A critique of models of political ideologies

Mathematical Philosophy essay

https://mathematicalphilosophy.github.io/site/index.html

Throughout the history, people have tried not only to classify their political views and put themselves and others into suitable categories based on views, but also relate these categories amongst each other. However, most of the resulting models perform badly in describing the real state of affairs.

We point out the flaws and attempt to find the cause of this bad performance. Afterwards, we suggest a better explanation and attempt to put forward a new model (or, more precisely, a class of models) which seem to perform better.

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Introduction

Since the ancient times, people were trying to describe their views, philosophical, moral and, most importantly for us, political ones. As a result, today we recognize a variety of political ideologies, some of which are anarchism, communism, democracy, fascism, liberalism, nationalism, progressivism, syndicalism and many more.

Naturally, people made attempts to classify them. The most popular attempt to do so is the well known left-right political spectrum. Other popular model is the two-axis political compass, which aims to separate social and economic views. Both of those perform exceptionally badly to describe the reality of political views held by people. It is worth noting that these two models are very simplistic and it is, indeed, not surprising that they underperform and are criticized by the majority of political scholars.

1 Basic models and their problems

"The validation of a model is not that it is 'true' but that it generates good testable hypotheses relevant to important problems." -Richard Levins

We shall, firstly, begin our analysis by looking at the problems with some simple models, namely *left-right* spectrum and political compass, while trying to find hints which could direct us towards the root of the problem.

1.1 Left-right spectrum

The *left-right* spectrum is the simplest of models of political ideologies. The origin of political terms of *left* and *right* are traced back to the French Revolution, where the supporters of the aristocracy sat on the right side and the supporters of the revolution sat on the left side of the National Assembly (this will be relevant later). Obviously, this classification which separates people into supporters of aristocracy and supporters of the revolutions is relatively outdated, so the intended interpretation of *left* and *right* has changed over time. But changed to what?

This model is, by far, the most widespread, at least in the public discourse (one would be dishonest to say that serious political scholars consider this model, as presented in public). Due to its casual use, not much thought is given to what terms *left* and *right* actually mean or should mean. When asked to place a certain individual or a group somewhere within the spectrum people may give a wide variety of answers, most of them inconsistent with each other, relatively biased and inaccurate in their description of political views. Let us illustrate this with an example.

Example 1 (Intuitive descriptions are inaccurate)**.** *Let us consider the problem of the term left in public discourse More specifically, of the term socialist, which is widely considered to imply being a leftist. When talking about socialists, people which come to mind for most of us are historical figures such as Vladimir Lenin or Joseph Stalin. Contemporary example known to most people might be a United States politician Bernie Sanders. He called himself a socialist and people tend to call him as such.*

Not only people are said to be socialist, however. Soviet Union was probably the most well known socialist country. Consider the contemporary examples such as Scandinavian countries, which are often called socialist, by both their citizens and foreigners alike. The People's Republic of China is a country which is, likewise, often called socialist (although usually followed by "with Chinese characteristics"), as is the country of Cuba.

Of course, no discussion of politics is interesting if the Nazis are not mentioned, so let's mention that the Nazi Party also called themselves (National)socialist and some people, consequently, refer to them as socialist.

So the question we need to ask is what do Lenin, Stalin, Bernie Sanders, USSR, Scandinavian countries, China, Cuba and the Nazis have in common? Nothing, actually. So what exactly does then the term socialist represent? The descriptions people give of socialists and, by extension, the left is inaccurate.

This example illustrates how intuitively grouping people and groups together does not work, since the group (the *left* in the case of the above example) may have nothing in common. So, in order to group them, we need some objective criteria according which we may place a person or a group within the spectrum. This is what some more serious attempts at public discourse attempt to do. However, when asked about the intended meaning of *left* and *right*, people may give a wide variety of answers, similarly as with placing individuals and groups within the spectrum. Most of the criteria given, however, are unable to model political ideologies within society, as we hope to demonstrate below.

We shall use some guidelines in our examination. These are, indeed, needed, since we give no definite meaning to the *left* and the *right* at this moment, but we need some references to provide counterexamples for certain interpretations. For example, people widely agree that anarchists and communists belong to the left, as well as that fascists belong to the right. These seem like reasonable claims, which we use in further discussion.

One of the proposed interpretations of *left* and *right* is that the *left* advocates for a bigger state and *right* advocates for a smaller state, i.e. the spectrum models the size of government or, alternatively, it does not represent the state itself, but its intervention in the economy. But this does not describe reality.

Example 2 (The state as a criterion is inaccurate)**.** *Consider fascism and anar-*

chism. We may often see that fascists, which are right wing, support increases in military spending and expansions (which are usually controlled by the state), while left wing anarchists want to abolish the state completely. Similarly, the "next best thing", namely state intervention in the economy, instead of its size, is inadequate, too, due to anarchists having no state to intervene in the economy, while fascists usually advocate for often state interventions.

Some propose that this spectrum does not describe anything state-related. Instead, it describes the tendency of the *left* to propose change and the tendency of the *right* to propose conservation. But this is no better.

Example 3 (Tendency to propose change as a criterion is inaccurate)**.** *Consider, for example, proposition of these laws: a law which limits the influence of syndicates (who are protecting worker's rights), a law which obliges workers to work more hours (more than* 40 *a week) and a law which prohibits abortions (which are currently available). These propositions would all be opposed by communists, i.e. communists would advocate for conservation, while the right would advocate for change, in opposition.*

A common suggestion is that this spectrum models equality and hierarchy, with the *left* advocating for equality and the *right* advocating for hierarchy. This, too is questionable.

Example 4 (Equality versus hierarchy as a criterion is inaccurate)**.** *Consider the right to bear arms and the distribution of guns among the citizens. Communists and anarchists are very pro-gun, but we tend to see moderate rightists also as pro-gun. But the centrists are often anti-gun in all cases. Likewise, distribution of guns to the people, and thus making them equal (as opposed to only arming the army or the police force), has been used for both, liberation and occupation, i.e. in the name of equality and in the name of hierarchy. So, the spectrum ceases to be a spectrum, within the intended meaning, since the middle opposes something the extremes advocate for.*

This counterexample for the proposition of equality versus hierarchy was somewhat more difficult to find and is less universal compared to earlier ones, but, as a result, it may hint at a potential solution. Since guns are used for liberation and occupation, for equality or hierarchy, maybe the mere propositions are not what matters, but the reasons behind them are. We shall keep this in mind for further discussion.

Let us consider another notion of globalism and openness towards immigration as opposed to nationalism and opposition to immigration, with the *left* being pro-immigration and the *right* being anti-immigration. This is also problematic.

Example 5 (Globalism versus nationalism as a criterion is inaccurate)**.** *Consider right wing market and labor policies, which, in support of cheap labor tend to be very pro-immigration when in need of such cheap labor. Similarly with globalization in general, which is used for cheap labor outside of the country's borders.*

We may again sense that the reasons behind the stances may play a greater role than just the stances by themselves.

A similar issue arises when the idea of individualism versus collectivism is proposed.

Example 6 (Individualism versus collectivism as a criterion is inaccurate)**.** *Consider communism and fascism. Both of these ideologies encourage collectivism. While communists unite and develop communities over their social class, the fascists do so, but with respect to their ethnicity or nation. Individualists, however, usually fall somewhere in between.*

To conclude, this model of *left* and *right* and an entire spectrum between them fails to accurately describe the positions of some of the most agreed upon ideologies, e.g. anarchism, communism and fascism, no matter which of the proposed interpretations we assign to it. We have hinted at motivations and/or goals of certain groups of people and at their possible role in creating a more accurate model.

1.2 Political compass

We have seen some of the problems with the *left-right* spectrum model and, of course, we are not the first to do so. Likewise, we are not the first who will attempt to create a better model. One of existing attempts is the political compass. It adds another axis to the *left-right* one and labels it as *authoritarian-libertarian*. The intended interpretation is as follows: the *left-right* axis describes economic policies and *authoritarian-libertarian* axis describes social policies a person or a group holds.

The economic axis should measure one's opinion on how the economy should be run, with the *left* proposing collectively and cooperatively run economy and the *right* proposing individually and competitively run economy. On the other hand, the social axis should measure one's opinion on the amount of personal freedom that should be allowed, with the *libertarian* proposing the maximization of personal freedom and *authoritarian* proposing obedience to authority.

This model divides people into four quadrants, *authoritarian left* (representing ideologies similar to communism and socialism), *authoritarian right* (representing ideologies similar to conservatism), *libertarian right* (representing ideologies

similar to anarcho-capitalism), and *libertarian left* (representing ideologies similar to anarchism). Of course, these are not separate categories, but regions of a continuum.

This is a model which many who are trying to get into politics often use, as it seems like an improvement over the *left-right* spectrum. But it comes with its own problems.

The vertical *authoritarian-libertarian* axis is problematic, for a few reasons. Firstly, those who are placed on the same side of it tend to disagree.

Example 7 (Libertarians see each other as authoritarians)**.** *Consider the right of a person to sell themselves into slavery. Some right libertarian ideologies view this as legitimate, if the relation of master and slave is agreed upon by consenting adults.*

On the other hand, some left libertarian ideologies consider slavery to be authoritarian, as an example of the ultimate domination over another human being, even when it is consentual.

But is this not denying the ability of consenting adults to form consentual contracts? This would be seen as authoritarian by the libertarian right.

This is troublesome, since libertarians should agree on societal problems, while disagreeing on economic issues. But we see that they disagree on social issues, and even hold opposing views. A quite different situation arises if we consider the *right*.

Example 8 (Agreement on the *right*)**.** *Regardless of their placement on the vertical axis, people on the right tend to share their political circles, consistently uniting to oppose the left and adopt each other's talking points.*

The similar thing happens with anarchists and communists, which both share their economic axis placement, but are placed differently on the social axis.

Example 9 (Agreement of the *left*)**.** *Anarchists and communists agree, for the most part. In fact, one could argue that they want the exact same thing, just disagree about the role of the state in achieving that goal. Whereas anarchists want to abolish the state immediately, communists think the state has to "wither away" after it fulfills its role of defending the interests of the working class.*

Even though the asymmetry within the axes is troublesome by itself, there is even a greater problem which we may now notice. The above example actually gives us a strong condition on the vertical *authoritarian-libertarian* axis, as there is only this one thing on which communists and anarchists agree. The state. One might be tempted to repeat the arguments from the critique of the *left-right* spectrum, but here that argument will not work, since fascists are *right authoritarians*

and anarchists *left libertarians*. We might use communism in the place of anarchism, stating that their goal is to achieve the classless society. Since their means do include the state, this might be a slightly flawed approach. This is the reason why we give a different example. As we have seen, the *authoritarian-libertarian* axis may only represent pro-state or anti-state stances (with *authoritarian* being pro-state and *libertarian* being anti-state), for otherwise anarchists and communists are not properly placed.

Example 10 (The state as a criterion is inaccurate)**.** *Consider the anarcho-capitalism ideology, which is right libertarian. This ideology legitimizes privately owned land, roads, police force, army, bureaucratic apparatus to manage the owned land and collect payments from residents in exchange for protection or use of the land. Likewise, it is legitimate to hold a monopoly on violence, enforced by private police and army. Even though it is private, this is a state. Which means that extremely right libertarian ideologies are pro-state.*

This means that *authoritarian-libertarian* axis representing pro-state or anti-state is inaccurate in describing the political ideologies in reality.

We may not conclude that the *authoritarian-libertarian* axis cannot describe anything, since if we interpret it as anything other than pro-state or anti-state, it is inaccurate on the *left*, but if we label it as such, it is inaccurate on the *right*. Therefore, we conclude, that the political compass, although popular in public discourse, is not an accurate representation of human political views.

2 Why do some models fail?

"Model building is the art of selecting those aspects of a process that are relevant to the question being asked." -John H. Holland

We saw the two most commonly used models of political ideologies and we have illustrated that they do not model human politics accurately. However, politics is not the only area of study where something similar happens. We shall note some other disciplines in which models fail and see if we can gain any useful insights by studying them.

2.1 Linguistics and large language models

Recently, linguistics has been able to produce computational models capable of language generation. These models use artificial neural networks to create, within their memory, models of human language commonly based on statistical analysis of a large quantity of data given to them. Various methods are used to achieve better and better results, but we shall not attempt at listing or explaining them, since it is of no interest to us currently. We note, however, that a lot of money and resources is funneled into computational linguistics in order to create better and better models.

What is of interest to us is that these models, even when given a lot of data, after many years of optimization and development, are sometimes very inaccurate. One of the examples is the conflation of words *actually* and *not* when attempting to "fill in the blanks" of a sentence.

Example 11. *Given a sentence*

"Even though it was not moral back then, slavery was _____ legal.",

where the correct answer was the word 'actually', models would often predict that the missing word is 'not'. Note that the sentence, by the virtue of context, does not even make (intuitive) sense is the word 'not' is used. The meaning may still be inferred, but the sentence itself is not constructed correctly.

To briefly note another mismatch of language models and human language, language models use the word *delve* significantly more than humans, which resulted in our ability to analyze academic papers and see (some of those) which ones were written by (or the writing of which was assisted by) artificial intelligence.

One of the criticisms often repeated is the simple notion that these models use models of human language which is structurally different from the way humans understand and use language. Namely, some of them use *word embeddings*, which are representations of words within a space spanning hundreds and thousands of

dimensions. Within this space, closer words are more similar in meaning. Humans, however, do not learn language that way. Some think that this is the core issue. We use already existing model, the mathematical notion of *vector space* to model the human use of language, which does not use *word embeddings* to learn language. In a way, we have the abstract model, and we try to fit the data within it, even though it is not suited to the data in question. This feels eerily similar to our study of models of political ideologies, does it not?

If we accept this criticism, then the next step would be for linguists and neuroscientists to study how language is interpreted by human brain and create new language models based on their insights, instead of using already existing models which are insufficient. Likewise, in order to model political ideologies, we need to study political views, and why and how they arise within human minds while hoping that our endeavor will be more successful than the linguistic one.

2.2 Physics and generalizations

A good area to look at for another example of models being wrong is physics, where new models are often introduced too generalize previous models. Understanding (roughly) how physicists study the nature using mathematical models may help us apply the same ideas to our problem of modeling political ideologies.

Firstly notice one, arguably the most famous, example of how a new model was invented. We are, of course, talking about Isaac Newton who, in the absence of adequate mathematical theory to model phenomena which he observed, invented calculus, an entirely new branch of mathematics. Note that if he did try to model his observations via already existing mathematics, it is absent from public historical knowledge. This approach of inventing a new branch of mathematics is nowadays very hard to do, since mathematics is very developed and many new ideas already fit within one of already existing areas. We do, however, note the idea of not using an existing inadequate model to force our observations within it, but, instead, creating a model based on those observations.

Let us now turn to the topic of generalizations, i.e. creating, from an existing model, a new one which explains everything the existing one does, equally good or better, but, additionally, explains something the existing model does not account for.

Example 12. *Consider Galilean relativity theory. It describes, via mathematical equations, relativity of velocity relative to the observer. His theory is inaccurate, as seen by Einstein's relativity theory. However, when low values are used within Einstein's equations, they approximate Galilean equations. Which means that Einstein's relativity generalizes Galilean relativity.*

This model gives an example how two models coincide, given certain circumstances, but one gives accurate predictions even in absence of those circumstances, while the other does not. Specifically, Galilean relativity and Einstein's relativity coincide when velocities are low, but Einstein's relativity is accurate even when the velocities are high, while Galilean is not.

We may think of this as a property of a greater model to contain, in a sense, a smaller one (greater and smaller in terms of explanatory power). If the greater model is accurate, then, the smaller one contained within it is accurate. Likewise, if the smaller one is inaccurate, then the greater one, containing it, is also inaccurate. This fact is of the utmost importance, since it allows us to conclude that a model is inaccurate based on it containing a smaller inaccurate model. This rules out many complicated models proposed by various people (since most of them are attempting to generalize the flawed political compass model).

3 Putting everything together

"The purpose of models is not to fit the data but to sharpen the questions."
-Samuel Karlin

Now it is time for us to attempt to create a new, better model. We shall attempt to use all of the insights we have collected so far, from the first notions of *left* and *right*, our considerations of goals behind the stances, instead of stances in isolation and out observations from other sciences. We can then test the model we created and formalize it mathematically. This allows us explore generalizations of such models, for future consideration, if an opportunity arises.

3.1 Abstract versus material

By drawing inspiration from other sciences, like linguistics and physics, we have seen that a procedure of creating a model which starts from abstractions, like the mathematical concept of *vector space* and tries to "fit reality into the model" results in some problems. Hence, we propose a different approach, similar to one used by Newton. We start from the material, from that which is real, and try to infer facts about the structure, instead of trying to interpret reality within some nice abstraction.

We, firstly, remind ourselves and the reader of the origins of the terms *left* and *right*, the French Revolution. What did those terms mean there and then? The answer is that one term represented a group of supporters of aristocracy and another represented a group of supporters of revolution. This will be our starting point. What were the material conditions back then? Why did the revolution come about in the first place? Somewhat simplified, but sufficient for our purposes, reason was that traders, merchants, manufacturers, etc. were financially powerful and organized class of society, but were excluded from political power. Hence, one may consider exactly this paradigm; conflict of a new, rising class and old, politically influential class.

From this, newly acquired perspective, we might hypothesize that the term *left* should mean something akin to *"supporter of a new, rising class"*, whilst *right* should, similarly, hold the meaning of *"supporter of an old, politically powerful class"*. Or, perhaps, to distance ourselves from terms themselves, since today's conditions may differ, we hypothesize that our model should be based on classification of all *"politically relevant"* social or socioeconomic classes. This gives us a real and material basis for our model (assuming a suitable notion of *class*), instead of trying to classify political views as degrees of some abstract idea, like role of the state or hierarchy versus equality.

3.2 Classes and their goals

We must now consider which social classes are *"politically relevant"*. In modern society and throughout the history, there are and there were many classes, such as workers, nobility, clergy, capitalists, slaves, etc. Some of them no longer exist, but we consider them and attempt to infer, from their historical roles, their relevance. Since the political models we consider first appeared to classify supporters of aristocracy and supporter of the revolution, we start there and consider the relevant classes, the aristocratic nobility and the revolutionary capitalists.

Nobility was the class which held most, if not all, political power. On the other hand, the capitalist class wanted to gain said political power. After the revolution, they established capitalism, a system which allowed them to acquire that political power. It seems reasonable to consider classes which want to gain, regain or keep political superiority as *relevant*.

Let us briefly reflect upon historical changes of power similar to French Revolution. The French Revolution caused a change from feudalism, where nobility held the power into capitalism, where capitalists held the power. Similarly, the replacement of slavery of the Roman Empire with serfdom, was a change from slave-based economy into feudalism, where the class of slave masters ceased to exist and nobility arose as a class which took the power. A notable example are the numerous socialist revolutions throughout the world, where the working class wanted to take political power from the capitalist class.

Now that we have more examples, we have a rough idea of what are we trying to describe. We want to describe up to which extent does an individual or a group of individuals support the goals of a certain class (ruling or potentially ruling).

What is left to do is to state a criterion upon which to judge is the class *relevant*. Take as an example the peasant class. It was relevant during Russian and Chinese socialist revolution, but not during French revolution. Why is that so? One (although possibly oversimplified) reason might be that Russian and Chinese peasants were introduced to the idea of socialism by people such as Lenin and Mao and were able to exchange ideas and, consequently, create their ideology. The peasant class within revolutionary France had no such opportunity to develop their ideology. Hence the *relevant* class is a class which may, through the articulation and exchange of their ideas, create their ideology, which (in a way) represents their goals (within the context of certain material conditions). For nobility, this was feudalism, for capitalists, this is capitalism and, for workers, this is socialism.

Now we have a fully developed idea on how to describe our political model. Firstly, enumerate all the socioeconomic classes which have developed their ideology. Secondly, given an individual or a group, analyze their views and approval of realization of any of those goals. This gives us the proximity of the views held by an individual or a group to the full realization of goals of a certain class.

3.3 Our proposed model

We now work towards the formalization of the above conclusion. But before we do so, let's consider how would we model modern politics (of North America and Europe). Within the countries of the "developed world", there are two classes which fit our criteria, the capitalist class and the working class. Although some countries are formally monarchies, where nobility exists, they are not the same type of nobility as the one which played a role in the French Revolution, since they do not want feudalism and play more of a traditional and cultural role, instead of a political one. There are classes which do not have the political ideology of their own, e.g. homeless, jobless people, who are neither workers or capitalists, or the clergy, which historically, always stood with the class who held power while never developing their own political ideology (though they do have developed a philosophical one).

Since we have only two contending classes, for any reasonable political proposition, one might ask which of these two classes benefits from it. And we might ask, for an individual, how many (percentage wise) of these propositions benefiting a certain class does this individual hold. The answer places this individual somewhere between fully supporting the working class and fully supporting the capitalist class. Notice that this seems light *left-right* spectrum. Indeed, for this situation, since we are considering only two classes, we have a line (or a spectrum) between supporting goals of the working class on the *left* and the goals capitalist class on the *right*. Let's consider all of the examples where the *left-right* spectrum failed to see if our model has similar problems.

We had problems with placing fascism, anarchist and communist on the appropriate end of the spectrum. But we have no such problem here. Indeed, communism is based upon the rights of the workers, with the goal similar as anarchism, while (historically) the goal of fascism is to act as a reactionary force to stop socialist revolution and keep the capitalist class in power. Similarly, problems like immigration is not a problem here, since we consider the reasons. Immigration may be used in order to gain access to cheap labor or in order to gain access to larger workforce in order to decrease working hours. So no objections we brought up in our critique of *left-right* spectrum are a problem for this new model.

What is left to do is to describe where is the line of *left* and *right*. We propose the following: those on the *left* support socialism and those on the *right* support capitalism, as the system which gives more political power to a certain class. Within the *left* or *right*, one is closer to center if within this system one supports policies which are associated with the opposing system. For example, a *centre-right* individual supports capitalism, but might be in favor of free healthcare, accessible education for everyone, worker unions, etc. On the other hand, *centre-left* individual supports socialism, but is in favor of market economy. The former

would be a social democracy and the latter would be market socialism, ideologies which are well established. Some issues would be dependent on the reason behind them, e.g. immigration or welfare.

3.4 Mathematical model(s)

Finally, we turn to the formalization of our model. Assume there are $n+1$ relevant classes within a certain society. Denote these classes as $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n$ and denote a *set of all classes*

$$\mathfrak{C} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_n\}.$$

Next, we consider a set of *propositions* P, the interpretation of which is a set of all policies, laws or stances an individual may potentially hold (e.g. pro-immigration, pro-union, anti-abortion, anti-equality, etc.). Next, we adjoin this data with a *benefit* function

$$\beta : \mathfrak{C} \to \mathcal{P}P,$$

from the set $\mathfrak{C}$ to the set of subsets of P the interpretation of which maps a class to a set of propositions which benefit that particular class.

Definition 13 (Political structure)**.** *Let $\mathfrak{C}$ be a set of $n+1$ classes, P a set of propositions and $\beta : \mathfrak{C} \to \mathcal{P}P$ a benefit function. The triple $\mathfrak{S} = (\mathfrak{C}, P, \beta)$ is a* political structure *or simply a* structure *of size $n+1$.*

What we need is to do next is to take an individual and the set of propositions $X \subseteq P$, which this individual supports and interpret that data.

Definition 14 (Individual)**.** *Let P be a set of propositions. An* individual *is a subset $X \subseteq P$.*

We need a generalization of a spectrum (i.e. a generalization of a continuous segment), which is a *simplex*. Let us first define it, and then describe how we interpret the data. Note that this notion is not key to understanding the model, but it gives us a way to neatly visualize where some people are in relation to others, politically, for up to four relevant classes (afterwards we need four spatial dimensions to create an image, which is not very useful).

Simplices

This is where we define a simplex, which is to be used as an interpretation of our political spectrum. We start by defining a notion of independant vectors.

Definition 15 (Affinely independant vectors)**.** *Given $n+1$ vectors $u_0, \ldots, u_n \in \mathbb{R}^n$, we say that they are* affinely independent *if $n = 0$ or $n > 0$ and vectors $u_1 - u_0, \ldots, u_n - u_0$ are linearly independent in $\mathbb{R}^n$.*

We are now ready to define a simplex.

Definition 16. *Given $u_0, \ldots, u_n \in \mathbb{R}^n$, an affinely independent set of vectors, we define an* n-simplex σ *as a set*

$$\sigma = \left\{ \sum_{i=0}^{n} \theta_i u_i : \theta_0, \ldots, \theta_n \geq 0, \sum_{i=0}^{n} \theta_i = 1 \right\},$$

with vertices $u_0, \ldots, u_n$. *We denote* $\sigma = [u_0, \ldots, u_n]$.

Next, we may see that each point (element) of a simplex corresponds to a unique choice of $\theta_1, \ldots, \theta_n \in \mathbb{R}$ (because, due to the condition $\sum_{i=0}^{n} \theta_i = 1$, any choice of the $\theta_1, \ldots, \theta_n$ uniquely determines θ_0). We refer to $\theta_0, \ldots, \theta_n$ as *barycentric coordinates*.

Now we are ready to interpret individual's stances in order to place him within the political spectrum.

Interpretation

The simplest way, given a political structure $(\mathfrak{C}, P, \beta)$ we might go about placing an individual X on the political spectrum is to just look at a percentage of individual preferences which correspond to elements of sets $\beta(\mathcal{C}_0), \ldots, \beta(\mathcal{C}_n)$ and use those as barycentric coordinates.

This is where we run into a problem if one proposition benefits more than one class. Then, obviously percentages might not add up to 1. Consider a case where there are two classes, $\mathcal{C}_i$ and $\mathcal{C}_j$ such that there is a proposition p such that $p \in \beta(\mathcal{C}_i) \cap \beta(\mathcal{C}_j)$. Assuming an individual X which is equal $X = \beta(\mathcal{C}_i)$, e.g. this individual fully supports all goals which benefit $\mathcal{C}_i$ and only those goals. Then the percentages add up to a number greater then 1.

Another problem is when there is a proposition which does not benefit any class. In such a case, the percentages might add up to less than 1.

Therefore we define some special types of structures.

Definition 17. *Let $\mathfrak{S} = (\mathfrak{C}, P, \beta)$ be a political structure. We say that $\mathfrak{S}$ is*

1. disjoint *if for all $i \neq j$, $\beta(\mathcal{C}_i) \cap \beta(\mathcal{C}_j) = \emptyset$,*
2. full *if for all $p \in P$ there exists at least one $\mathcal{C}_i$ such that $p \in \beta(\mathcal{C}_i)$.*

This, although not really optimal, is a good starting point. We shall connect these problems with our earlier observations and see some potential (and very natural) resolutions. But first, as a demonstration, define a position of an individual on a simplex in the following way.

Definition 18. *Let $\mathfrak{C}$ be a full and disjoint political structure of size $n+1$. Let $\sigma = [u_0, \ldots, u_n]$ be a simplex. A* political position function $\pi : \mathcal{P}P \to \sigma$ *is a function such that:*

$$\pi(X) = \sum_{i=0}^{n} \alpha_i u_i, \text{ where } \alpha_i := \frac{|X \cap \beta(\mathcal{C}_i)|}{|X|}.$$

Proposition 19. *Political position function is well defined, i.e. $\alpha_i \geq 0$, for all indices i and $\sum_{i=0}^{n} \alpha_i = 1$.*

Proof. Since cardinalities are positive integers, the first claim is trivial, since ratio of positive numbers is positive.

For the second claim We may use tehe disjointness property of sets $\beta(\mathcal{C}_i)$ to conclude that sets $X \cap \beta(\mathcal{C}_i)$ are also disjoint and, therefore

$$\sum_{i=0}^{n} \frac{|X \cap \beta(\mathcal{C}_i)|}{|X|} = \frac{1}{|X|} \left| \bigcup_{i=0}^{n} (X \cap \beta(\mathcal{C}_i)) \right| = \frac{1}{|X|} \left| X \cap \bigcup_{i=0}^{n} \beta(\mathcal{C}_i) \right| = \frac{|X|}{|X|} = 1,$$

where we also use the property that the union of all $\beta(\mathcal{C}_i)$ is equal to P, since every proposition is contained in at least one $\beta(\mathcal{C}_i)$. □

Now we have a model where we can interpret some simple political situations.