Final Review – Chapter 3

1. Given $f(x) = 3x - x^2$, find $f'(x)$ *using the definition of the derivative.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h) - (x+h)^2] - (3x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h} = \lim_{h \to 0} \frac{3h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} 3 - 2x - h = \underline{3 - 2x = f'(x)}$$

2. Find dy/dx for each of the following.

(a) $y^2 e^x + 3 = x^2 + y$

$$2y\,y'\,e^x + y^2 e^x = 2x + y'$$

$$2yy'e^x - y' = 2x - y^2 e^x$$

$$y' \cdot (2ye^x - 1) = 2x - y^2 e^x$$

$$y' = \frac{2x - y^2 e^x}{2ye^x - 1}$$

(b) $y = (\sin(x))^x$

$$\ln y = x \ln(\sin(x))$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln(\sin(x)) + x \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = y\left[\ln(\sin(x)) + x\cot(x)\right]$$

$$= \left[(\sin x)^x\right]\left[\ln(\sin x) + x\cot x\right]$$

3. Assume that the number of yeast cells in a laboratory culture is modeled by the function

$$n(t) = \frac{1000}{1+4e^{-t}}, \quad = 1000\left(1+4e^{-t}\right)^{-1}$$

where t is measured in hours.

(a) Find the average rate of change of the population in the first hour. You do not have to simplify your answer but you do have to give units.

$$\text{avg r.o.c} = \frac{n(1)-n(0)}{1-0} = \frac{\frac{1000}{1+4/e} - \frac{1000}{1+4}}{1} = 1000\left[\frac{1}{1+4/e} - \frac{1}{5}\right] \text{ cells/hr}$$

(b) Find $n'(t)$.

$$n'(t) = -1000\left(1+4e^{-t}\right)^{-2}\left(-4e^{-t}\right) = \frac{4000}{e^{t}\left(1+4e^{-t}\right)^{2}}$$

(c) Find $n'(1)$ and interpret it in the context of the problem. (You can use your calculator to get a decimal approx.)

$$n'(1) = \frac{4000}{e\left(1+4/e\right)^{2}} \text{ cells/hr.} \approx 241 \text{ cells/hr}$$

At hour 1 (when $t=1$) the number of cells increasing at a rate of cells each hour.

(d) Find and interpret $\lim_{t\to\infty} n(t)$.

$$\lim_{t\to\infty} \frac{1000}{1+4e^{-t}} = \frac{1000}{1+0} = 1000 \quad (\text{b/c as } t\to\infty \;\; e^{-t}\to 0)$$

In the longterm, the population of yeast cells approaches 1000 cells.

(e) Find and interpret $\lim_{t\to\infty} n'(t)$.

$$\lim_{t\to\infty} \frac{4000}{e^{t}\left(1+4e^{-t}\right)^{2}} = 0 \quad \text{b/c as } t\to\infty,\; 1+4e^{-t}\to 1 \text{ and } e^{t}\to\infty.$$

In the longterm, the rate of change of the population approaches 0. That is, the population remains steady.