Cross Sections and Reference Frames

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1 Introduction

The essence of computational physics is to apply the powerful numerical analysis techniques of computers to the data analysis and extraction of theoretical predictions from mathematically sophisticated models . This paper describes my attempt to utilize this exciting area of physics by asking a theoretical question, which occurred to me in the midst of taking a particle physics course this past semester, and subsequently asking if I could make a program which would make performing this calculation of the answer much more trivial.

1.1 Project Goal

For this project I wanted to address the following question: How does one transform the differential scattering cross section calculated in the center of mass frame to the lab frame? Given the answer to this question, I then set out to see if I could write a program which would perform the necessary computations to make such a transformation on any data set.

1.2 Motivation

In addressing this question, the models I must with are those of quantum field theory, in the context of particle physics, and special relativity. In the discipline of particle physics a main source of experimental confirmation for these models come from scattering processes in which, usually, 2 particle are collided with each other and the products of these collisions are analyzed and compare with the results of quantum field theory.

A fundamental quantity in particle physics which quantum field theory tells us how to calculate is the scattering cross section. It is a quantity which specifies the effective area, traverse to the colliding particles motion, they must enter in order for the particle to interact (scatter) via the forces which influence them. Abstractly it is then an effective interaction region within which, two particles of a particular type entering ensures their scattering.

A differential cross section is a somewhat more abstract quantity compared to a total cross section. It quantifies the angular dependence of a total scattering cross section and indicates how the effective interaction region varies with the direction from which one is attempting to detect scattered particles.

In all experiments, one is measuring all quantities in the lab frame which is usually taken to be at rest. However, in many instances a scattering cross section is most easily calculated in the center of mass reference in which the kinematics of the scattering events greatly simplify. Therein lies the main motivation for completing this project: if we measure everything in the lab frame, but calculate the theoretical prediction in the center of mass frame, how can we compare the experimental data to the theoretical prediction? The answer is we have to Lorentz transform the differential cross section from the center of mass reference frame to the lab reference frame where we can then compare experimental results with theoretical predictions.

2 Theory

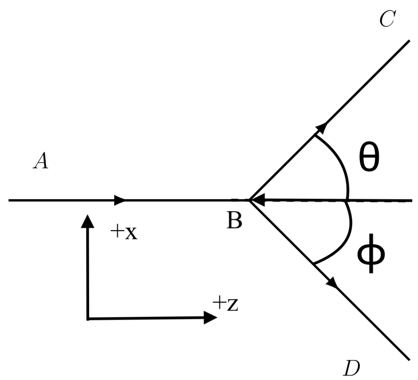

Figure 1: 2-body scattering process in the lab frame.

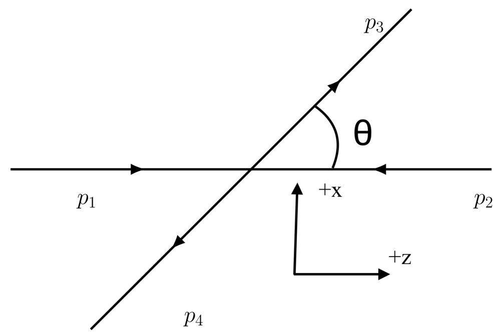

Figure 2: 2-body scattering process in the lCM frame.

Consider Figure 1 which depicts a typical 2-body scattering event in the lab frame. Our goal is to analyze a scattering system in which the initial trajectories of each particle (A and B) are collinear and thereby define an axis for the system. We can choose for this axis to be whichever direction we want and by convention we choose it to be the z-axis while taking the direction in the plane orthogonal to it to be the x-axis. The angle θ and ϕ represent the scattering angles of the product particles C and D, respectively. This is the frame in which a typical scattering events if measured.

In Figure 2 is depicted the same scattering event in the CM frame where each particle is labeled by its initial four momenta. By definition, the CM frame is the one in which the total initial momentum of the system is defined to be zero and as a result, by momentum conservation, is zero after as well. Therefore there is only one scatting angle defined as θ. We can now state our goal as follows: given a differential cross section calculated in the center of mass frame, how does it transform under a boost along the z-axis to the lab frame? Or, mathematically,

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{CM} \xrightarrow{\Lambda} \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{Lab} \tag{1}$$

We now define the CM and lab frame as

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{CM} := \frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'} \quad \text{and} \quad \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{Lab} := \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{2}$$

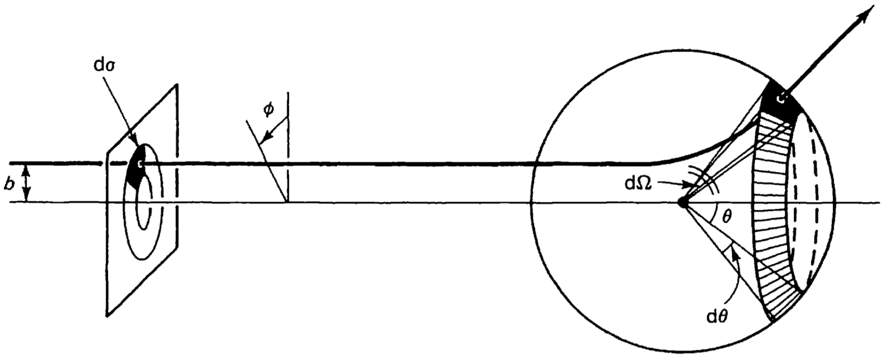

Figure 3: Differential scattering cross section and differential solid angle element as seen in [1] pg 201.

2.1 Derivation of Lorentz Transformed Differential Cross Section

To go about addressing this question we first need to ask which quantities transform under a Lorentz boost. For a general boost it is easy to see from Figure 3 that the total cross section σ would have to transform since it is composed of an area. However, there is one particular boost direction under which the cross section doesn't transform and this is when the boost is along the direction of initial propagation or, as we have defined it, the z-axis. We can see this is true based on the fact that the differential cross section (Figure 3) is an areal quantity composed of parts orthogonal to the direction of propagation and, as we know from Lorentz transformations, distances orthogonal to the boost direction to not get affected by the boost.

Angular quantities will also, in general, be affected by an arbitrary boost. In particular we can see that the scattering angle θ must change under a boost along the $\pm z$-direction because for any four vector $a = (a^0, \vec{a})$ making the angle θ with respect to the z-axis, θ is given by

$$\cos\theta = \frac{a_z}{a_x} \tag{3}$$

and when a boost is performed on a, the z-component will change causing the θ to change. Hence, $\cos\theta' \neq \cos\theta \rightarrow d\cos\theta' \neq d\cos\theta$. However, the angle ϕ any four vector $b = (b^0, \vec{b})$ makes in the xy-plane is given by

$$\tan\phi = \frac{b_y}{b_x} \tag{4}$$

which are both components of b orthogonal to the boost direction implying ϕ doesn't change under a boost along the z-axis. Thus, $d\phi = d\phi'$. Now recalling the formula for the differential solid angle given by

$$d\Omega = \sin\theta d\theta d\phi = d\cos\theta d\phi \tag{5}$$

we can see based on the transformation properties of θ and ϕ that the differential solid angles in each reference frame are related by

$$\frac{d\Omega'}{d\Omega} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'}\frac{\mathrm{d}\phi'}{\mathrm{d}\phi} = \frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta} \tag{6}$$

Now, using the chain rule, we can use this result to relate the differential cross sections in each reference frame as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'}\frac{\mathrm{d}\Omega'}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'}\frac{\mathrm{d}\cos\theta'}{\mathrm{d}\cos\theta} \tag{7}$$

Thus, if we can find out how to relate the CM frame quantity $\cos\theta'$ to the corresponding lab frame quantity $\cos\theta$, we will have our formula for transforming the differential cross section. We can find this relation by considering the kinematics of the 2-body scattering process in each reference frame and utilizing their relationship under a boost along the z-axis.

In the lab frame the four momentum of each particle takes the form

$$\begin{aligned} p_A^\mu &= (E_A, 0, 0, p_A) \\ p_B^\mu &= (E_B, 0, 0, -p_B) \\ p_C^\mu &= (E_C, p_C\sin\theta, 0, p_C\cos\theta) \\ p_D^\mu &= (E_D, -p_D\sin\phi, 0, p_D\sin\phi) \end{aligned} \tag{8}$$

and in the center of mass they take the form

$$\begin{aligned} p_A^{'\mu} &= (E'_A, 0, 0, p_A) \\ p_B^{'\mu} &= (E'_B, 0, 0, -p_A) \\ p_C^{'\mu} &= (E'_C, p_C\sin\theta', 0, p_C\cos\theta') \\ p_D^{'\mu} &= (E'_D, -p_D\sin\theta', 0, -p_D\sin\theta') \end{aligned} \tag{9}$$

Under an arbitrary boost along the $z-$axis from the CM frame to the lab frame given by

$$\Lambda(\beta, \hat{\boldsymbol{z}}) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \quad \gamma - \frac{1}{\sqrt{1-\beta^2}} \tag{10}$$

The x and z components of $p_C^{'\mu}$ are related to p_C^μ by

$$p'_{C,x} = \gamma(p_{C,x} - \beta E_C) \Rightarrow \mathbf{p'}_C\cos\theta' = \gamma(\mathbf{p}_C\cos\theta - \beta E_C) \tag{11}$$

and

$$p'_{C,y} = p_{C,y} \Rightarrow \mathbf{p'}_C \sin\theta' = \mathbf{p}_C \sin\theta \tag{12}$$

Dividing the result of (12) by that of (11) we obtain

$$\tan\theta' = \frac{\mathbf{p}_C \sin\theta}{\gamma(\mathbf{p}_C\cos\theta - \beta E_C)} = \frac{\sqrt{1-\cos^2\theta}}{\gamma(\cos\theta - \beta\frac{E_C}{\mathbf{p}_C})} = \frac{\sqrt{1-\cos^2\theta}}{\gamma(\cos\theta - \frac{\beta}{\beta_C})} \tag{13}$$

where we've used the result $\beta_C = \frac{|\mathbf{p}_C|}{E_C}$. Now we can use the following trig identity

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} \tag{14}$$

to obtain

$$\cos\theta' = \frac{1}{\sqrt{1+\left[\frac{\sqrt{1-\cos^2\theta}}{\gamma(\cos\theta-\frac{\beta}{\beta_C})}\right]^2}} = \frac{\gamma(\cos\theta - \frac{\beta}{\beta_C})}{\sqrt{\gamma^2(\cos\theta-\frac{\beta}{\beta_C})^2 + 1 - \cos^2\theta}} \tag{15}$$

Relabeing $\frac{\beta}{\beta_C} := \beta_0$ we differentiate (15) with respect to $\cos\theta := x$ so

$$\begin{aligned}
\frac{\mathrm{d}\cos\theta'}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\gamma(x-\beta_0)}{\sqrt{\gamma^2(x-\beta_0)^2+1-x^2}}\right] \\
&= \frac{\gamma(\gamma^2(x-\beta_0)^2+1-x^2)^{1/2} - \gamma(x-\beta_0)\frac{1}{2}(\gamma^2(x-\beta_0)^2+1-x^2)^{-1/2}(2\gamma^2(x-\beta_0)-2x)}{(\gamma^2(x-\beta_0)^2+1-x^2)} \\
&= \frac{\gamma(\gamma^2(x-\beta_0)^2+1-x^2) - \gamma(x-\beta_0)(\gamma^2(x-\beta_0)-x)}{(\gamma^2(x-\beta_0)^2+1-x^2)^{3/2}} \\
&= \frac{\gamma^3(x-\beta_0)^2+\gamma(1-x^2)-\gamma^3(x-\beta_0)^2+\gamma(x^2-x\beta_0)}{(\gamma^2(x-\beta_0)^2+1-x^2)^{3/2}} \\
&= \frac{\gamma(1-x\beta_0)}{(\gamma^2(x-\beta_0)^2+1-x^2)^{3/2}} \\
&= \frac{\gamma(1-\cos\theta\frac{\beta}{\beta_C})}{(\gamma^2(\cos\theta-\frac{\beta}{\beta_C})^2+1-\cos^2\theta)^{3/2}}
\end{aligned} \tag{16}$$

Inserting this result into (7) we obtain the formula relating the CM differential cross section to the lab frame differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\gamma(1-\cos\theta\frac{\beta}{\beta_C})}{(\gamma^2(\cos\theta-\frac{\beta}{\beta_C})^2+1-\cos^2\theta)^{3/2}}\frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'} \tag{17}$$

3 Computation

I now wanted to take the result of (17) and create code which would take as input a CM differential cross section, initial and final lab frame momentum vectors, quantities measured in a typical scattering experiment, and array of angles θ which would reproduce the corresponding lab frame cross section values.

Although detailed in the Python notebook, I will briefly describe the structure of the code. First, given the required inputs, it calculates the CM velocity which is then inserted into a Lorentz transformation function to define the CM frame four vectors in terms of the lab frame quantities. Then, these four momenta are inserted into a CM differential cross section function, multiplied by the Lorentz transformation coefficient (16), which in general will take as input 4 four momenta and array of cosine values which is calculated using (15). These quantities can then be used to calculate the cross section values in the lab frame according to (17).

3.1 Challenges

To deem whether or not my code was successful in producing the desired results for arbitrary differential cross sections, I wanted to test it against actual data collected by a scattering experiment.

Given this transformation formula relies heavily on the change in the scattering angle θ, to test out the full apparatus of the code I focused on finding an experiment which measured a differential scattering cross section which had angular dependence. As long as the center of mass frame is sufficiently distinct from the lab frame, this should not have been to hard to find.

I soon realized in my searches, which mainly took place on the website `https://www.hepdata.net`, that many scattering events are such that the CM frame and lab frames are not distinct! This is mainly due to the fact that scattering experiments are mostly conducted with stable particle to comprise a beam so that one may create many scattering events. With the only stable particles being electron, proton, neutron, and their anti-particles (somewhat), this means in many experiments a particle is collided with a particle of the same type and the beams are generated in such a way that they have the same energy and therefore the particles of each beam have the same momentum which is, by definition, the CM frame. Aside from this caveat, finding experiments in general which measured the angular distribution of a differential scattering cross section was difficult to find.
To get around this problem I decided I could still test my code on a scattering experiment in which the CM frame was equal to the lab frame to see if it produces the same results, as it should. I chose the scattering process $ee^+ \to \mu\mu^+$ which has a CM differential cross section given by

$$\frac{\mathrm{d}\sigma'}{\mathrm{d}\Omega'} = \frac{4\alpha}{s}(1 + \cos^2\theta') \tag{18}$$

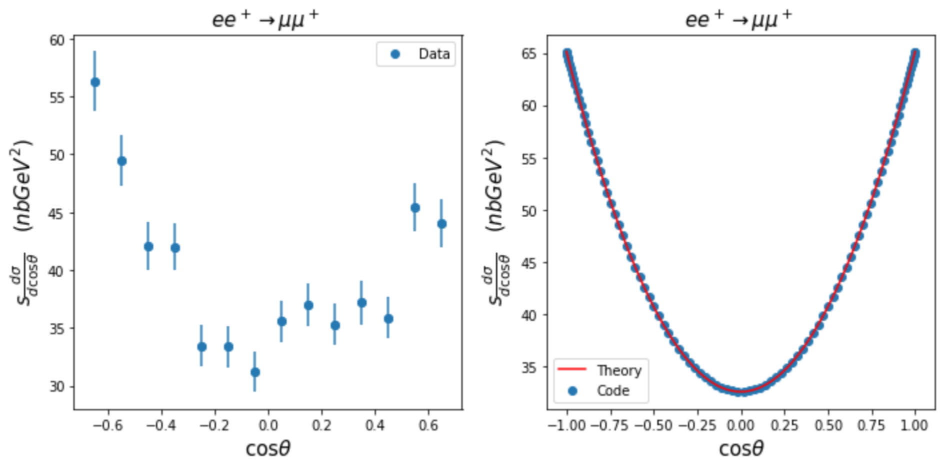

Figure 4: Results from comparison of code with that of experimental results contained in [1].

where $s := (p_1^\mu + p_2^\mu)^2$ is the total energy in the CM frame. I then found an experiment ([2]) which measured this differential cross section and provided their explicit data.

3.2 Results

In [2] they provide the value of $\sqrt{s}$ from which I was able to attain the initial and final momentum of each particle given the CM frame and lab frames are equal. These values could then be applied to the transformation function and thus produce the values of the differential cross section calculated in the lab frame, which should in fact be equal to the lab frame cross section.

The results are displayed in Figure 4 where it can be seen that on the y-axis are the values

$$s\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} \tag{19}$$

instead of

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \tag{20}$$

The former can be obtained from the latter by first multiplying (18) by $sd\phi$ to obtain

$$s\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\frac{\mathrm{d}\phi}{\mathrm{d}\phi} = 4\alpha(1+\cos^2\theta)d\phi \tag{21}$$

Integrating over ϕ we then obtain the plotted formula:

$$s\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = 4\alpha\int_0^{2\pi}(1+\cos^2\theta)d\phi = 2\pi(4\alpha)(1+\cos^2\theta) \tag{22}$$

I was able to reproduce the result of the experiment proving my code performs correctly for the trivial case in which it should. However, it remains to prove itself in the determination of a lab frame differential cross section which is distinct from the CM frame formula.

References

[1] David Griffiths. *Introduction to Elementary Particles.* Wiley-VCH, 2nd edition, 2008.

[2] M. E. Levi et al. Weak Neutral Currents in e+ e- Collisions at s**(1/2)=29-GeV. *Phys. Rev. Lett.*, 51:1941, 1983.