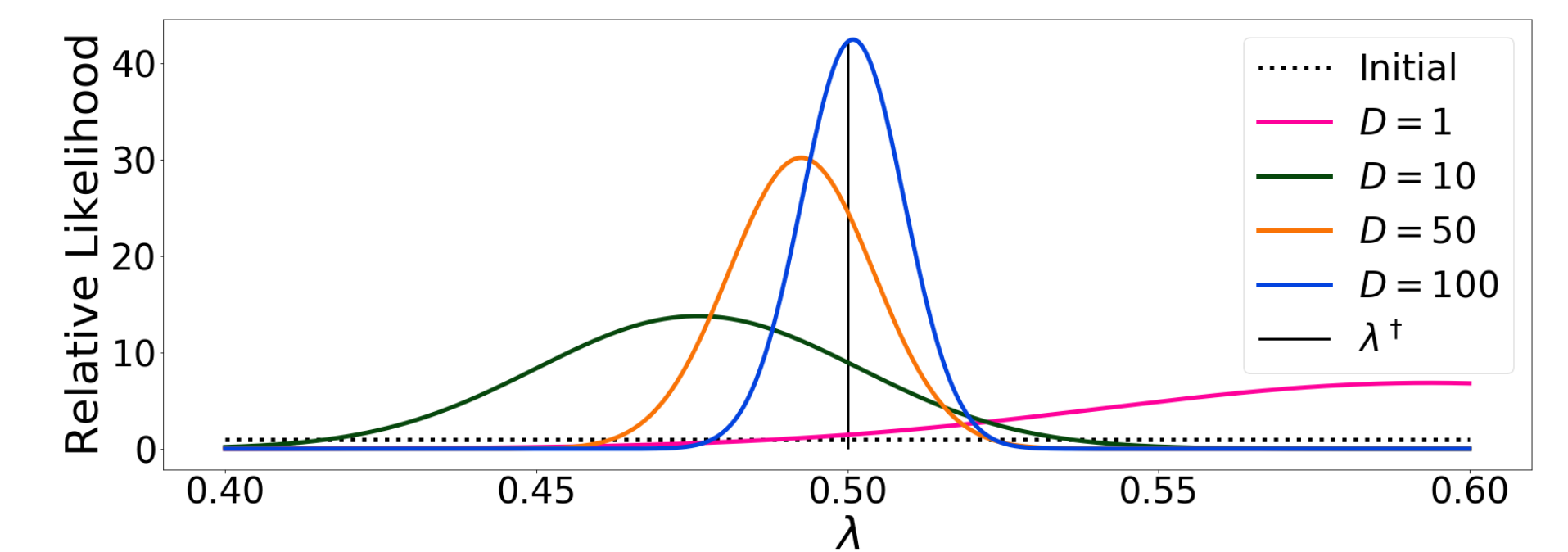
## Example

Consider an exponential decay problem with uncertain initial condition:

$$\partial_t u = -u, \quad u(t_0) = \lambda^\dagger = 0.5$$

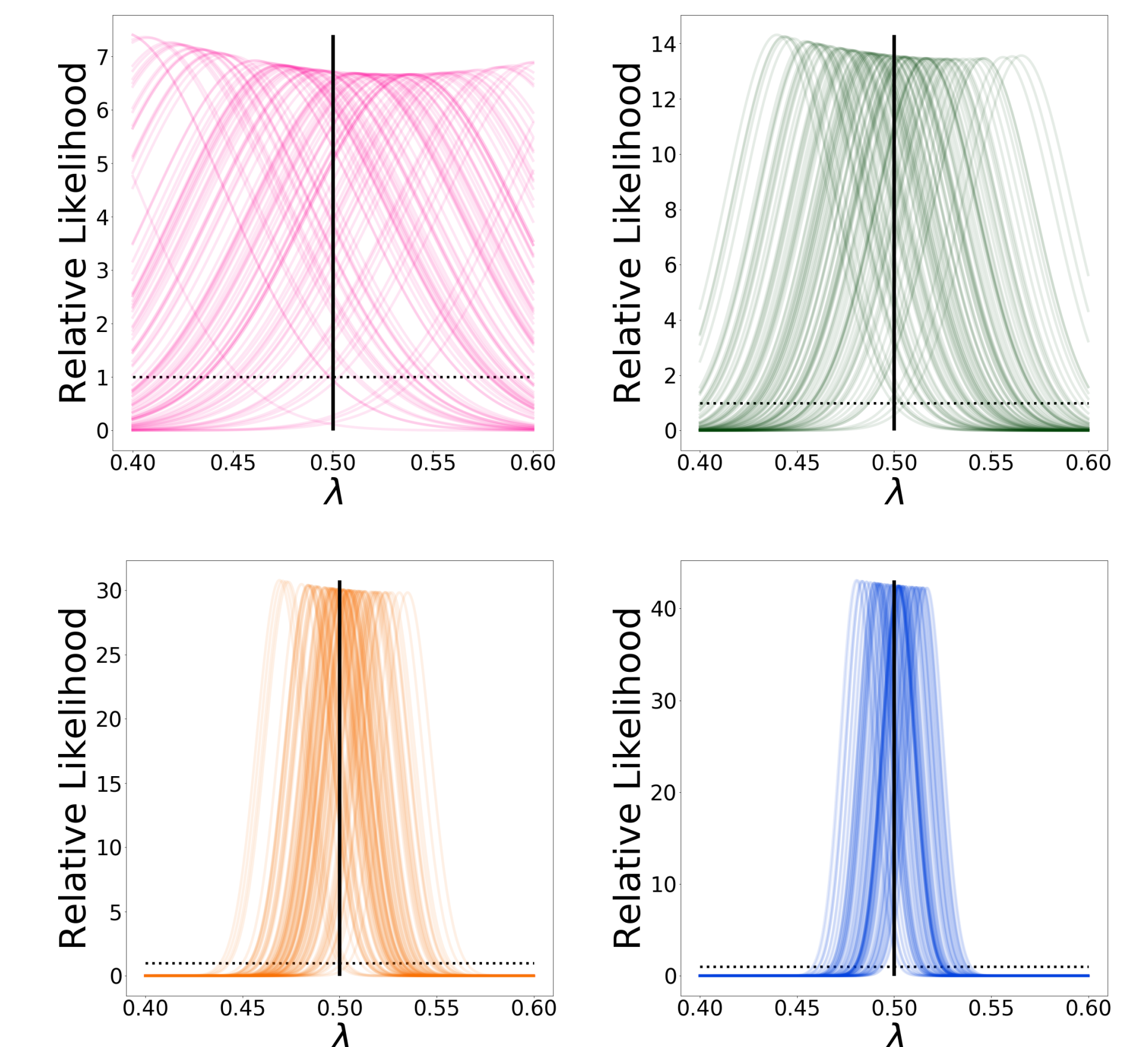


### Convergence



$\lambda^\dagger$  and  $\pi_{\text{pre}}$  for  $D = 1, 10, 50, 100$  for  $N = 1000$

### Stability



$\pi_{\text{pre}}$  for one hundred realizations of  $\xi^\dagger$  for  $D = 1, 10, 50, 100$