LECTURE NOTES 2-2: THE LIMIT OF A FUNCTION

Things to Know:

- The intuitive definitions of a *limit* and a *one-sided limit*.
- How to find a (one-sided) limit using a calculator or the graph of the function, including infinite limits.
- How to find limits for piecewise-defined functions.
- How to distinguish between the various ways a limit may *not* exist.
- Understand how using a calculator can give an incorrect answer when evaluating a limit.

Intuitive Idea and Introductory Examples

(Note that this is motivated by our discussion of tangent lines and instantaneous velocity.)

Say: "the limit of $f(x)$, as x approaches a is L"

Write: $\lim_{x \to a} f(x) = L$

It means: as x gets closer and closer to a, $f(x)$ can be made arbitrarily close to the number L.

$\boxed{\text{EXAMPLE:}}$ Use calculation to guess $\lim_{x \to 2} \frac{x-2}{x^2-x-2}$.

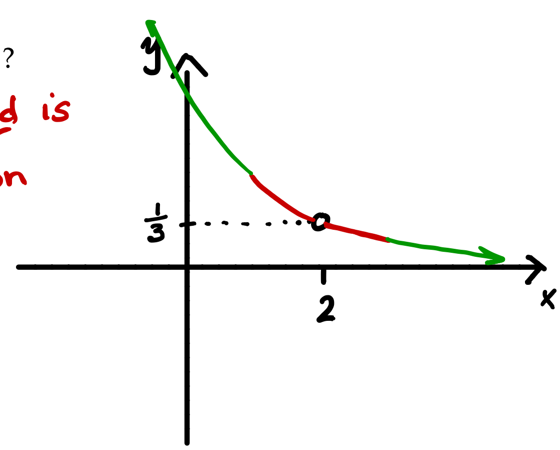
x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$f(x)$	0.5	0.4	0.34483	0.33445	0.33344	DNE	0.33322	0.33223	0.32258	0.28571	0.25

GUESS: $\lim_{x \to 2} \frac{x-2}{x^2-x-2} = 0.3333... = 1/3$.

What does the table above tell you about the *graph* of $y = \frac{x-2}{x^2-x-2}$?

While there is a "hole" at $x=2$, close to $x=2$ the y-values get close to $1/3$.

- the red is what our calculation tells us.
- the green is obtained by graphing more points



$\boxed{\text{EXAMPLE:}}$ [Why do all the calculation? Just pick a number really close to "a," right???!!]

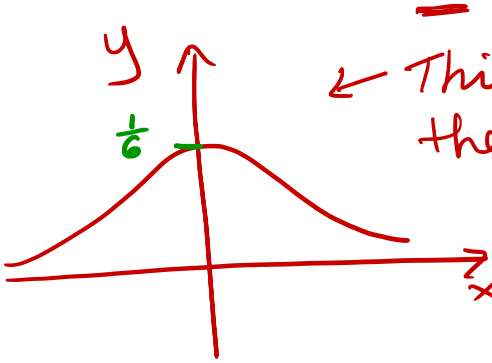
$$\text{Use calculation to guess } \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}.$$

Jill just picks numbers super-close to $a = 0$, say ± 0.000001 :

t	-0.000001	0	0.000001
$f(t)$	0	DNE	0

GUESS: $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = 0.$ ← wrong!!

Hint: Never believe Jill! Why can't this be right and what went wrong?



← This is what the graph looks like. Also the numerator isn't ever exactly zero!

The numerator gets so small my calculator thinks it's zero.

hmm.... what SHOULD this limit be? $L = 1/6$ is my guess using the graph... ?

$\boxed{\text{EXAMPLE:}}$ [Sample points may not illustrate the big picture. Theory will be useful.]

$$\text{Use calculation to guess } \lim_{\theta\to 0}\sin\left(\frac{\pi}{\theta}\right).$$

θ	-0.1	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(\theta)$	0	0	0	dne	0	0	0

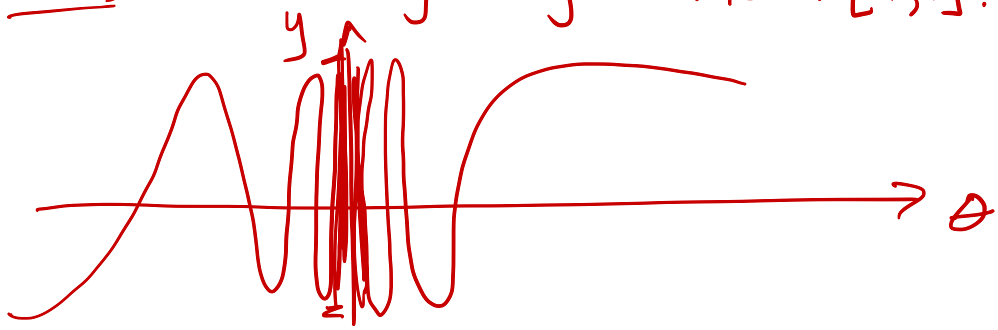
GUESS: $\lim_{\theta\to 0}\sin\left(\frac{\pi}{\theta}\right) = 0$

Do you believe your answer? No way!

(I use my numerical sense...) If I choose θ to be really small positive numbers, $\frac{\pi}{\theta}$ will get really big (ie head to ∞.) So..., $\sin(\frac{\pi}{\theta})$ should oscillate!

So $\sin\left(\frac{\pi}{\theta}\right)$ SHOULD be oscillating along the interval $[-1,1]$.

Also the graph looks like:



Practice Problems

1. For each problem below, fill out the chart of values, then use the values to *guess* the value of the limit. Finally rate your confidence level on a 0 to 3 scale where (0 = I'm sure this is wrong) and (3 = I'm sure this is right.)

(a) $\lim\limits_{\theta \to 0} = \dfrac{\sin\theta}{\theta} = 1$ confidence? _______

x	-1	-0.5	-0.1	-0.001	0	0.001	0.1	0.5	1
$f(x)$	0.8415	0.9589	0.9983	0.9999	DNE	0.9999	0.9983	0.9589	0.8415

(b) $\lim\limits_{x \to 2} f(x) = \boxed{\text{DNE}}$ where $\begin{cases} |x-1| & x \le 2 \\ x+1 & x > 2 \end{cases}$ confidence? _______

x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$f(x)$	0	0.5	0.9	0.99	0.999	1	3.001	3.01	3.1	3.5	4

(c) $\lim\limits_{x \to 0} \dfrac{e^{2x}-1}{x} = 2$ confidence? _______

x	-0.5	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1	0.5
$f(x)$	1.264	1.813	1.98	1.998	1.9998	dne	2.0002	2.002	2.02	2.214	3.44

DEFINITIONS:

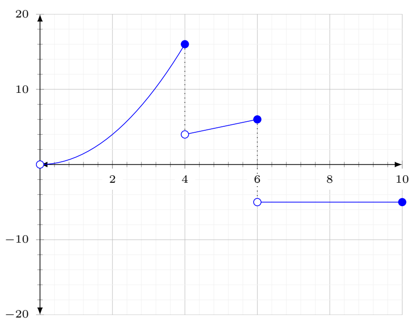
Say: "the limit as x approaches a *on the left* is L"; Write: $\boxed{\lim_{x \to a^-} f(x) = L}$

It means as x approaches a *from below* or *for x's smaller than a*, $f(x)$ can be made arbitrarily close to L.

Say: "the limit as x approaches a *on the right* is L"; Write: $\boxed{\lim_{x \to a^+} f(x) = L}$

It means as x approaches a *from above* or *for x's larger than a*, $f(x)$ can be made arbitrarily close to L.

EXAMPLE: The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x \to 4^-} f(x) = 16$

(b) $\lim_{x \to 4^+} f(x) = 4$

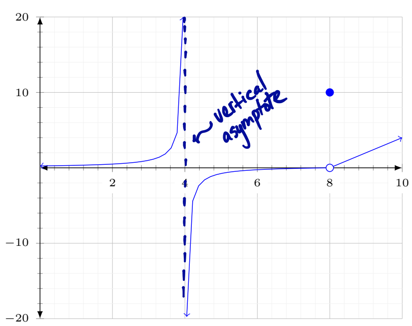
(c) $\lim_{x \to 4} f(x) = DNE$

(d) $f(4) = 16$

(e) $\lim_{x \to 8} f(x) = -5$

(f) $f(8) = -5$

EXAMPLE: The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x \to 4^-} f(x) = \infty$

(b) $\lim_{x \to 4^+} f(x) = -\infty$

(c) $\lim_{x \to 4} f(x) = DNE$

(d) $f(4) = DNE$

(e) $\lim_{x \to 8} f(x) = 0$

(f) $f(8) = 10$

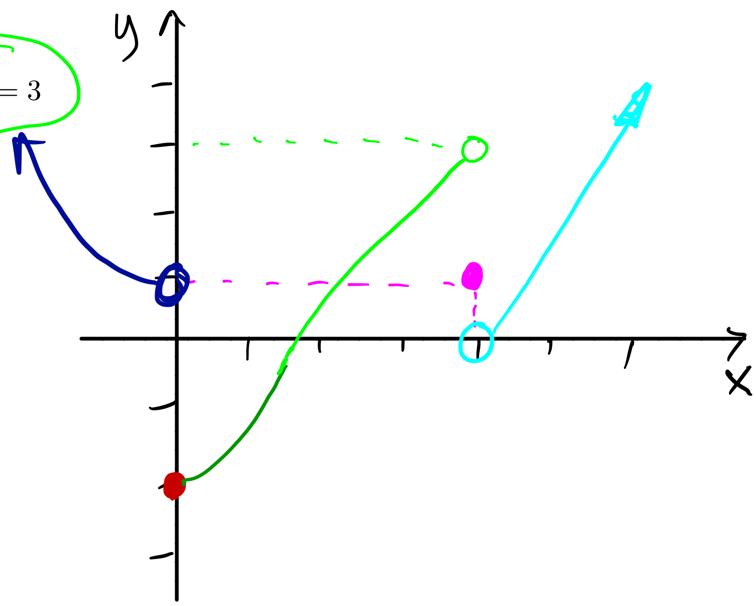
Write the equation of any vertical asymptote:

$x = 4$

2. Sketch the graph of an function that satisfies *all* of the given conditions. Compare your answer with that of your neighbor.

$\lim_{x\to 0^-} f(x) = 1$ $\quad \lim_{x\to 0^+} f(x) = -2$ $\quad \lim_{x\to 4^-} f(x) = 3$

$\lim_{x\to 4^+} f(x) = 0$ $\quad f(0) = -2$ $\quad f(4) = 1$



Many different answers here.

3. Determine the limit. Explain your answer.

(a) $\lim_{x\to 5^+} \frac{2+x}{x-5} = \infty$

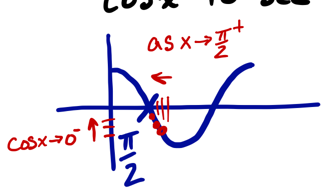
Explanation: As $x \to 5^+$, the numerator $2+x$ approaches 7. The denominator, $x-5$ approaches 0 but is always positive (a little larger than zero). Thus the quotient (a fixed positive number / a very small positive number) approaches to infinity.

(b) $\lim_{x\to 5^+} \frac{2+x}{5-x} = -\infty$

Explanation: In this case, the denominator approaches 0 but is always negative. Thus the quotient is negative.

(c) $\lim_{x\to(\pi/2)^+} \frac{\sec x}{x} = \lim_{x\to(\pi/2)^+} \frac{1}{x\cos x} = \boxed{-\infty}$

Explanation: As x get closer to $\frac{\pi}{2}$ from above, I can use the graph of $\cos x$ to see that $\cos x$ will approach 0 but be negative.



So $x\cdot\cos x$ will approach 0 and be negative.

So the quotient: $\frac{1}{x\cos x}$, will approach $-\infty$.

Principle: When considering the limit of a quotient $\left(\frac{h_1(x)}{h_2(x)}\right)$, if the numerator approaches a fixed nonzero constant and the denominator approaches zero (maybe +'ve, maybe -'ve) then the limit of the quotient will be unbounded. (ie heading toward $+\infty$ or $-\infty$. direction matters)