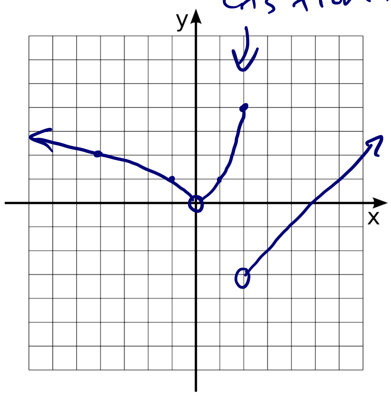
Final Review - Chapter 2
(Limits, + Continuity + L'Hospital's Rule)

Example 1: Sketch the graph of $f(x) = \begin{cases} \sqrt{-x}, \text{ if } x < 0 \\ x^2 \text{ if } 0 < x \leq 2 \\ x - 5, \text{ if } x > 2 \end{cases}$ and give the interval on which f is continuous. At what numbers is f continuous from the right, left or neither?

$(-\infty, 0) \cup (0,2) \cup (2,\infty)$

cts from left!



a) $\lim_{x \to 0^-} f(x) = 0$

b) $\lim_{x \to 0^+} f(x) = 0$

c) $\lim_{x \to 0} f(x) = 0$

d) $\lim_{x \to 2^-} f(x) = 4$

e) $\lim_{x \to 2^+} f(x) = -3$

f) $\lim_{x \to 2} f(x) = \text{DNE}$

- Find limits using factoring, algebra, conjugates.

Example 2: Find the following limits:

a) $\lim_{x \to -1^-} f(x)$ for $f(x) = \begin{cases} x^2 - 1 & \text{for } x < 1 \\ 2x + 3 & \text{for } x \geq 1 \end{cases}$

$\lim_{x \to -1^-} f(x) = f(-1) = (-1)^2 - 1 = \boxed{0}$

Example 3: Find the following limits:

a) $\lim_{x \to 1} e^{x-1} \sin\left(\frac{\pi x}{2}\right)$

$= e^{1-1} \sin\left(\frac{\pi}{2}\right)$

$= e^0 \cdot 1 = \boxed{1}$

b) $\lim_{x \to 0} \frac{5x^2}{1 - \cos x}$

$\frac{0}{0}$

$\overset{L.H.}{=} \lim_{x \to 0} \frac{10x}{\sin x}$

$\overset{L.H}{=} \lim_{x \to 0} \frac{10}{\cos x} = \boxed{10}$

Example 4: Find the following limits:

a) $\lim_{x\to 3} \frac{2x^2-18}{x^2+x-12}$ $\frac{0}{0}$

$= \lim_{x\to 3} \frac{2(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+4)}$

$= \frac{2(3+3)}{3+4} = \boxed{\frac{12}{7}}$

b) $\lim_{h\to 0} \frac{(4+h)^3-64}{h}$

$= \lim_{h\to 0} \frac{\cancel{64}+16h+4h^2+h^3-\cancel{64}}{h}$

$= \lim_{h\to 0} 16+4h+h^2 = 16+0+0$

$\boxed{=16}$

Example 5: Find the following limits:

a) $\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} \cdot \frac{4}{4}$

$= \lim_{x\to -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x\to -4} \frac{1}{4x} = \boxed{-1/16}$

- Find infinite limits. As in the limit is equal to plus or minus infinity or has an infinite discontinuity.
- Find limits at infinity. This means x goes to plus or minus infinity.

Example 6: Find the following limits:

a) $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$ $\approx \frac{e^5}{-\text{"}0\text{"}}$

$-\infty$

b) $\lim_{x\to \pi^-} \cot x$ $= \lim_{x\to \pi^-} \frac{\cos x}{\sin x}$ $\frac{-1}{0^+}$

$= \boxed{-\infty}$

Example 7: Find the following limits.

a) $\lim_{x\to\infty} \frac{4x^4+5}{(x^2-2)(2x^2-1)}$ $\frac{\cdot 1/x^4}{\cdot 1/x^4}$

$= \lim_{x\to\infty} \frac{4+5/x^4}{(1-2/x^2)(2-1/x^2)}$

$= \frac{4}{1\cdot 2} = \boxed{2}$

b) $\lim_{x\to-\infty} \frac{\sqrt{9x^6-x}}{x^3+1}$

$= \lim_{x\to\infty} \frac{\sqrt{9x^6+x}}{-x^3+1} \frac{\cdot 1/x^3}{\cdot 1/x^3}$

$= \lim_{x\to\infty} \frac{\sqrt{9+x/x^6}}{-1+1/x^3} = \frac{\sqrt{9}}{-1}$

$\boxed{=-3}$

Example 9: Find the following limits.

a) $\lim_{x\to\infty} \sec\left(\frac{x^2}{x^3-2}\right)$ $\frac{\cdot 1/x^2}{\cdot 1/x^2}$

$= \lim_{x\to\infty} \sec\left(\frac{1}{x-2/x^2}\right)$

$= \sec 0 = \boxed{1}$

b) $\lim_{x\to 0^+} \arctan(1/x)$

$= \pi/2$

$\left(\lim_{x\to\infty} \arctan x = \pi/2\right)$

Example 10: Find the following limits using l'Hospital's rule. I won't tell you explicitly to do this on the exam. You will have to know when you can/ cannot apply this rule.

a) $\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$ $\frac{-\infty}{\infty}$

$\overset{L.H.}{=} \lim_{x\to\infty} \frac{-e^x}{2e^x} = \boxed{-2}$

b) $\lim_{h\to 0} \frac{\sin h}{h\cos h}$ $\overset{L.H.}{=} \lim_{h\to 0} \frac{\cos h}{\cos h - h\sin h}$

$= \frac{1}{1-0} = \boxed{1}$

- Know and apply the defintion of continuity.
- Determine where a function is discontinuous and why. continuous.

Definition of Continuity A function f is continuous at c if the following three conditions are met:

1. $x=c$ is in domain of f ($f(c)$ exists)
2. $\lim_{x \to c} f(x)$ exists
3. $f(c) = \lim_{x \to c} f(x)$

Example 11: Find all points of discontinuity of $h(x) = \dfrac{x-4}{x^2-x-12}$ and explain why the points are discontinuous and state if they are removable or non-removable.

$h(x) = \dfrac{x-4}{(x-4)(x+3)}$

discontinuous @ $x = -3, 4$ ($\div 0$)

$x=4$ is removable

$x=-3$ is <u>not</u> removable

Final Review - Chapter 3
(Derivative rules)

- Find derivatives using the limit defintion.
- Know how to apply the sum, difference, product, quotient, and chain rules.
- Know when to use logarithmic differentiation to find a derivative.

Example 1: Find the derivative of $f(x) = 9 + x - 2x^2$ using the definition of the derivative. Then find an equation of the tangent line at the point $(2, 3)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{9 + (x+h) - 2(x+h)^2 - 9 - x + 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{9} + \cancel{x} + h - \cancel{2x^2} - 4xh - 2h^2 - \cancel{9} - \cancel{x} + \cancel{2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{h - 4xh - 2h^2}{h} = \lim_{h \to 0} 1 - 4x - 2h = \underline{1 - 4x}$$

$m = 1 - 4(2) = -7$

$$\boxed{y - 3 = -7(x-2)}$$

Example 2: Calculate y'.

a) $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$

$= x^{-1/2} - x^{-3/5}$

$y' = -\frac{1}{2}x^{-3/2} + \frac{3}{5}x^{-8/5}$

b) $y = \dfrac{\tan x}{1 + \cos x}$

$y' = \dfrac{(1+\cos x)\sec^2 x - \tan x(-\sin x)}{(1+\cos x)^2}$

Example 3: Calculate y'.

a) $y = x \cos^{-1} x$

$y' = \cos^{-1} x - \dfrac{x}{\sqrt{1-x^2}}$

b) $y = (\arcsin(2x))^2$

$y' = 2\arcsin(2x) \cdot \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2$

$= \dfrac{4\arcsin(2x)}{\sqrt{1-4x^2}}$

Example 4: Calculate y'.

a) $y = e^{x \sec x}$

$y' = e^{x\sec x} \cdot (\sec x + x \sec x \tan x)$

b) $y = 10^{\tan(\pi\theta)}$

$y' = 10^{\tan(\pi\theta)} \cdot \ln 10 \cdot \sec^2(\pi\theta) \cdot \pi$

Example 6: Find $\frac{dy}{dx}$.

a) $y = \arcsin(e^{2x})$

$y' = \dfrac{1}{\sqrt{1-(e^{2x})^2}} \cdot e^{2x} \cdot 2$

$= \dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

b) $y = \displaystyle\int_{x^2}^{3} \frac{t+4}{\cos t}\,dt$

$y = -\displaystyle\int_3^{x^2} \frac{t+4}{\cos t}\,dt$

$\boxed{y' = -\dfrac{x^2+4}{\cos(x^2)} \cdot 2x}$

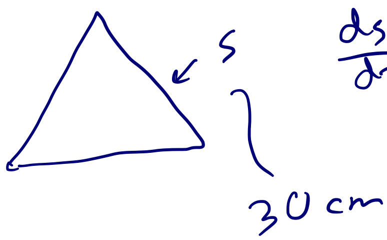
Example 7: Find the derivative of $h(x) = \ln\left(\frac{x^2-4}{2x+5}\right)$ $= \ln(x^2-4) - \ln(2x+5)$

$$h'(x) = \frac{1}{x^2-4} \cdot 2x - \frac{1}{2x+5} \cdot 2$$

$$= \frac{2x}{x^2-4} - \frac{2}{2x+5}$$

- Solve related rates problems.

Example 11: The sides of an equilateral triangle are increasing at a rate of 10 cm/min. At what rate is the area of the triangle increasing when the sides are 30 cm long? ($A = \frac{\sqrt{3}}{4}(\text{side})^2$)



$\frac{ds}{dt} = 10$ cm/min

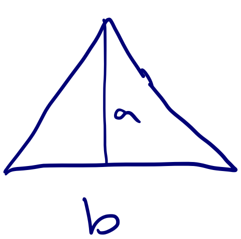
$$A = \frac{\sqrt{3}}{4} s^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} s \cdot \frac{ds}{dt}$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}(30)(10) = \boxed{150\sqrt{3} \text{ cm}^2/\text{min}}$$

Example 12: The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

$\frac{db}{dt}$?



$$A = \frac{1}{2}ab$$

$$\frac{dA}{dt} = \frac{1}{2}\left(a\frac{db}{dt} + b\frac{da}{dt}\right)$$

$$2 = \frac{1}{2}\left(10 \cdot \frac{db}{dt} + 20(1)\right)$$

$\frac{da}{dt} = 1$ cm/min

$\frac{dA}{dt} = 2$ cm^2/min

$\Rightarrow \boxed{\frac{db}{dt} = -1.6 \text{ cm/min}}$

decreasing!

$a = 10$

$A = 100 \Rightarrow 100 = \frac{1}{2}(10)(b) \Rightarrow b = 20$