TSRT78 - HOMEWORK 2, MATHEUS BERNAT (MATVI959)

GIVEN: $y_k = s_k + e_k$

$s_k - 0{,}7\, s_{k-1} = v_k - 0{,}2\, v_{k-1}$

v_k & e_k independent noise sequences, $\sigma_v^2 = 2$ $\sigma_e^2 = 2$

a) GOAL: Determine filter $H(q)$ from observations y_l, $-\infty \le l \le \infty$ that provides best estimate $\hat{s}_k$ of s_k.

SOLUTION:

(i) The filter $H(q)$ that minimizes the loss function $V(h) \overset{\text{p. 272}}{=} E\left[(s(t)-\hat{s}(t))^2\right] = E\left[\left(s(t) - \sum_{k\in I} h(k)\cdot y(t-k)\right)^2\right]$,

where $I = (-\infty, \infty)$, is the non-causal Wiener filter.

Write the Wiener-Hopf equation in (7.18): $\sum_{i \in I} h(i)\cdot R_{yy}(k-i) = R_{sy}(k)$, $k \in I$

This is equivalent to: $R_{yy}(k) * h(k) = R_{sy}(k)$ $(*)$ from (7.26).

DTFT of $(*)$: $H(e^{i\omega})\cdot \Phi_{yy}(e^{i\omega}) = \Phi_{sy}(e^{i\omega})$. So, the non-causal Wiener filter is given by

$$H(e^{i\omega}) = \frac{\Phi_{sy}(e^{i\omega})}{\Phi_{yy}} \Leftrightarrow \Big/\!\!\Big/ \Phi_{sy}(e^{i\omega}) = E[s(t)\cdot y(t)] = E[s(t)\cdot(s(t)+e(t))] = E[s(t)\cdot s(t)] + \underbrace{E[s(t)\cdot e(t)]}_{=0} = \Phi_{ss}(e^{i\omega}) \Big/\!\!\Big/ \Leftrightarrow$$

$$\Leftrightarrow H(e^{i\omega}) = \frac{\Phi_{ss}(e^{i\omega})}{\Phi_{ss}(e^{i\omega}) + \Phi_{ee}(e^{i\omega})} \xrightarrow{z = e^{i\omega}} H(z) = \frac{\Phi_{ss}(z)}{\Phi_{ss}(z) + \Phi_{ee}(z)}$$

(ii) Find now $\Phi_{ss}(z)$ and $\Phi_{ee}(z)$. (See e.g. p. 266)

From the exercise: $\Phi_{ee}(z) = 2$.

Find $\Phi_{ss}(z)$: $s_k - 0{,}7\cdot s_{k-1} = v_k - 0{,}2\cdot v_{k-1} \Leftrightarrow s_k - 0{,}7\cdot s_k\cdot q^{-1} = v_k - 0{,}2\cdot v_k \cdot q^{-1} \Leftrightarrow$

$$\Leftrightarrow s_k(1 - 0{,}7\cdot q^{-1}) = v_k(1-0{,}2\cdot q^{-1}) \Leftrightarrow s_k = \frac{1-0{,}2\, q^{-1}}{1-0{,}7\cdot q^{-1}}\cdot v_k \Rightarrow F_s(q) = \frac{1-0{,}2\cdot q^{-1}}{1-0{,}7\cdot q^{-1}} = \frac{q-0{,}2}{q-0{,}7}$$

Therefore:

$$\Phi_{ss}(z) = |F_s(z)|^2\cdot \sigma_v^2 = \frac{1-0{,}2\cdot z}{1-0{,}7\cdot z}\cdot\frac{1-0{,}2\cdot z^{-1}}{1-0{,}7\cdot z^{-1}}\cdot 2 = 2\cdot\frac{1-0{,}2 z}{1-0{,}7\cdot z}\cdot\frac{\frac{z-0{,}2}{z}}{\frac{z-0{,}7}{z}} = 2\cdot\frac{(1-0{,}2z)\cdot(z-0{,}2)}{(1-0{,}7z)\cdot(z-0{,}7)} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\Phi_{ss}(z) = \frac{2\cdot(-0{,}2z^2 + z + 0{,}04z - 0{,}2)}{-0{,}7z^2 + 0{,}49z + z - 0{,}7} = 2\cdot\frac{(-0{,}2\cdot z^2 + 1{,}04z - 0{,}2)}{-0{,}7z^2 + 1{,}49z - 0{,}7} = \frac{2\cdot(z-5)(z-0{,}2)}{(z-1{,}43)\cdot(z-0{,}7)}}$$

So: $\boxed{\Phi_{yy}(z) = \Phi_{ss}(z) + 2 = 2\cdot\left(\frac{(z-5)(z-0{,}2) + (z-1{,}43)(z-0{,}7)}{(z-1{,}43)\cdot(z-0{,}7)}\right) = 2\cdot\frac{(z-3{,}37)\cdot(z-0{,}3)}{(z-1{,}43)\cdot(z-0{,}7)}}$

(iii) Get $H(q)$:

$$H(q) = \frac{\Phi_{ss}(z)}{\Phi_{yy}(z)} \overset{\cdot z/z}{=} z\left(\frac{(z-5)(z-0{,}2)}{z(z-3{,}37)\cdot(z-0{,}3)}\right) = z\cdot\left(\frac{1}{z} + \frac{-0{,}5}{z-3{,}37} + \frac{0{,}5}{z-0{,}3}\right) = 1 + \frac{-0{,}5z}{z-3{,}37} + \frac{0{,}5z}{z-0{,}3} \Leftrightarrow$$

$$\Rightarrow \boxed{H(q) = \underbrace{\frac{0{,}5\cdot q - 3{,}37}{q-3{,}37}}_{\text{anti-causal}} + \underbrace{\frac{0{,}5\cdot q}{q-0{,}3}}_{\text{causal}}}$$

(implemented as $\hat{s}(t) = H(q)\cdot y(t) = \hat{s}^+(t) + \hat{s}^-(t)$)

■

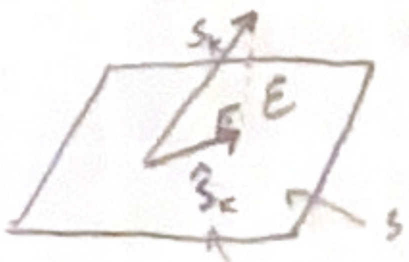
B) goal: Determine filter $H(q)$ from observations y_l, $l=\{k-5,\dots,k+5\}$ gives best estimate $\hat{s}_k$ of s_k.

SOLUTION:

PAGE 2

(i) We want to find a filter $H(q)$ that gives the $\hat{s}(k)$ that minimizes the loss function $V(h)=E\left[(s(k)-\hat{s}(k))^2\right]$, where $\hat{s}(k)=\sum_{l\in\lambda} h(l)\cdot y(k-l)$, $\lambda=[-5,-4,\dots,4,5]$ ☆

According to projection theory, in order for us to have $\hat{s}$ closest to s: $\langle s_k-\hat{s}_k , y_{k-j}\rangle=0 \quad \forall j\in\lambda$



$\hat{s}_k$ is a linear combination of $y_{k-j}\ \forall j\in\lambda$: $y_{k-5}, y_{k-4},\dots, y_k,\dots, y_{k+4}, y_{k+5}$

So: $\langle (s_k-\hat{s}_k), y_{k-j}\rangle=0 \iff y_{k-j}\, s_k - y_{k-j}\cdot\hat{s}_k=0 \Rightarrow$ // take expected value, since stochastic process // $\Rightarrow$

$$\Rightarrow E(y_{k-j}\cdot s_k)-E(y_{k-j}\cdot\hat{s}_k)=0 \iff E(y_{k-j}\cdot s_k)-E\Big(y_{k-j}\sum_{l\in\lambda} h_l\cdot y_{k-l}\Big)=0 \iff$$

$$\iff \underbrace{E(y_{k-j}\, s_k)}_{R_{sy}(j)}-\sum_{l\in\lambda} h_l\cdot \underbrace{E(y_{k-j}\cdot y_{k-l})}_{R_{yy}(j-l)}=0 \iff \boxed{R_{sy}(j)=\sum_{l\in\lambda} h_l\cdot R_{yy}(j-l) \quad \forall j\in\lambda} \quad \lambda=[-5,4,\dots,4,5]$$

From ☆ we see that the filter $H(q)$ is a non-causal filter with only 11 values (i.e. $h(-5), h(-4),\dots, h(0),\dots, h(4), h(5)$).

(ii) From $R_{sy}(j)=\sum_{l\in\lambda} h_l\, R_{yy}(j-l)\quad\forall j\in\lambda$, $\lambda=(-5,-4,\dots,4,5)$ we can find all the $h(-5)\dots h(5)$:

$$\begin{bmatrix} R_{sy}(-5)\\ \vdots \\ R_{sy}(0)\\ \vdots\\ R_{sy}(5)\end{bmatrix}=\begin{bmatrix} R_{yy}(0) & R_{yy}(-1) & \cdots & R_{yy}(-10)\\ & & & \vdots\\ R_{yy}(5) & \cdots\ R_{yy}(0) & & R_{yy}(-5)\\ \vdots & & \ddots & \vdots\\ R_{yy}(10)\cdots & R_{yy}(5) & \cdots & R_{yy}(0)\end{bmatrix}\cdot\begin{bmatrix} h(-5)\\ \vdots\\ h(0)\\ \vdots\\ h(5)\end{bmatrix}$$

where $R_{sy}(j)=E\left[s_k\cdot y_{k-j}\right]=E\left[s_k(s_{k-j}+e_{k-j})\right]=E\left[s_k\cdot s_{k-j}\right]+\underbrace{E\left[s_k\, e_{k-j}\right]}_{=0}=R_{ss}(j)$.

(iii) Now we want to calculate $R_{ss}(j)$.

$R_{ss}(0)=E(s_k\cdot s_k)=E((0{,}7\cdot s_{k-1}+v_k-0{,}2\cdot v_{k-1})^2) \iff$

$$\iff R_{ss}(0)=E(0{,}7^2\cdot s_{k-1}^2)+E(v_k^2)+E(0{,}2^2\cdot v_{k-1}^2)+\underbrace{E(2\cdot 0{,}7\cdot s_{k-1}\cdot v_k)}_{=0}+E(-2\cdot 0{,}7\cdot 0{,}2\cdot s_{k-1}\cdot v_{k-1})+\underbrace{E(-2\cdot 0{,}2\cdot v_k\cdot v_{k-1})}_{=0}\iff$$

$$\iff R_{ss}(0)=0{,}7^2\cdot R_{ss}(0)+R_{vv}(0)+0{,}2^2\cdot R_{vv}(0)-0{,}28\cdot R_{sv}(0) \iff \text{//}\ \cdot R_{vv}(0)=\sigma_v^2=2,\ \cdot R_{sv}(0)=E(s_k\cdot v_k)=E((0{,}7\cdot s_{k-1}+v_k-0{,}2\cdot v_{k-1})\cdot v_k) \text{//} \iff$$

$$\iff R_{ss}(0)=0{,}49\cdot R_{ss}(0)+2{,}08-0{,}28\cdot\Big(\underbrace{E(0{,}7\cdot s_{k-1}\cdot v_k)}_{=0}+\underbrace{E(v_k^2)}_{=2}+\underbrace{E(-0{,}2\cdot v_{k-1}\cdot v_k)}_{=0}\Big)\iff$$

$$\iff R_{ss}(0)=\frac{1}{0{,}51}\cdot(2{,}08-0{,}56)\iff$$

$$\iff \boxed{R_{ss}(0)=2{,}98}$$

(continues)

(continuation)

$$R_{ss}(m) = E(s_K \cdot s_{K+m}) = E(s_K \cdot (0{,}7 \cdot s_{K-1+m} + v_{K+m} - 0{,}2 \cdot v_{K-1+m})) \Leftrightarrow$$

$$\Leftrightarrow R_{ss}(m) = E(0{,}7 \cdot s_K \cdot s_{K-1+m}) + \underbrace{E(s_K \cdot v_{K+m})}_{=0} + \underbrace{E(-0{,}2 \cdot s_K \cdot v_{K-1+m})}_{=0} \Leftrightarrow$$

$$\Leftrightarrow \boxed{R_{ss}(m) = 0{,}7 \cdot R_{ss}(m-1)} \text{ for } m \geq 2$$

(for symmetry of R_{ss})

since indices of v are higher than indices of s. (independent white noise)

$$\text{And: } R_{ss}(1) = E(s_K \cdot s_{K+1}) = E(s_K \cdot (0{,}7 \cdot s_K + v_{K+1} - 0{,}2 \cdot v_K)) \Leftrightarrow$$

$$\Leftrightarrow R_{ss}(1) = E(0{,}7 \cdot s_K^2) + \underbrace{E(s_K \cdot v_{K+1})}_{=0} + E(-0{,}2 \cdot s_K \cdot v_K) \Leftrightarrow$$

$$\Leftrightarrow R_{ss}(1) = 0{,}7 \cdot R_{ss}(0) - 0{,}2 \cdot R_{sv}(0) = 0{,}7 \cdot 2{,}38 - 0{,}2 \cdot 2 \Leftrightarrow$$

$$\Leftrightarrow \boxed{R_{ss}(1) = 1{,}6863}$$

In summary, the covariance function R_{ss} is:

$$R_{ss}(j) = \begin{cases} 2.3803, & j=0 \\ 1.6863, & j=1 \\ 0{,}7 \cdot R_{ss}(j-1), & j \geq 2 \end{cases}$$

We have also that:

$$R_{yy}(j) = \begin{cases} R_{ss}(j) + \overbrace{R_{ee}(j)}^{=0} = R_{ss}(j), & j \neq 0 \\ R_{ss}(j) + \underbrace{R_{ee}(j)}_{=\sigma_e^2} = R_{ss}(j) + 2, & j = 0 \end{cases}$$

Implementing the linear equation system in MATLAB we get the coefficients $h(-5), h(-4), \ldots, h(5)$, and (from page 272), $H(q)$ is:

$$H(q) = h(-5) \cdot q^5 + h(-4) \cdot q^4 + \ldots + h(0) + \ldots + h(4) \cdot q^{-4} + h(5) \cdot q^{-5}$$

Obs: for the 1st filter, to simulate it using filtfilt we just send the causal part of $H(z)$ to filtfilt:

$$H(z) = \frac{\dfrac{2 \cdot (1-0{,}2z) \cdot (1-0{,}2z^{-1})}{\cancel{(1-0{,}7z) \cdot (1-0{,}7z^{-1})}}}{\dfrac{2 \cdot (1-0{,}2z) \cdot (1-0{,}2z^{-1}) + 2 \cdot (1-0{,}7z)(1-0{,}7z^{-1})}{\cancel{(1-0{,}7z)(1-0{,}7z^{-1})}}} = \frac{(1-0{,}2z)(1-0{,}2z^{-1})}{(1-0{,}2z)(1-0{,}2z^{-1}) + (1-0{,}7z)(1-0{,}7z^{-1})} \Leftrightarrow$$

$$\Leftrightarrow H(z) = \frac{(1-0{,}2z) \cdot (z-0{,}2)}{(1-0{,}2z)(z-0{,}2) + (1-0{,}7z)(z-0{,}7)} \overset{\text{MATLAB}}{=} \frac{(1-0{,}2z) \cdot (z-0{,}2)}{-(z-2{,}39)(z-0{,}42)} \overset{(z^{-1}/z^{-1})}{=} \frac{(z^{-1}-0{,}2) \cdot (z-0{,}2)}{(1-2{,}39z^{-1}) \cdot (z-0{,}42)} \Leftrightarrow$$

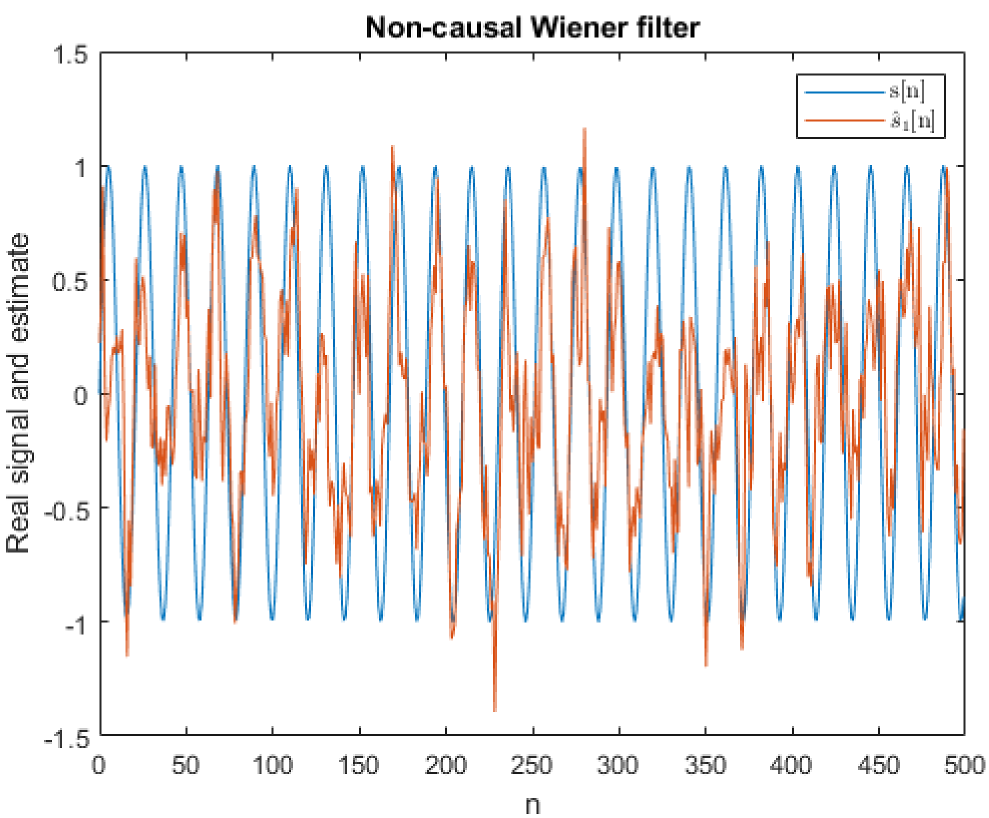
$$\Leftrightarrow H(z) = \frac{1}{2{,}39} \frac{(z^{-1}-0{,}2) \cdot (z-0{,}2)}{(z^{-1}-0{,}42) \cdot (z-0{,}42)} = \underbrace{\sqrt{\frac{1}{2.39}} \cdot \frac{(z^{-1}-0{,}2)}{(z^{-1}-0{,}42)}}_{\tilde{H}(z)_-} \cdot \underbrace{\sqrt{\frac{1}{2.39}} \frac{(z-0{,}2)}{(z-0{,}42)}}_{\tilde{H}(z)_+}$$

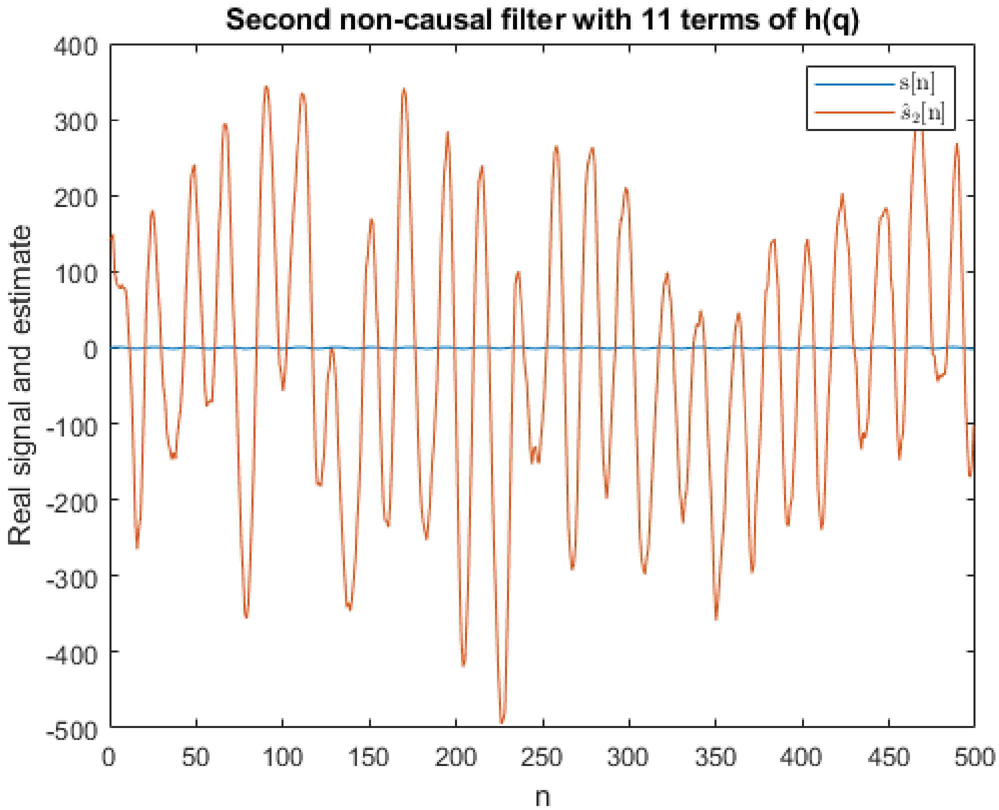
```
% Homework 2

clear;clf;

% -------------------- Calculate Rss and Ryy ---------------------

% Get the auto-covariance function for Rss. It is symmetric so just
% calculate from 0 to 5.
Rss = zeros(1,11);
Rss(1) = 2.9804; % Rss_0
Rss(2) = 1.6863; % Rss_1
for k = 3:11
    Rss(k) = 0.7*Rss(k-1);
end

% Get the auto-covariance function for Ryy.
Ree_0 = 2;
Ryy = zeros(1,11);
Ryy(1) = Rss(1) + Ree_0;
for k = 2:11
    Ryy(k) = Rss(k);
end

% Solve the linear equation system to get h(-5)...h(5)
% As in page 273
A = toeplitz(Ryy(1:11),Ryy(1:11));
b = [Rss(6) Rss(5) Rss(4) Rss(3) Rss(2) Rss(1:6)]';
h = A\b % = inv(A)*b

% ANSWER:
% h = 0.0040 0.0084 0.0197 0.0470 0.1123 0.4910 0.1123 0.0470 0.0197 0.0084 0.0040

% ANSWER: The values of h found will be used to compute H(q). See the
% continuation in the solution by hand.

% ---------------------------- Simulate the signals -----------------------

% Create real signal
N = 500;
n = 0:N-1;
sigmaNoise = 1;
f = 0.3;
s = sin(f*n);                    % Interesting signal
y = s + sigmaNoise*randn(1,N);   % Signal + noise

% -------- Filter y using the optimal non-causal Wiener filter ------------
% Anti-causal part
%a_ac = [1 -3.37];
%b_ac = [0.5 -3.37];
%sHat_ac = filtfilt(b_ac, a_ac, y);

% Causal part
a_c = [1 -0.3];
b_c = [0.5 0];
sHat_c = filtfilt(b_c, a_c, y);

sHat = sHat_c;

MSE_1 = sum((s - sHat).^2)/length(s) % MSE_1 = 0.2621
```

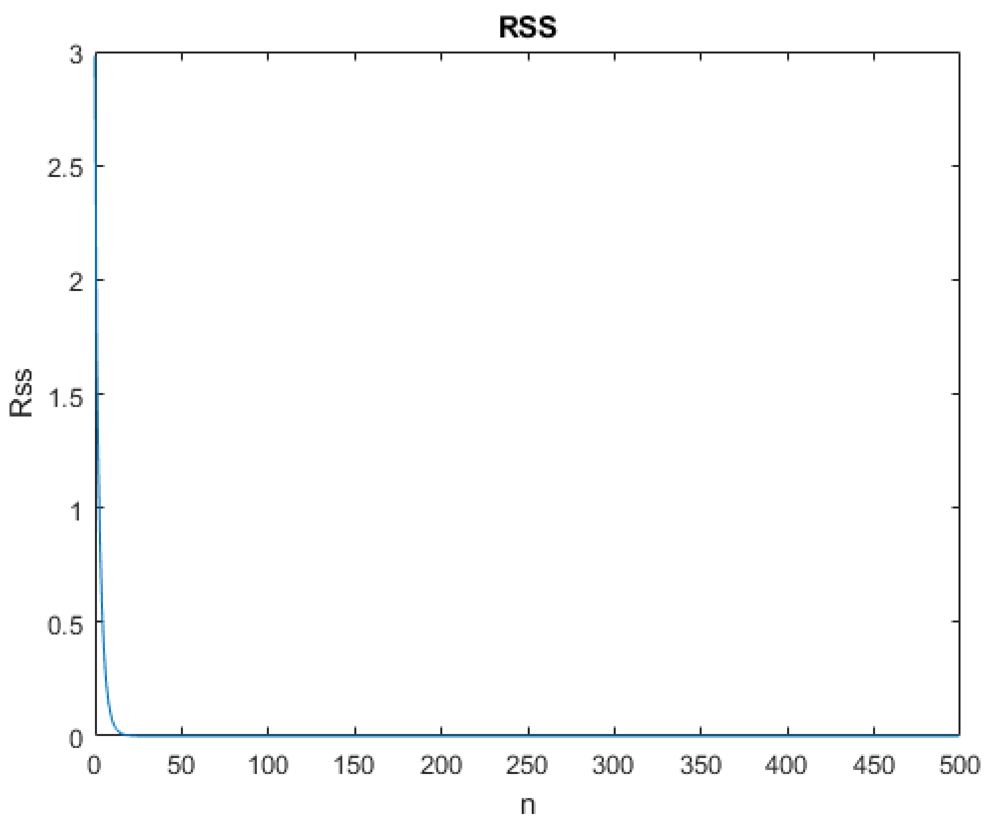
```
% ANSWER: MSE for non-causal filter = 0.2621

% Plot
figure(1);
plot(n, s); hold on; plot(n, sHat);
legend('s[n]','$\hat{s}_1$[n]','Interpreter','latex');
xlabel('n');ylabel('Real signal and estimate');
title("Non-causal Wiener filter");

% ---------------- Filter y using the h(-5),...,h(5) --------------------

a = 1;
a = h(6:11);
% Just send the causal part to filtfilt
sHat_2 = filtfilt(b,a,y);
figure(2);
plot(n, s); hold on; plot(n, sHat_2);
legend('s[n]','$\hat{s}_2$[n]','Interpreter','latex');
xlabel('n');ylabel('Real signal and estimate');
title("Second non-causal filter with 11 terms of h(q)");

MSE_2 = sum((s - sHat_2).^2)/length(s) % 3.65e4

% ANSWER: MSE for second filter = 3.65e4, not reasonable


% --------------- Plot Rss ---------------

Rss = zeros(1,length(y));
Rss(1) = 2.9804; % Rss_0
Rss(2) = 1.6863; % Rss_1
for k = 3:length(y)
    Rss(k) = 0.7*Rss(k-1);
end
figure(3);
plot(n,Rss);
xlabel("n");ylabel("Rss");title("RSS");

% ANSWER: I didn't have the time to plot the second filter so I can't
% really explain why their performances are alike.
```

```
h =

    0.0040
    0.0084
    0.0197
    0.0470
    0.1123
    0.4910
    0.1123
    0.0470
    0.0197
    0.0084
    0.0040


MSE_1 =
```

```
    0.2515


MSE_2 =

   3.1818e+04
```


Non-causal Wiener filter
s[n]
$\hat{s}_1$[n]
Real signal and estimate
n
1.5
1
0.5
0
-0.5
-1
-1.5
0
50
100
150
200
250
300
350
400
450
500





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