HOMEWORK 3 - MATHEUS BERNAT (MATVI 359)

QUESTION 2

GIVEN:
$$\begin{cases} x_{k+1} = 0{,}8 \cdot x_k + w_k & (0.1) \\ y_k = x_k + v_k & (0.2) \end{cases}$$
, w_k & v_k independent white noise processes, $\sigma_w^2 = 0{,}36$, $\sigma_v^2 = 1$
$F_s = 5$ [Hz]

I. GOAL: Transfer function of stationary one-step ahead (predictor) causal Wiener filter. Plot frequency & phase response.

SOLUTION:

From (7.22): causal WF for 1-step prediction: $H(q) = h_1 \cdot q^{-1} + h_2 \cdot q^{-2} + \ldots$

Follow algorithm 7.1 to find this filter

$$y_k \longrightarrow \boxed{\frac{1}{T(q)}} \xrightarrow{e} \boxed{\frac{1}{q} \cdot \left[\frac{q \cdot \Phi_{xy}(q)}{\sigma_e^2 \cdot T(q^{-1})}\right]_+} \longrightarrow \hat{y}(t+1)$$

(i) Factorize $\Phi_{yy}(z)$ into $\Phi_{yy}(z) = \sigma_e^2 \cdot T(z) \cdot T(z^{-1})$

(7.3) From (2): $\Phi_{yy}(z) = \Phi_{xx}(z) + \Phi_{vv}(z)$ (1.1), where $\Phi_{vv}(z) = \sigma_v^2 = 1$

WHY
$y = x + n$
$\Downarrow$
$\Phi_{yy}(z) = \Phi_{xx}(z) + \Phi_{nn}(z)$?

Find $\Phi_{xx}(z)$ first:

(7.8)
$$\Phi_{xx}(z) = \underbrace{|F_x(z)|^2}_{=?} \cdot \underbrace{\Phi_{ww}(z)}_{=1} \quad (1.2) \text{, where } \Phi_{ww}(z)^2 = \sigma_w^2 = 1$$

Find $F_x(z)$:

From (1): $x_k \cdot q - 0{,}8 \cdot x_k = w_k \Leftrightarrow x_k(q - 0{,}8) = w_k \Leftrightarrow x_k = \frac{1}{q - 0{,}8} \cdot w_k \Rightarrow \boxed{F_x(z) = \frac{1}{z - 0{,}8}}$

Insert $F_x(z)$ in (1.2) to get $\Phi_{xx}(z)$:

$$\Phi_{xx}(z) = |F_x(z)|^2 \cdot \Phi_{ww}(z) = F_x(z) \cdot F_x(-z) \cdot \sigma_w^2 = \frac{0{,}36}{z - 0{,}8} \cdot \frac{1}{z^{-1} - 0{,}8} \overset{\cdot \frac{z}{z}}{=} \frac{1}{z - 0{,}8} \cdot \frac{z}{1 - 0{,}8z} \overset{\cdot \frac{0{,}8}{0{,}8}}{=} \frac{0{,}36}{0{,}8} \cdot \frac{(-1) \cdot z}{(z-0{,}8) \cdot (z - 1{,}25)} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\Phi_{xx}(z) = \frac{-0{,}45z}{(z-0{,}8) \cdot (z-1{,}25)}}$$

Use $\Phi_{xx}(z)$ in (1.1) to finally find $\Phi_{yy}(z)$:

MATLAB

$$\Phi_{yy}(z) = \Phi_{xx}(z) + \Phi_{vv}(z) = \frac{-0{,}45z}{(z-0{,}8)(z-1{,}25)} + 1 = \frac{-0{,}45z + (z-0{,}8)(z-1{,}25)}{(z-0{,}8)(z-1{,}25)} = \frac{z^2 + z \cdot (-0{,}8 - 1{,}25 - 0{,}45) + 0{,}8 \cdot 1{,}25}{(z-0{,}8) \cdot (z-1{,}25)} \Leftrightarrow$$

$$\Leftrightarrow \Phi_{yy}(z) = \frac{(z-0{,}5) \cdot (z-2)}{(z-0{,}8) \cdot (z-1{,}25)} \overset{\cdot z^{-1}/z^{-1}}{=} \frac{(z-0{,}5) \cdot (1 - 2 \cdot z^{-1})}{(z-0{,}8) \cdot (1 - 1{,}25 \cdot z^{-1})} = \frac{2}{1{,}25} \cdot \frac{(z-0{,}5) \cdot (z^{-1} - 0{,}5)}{(z-0{,}8) \cdot (z^{-1} - 0{,}8)} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\Phi_{yy}(z) = \underbrace{1{,}6}_{\sigma_e^2} \cdot \underbrace{\frac{(z-0{,}5)}{(z-0{,}8)}}_{T(z)} \cdot \underbrace{\frac{(z^{-1}-0{,}5)}{(z^{-1}-0{,}8)}}_{T(z^{-1})}}$$

$$, \quad T(z^{-1}) = \frac{(z^{-1}-0{,}5)}{(z^{-1}-0{,}8)} = \frac{(1-0{,}5z)}{(1-0{,}8z)} = \frac{1}{2} \cdot \frac{1}{0{,}8} \cdot \frac{(z-2)}{(z-1{,}25)} = \frac{1}{1{,}6} \cdot \frac{(z-2)}{(z-1{,}25)}$$

(ii) Compute the filter $H_e(z) = \frac{z \cdot \Phi_{xy}(z)}{\sigma_e^2 \cdot T(z^{-1})}$ and take its causal part $[H_e(z)]_+$.

x and v uncorrelated
$R_{xy} = R_{xx}$

OBS: $\Phi_{xy}(z) = \Phi_{xx}(z)$

$$H_e(z) = \frac{z \cdot \Phi_{xx}(z)}{\sigma_e^2 \cdot T(z^{-1})} = \frac{z \cdot \left(\frac{-0{,}45z}{(z-0{,}8)(z-1{,}25)}\right)}{1{,}6 \cdot \frac{1}{1{,}6} \frac{(z-2)}{(z-1{,}25)}} = \frac{-0{,}45 \cdot z^2 \cdot \cancel{(z-1{,}25)}}{(z-0{,}8)\cancel{(z-1{,}25)}(z-2)} = -0{,}45 \cdot z \cdot \frac{z}{(z-0{,}8)(z-2)} \Leftrightarrow$$

MATLAB residue

$$\Leftrightarrow H_e(z) = \underbrace{\frac{0{,}3 \cdot z}{z - 0{,}8}}_{[H_e(z)]_+} + \underbrace{\frac{-0{,}75z}{z-2}}_{[H_e(z)]_-} \Rightarrow \boxed{[H_e(z)]_+ = \frac{0{,}3 \cdot z}{z - 0{,}8}}$$

(iii) The total WF is now (due to cascade connection):

(pg. 283) $$H(z) \stackrel{\downarrow}{=} \frac{1}{z \cdot T(z)} \cdot \left[H_e(z) \right]_+ = \frac{1}{z} \cdot \frac{\cancel{(z-0{,}8)}}{(z-0{,}5)} \cdot \frac{0{,}3 \cdot \cancel{z}}{\cancel{(z-0{,}8)}} \Leftrightarrow$$

$$\boxed{H(z) = \frac{0{,}3}{z-0{,}5}} \Rightarrow H(q) = \frac{0{,}3}{q-0{,}5} = \frac{0{,}3 \cdot q^{-1}}{1-0{,}5 \cdot q^{-1}} \longrightarrow \text{WF for 1-step ahead prediction!}$$

ANSWER: The transfer function of the causal WF for 1-step ahead prediction is:

$$H(z) = \frac{0{,}3}{z-0{,}5} \quad \blacksquare$$

(See plot of frequency & phase response in MATLAB)

II. GOAL: Transfer function of stationary 1-step ahead KF predictor for x_k. Plot frequency & phase response.

SOLUTION: System:
$$\begin{cases} x_{k+1} = 0{,}8 \cdot x_k + w_k \\ y_k = x_k + v_k \end{cases}$$

(8.1c)

Identify: $A = 0{,}8,\ B = 1,\ C = 1,\ D = 1,\ Q \stackrel{\downarrow}{=} \mathrm{cov}(w(t)) = \sigma_w^2 = 0{,}36,\ R = \mathrm{cov}(v(t)) = \sigma_v^2 = 1$

From (8.27): (write time indices as "t" instead of "k" to not confuse with filter gain $K(t)$)

(i) $\hat{x}(t+1|t) = (A - A \cdot K(t) \cdot C) \cdot \hat{x}(t|t-1) + A \cdot K(t) \cdot y(t) \Leftrightarrow$

$$\Leftrightarrow \boxed{\hat{x}(t+1|t) = (0{,}8 - 0{,}8 \cdot K(t)) \cdot \hat{x}(t|t-1) + 0{,}8 \cdot K(t) \cdot y(t)}$$

(ii) $$P(t+1|t) = A \cdot P(t|t-1) A^T - \frac{A \cdot P(t|t-1) \cdot C^T \cdot C \cdot P(t|t-1) \cdot A^T}{C P(t|t-1) C^T + R} + Q \Leftrightarrow // A = 0{,}8;\ C = 1;\ R = 1;\ Q = 0{,}36 // \Leftrightarrow$$

$$\Leftrightarrow P(t+1|t) = 0{,}64 \cdot P(t|t-1) - \frac{0{,}64 \cdot P(t|t-1)^2}{P(t|t-1)+1} + 0{,}36$$

In stationarity $P(t) \to \bar{P}$ when $t \to \infty$. So:

$$\bar{P} = 0{,}64 \bar{P} - \frac{0{,}64 \bar{P}^2}{\bar{P}+1} + 0{,}36 \Leftrightarrow (0{,}36 \bar{P} - 0{,}36)(\bar{P}+1) = -0{,}64 \bar{P}^2 \Leftrightarrow 0{,}36 \bar{P}^2 + \cancel{0{,}36 \bar{P}} - \cancel{0{,}36 \bar{P}} - 0{,}36 + 0{,}64 \bar{P}^2 \Leftrightarrow \bar{P}^2 = 0{,}36 \Rightarrow \boxed{\bar{P} = 0{,}6}$$

(iii) $$K(t) = \frac{P(t|t-1) \cdot C^T}{C \cdot P(t|t-1) C^T + R} = \frac{P(t|t-1)}{P(t|t-1)+1}$$

In stationarity: $K(t) \to \bar{K}$:

$$\bar{K} = \frac{\bar{P}}{\bar{P}+1} = \frac{0{,}6}{1{,}6} \Leftrightarrow \boxed{\bar{K} = 0{,}375}$$

(iv) Therefore: $\hat{x}(t+1) = (0{,}8 - 0{,}8\bar{K}) \hat{x}(t) + 0{,}8 \cdot \bar{K} \cdot y(t) \Leftrightarrow q \cdot \hat{x}(t) - (0{,}8 - 0{,}8\bar{K}) \hat{x}(t) = 0{,}8 \cdot \bar{K} \cdot y(t) \Leftrightarrow$

$$\Leftrightarrow \hat{x}(t) = \frac{0{,}8 \cdot \bar{K}}{q - 0{,}8 + 0{,}8 \cdot \bar{K}} \cdot y(t) = \frac{0{,}3}{q-0{,}5} \cdot y(t)$$

ANSWER: The transfer function for the stationary 1-step ahead KF predictor for x_k is: $$H(z) = \frac{0{,}3}{z-0{,}5} \quad \blacksquare$$ (same as non-causal 1-step ahead predictor WF)

(See plot of frequency & phase response in MATLAB)

$\boxed{\text{QUESTION 1}}$

GIVEN: $y(t)$ has auto-correlation function $R_{yy}(\tau)$. Sampled to achieve: $y[k] = y(k\cdot T_s)$

Wish to up-sample signal by factor 3 by interpolating it.

GOAL: Determine interpolation $\begin{cases} \hat{y}\left(k\cdot T_s + \frac{T_s}{3}\right) = a_1\cdot y_k + a_2 \cdot y_{k+1} \\ \hat{y}\left(k\cdot T_s + \frac{2}{3}\cdot T_s\right) = a_3\cdot y_k + a_4\cdot y_{k+1} \end{cases}$ that minimizes $E\left\{\left(y\left(k\cdot T_s + \frac{m}{3}T_s\right) - \hat{y}\left(kT_s + \frac{m}{3}T_s\right)\right)^2\right\}$, $m = \{1,2\}$.

SOLUTION:

$$\star = E\left[\left(y\left(k\cdot T_s + \frac{m}{3}\cdot T_s\right) - \hat{y}\left(k\cdot T_s + \frac{m}{3}T_s\right)\right)^2\right] = E\left[y\left(k\cdot T_s + \frac{m}{3}T_s\right)^2\right] + E\left[\hat{y}\left(k\cdot T_s + \frac{m}{3}T_s\right)^2\right] + E\left[-2\cdot y\left(kT_s + \frac{m}{3}T_s\right)\cdot \hat{y}\left(kT_s + \frac{m}{3}T_s\right)\right]$$

(i) $m=1$:

$$\star = \underbrace{E\left[y\left(kT_s + \frac{T_s}{3}\right)^2\right]}_{R_{yy}(0)} + E\left[(a_1\cdot y_k + a_2\cdot y_{k+1})^2\right] + E\left[-2\cdot y\left(k\cdot T_s + \frac{T_s}{3}\right)\cdot (a_1\cdot y_k + a_2\cdot y_{k+1})\right] \Leftrightarrow$$

$$\Leftrightarrow \star = R_{yy}(0) + \underbrace{E\left[a_1^2\cdot y_k^2\right]}_{a_1^2\cdot R_{yy}(0)} + \underbrace{E\left[a_2^2\cdot y_{k+1}^2\right]}_{a_2^2\cdot R_{yy}(0)} + E\left[2a_1a_2\cdot y_k\cdot y_{k+1}\right] + E\left[-2\cdot y\left(kT_s + \frac{T_s}{3}\right)\cdot a_1\cdot y_k\right] + E\left[-2y\left(kT_s+\frac{T_s}{3}\right)\cdot a_2\cdot y_{k+1}\right] \Leftrightarrow$$

$$\Leftrightarrow \star = R_{yy}(0)\cdot(1+a_1^2+a_2^2) + E\left[2\cdot a_1\cdot a_2\cdot y(k\cdot T_s)\cdot y(k\cdot T_s + T_s)\right] + E\left[-2y\left(kT_s+\frac{T_s}{3}\right)\cdot a_1\cdot y(kT_s)\right] + E\left[-2y\left(kT_s+\frac{T_s}{3}\right)\cdot a_2\cdot y(kT_s+T_s)\right] \Leftrightarrow$$

$$\Leftrightarrow \star = R_{yy}(0)\cdot(1+a_1^2+a_2^2) + 2\cdot a_1 a_2\cdot R_{yy}(T_s) - 2\cdot a_1\cdot R_{yy}\left(\frac{T_s}{3}\right) - 2a_2\cdot R_{yy}\left(\frac{2T_s}{3}\right)$$

We want to minimize "$\star$" with regards to a_1 and a_2.

$$\begin{cases} \dfrac{d\star}{da_1} = 2a_1\cdot R_{yy}(0) + 2a_2\cdot R_{yy}(T_s) - 2\cdot R_{yy}\left(\frac{T_s}{3}\right) = 0 \Leftrightarrow a_1 = \dfrac{-\cancel{2}a_2\cdot R_{yy}(T_s) + \cancel{2}R_{yy}\left(\frac{T_s}{3}\right)}{\cancel{2}R_{yy}(0)} \quad (1) \\[2ex] \dfrac{d\star}{da_2} = 2a_2\cdot R_{yy}(0) + 2a_1\cdot R_{yy}(T_s) - 2\cdot R_{yy}\left(\frac{2T_s}{3}\right) = 0 \quad (2) \end{cases}$$

Put (1) in (2): $2\cdot a_2\cdot R_{yy}(0) + \cancel{2}\cdot\left(\dfrac{-2a_2\cdot R_{yy}(T_s) + 2R_{yy}\left(\frac{T_s}{3}\right)}{\cancel{2}R_{yy}(0)}\right)\cdot R_{yy}(T_s) - 2R_{yy}\left(\frac{2T_s}{3}\right) = 0 \Leftrightarrow$

$$\Leftrightarrow 2a_2\cdot R_{yy}(0) \dfrac{-2a_2\cdot R_{yy}(T_s)^2 + 2\cdot R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)} = 2R_{yy}\left(\frac{2T_s}{3}\right) \Leftrightarrow$$

$$\Leftrightarrow a_2\cdot\left(\cancel{2}\cdot R_{yy}(0) - \cancel{2}\cdot\dfrac{R_{yy}(T_s)^2}{R_{yy}(0)}\right) = \cancel{2}R_{yy}\left(\frac{2T_s}{3}\right) - \cancel{2}\dfrac{R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)} \Leftrightarrow$$

$$\Leftrightarrow a_2 = \dfrac{\dfrac{R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{\cancel{R_{yy}(0)}}}{\dfrac{R_{yy}(0)^2 - R_{yy}(T_s)^2}{\cancel{R_{yy}(0)}}} \Leftrightarrow \boxed{a_2 = \dfrac{R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2}}$$

And a_1:

$$a_1 = \dfrac{-\cancel{2}\cdot\left(\dfrac{R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2}\right)\cdot R_{yy}(T_s) + \cancel{2}\cdot R_{yy}\left(\frac{T_s}{3}\right)}{\cancel{2}R_{yy}(0)}$$

(continues)

$$a_1 = \frac{-R_{yy}\left(\frac{2T_s}{3}\right)\cdot \cancel{R_{yy}(0)}\cdot R_{yy}(T_s) + \cancel{R_{yy}\left(\frac{T_s}{3}\right) R_{yy}(T_s)^2} + R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(0)^{\cancel{2}} - \cancel{R_{yy}\left(\frac{T_s}{3}\right) R_{yy}(T_s)^2}}{\cancel{R_{yy}(0)}\cdot\left(R_{yy}(0)^2 - R_{yy}(T_s)^2\right)}$$

$$\Leftrightarrow \boxed{a_1 = \frac{-R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(T_s) + R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(0)}{R_{yy}(0)^2 - R_{yy}(T_s)^2}}$$

Run a D-test to check if the value given by a_1 & a_2 really is a min-value at $\star(a_1, a_2)$:

$$\frac{d^2\star}{d^2a_1} = 2\cdot R_{yy}(0) \qquad \frac{d^2\star}{d^2a_2} = 2R_{yy}(0) \qquad \frac{d^2\star}{da_1a_2} = 2\cdot R_{yy}(T_s)$$

$$D = \frac{d^2\star(a_1,a_2)}{d^2a_1}\cdot\frac{d^2\star(a_1,a_2)}{d^2a_2} - \left[\frac{d^2\star(a_1,a_2)}{da_1a_2}\right]^2 = 4\cdot R_{yy}(0)^2 - 4\cdot R_{yy}(T_s)^2$$

To have a minimum at (a_1, a_2), D must be positive, so $R_{yy}(0)^2$ must be greater than $R_{yy}(T_s)^2$.

(ii) $\underline{m=2}$:

$$\star = E\left[y\left(kT_s + \frac{2T_s}{3}\right)^2\right] + E\left[\left(a_3\cdot y_k + a_4\cdot y_{k+1}\right)^2\right] + E\left[-2y\left(kT_s + \frac{2T_s}{3}\right)\cdot\left(a_3\cdot y_k + a_4\cdot y_{k+1}\right)\right] \Leftrightarrow$$

$$\Leftrightarrow \star = R_{yy}(0) + E\left[a_3^2\cdot y_k^2\right] + E\left[a_4^2\cdot y_{k+1}^2\right] + E\left[2\cdot a_3 a_4 \cdot y_k\cdot y_{k+1}\right] + E\left[-2a_3\, y\left(kT_s + \frac{2T_s}{3}\right)\cdot \overset{=y(kT_s)}{y_k}\right] + E\left[-2a_4\, y\left(kT_s + \frac{2T_s}{3}\right)\cdot \overset{=y(kT_s+T_s)}{y_{k+1}}\right] \Leftrightarrow$$

$$\Leftrightarrow \star = R_{yy}(0)\cdot\left(1 + a_3^2 + a_4^2\right) + 2\cdot a_3 a_4 \cdot R_{yy}(T_s) - 2a_3\cdot R_{yy}\left(\frac{2T_s}{3}\right) - 2a_4\cdot R_{yy}\left(\frac{T_s}{3}\right)$$

We want to minimize "$\star$" with regards to a_3 and a_4. Notice that the expression for "$\star$" now is similar to the one when $m=1$, replacing a_3 by a_2 and a_4 by a_1. So we get:

$$\boxed{a_3 = \frac{R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2}} \quad \text{and} \quad \boxed{a_4 = \frac{R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2}}$$

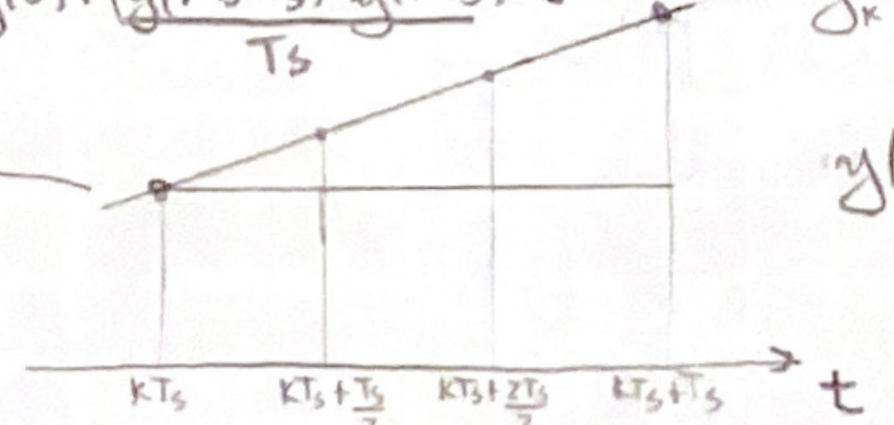
Equivalent to the case $m=1$, (a_3, a_4) minimizes "$\star$" if $|R_{yy}(0)| > |R_{yy}(T_s)|$.

<u>ANSWER</u>:

$$\left|\begin{array}{l} a_1 = a_4 = \dfrac{R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2} \\[2ex] a_2 = a_3 = \dfrac{R_{yy}\left(\frac{2T_s}{3}\right)\cdot R_{yy}(0) - R_{yy}\left(\frac{T_s}{3}\right)\cdot R_{yy}(T_s)}{R_{yy}(0)^2 - R_{yy}(T_s)^2} \end{array}\right. \quad , \text{ for } |R_{yy}(0)| > |R_{yy}(T_s)|$$

<u>Note</u>: In order for the optimal solution calculated above to be a linear interpolation, the interpolated points must be in the same line as the known points. So:

$$y(t) = y(0) + \frac{\left(y(kT_s+T_s) - y(kT_s)\right)}{T_s}\cdot t$$



$$y_k = y(0) + \frac{\left(y_{k+1} - y_k\right)}{T_s}\cdot k\cdot T_s = y(0) + \left(y_{k+1} - y_k\right)\cdot k \quad \& \quad y_{k+1} = \dots = y(0) + \left(y_{k+1} - y_k\right)(k+1)$$

$$y\left(kT_s + \frac{T_s}{3}\right) = \alpha\cdot y_k + \beta\cdot y_{k+1} = y(0)\cdot\underbrace{(\alpha+\beta)} + \alpha\left(y_{k+1} - y_k\right)k + \beta\left(y_{k+1} - y_k\right)(k+1) = \underbrace{y(0)} + \frac{\left(y_{k+1} - y_k\right)}{T_s}\cdot\left(kT_s + \frac{T_s}{3}\right)$$

(continues)

So: $\begin{cases} \alpha+\beta=1 \Rightarrow \alpha = 1-\beta \\ \alpha k + \beta(k+1) = k+\frac{1}{3} \end{cases} \rightarrow (1-\beta)k+\beta(k+1) = k+\frac{1}{3} \Leftrightarrow \cancel{k}-\cancel{k\beta}+\cancel{\beta k}+\beta=\cancel{k}+\frac{1}{3} \Leftrightarrow \boxed{\beta=\frac{1}{3}} \Rightarrow \boxed{\alpha=\frac{2}{3}}$

Therefore, for the optimal solution to be an interpolation:

$y(kT_s+\frac{T_s}{3}) = \frac{2}{3}\cdot y_k+\frac{1}{3}\cdot y_{k+1}$, i.e., $\boxed{a_1=\frac{2}{3}}$ and $\boxed{a_2=\frac{1}{3}}$

Equivalently:

$y(kT_s+\frac{2T_s}{3}) = \alpha\cdot y_k+\beta y_{k+1} = y(0)\cdot(\alpha+\beta)+(y_{k+1}-y_k)\cdot(\alpha k+\beta(k+1)) = y(0)+\frac{(y_{k+1}-y_k)}{\cancel{T_s}}\cdot(k\cdot \cancel{T_s}+\frac{2\cancel{T_s}}{3}) \Leftrightarrow$

$\Leftrightarrow \begin{cases} \alpha+\beta=1 \Leftrightarrow \alpha=1-\beta \\ \alpha k+\beta(k+1)=k+\frac{2}{3} \end{cases} \rightarrow (1-\beta)k+\beta(k+1)=k+\frac{2}{3} \Leftrightarrow \cancel{k}-\cancel{\beta k}+\cancel{\beta k}+\beta=\cancel{k}+\frac{2}{3} \Rightarrow \beta=\frac{2}{3}$ & $\alpha=\frac{1}{3}$

So. $\boxed{a_3=\frac{1}{3}}$ and $\boxed{a_4=\frac{2}{3}}$

In summary, the conditions $a_1=a_4=\frac{2}{3}$ and $a_2=a_3=\frac{1}{3}$ must be fulfilled for linear interpolation to be optimal in the MSE sense. These are requirements on the auto-correlation function, since a_1, a_2, a_3 and a_4 are functions of $R_{yy}(\tau)$. ∎

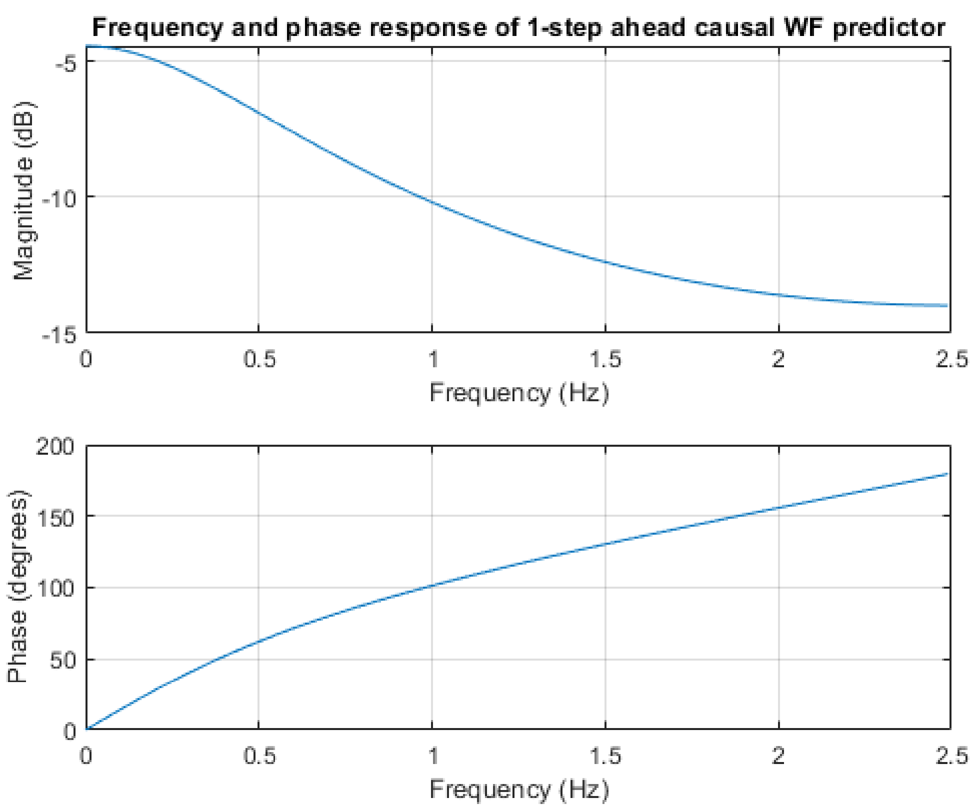
Contents


```
% Homework 3

clear;
```

Causal Wiener filter for 1-step ahead prediction

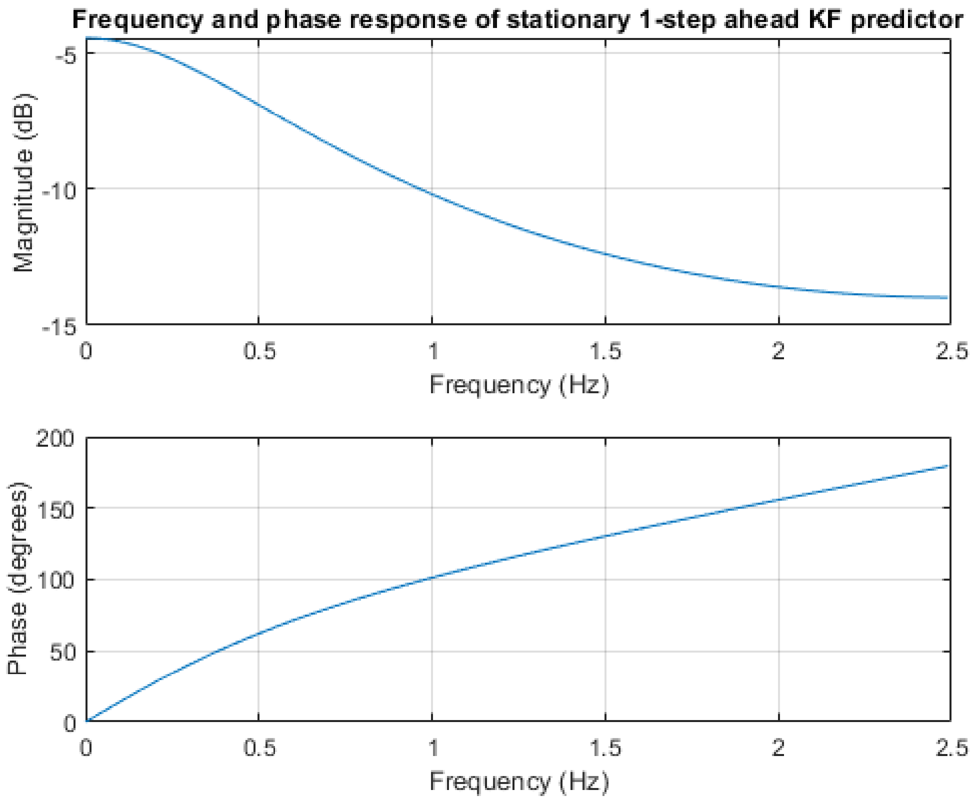
```
% Plot frequency and phase spectrum
fs = 5; % [Hz]
nrPoints = 256; % standard
b = 0.3;
a = [-0.5 1];
freqz(b,a,nrPoints,fs);
title('Frequency and phase response of 1-step ahead causal WF predictor');
```



Stationary 1-step ahead Kalman fitler predictor

```
% Plot frequency and phase spectrum
fs = 5; % [Hz]
nrPoints = 256; % standard
b = 0.3;
a = [-0.5 1];
freqz(b,a,nrPoints,fs);
title('Frequency and phase response of stationary 1-step ahead KF predictor');
```

```
% Reference:
% https://se.mathworks.com/help/signal/ref/freqz.html#bt8l9fo-1
```



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