§5.2 Notes

Preliminaries

① summation notation

$$1^2+2^2+3^2+4^2+\dots+100^2 = \sum_{k=1}^{100} k^2$$

$$x_1+x_2+x_3+\dots+x_{51} = \sum_{k=1}^{50} x_k$$

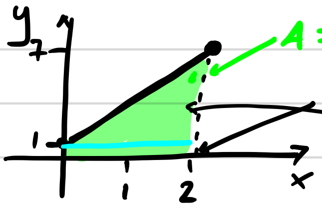
② useful fact $\displaystyle\sum_{k=1}^{n} k = 1+2+3+\dots+(n-1)+n = \frac{n(n+1)}{2}$

$$\begin{array}{r} 1+2+3+\dots+99+100 \\ 100+99+98+\dots+2+1 \\ \hline 101+101+101+\dots+101+101 = 100(101) \end{array}$$

TWO sums or double the amount

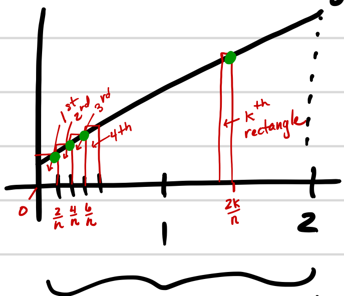
So $1+2+3+\dots+99+100 = \dfrac{100(101)}{2}$

Ex Use the definition of the definite integral to find the area under $f(x) = 3x+1$ on $[0,2]$.



$A = \underset{\substack{\uparrow \\ \text{rectangle}}}{2} + \underset{\substack{\uparrow \\ \text{triangle}}}{\tfrac{1}{2}\cdot 2\cdot 6} = 8$ (geometry)

- Estimate area using rectangles
- More rectangles improves estimation.
- Use $\boxed{n}$ rectangles.



n sub intervals.

width: $\frac{2}{n}$

endpoints: $0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \ldots, \frac{2k}{n}, \ldots, \frac{2n}{n}$

height (use right-hand endpts)

$f\left(\frac{2}{n}\right), f\left(\frac{4}{n}\right), \ldots, f\left(\frac{2k}{n}\right), \ldots$

areas of rectangles $= \frac{2}{n}\left(f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + \cdots + f\left(\frac{2k}{n}\right) + \cdots + f\left(\frac{2n}{n}\right)\right]$

$$= \frac{2}{n}\left(\left[3\left(\frac{2}{n}\right)+1\right] + \left[3\left(\frac{4}{n}\right)+1\right] + \ldots + \left[3\left(\frac{2k}{n}\right)+1\right] + \ldots + \left[3\left(\frac{2n}{n}\right)+1\right]\right)$$

$$= \frac{2}{n}\left(3\left(\frac{2}{n} + \frac{4}{n} + \ldots + \frac{k}{n} + \ldots + \frac{2n}{n}\right) + \underbrace{(1+1+\ldots+1)}_{n}\right]$$

$$= \frac{2}{n}\left[\frac{3\cdot 2}{n}(1+2+\ldots+k+\ldots+n) + n\right] = \frac{12}{n^2}\cdot\frac{n(n+1)}{2} + 2$$

$$\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+2=\frac{6(n^2+n)}{n^2}+2$$ ← "closed form." No $+\dots+$.

How can we formalize the notion of adding more & more rectangles?

Let $n\to\infty$.

So $A=\lim\limits_{n\to\infty}\left[\frac{6(n^2+n)}{n^2}+2\right]=6+2=8$ ✓ *Rewrite w/ Σ

__def__: Definite Integral

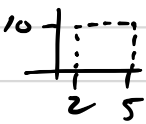
$$\int_a^b f(x)\,dx=\underbrace{\lim_{n\to\infty}}_{\text{more \& more rectangles}}\overbrace{\sum_{i=1}^{n}}^{\text{add up the areas of all the rectangles.}}\underbrace{f(x_i^*)}_{\text{height of rectangle}}\underbrace{\Delta x}_{\text{width of rectangle}}$$

picture

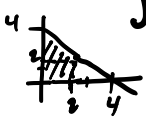
__Questions__: • What happens if the graph of $f(x)$ is below the x-axis?

• What happens if $b<a$?

__Examples__: $\int_2^5 10\,dx=3\cdot10=30$



$\int_0^2(4-x)\,dx=4+\frac{1}{2}\cdot2\cdot2=5$



$\int_0^5(4-x)\,dx=\frac{1}{2}\cdot4\cdot4-\frac{1}{2}\cdot1\cdot1=\frac{16}{2}-\frac{1}{2}=\frac{15}{2}$