Observations of Dihedral Group Actions

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I. Introduction

There are many connections between math and music. Theorists such as David Lewin, Richard Cohn, and their contemporaries have exemplified a variety of opportunities to look at post-tonal and neo-Riemannian theory in a mathematical context. Additionally, mathematicians as early as Milton Babbitt and more currently, Alissa Crans, Thomas Fiore, and Ramon Satyendra have also approached ways to explain music and math connections. With the help of these phenomenal minds, I will give my observations and musical synthesis of Crans et al., a more mathematically based paper, and propose the research I plan to conduct in the future.

With the help of my mathematics professor, Dr. McFaddin, I am working through *Abstract Algebra* by David Dummit and Richard Foote, learning the basics of group theory to obtain a better grasp on the topics presented in both music theory and mathematical papers. My approach will be to present the information in a more pedagogical way, outlining the information with examples and offering a more thorough explanation of the concepts.

II. Preliminaries

The main focus of Crans et al., are the actions of the dihedral group of order 24 on the set of major and minor triads. In this section, I define the notion of a group, the example of dihedral groups, group actions, and a few other topics which will be useful later in this manuscript. Additional definitions will be presented within subsequent sections.

Definition. A *group* is an ordered pair, $(G,*)$ where G is a set and $*$ is a binary operation on G satisfying the following axioms:

(i) $(a * b) * c = a * (b * c)$, for all $a, b, c \in G$, i.e. is *associative,*
(ii) there exists an element e in G called an *identity* of G, such that for all $a \in G$ we have $a * e = e * a = a$,
(iii) for each $a \in G$ there is an element a^{-1} of G, called an *inverse* of a such that $a * a^{-1} = a^{-1} * a = e$ **[1, pp. 16-17]**.

Examples
(1) Z_{12} under addition. This group is used throughout post-tonal theory because there are 12 pitch-classes or integers, {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, in the twelve-tone system. We use modular arithmetic to stay within the confinements of these twelve integers. For instance, 13 $(mod$ 12) is equal to 1 $(mod$ 12) **[1, 17]**.

(2) Let D_n be the set of *symmetries* of a regular n-gon, where a symmetry is any rigid motion of the n-gon. Such a symmetry can be realized by taking a copy of the n-gon, moving this copy in n-space and placing the copy back on the original n-gon, so it exactly covers it. **[1, 23]**
Its order is given by:

$$|D_n| = 2n$$

The dihedral group, D_n, can be presented by two generators, $< s, t >$ subject to the relations

$$D_n =< s, t \mid s^n = t^2 = 1, st = ts^{-1} >$$

These generators will appear again later in **Section IV.**

In terms of rigid motions of an n-gon, s denotes rotation of the n-gon whereas t denotes the reflections about the lines connected to the vertices of the n-gon. The generators can be related to *transpositions and inversions*, which will also be discussed in **Section IV [1, 24-26]**.

(3) The Symmetric group, S_n. Recall that a *function* $f: A \to B$ from a set A to a set B is a rule associating any element $a \in A$ with a unique element $b \in B$. A function is *injective* if $f(a_1) = f(a_2)$ implies $a_1 = a_2$. It is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. It is *bijective* if it is both injective and surjective **[1, 2]**. Let A be any nonempty set. The set of all *bijections* from A to itself forms a group $Sym(A)$ with the group operation given by function composition. If $A = \{1, \dots, n\}$, we write S_n for $Sym(A)$ **[1, 29]**.

In addition to functions on sets, *group homomorphisms* are the functions of primary interest in group theory and are important for musical applications.

Definition. Let $(G,*)$ and $(H, \bullet)$ be groups. A map $\varphi: G \to H$ satisfying

$$\varphi(x * y) = \varphi(x) \bullet \varphi(y) \text{ for all } x, y \in G$$

is called a *homomorphism* **[1, 36]**. The $*$ and $\bullet$ denote group operations which may be taken to addition, multiplication, etc. The important component of this definition is the left-side operation, $*$, is computed in G and the right-side operation, $\bullet$, is computed in H.

Definition. Given the groups G and H, a homomorphism $\varphi: G \to H$ is an *isomorphism* or G and H are said to be *isomorphic*, written $G \cong H$, if
(1) φ is a homomorphism (i.e. $\varphi(x * y) = \varphi(x) \bullet \varphi(y)$), and
(2) φ is a bijection **[1, 37]**.

We now come to the definition of a group action:

Definition. A group action is a map $G \; x \; A \to A$ given by $(g, a) \to g \cdot a$, for $g \in G, a \in A$ which satisfies the following properties:

(i) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, for all $g_1 g_2 \in G$, $a \in A$, and
(ii) $1 \cdot a = a$, for all $a \in A$ **[1, 41]**.

An action will take an element in A and map it to another element in A (i.e. the action is *permutation* of A). The focus of this paper is on particularly nice actions, specifically actions that are *faithful*. An action is faithful if G acts on a set A and distinct elements of G create *distinct* permutations of A **[1, 43]**.

Knowing and understanding these definitions will be useful for the rest of the content in this paper. For more information about these terms and concepts, there are a variety of resources both online and in print including the *Abstract Algebra* textbook by Dummit and Foote.

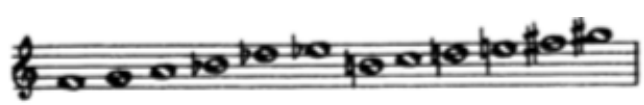
III. Rotations in Ernst Krenek's music

When defining pitch-classes as integers, our space of pitch-classes is given by Z_{12} (viewed as a set). The symmetric group S_{12} acts on Z_{12}. To understand this action and the arithmetic of S_{12}, we consider the more general case of S_n. The group operation in the symmetric group can be understood by using cycle notation. For instance (1 2 3) is a three-cycle which takes 1→2, 2→3, and 3→1. On the other hand, (1 2) is a two-cycle taking 1→2 and 2→1. We can easily multiply or compose these bijections using this notation. When given a set which does not consist of numbers, for example the set S of major and minor triads (used in Crans et al.), one can still define the group $Sym(S)$ from the set, as above. The group $Sym(S)$ is made up of bijections between the elements of S.

Because the objects inside of an element of S_n can be arbitrary, this is precisely why we can relate group actions to major and minor triads. By taking the dihedral group of order 24 and acting on the set S, we can produce numerous mappings mathematically and musically. Here we focus on cycle notation and how the specific objects within an element of S_6 map from one object to another in the musical example provided by Ernst Krenek.

Krenek focuses on hexachordal structures in creating a twelve-tone series for his choral work, *Lamentatio Jeremiae Prophetae.* Presented in his first example, Krenek depicts a twelve-tone row, clearly divided into two sets of six due to the 6-note ascension to Eb, dropping a tritone down to B-natural and continuing another 6-note ascension. After **Figure 1**, Krenek notes that the modifications or "rotations" are made by "making the first tone the last." Numbering the row 1-12, we can note the change by the placement of the "first tone." For instance, the first rotation moves 1, F, to the original placement of 6, Eb. Therefore, our original hexachord, 1 2 3 4 5 6 is now 2 3 4 5 6 1. Krenek continues the example until 6 is placed in the first position of the row. The same method is used in the second hexachordal set, which can be labeled as 7 8 9 10 11 12 **[2, 212]**.

Ex. 1



1 2 3 4 5 6 7 8 9 10 11 12

Figure 1. Prime form of musical line.

Figure 2. Permutations of Example 1.

After hearing a lecture by R. P. Blackmur, Krenek developed an interest in sestina form and medieval poetry. He composed his own *Sestina* with his own text with a similar model to twelfth-century troubador, Arnaut Daniel **[3, 39]**. Of course, Krenek's poem was also serialised to fit the esthetics of the time, utilizing another form of rotation. The rule for this specific rotation is "switching the position of every two keywords equidistant from the center of the series, proceeding from the end toward the middle." For example, beginning with the words (shown in numbers) 1 2 3 4 5 6, the second order would be 6 1 5 2 4 3. **[2, 223]**.

Figure 3. *Sestina* prime form and first iteration.
(Note the numbers from Krenek's Examples 10 and 11 are intervallic counts and are *not* the count of the pitches.)

To find our initial permutation, or σ, we will see where each note/integer in Example 10 maps. $1\to6$, $2\to1$, $3\to5$, $4\to2$, $5\to4$, $6\to3$. In cycle notation, this is given by (1 6 3 5 4 2).

Let $\sigma = (1\ 6\ 3\ 5\ 4\ 2) \in S_6$. Because S_6 is order 6!, there are numerous lengths and possibilities to choose from. In this case, σ is a six-cycle, which is the longest possible length of a cycle in S_6. Because σ is a six-cycle, it has order six **[1, 32]**. When σ is iterated six times, it produces the identity, or it returns back to its original form. To move from one permutation to the next as Krenek does visually on the staff, simply multiply σ by itself resulting in σ^2. Upon multiplication, σ^2 produces two three-cycles:

$$\sigma^2 = (1\ 6\ 3\ 5\ 4\ 2)(1\ 6\ 3\ 5\ 4\ 2)$$
$$= (1\ 3\ 4)(2\ 6\ 5).$$

To multiply two cycles together, start with the right-most cycle, and follow where 1 maps. Then follow where the second integer maps in the next cycle to the left. Repeat the process until you return to your original integer, 1. For instance, using the cycles above, we have 1→6, 6→3, producing (1 3 . We then determine where 3 maps, and we have 3→5, 5→4, producing (1 3 4 . We then proceed to determine where 4 is mapped, and we produce 4→2, 2→1, concluding this cycle, (1 3 4). Repeat this process until all objects have been used.

To produce the third iteration of σ, multiply σ^2 with the original six-cycle. Continue until σ^6 which reproduces σ the identity bijection. Another way to analyze this permutation is by group actions. σ which is an element of S_6, acts on the set A, as given in Krenek's musical example.

$$\varphi\colon S_6\ x\ A \to A$$
$$\sigma \longmapsto \sigma \cdot a \text{ for } \sigma \in S_6,\ a \in A$$

Operating purely on music and the beauty of its sound, we can be extremely limited. However, with the tools of mathematics, we can note that within the group S_6, there are 6! other sets at our disposal. Suppose we chose a three-cycle in S_6; then we could manipulate the three pitches, a triad if considered musically, to create new mappings and produce different permutations.

(1 2 3)(1 2 3) ⟹1→2, 2→3, 3→1 produces (1 3. 2→3, 3→1 completes the cycle, (1 3 2).
= (1 3 2).

Again, because there are 6! elements within S_6, the possibilities abound, allowing both mathematicians and musicians alike the freedom of choice and variability in their cycle notation!

IV. Observations from "Musical Actions of Dihedral Groups"

Using dihedral groups, group actions, and the music theory described above, one may follow my observations of the article written by Alissa Crans, Thomas Fiore, and Ramon Satyendra (Crans et al.). Their focus is on how the dihedral group of order 24 (the group containing the twelve pitch-classes or Z_{12}) acts on the set of major and minor triads. There are two (a priori) subgroups of the dihedral group order 24 (D_{12}) which are of interest to both Crans et al. and musicians alike: the groups referred to as the T_n/I_n-group and the PLR-group. Note that these subgroups are shown to be isomorphic to D_{12}. For the purpose of these observations, we will use 0 to represent pitch-class C for all examples.

Shown below, the group D_{12} is given by the rigid symmetries of a 12-gon which is generated by s and t, satisfying the equations

$$s^{12} = 1, t^2 = 1, tst = s^{-1}$$ **[7, 1]**.

Post-Tonal theory generally uses a clock-face to represent the twelve-pitch classes, but in reality, it is a 12-gon.

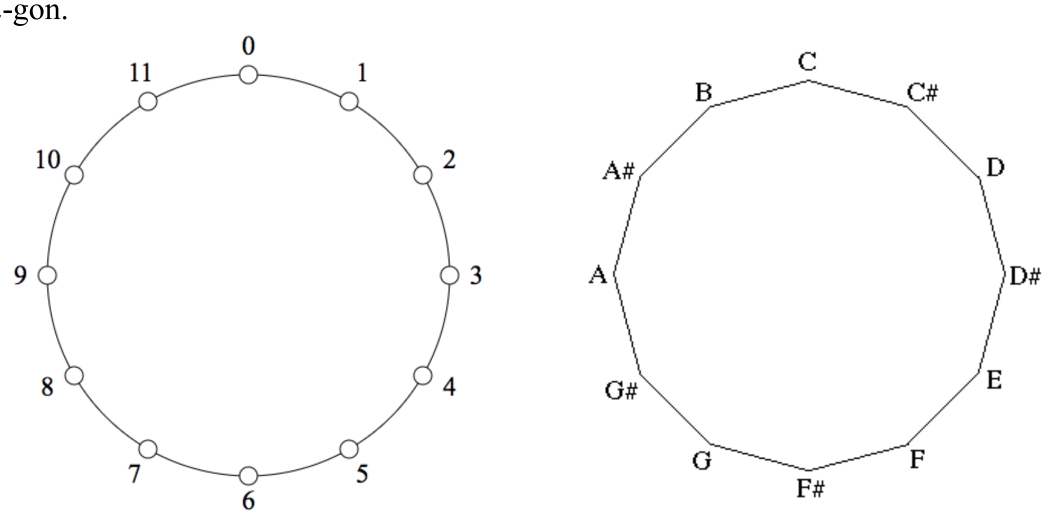


Figure 4: Clock-face versus 12-gon[1]

T_n/I_n-Group

The first group, T_n/I_n involves transposition and inversion.

Definition. Transposition refers to rotating about the clock-face while maintaining the structure or the intervallic relations of the particular object. For example, transposing the C major triad, {C, E, G} = {0, 4, 7} by one-semitone, i.e. by T_1, will produce a triad one-semitone clockwise rotation from {C, E, G}. See **Figure 5** for a visual representation.

$$T_1 \{0, 4, 7\} = \{1, 5, 8\}$$

In general, $\mathbf{T_n (x) = y}$ where n is the number of semitones, x is the original starting pitch or set, and y is the destination pitch or set. To calculate this, simply use addition: $n + x = y$.
To find n, compute $y - x$. **[6, 53]**.

[1] The left example is generated by **[4]** and right example is from **[5]**.

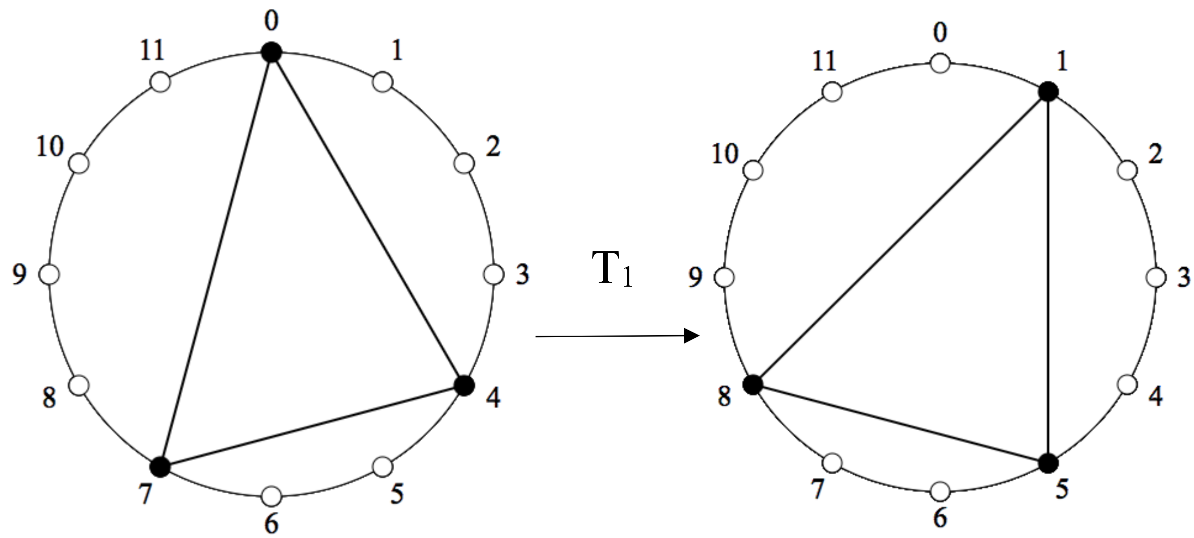


Figure 5: $T_1\{0, 4, 7\} = \{1, 5, 8\}$[2]

Definition. Inversion reflects an object or a group of objects over a fixed axis. An axis is produced by one of the tritone relationships (for example 0-6) within Z_{12}. There are a few ways to calculate inversions.

First, if $\mathbf{I_n(x) = y}$**, then** $\boldsymbol{y = n - x}$ **[6, 53]**. Secondly, another calculation of inversion is I_y^x. To determine n, compute $x + y$ **[6, 58]**. This method shows the map of x to y. Each x will map to a specific y, which relates back to the earlier definition of injectivity. Because we can either solve for y or solve for n, these operations are *simply transitive*.

As alluded to above, the group D_{12} acts on the set of pitch-classes in Z_{12}. A more systematic approach to the group T_n/I_n is graphically representing their mappings:

$$T_n: Z_{12} \rightarrow Z_{12}$$
$$T_n(x): x + n \ (mod\ 12)$$

$$I_n: Z_{12} \rightarrow Z_{12}$$
$$I_n(x): -x + n \ (mod\ 12)$$

Other modular groups are used throughout music. For instance, Crans et al. gives the example of Bach working in *modulo* 7 space or Z_7 by focusing on diatonic scales. The mappings above are still applicable but Z_{12} would become Z_7 for both T_n and I_n **[7, 482]**.

For the T_n/I_n-group to be transitive, for any consonant triads Y and Z there is a unique element g of the T_n/I_n-group such that $gY = Z$ **[7, 483]**.

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[2] Generated by **[4]**.

To show this, Crans et al. created a table for the mapping I_0 on every single major triad. For this paper, we can look at it more generally and provide one example for both T_n and I_n.

$$T_n \{0, 4, 7\} = \{T_n(0), T_n(4), T_n(7)\} \tag{1}$$

$$I_n \{0, 4, 7\} = \{I_n(0), I_n(4), I_n(7)\} \tag{2}$$

PLR-Group

The second group which acts on the set of major or minor triads is the PLR-group. Hugo Riemann, a 19th-century music theorist, developed the PLR-group method. Recently, when David Lewin wrote about transformational theory in his 1982 essay, “A Formal Theory of Generalized Tonal Functions,” he proposed the use of Riemannian theory **[8, 170]**. From there, Neo-Riemannian Theory has been utilized in the analysis of composers such as Wagner and Liszt and has offered a “response to analytical problems posed by chromatic music that is triadic but not altogether tonally unified” **[8, 167]**.

Joseph Straus defines each of the transformations of the PLR group in his book, *Introduction to Post-Tonal Theory*. There are three additional transformations, P’, L’, and R’ which are associated with Neo-Riemannian Theory, but these will be omitted because they are not found in the Crans et al. article. Each of these transformations involve “common-tone-preserving contextual inversion and voice leading parsimony” **[6, 188]**. P, L, and R all preserve two common tones in their transformations, creating efficient voice-leading parsimony, or a smooth connection between the triads with the smallest amount of movement possible **[6, 188]**.

The operation P, which stands for *parallel*, moves a major triad to a minor triad with the same root (or vice versa). The contextual inversion is “around the shared perfect fifth.” For each of these examples, **+** denotes positive parity or major whereas **-** denotes negative parity or minor **[6, 188]**.

For instance, C+ → C-

The transformation L, which stands for *leading-tone exchange*, redefines the third of a major triad to be the root of the minor triad **[6, 188]**. Moving from major triad to minor triad, the root moves up a major third whereas moving from minor to major the root moves down a major third. The contextual inversion here is an inversion “around the shared minor third.”

Example: C+ → E-

The final transformation, R, stands for *relative*. “The root of a major triad becomes the third of a minor triad” **[6, 188]**. Another way to consider this movement is the major triad shifts to its relative minor, three semitones lower than the root. This contextual inversion inverts “around the shared major third” **[6, 188]**.

Example: C+ → A-

Each of these operations can be represented geometrically on a *Tonnetz* graph or "tone network" depicted by **Figure 6 [7, 487]**. **Figure 7** represents each PLR transformation on C+. Below, the transformations will be represented in function notation.

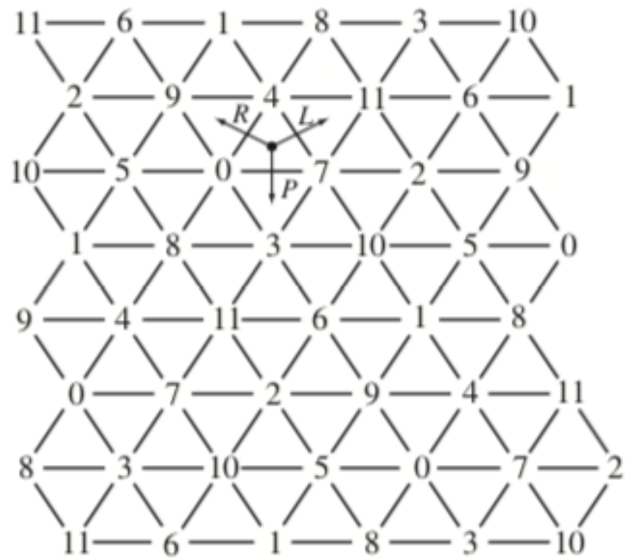

Figure 6. Tonnetz Graph, representing PLR transformations.[3]

[3] From **[7]**.

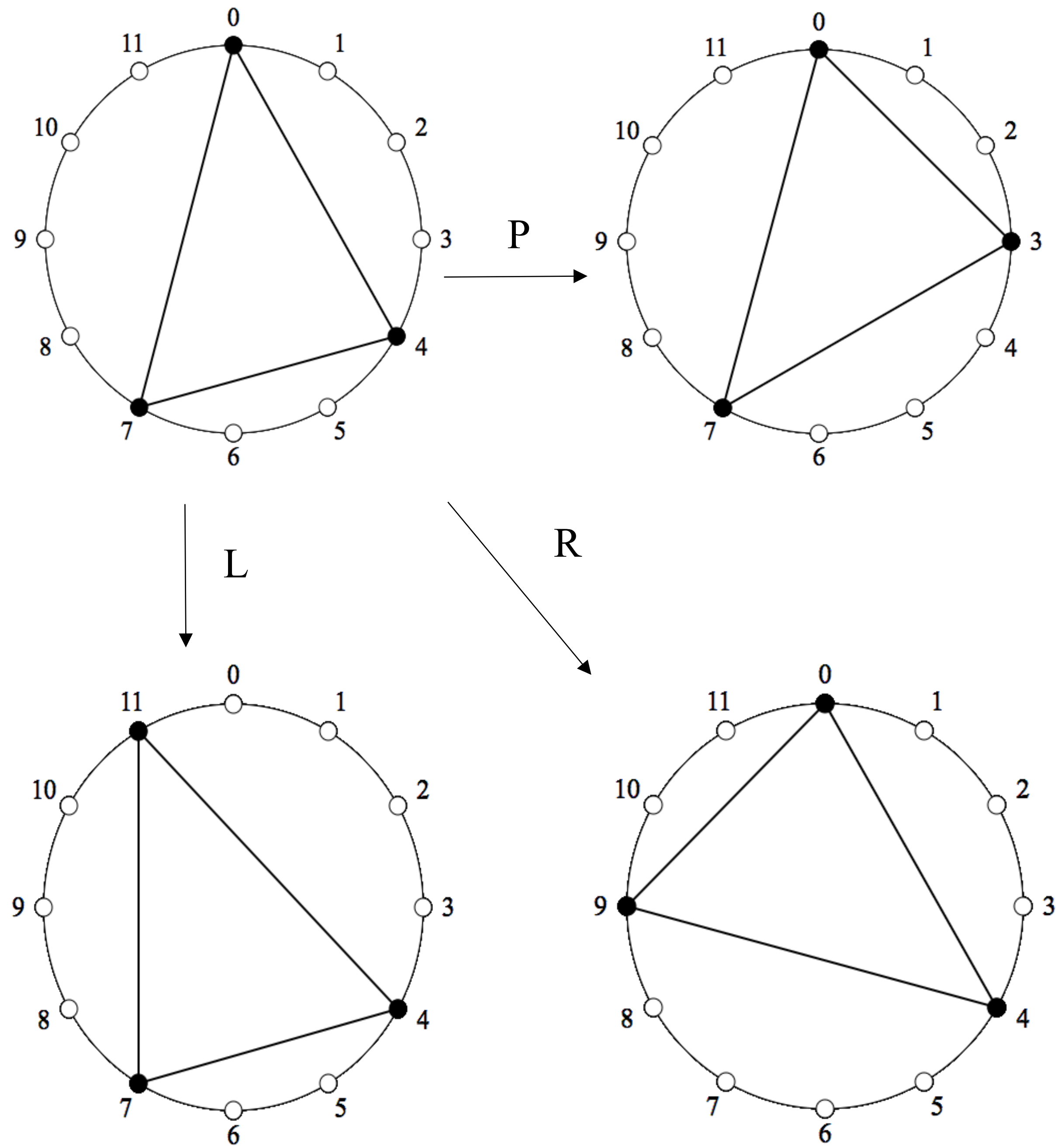

Figure 7. PLR-group transformations on {0, 4, 7}.[4]

[4] Generated by **[4]**.

Crans et al. defines the elements of the PLR-group explicitly as a function:

Consider the three functions P, L, R: $S \rightarrow S$ defined by

$$P\,\{y_1, y_2, y_3\} = I_{y_1+y_3}\,\{y_1, y_2, y_3\} \tag{3}$$
$$L\,\{y_1, y_2, y_3\} = I_{y_2+y_3}\,\{y_1, y_2, y_3\} \tag{4}$$
$$R\,\{y_1, y_2, y_3\} = I_{y_1+y_2}\,\{y_1, y_2, y_3\} \tag{5}$$

These functions are "*contextual inversions"* because "the axis of inversion" is dependent upon its input triad. These differ from I_n, however, because the axis of inversion is "independent of the input" **[7, 485]**.

Function	Axis of Inversion Spanned by
P	$\frac{y_1+y_3}{2}, \frac{y_1+y_3}{2}+6$
L	$\frac{y_2+y_3}{2}, \frac{y_2+y_3}{2}+6$
R	$\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}+6$

Figure 8. Axis of Inversion for PLR-Group **[7]**.
(The axis of inversion for I_n is **n/2 – n/2 + 6**.)

For example, let us map the C-major triad, {0, 4, 7}, through each of the PLR transformations.

$$P\,\{0, 4, 7\} = I_7\,\{0, 4, 7\} = \{7, 3, 0\} = \{0, 3, 7\}$$
$$C+ \rightarrow C-$$
$$L\,\{0, 4, 7\} = I_{11}\,\{0, 4, 7\} = \{11, 7, 4\} = \{4, 7, 11\}$$
$$C+ \rightarrow E-$$
$$R\,\{0, 4, 7\} = I_4\,\{0, 4, 7\} = \{4, 0, 9\} = \{9, 0, 4\}$$
$$C+ \rightarrow A-$$

The visual representations of the axes of inversion for each PLR function can be found in **[7]**.

Centralizer of PLR-Group and T_n/I_n-Group.

An action of a group G on a set S will produce the same data as a homomorphism, i.e.

$$G \rightarrow Sym(S)$$

If this action is *faithful*, this homomorphism is injective. Thus, we can view $G \subseteq Sym(S)$. When relating this to D_{12}, the two actions of D_{12} on triads give two different embeddings or injective maps of D_{12} into $Sym(S)$, producing two subgroups of $Sym(S)$. It so happens that these two subgroups are intimately related to one another.

More precisely, since the PLR-group and T_n/I_n-group act differently on the set of major and minor triads, they sit differently in S_{24}. i.e., we have two *distinct* injective homomorphisms:

$$\varphi_{PLR}: D_{12} \to S_{24}$$

$$\varphi_{T_n/I_n}: D_{12} \to S_{24}$$

We will denote the image of φ_{PLR} as H_1 and φ_{T_n/I_n}as H_2. Both of these groups are *isomorphic* to D_{12} but sit inside S_{24} in different ways. Naturally, the question arises: how are they related? Crans et al. noticed a relationship between them using the notion of *centralizers* **[7]**:

Let A be a nonempty subset of G.

Definition. Define $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. This subset of G is called the *centralizer* of A in G. Since $gag^{-1} = a$ if and only if $ga = ag$, $C_G(A)$ is the set of elements of G which commute with every element of A **[1, 49]**. We can use the centralizer in relating our two groups: $C_{S_{24}}(H_1) = H_2$ and $C_{S_{24}}(H_2) = H_1$. To see this proof, see **[7]**.

V. Conclusion

My hope with this paper is to display my observations and understandings of both the music theory and mathematical realms. While these observations only cover a portion of Crans et al., it gives a basis of the paper's main topic: group actions. I hope to continue developing this manuscript and creating more connections between my two fields of interest. For example, the Tonnetz graph provides a connection to another field of mathematics, topology, which can be applied directly to music theory.

VI. Resources

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