# Integration Bee Kaizo

March 1, 2025

## Qualifying Round

1.
$$\int_{-1/3}^{1/3} \cot(3\cos^{-1}(x))\,\mathrm{d}x$$

2.
$$\int \frac{\sin(4x)}{4+\sin^4(x)}\,\mathrm{d}x$$

3.
$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{2\sqrt{x}-x}+\sqrt{\sqrt{x}-x}}$$

4.
$$\int_{-\infty}^{\infty} \left\lfloor \frac{3}{x^2+1} \right\rfloor \mathrm{d}x$$

5.
$$\int \frac{\mathrm{d}x}{\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}-x^2}}$$

6.
$$\int_{-1}^{1} \frac{\mathrm{d}x}{e^{\sqrt[3]{x}}-1}$$

7.
$$\int_0^1 \left(\frac{\tan^2 x}{\tan x - x} - \frac{3}{x}\right)\mathrm{d}x$$

8.
$$\int_0^{\infty} e^{-|\ln(e^x-1)|}\,\mathrm{d}x$$

9.
$$\int_0^1 e^{\sin^{-1}(x)} \ln(x+\sqrt{1-x^2})\,\mathrm{d}x$$

- $I_1 = 0$ since $\cot(3\cos^{-1}(-x)) = \cot(3\pi - 3\cos^{-1}x) = -\cot(3\cos^{-1}x)$

- $I_2 = \int \frac{4\cos x \sin x(1-2\sin^2 x)}{4+\sin^4 x}\,dx \underset{s=\sin^2 x}{=} 2\int \frac{1-2s}{4+s^2}\,ds = \tan^{-1}\left(\frac{\sin^2 x}{2}\right) - 2\log(4+\sin^4 x)$

$e^{4ix} = \sim + i(4\cos^3 x \sin x - 4\cos x \sin^3 x)$

$1-(1-x)^2 \qquad \frac{1}{4}-(\frac{1}{2}-x)^2$

- $I_3 = \int_0^1 \frac{1}{\sqrt{x}}\cdot\left(\sqrt{2\sqrt{x}-x} - \sqrt{\sqrt{x}-x}\right)dx \underset{x\to x^2}{=} 2\int_0^1\left(\sqrt{2x-x^2} - \sqrt{x-x^2}\right)dx$

$= 2\left[\int_0^1 \sqrt{1-x^2}\,dx - \frac{1}{2^2}\int_{-1}^1 \sqrt{1-x^2}\,dx\right] = \int_0^1\sqrt{1-x^2}\,dx = \frac{\pi}{4}$ Area of 1/4 of circle w/ radius 1

For 2nd take $\frac{1}{2}-x \to \frac{1}{2}x$

- $I_4 = 2\int_0^{+\infty}\left\lfloor\frac{3}{x^2+1}\right\rfloor dx$ $=0,\ x\geq\sqrt{2}$ $= 2\int_0^{\sqrt{2}}\left\lfloor\frac{3}{x^2+1}\right\rfloor dx = 2\left(\int_0^{\sqrt{\frac{3}{2}-1}} 2 + \int_{\sqrt{\frac{3}{2}-1}}^{\sqrt{\frac{3}{1}-1}} 1\right) = 3\sqrt{2}$

- $I_5 = \int \frac{dx}{\sqrt{x^{\underbrace{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}_{15/8=2-\frac{1}{8}}}-x^2}} \underset{x=t^{-8}}{=} -8\int \frac{t^{8-9}}{\sqrt{t-1}}\,dt = -8\cdot 2\int\frac{1}{1+u^2}\,du = -16\tan^{-1}(x^{-\frac{1}{8}}-1)$

$\sqrt{t-1} = u$

- $I_6 = \int_0^1\left(\frac{1}{e^{3\sqrt{x}}-1} + \frac{1}{e^{-3\sqrt{x}}-1}\right)dx = -1$

- $I_7 = \log\left|\frac{\tan x - x}{x^3}\right|\Big|_0^1 = \log(\tan(1)-1) + \log(3)$

$D(\tan x - x) = \tan^2 x$

$\tan x \sim x + \frac{x^3}{3}$

- $I_8 = \int_{-\infty}^{\infty} e^{-|x|}\frac{e^x}{e^x+1}\,dx = \int_0^\infty e^{-x}\left(\frac{e^x}{e^x+1} + \frac{1}{1+e^x}\right)dx = 1$

$e^x - 1 \to e^x$

- $I_9 \underset{x\to\sin x}{=} \int_0^{\pi/2} e^x \ln(\sin x + \cos x)\cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x)\ln(\sin x + \cos x) - \frac{1}{2}e^x\cos x\Big|_0^{\pi/2}$

$= \frac{1}{2}$

Obs: $D(e^{(1+i)x}\ln(\sin x + \cos x)) = (1+i)e^{(1+i)x}\ln(\sin x+\cos x) + e^{(1+i)x}\frac{\cos x - \sin x}{\sin x + \cos x}$

$\to \int e^{(1+i)x}\ln(\sin x + \cos x) = e^x\frac{e^{ix}}{1+i}\ln(\sin x + \cos x) - \int e^x\frac{e^{ix}}{1+i}\frac{\cos x - \sin x}{\sin x + \cos x}$

Also $\frac{e^{ix}}{1+i} = \frac{1}{2}(\cos x + \sin x) + i\,\mathrm{Im}(-)$

$D e^x \cos x = e^x(\cos x - \sin x)$

10.
$$\int (\tan x - \cot x)^3 (\tan x + \cot x)^2 \, \mathrm{d}x$$

11.
$$\int \sqrt{\frac{3^x + 2^x}{3^x - 2^x}} \, \mathrm{d}x$$

12.
$$\int_1^{\sqrt{2}} \frac{(x^2 - 1)\ln(x^2 - 1)}{\sqrt{2 - x^2}} \, \mathrm{d}x$$

13.
$$\int \overbrace{(1 + \ln x)^2}^{v'} \overbrace{\tan^{-1}(\ln x)}^{u} \, \mathrm{d}x$$

14.
$$\int \frac{(\ln x)^{\ln x} \ln(\ln x)}{x^2} \, \mathrm{d}x$$

15.
$$\int_0^1 \sin^{-1}\left(\sqrt{\frac{\sqrt{x}}{\sqrt{x} + \sqrt{1 - x}}}\right) \mathrm{d}x$$

16.
$$\int_0^{\pi} \sqrt{2 + \cos x + \sqrt{5 + 4\cos x}} \, \mathrm{d}x$$

17.
$$\int \frac{\mathrm{d}x}{1 + \left(1 - \frac{1}{x}\right) \tan x}$$

18.
$$\int_0^{\sqrt{3}/2} \sqrt{\frac{x^2 + \sqrt{x^4 + 1}}{x^4 + 1}} \, \mathrm{d}x$$

19.
$$\int_0^{\pi/4} \frac{\cos(4x)}{\cos^6 x} \, \mathrm{d}x$$

20.
$$\int_0^1 \tan^{-1}(2^x - 1) \, \mathrm{d}x$$

$\bullet\ I_{10} = -\int 2^5 \frac{\cos^3 2x}{\sin^5(2x)}\,dx = -2^4\int \frac{1-s^2}{s^5}\,ds = 2^4\left(\frac{1}{4}\frac{1}{\sin^4 2x} - \frac{1}{2}\frac{1}{\sin^2 2x}\right)$

$= \frac{4}{\sin^4 2x} - \frac{8}{\sin^2 2x}$ //

$\bullet\ I_{11} = -2\int \frac{\cos\theta}{\cancel{\sin\theta}} \cdot \frac{2\sin\theta\cos\theta}{2\cos^2\theta - 1}\frac{d\theta}{\ln(2/3)} = -\frac{2}{\ln(2/3)}\left(\theta + \int \frac{1}{\cos 2\theta}\,d\theta\right) =$

$2^x = 3^x \cos 2\theta$

$dx = d\,\frac{\ln\cos 2\theta}{\ln(2/3)} = -\frac{\sin 2\theta}{\cos 2\theta}\cdot\frac{2}{\ln(2/3)}\,d\theta$

$-\frac{2}{\ln(2/3)}\left(\theta + \frac{1}{2}\ln(\sec 2\theta + \tan 2\theta)\right)$

$= -\frac{1}{\ln(2/3)}\left(\cos^{-1}\left(\frac{2}{3}\right)^x + \ln\left(\left(\frac{3}{2}\right)^x + \sqrt{\left(\frac{3}{2}\right)^{2x} - 1}\right)\right)$ //

$\bullet\ I_{12} = \int_{\pi/4}^{\pi/2} (2\sin^2\theta - 1)\ln(2\sin^2\theta - 1) = \int_0^{\pi/4} \underbrace{(2\cos^2\theta - 1)}_{\cos 2\theta}\ln\underbrace{(2\cos^2\theta - 1)}_{\cos 2\theta}\,d\theta = \frac{\sin 2\theta}{2}\ln\cos 2\theta + \int \frac{\sin 2\theta}{\cancel{2}}\frac{\sin 2\theta \cdot \cancel{2}}{\cos 2\theta}$

$\theta \to \frac{\pi}{2} - \theta$

$x = \sqrt{2}\sin\theta$

$= \frac{\sin 2\theta}{2}\ln\cos 2\theta + \frac{1}{2}\int \frac{s^2}{1-s^2}\,ds = \frac{\sin 2\theta}{2}\ln\cos 2\theta + \frac{1}{2}\left(-s + \frac{1}{2}\ln\frac{1+s}{1-s}\right)$

$= \frac{\sin 2\theta}{2}\ln\cos 2\theta + \frac{1}{4}\ln\left(\frac{1+\sin 2\theta}{1-\sin 2\theta}\right) - \frac{1}{2}\sin 2\theta \Big|_0^{\pi/4} = \frac{1}{4}\ln(2) - \frac{1}{2} + \lim_{\theta\to\frac{\pi}{4}} \frac{\sin 2\theta}{2}\ln\cos 2\theta - \frac{1}{4}\ln(1-\sin 2\theta)$

$= \frac{\ln 2 - 2}{4} + \underbrace{\lim_{\alpha\to 0} \frac{1}{2}\sin\left(2\alpha + \frac{\pi}{2}\right)\ln\cos\left(2\alpha + \frac{\pi}{2}\right) - \frac{1}{4}\ln\left(1 - \sin\left(2\alpha + \frac{\pi}{2}\right)\right)}$

$\alpha = \theta - \frac{\pi}{4}$

$= \frac{\ln 2 - 1}{2}$ //

$= \frac{1}{2}\underbrace{\cos 2\alpha}_{\approx 1} \cdot \ln \underbrace{\sin 2\alpha}_{\approx 2\alpha} - \frac{1}{4}\ln\underbrace{(1 - \cos 2\alpha)}_{\approx \frac{(2\alpha)^2}{2}}$

$\approx \frac{1}{2}\ln(2\alpha) - \frac{1}{2}\ln(2\alpha) + \frac{1}{4}\ln(2)$

$\bullet\ I_{13} = x((\ln x)^2 + 1)\tan^{-1}(\ln x) - x$ //

$V = \int (1 + \ln x)^2 = x(\ln x + 1)^2 - 2\int(\ln x + 1)$

$= x(\ln x + 1)^2 - 2x\ln x$

$= x((\ln x)^2 + 1)$

$D(\ ) = \frac{\ln\ln x - 1 + \frac{1}{x}}{x}$ " $= \frac{\ln\ln x}{x}$

$\bullet\ I_{14} = \int e^{\overbrace{\ln x(\ln\ln x - 1)}} \cdot \overbrace{e^{\ln x}}\frac{\ln\ln x}{x^{\cancel{2}}}\,dx = e^{(\ln x)(\ln\ln x - 1)}$

$= \frac{(\ln x)^{(\ln x)}}{x}$ //

• $I_{15} = \int_0^1 \sin^{-1}\left(\underbrace{\frac{\sqrt{1-x}}{\sqrt{x}+\sqrt{1-x}}}_{u}\right) dx = \frac{1}{2}\int_0^1 \underbrace{\sin^{-1}u + \sin^{-1}u'}_{\theta_1+\theta_2=\frac{\pi}{2}}\, dx = \frac{\pi}{4}$

$u^2 + u'^2 = 1$

• $I_{16} = \int_3^1 \sqrt{\frac{u^2+3}{4}+u}\cdot\frac{4\,du}{\sqrt{(u^2-1)(9-u^2)}} = 2\int_1^3 \frac{\sqrt{(u+3)(u+1)}}{\sqrt{(u+1)(3+u)(u-1)(3-u)}}\,du = 2\int_{-1}^1 \frac{du}{\sqrt{1-u^2}} = 4\sin^{-1}u\Big|_0^1 = 2\pi$

$\to u^2+4u+3$

$u-2\to u$

$-3+4u-u^2$

$\hookrightarrow 1-(u-2)^2$

$u=\sqrt{5+4\cos x} \Rightarrow$

$\cos x = \frac{u^2-5}{4} \to dx = \frac{-4\,du}{\sqrt{16-u^4+10u^2-25}}$

$10u^2-u^4-9 = (u^2-1)(9-u^2)$

• $I_{17} = \int \frac{x\cos x}{\underbrace{x\cos x+(x-1)\sin x}_{u}} = \int \frac{(u+du)/2}{u} = \frac{x}{2} + \frac{1}{2}\ln\left(x\cos x + (x-1)\sin x\right)$

$du = \cos x - x\sin x + \sin x + (x-1)\cos x$

$= x\cos x - (x-1)\sin x$

$\tan\theta_1 = \frac{3}{4} \to \cot\theta_1 = \frac{4}{3} \to \csc\theta_1 = \sqrt{1+\frac{16}{9}} = \frac{5}{3}$

• $I_{18} = \int_0^{\theta_1} \sqrt{\frac{\cancel{\tan x}+\cancel{\sec x}}{\cancel{\sec^2 x}}}\cdot 2\sqrt{\frac{\cancel{\cos x}}{\sin x}}\cdot\frac{1}{\cancel{\cos^2 x}}\cdot\frac{dx}{\cos x} = \int \sqrt{\frac{\sin x+1}{\sin x}}\cdot\frac{1}{1-\sin^2 x}\cos x\,dx$

$\sin x$

$x^2 \to \tan x$

$\frac{1+\sin x}{\sin x} = u$ : $-\int \sqrt{u}\,\frac{\cancel{(u-1)^2}}{\underbrace{(u-1)^2-1}_{u(u-2)}}\cdot\frac{du}{\cancel{(u-1)^2}} = -\int 2\,\frac{1}{z^2-2}\,dz = +\frac{2}{2\sqrt{2}}\ln\left|\frac{z+\sqrt{2}}{z-\sqrt{2}}\right|$

$\hookrightarrow \sin x = \frac{1}{u-1}$

$u = z^2$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sqrt{\frac{1+\sin x}{\sin x}}}{\sqrt{2}-\sqrt{\frac{1+\sin x}{\sin x}}}\right| = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2\sin x}+\sqrt{1+\sin x}}{\sqrt{2\sin x}-\sqrt{1+\sin x}}\right|\Bigg|_0^{\theta_1}$

$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+2\sqrt{2}/\sqrt{3}}{\sqrt{2}-2\sqrt{2}/\sqrt{3}}\right| = \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{3}+2}{2-\sqrt{3}}\right) = \sqrt{2}\ln(2+\sqrt{3})$

• $I_{19} = \int_0^{\pi/4} \frac{\cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x}{\cos^4 x\cdot\cos^2 x}\,dx = \int_0^1 (1-6t^2+t^4)\,dt = 1 - \frac{6}{3} + \frac{1}{5} = -\frac{4}{5}$

$t = \tan x$

• $I_{20} = \int_0^1 \tan^{-1}(\underbrace{2^{1-x}-1}_{v})\,dx = \frac{1}{2}\int_0^1 \tan^{-1}\frac{u+v}{1-uv}\,dx = \frac{\pi}{8}$

$u = 2^x - 1$

$1-uv = 1-(2-2^x-2^{1-x}+1) = 2^x+2^{1-x}-2 = u+v$