MUS420/EE367A Lecture 7
Wave Digital Filters

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Wave Digital Filters

A *Wave digital filter* (WDF) is a particular kind of *digital filter* (or finite difference scheme) based on physical modeling principles.

- Developed to digitize lumped *electrical* circuit elements:
  - inductors
  - capacitors
  - resistors
  - gyrators, circulators, etc., (classical circuit theory)
- Each element is digitized by the *bilinear transform*
- *Wave variables* are used in place of physical variables (new), yielding superior numerical properties.
- Element connections involve *wave scattering*

Wave Digital Filter (WDF) Construction

Wave digital elements may be derived from their describing differential equations (in continuous time) as follows:

1. Express forces and velocities as *sums of traveling-wave components* (*"wave variables"*):

$$\begin{aligned} f(t) &= f^{+}(t) + f^{-}(t) \\ v(t) &= v^{+}(t) + v^{-}(t) \end{aligned}$$

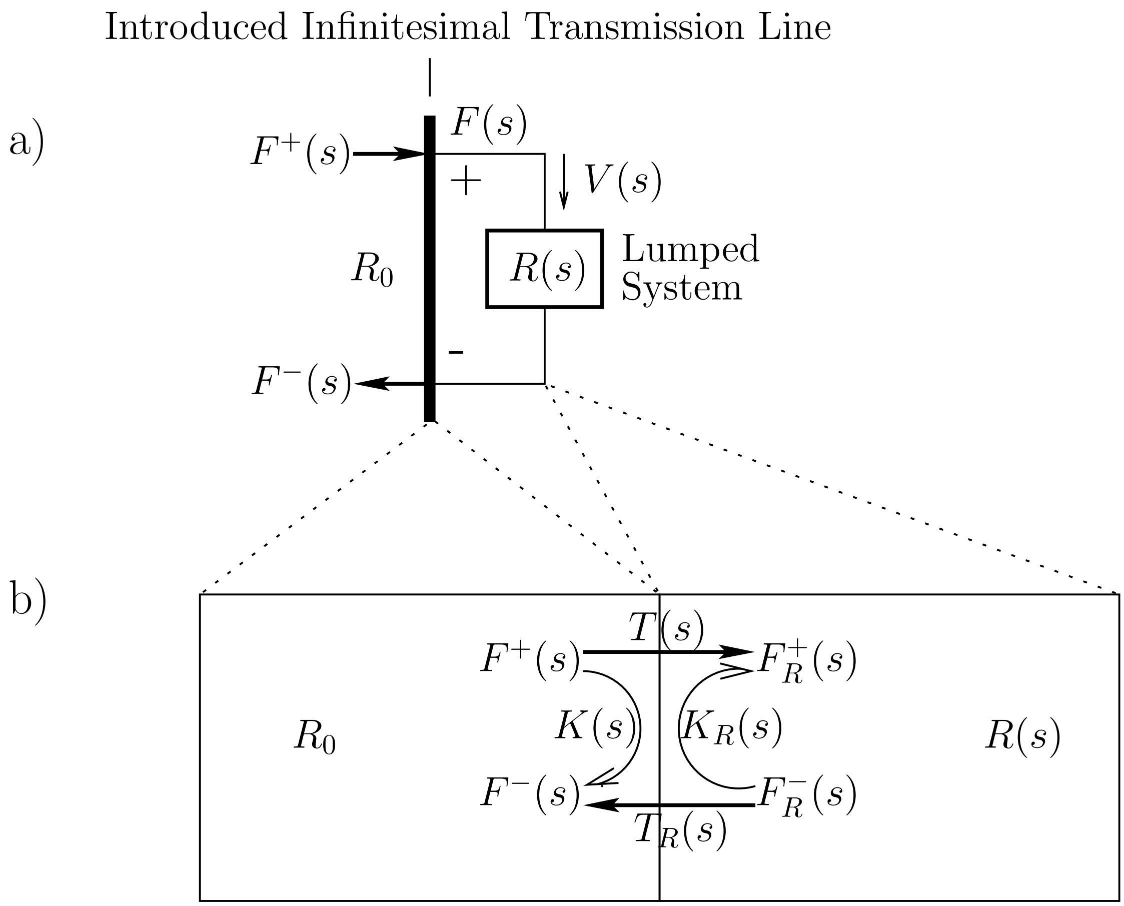
   The actual "travel time" is always *zero*.
   (For historical reasons, WDFs typically use traveling-wave components scaled by 2.)

2. Digitize via the *bilinear transform* (trapezoid rule)

3. Use *scattering junctions* ("*adaptors*") to connect elements together in

   - *series* (common velocity, summing forces), or
   - *parallel* (common force, summing velocities).

Wave Variable Decomposition



- The inserted waveguide impedance R_0 is *arbitrary* because it was *physically introduced*.
- The element now interfaces to other elements by abutting its waveguide (transmission line) to that of other element(s).
- Such junctions involve *lossless wave scattering*:

$$\begin{aligned} F_R^+(s) &= T(s)F^+(s) + K_R(s)F_R^-(s) \\ F^-(s) &= T_R(s)F_R^-(s) + K(s)F^+(s) \end{aligned}$$

Element Reflectance

Imposing *physical continuity constraints* across the junction:

$$\begin{aligned} F(s) &= F_R(s) \\ 0 &= V(s) + V_R(s) \end{aligned}$$

with

$$\begin{aligned} F(s) &= F^+(s) + F^-(s) \\ F_R(s) &= F_R^+(s) + F_R^-(s) \\ V(s) &= V^+(s) + V^-(s) = \frac{F^+(s)}{R_0} - \frac{F^-(s)}{R_0} \\ V_R(s) &= V_R^+(s) + V_R^-(s) = \left[\frac{F_R^+(s)}{R(s)} - \frac{F_R^-(s)}{R(s)}\right] \end{aligned}$$

we obtain the *reflection transfer function* ("reflectance") of the element with impedance $R(s)$:

$$\boxed{S_R(s) \stackrel{\Delta}{=} \frac{F^-(s)}{F^+(s)} = \frac{R(s) - R_0}{R(s) + R_0}}$$

This is the *impedance step over the impedance sum*, the usual force-wave reflectance at an impedance discontinuity, but now in the Laplace domain.

Reflectance of Ideal Mass, Spring, and Dashpot

For a mass m kg, the impedance and reflectance are respectively

$$\begin{aligned} R_m(s) &= ms \\ \Rightarrow \quad S_m(s) &= \frac{ms - R_0}{ms + R_0} \end{aligned}$$

This reflectance is a *stable first-order allpass filter*, as expected, since energy is not dissipated by a mass.

For a spring k N/m, we have

$$\begin{aligned} R_k(s) &= \frac{k}{s} \\ \Rightarrow \quad S_k(s) &= \frac{\frac{k}{s} - R_0}{\frac{k}{s} + R_0} \end{aligned}$$

also allpass as expected.

For a dashpot μ N s/m, we have

$$\begin{aligned} R_\mu(s) &= \mu \\ \Rightarrow \quad S_\mu(s) &= \frac{\mu - R_0}{\mu + R_0} \end{aligned}$$

Bilinear Transformation

To digitize via the bilinear transform, we make the substitution

$$s = c\frac{1 - z^{-1}}{1 + z^{-1}}$$

where c is any positive real constant (typically $2/T$).

For the ideal mass reflectance

$$S_m(s) = \frac{ms - R_0}{ms + R_0}$$

the bilinear transform yields

$$\tilde{S}_m(z) = \frac{p_m - z^{-1}}{1 - p_m z^{-1}}$$

with

$$p_m \stackrel{\Delta}{=} \frac{mc - R_0}{mc + R_0}$$

Note that $|p_m| < 1$ and $|\tilde{S}_m(e^{j\omega T})| = 1$. The stable allpass nature of the digitized mass reflectance is preserved by the bilinear transform, as always.

Important Observation:

If we choose $R_0 = mc$, then $p_m = 0$ and $\tilde{S}_m(z) = -z^{-1} \;\Rightarrow\;$ *no delay-free path through the mass reflectance*

Digitized Reflectances Without Delay-Free Paths

Plan:

1. Fix the bilinear-transform frequency-scaling parameter c once for the whole system (so there is only one frequency-warping)
2. Set the "connector" wave impedance R_0 separately for each circuit element to eliminate the delay-free path in its reflectance
3. We will then get scattering when we connect different elements together

This yields the following elementary reflectances:

Element		**Reflectance**
ideal spring (capacitor)	$\leftrightarrow$	*unit delay*
ideal mass (inductor)	$\leftrightarrow$	*unit delay and sign inversion*
ideal dashpot (resistor)	$\leftrightarrow$	0

The original element values remain only in the waveguide-interface impedances $R_0 = k/c, mc, \mu$

Wave Digital Elements

In summary, our chosen digital element reflectances (and their connecting wave impedances R_0) are

- "Wave digital mass" (interface impedance $R_0 = mc$)

  $$\boxed{\tilde{S}_m(z) = -z^{-1}} \quad \text{(mass reflectance)}$$

- "Wave digital spring" ($R_0 = k/c$)

  $$\boxed{\tilde{S}_k(z) = z^{-1}} \quad \text{(spring reflectance)}$$

- "Wave digital dashpot" ($R_0 = \mu$)

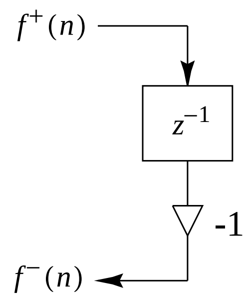
  $$\boxed{\tilde{S}(z) = 0} \quad \text{(dashpot [non-]reflectance)}$$

  (In this case, the interface is the element itself.)

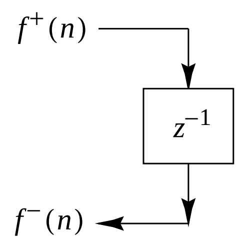
These are the *discrete-time reflectances* of the basic circuit building-blocks as seen from their interface-waveguides

We still have the usual freedom in choosing our bilinear-transform frequency-scaling constant c

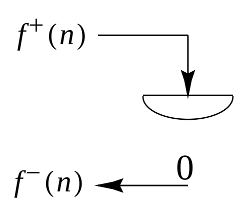
Elementary Wave Flow Diagrams



Wave digital mass



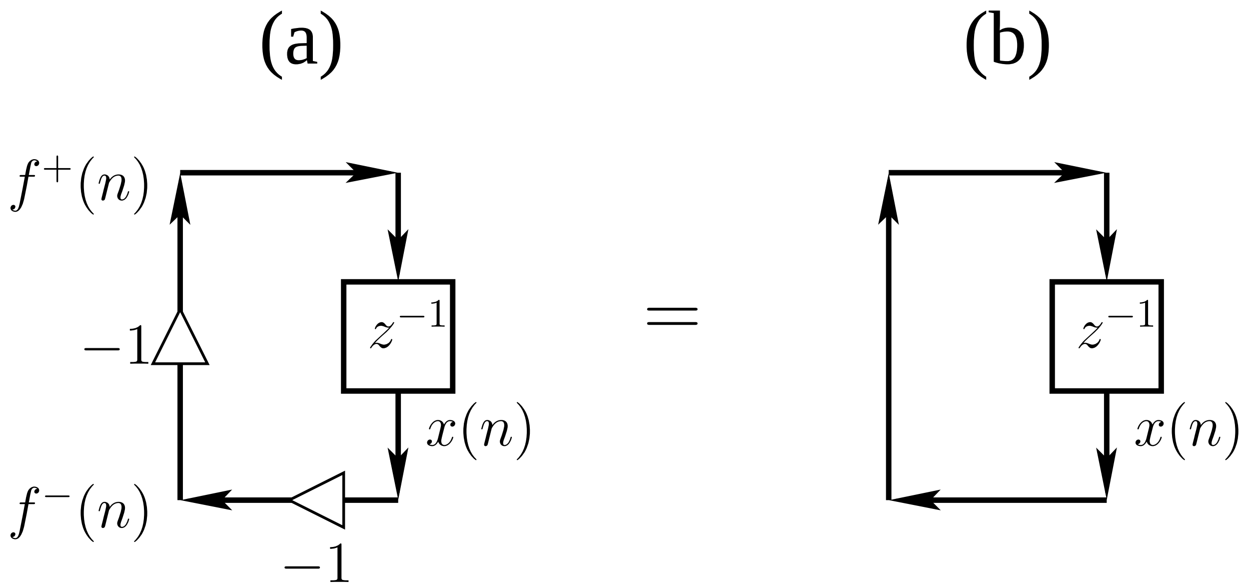
Wave digital spring



Wave digital dashpot

Example: "Piano hammer in flight"

Mass m at constant velocity, force-wave simulation:



- The reflecting termination on the left corresponds to zero force on the mass
- A nonzero state variable $x(n)$ corresponds to a nonzero *velocity* for the mass:

$$
\begin{aligned}
v(n) \;=\; v^+(n) + v^-(n) \;&=\; \frac{f^+(n)}{R_0} - \frac{f^-(n)}{R_0} \\
&=\; \frac{f^+(n)}{mc} + \frac{f^+(n-1)}{mc} \;=\; \frac{x(n+1)+x(n)}{mc} \\
&=\; \frac{2}{mc}x(n) \;=\; \frac{T}{m}x(n)
\end{aligned}
$$

when $c = 2/T$ is chosen for the bilinear transform

Mass Momentum and Energy

- Above we found the mass *velocity* to be
$$v(n) \;=\; \frac{2}{mc}x(n) \;=\; \frac{T}{m}x(n)$$
when $c = 2/T$ is chosen for the bilinear transform

- The *momentum* of the mass is therefore
$$p(n) \;\stackrel{\Delta}{=}\; m\,v(n) \;=\; \frac{2}{c}\,x(n) \;=\; T\,x(n)$$
when $c = 2/T$

- State variable $\boxed{x(n) = p(n)/T}$ is *mass momentum per sample*

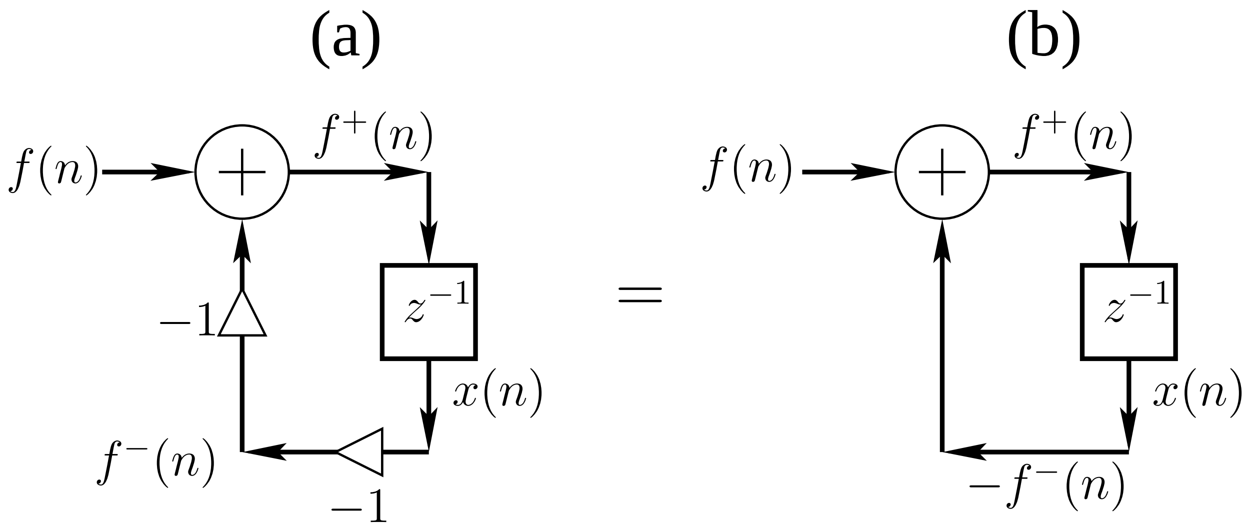
- Since momentum is conserved, *momentum waves* are good to consider in place of velocity waves

- The *kinetic energy* of the mass is given by
$$\mathcal{E}_m = \frac{1}{2}mv^2(n) = \frac{p^2(n)}{2m} = \frac{2}{mc^2}x^2(n) \rightarrow \frac{[T\,x(n)]^2}{2m}$$
for $c \rightarrow 2/T$

- The *potential energy* of the mass-in-flight is of course zero ($f(n) \equiv 0$)

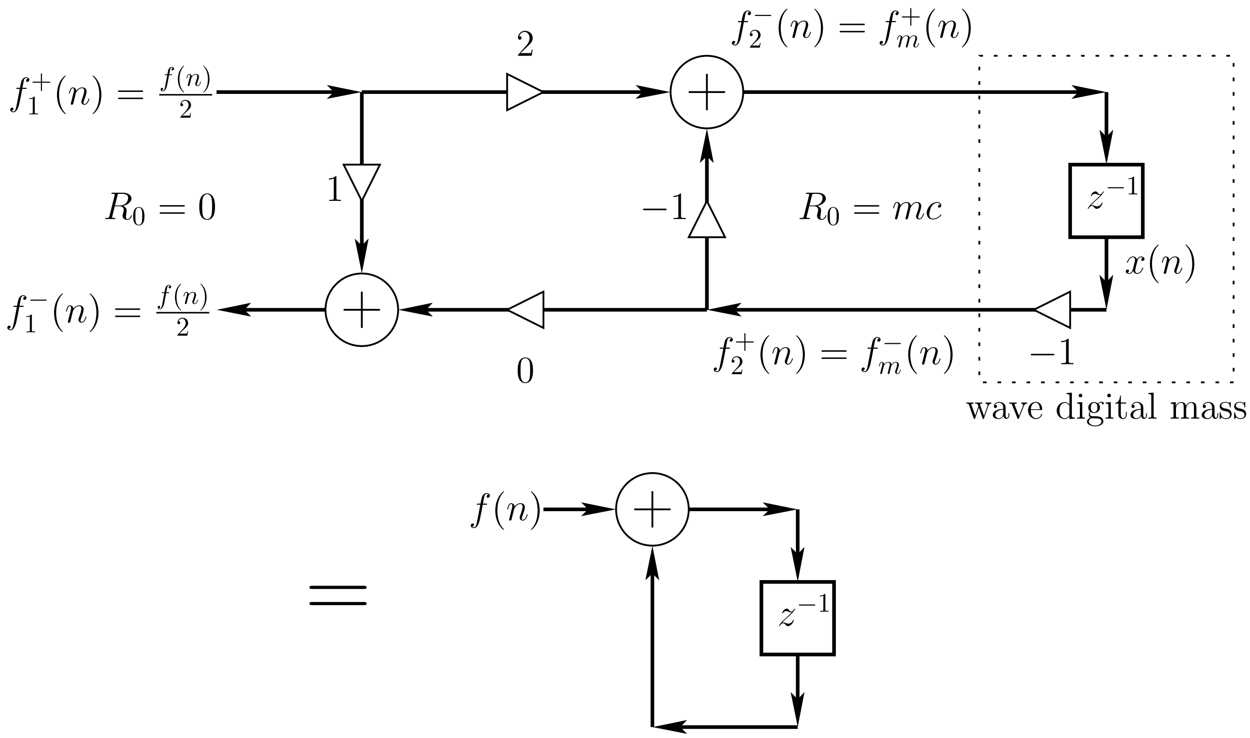
Force Driving a Mass

$$f(n) = f^+(n) + f^-(n) \quad \Rightarrow \quad f^+(n) = f(n) - f^-(n)$$



Wave digital mass driven by external force $f(n)$.

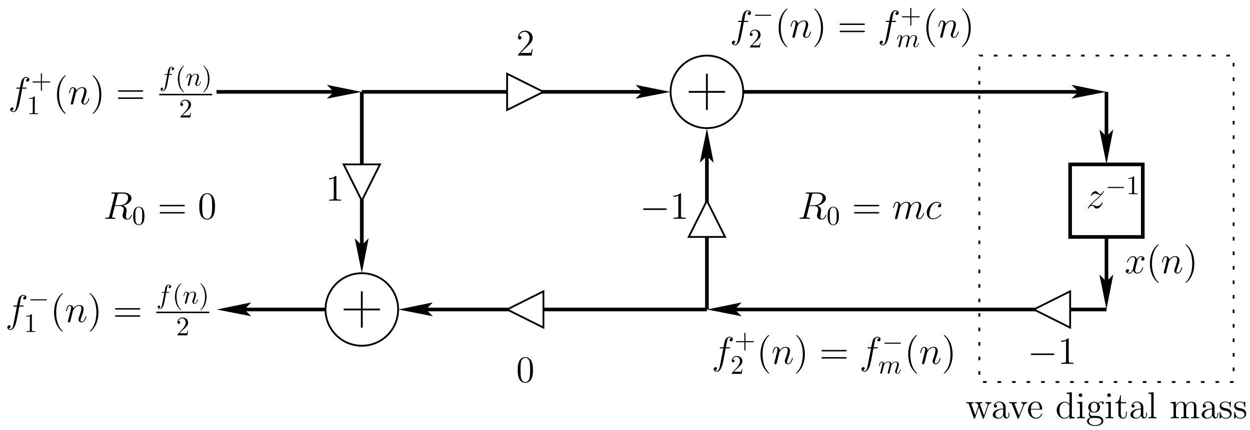
Traveling-Wave View of Driving Force



- Parallel junction with $R_0 = 0$ on the force side and $R_0 = mc$ on the mass side
- Impedance step over impedance sum is $R = (mc - 0)/(mc + 0) = 1$
- Obviously *non-physical* (see next page)

Zero Source-Impedances are Non-Physical

We postulated the following driving-source interface:



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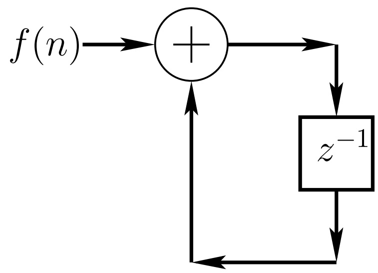
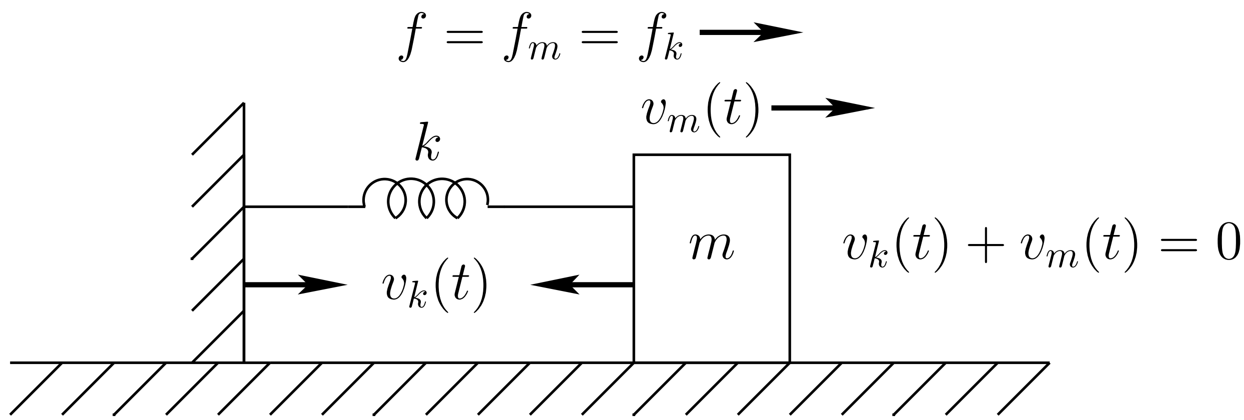


Non-physical because:

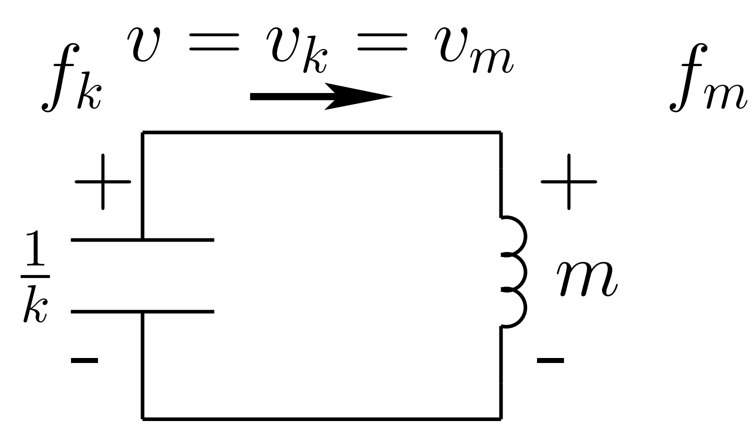
- Velocity transmission is zero $\Rightarrow$ *no power* delivered
- There can be no traveling force (voltage) wave in a zero impedance (which would "short it out")
- Recall power waves: $[f^+(n)]^2/R_0 = \infty$ if $f^+(n) \neq 0$
- Zero source-impedances can be a useful idealization, but be careful
- **Exercise:** Study the case of small $R_0 = \epsilon > 0$.

Spring-Driven Mass

To keep the model physical, let's use a pre-compressed *spring* as our force-source for driving the mass:



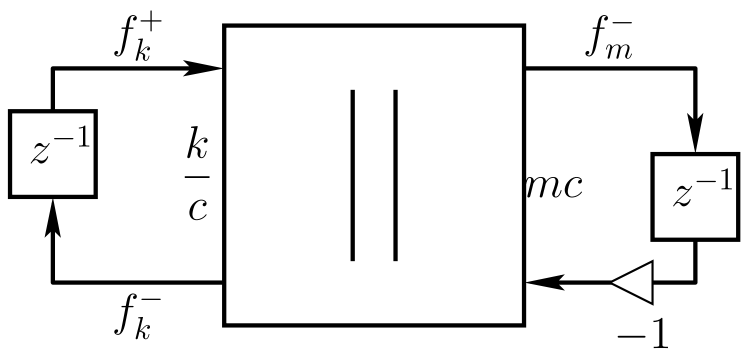
Physical Diagram



Equivalent Circuit

- The mass and spring form a *loop*, so the connection can be defined as either parallel or series (as determined by the element reference directions)

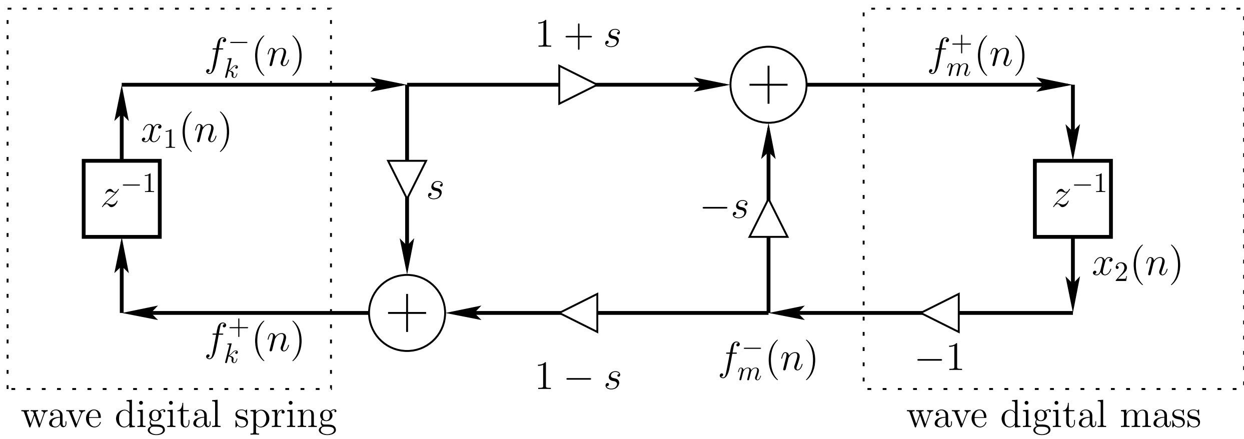
- We arbitrarily choose a *parallel* junction, giving the following physical constraints:
  - $f_k(n) = f_m(n)$ (common force)
  - $v_k(n) + v_m(n) = 0$ (sum of spring-compression-velocity and rightgoing-mass velocity is zero)
- **Exercise:** Work out the case for a series junction and verify everything comes out the same physically
- Connecting our wave digital spring and mass at a parallel force-wave junction is depicted as follows:



WDF Diagram

Note the WDF symbol "||" for a *parallel adaptor* (scattering junction)

Expanded Wave Digital Spring-Mass System



State variables labeled $x_1(n)$ and $x_2(n)$

Low-Frequency Analysis:

- Assume sampling rate $f_s = 1/T$ is large $\Rightarrow$
- Bilinear transform constant $c = 2/T$
- Frequency warping not an issue
- Physical simulation should be very accurate

The reflection coefficient for our parallel force-wave connection is given as usual by the *impedance step over the impedance sum:*

$$s = \frac{mc - k/c}{mc + k/c} = \frac{m2/T - kT/2}{m2/T + kT/2} = \frac{m - kT^2/4}{m + kT^2/4} \approx 1$$

We can now see what's going physically at low frequencies relative to the sampling rate:

Low-Frequency Spring-Driven-Mass Analysis

Referring to the previous figure:

- We found earlier that $x_2(n) \approx p_m(n)/T$ where $p_m(n)$ is the mass momentum at time n, and T is the sampling interval
- We similarly find that $x_1(n) = f_k^-(n) \approx f(n)/2$, so that the mass sees $(1+s)f(n)/2 \approx f(n)$ coming in each sample from the summer, *i.e.*,

$$\frac{p_m(n)}{T} \approx \frac{p_m(n-1)}{T} + f(n)$$

- Multiplying through by T gives the *momentum update* per sample:

$$p_m(n) \approx p_m(n-1) + f(n)T \stackrel{\Delta}{=} p_m(n-1) + \Delta p(n)$$

  where $\Delta p(n) \stackrel{\Delta}{=} f(n)T$ is the momentum transferred to the mass by constant force $f(n)$ during one sampling interval T

- This makes physical sense and suggests *momentum* and *momentum-increment* samples as an appealing choice of wave variables

Classic WDF Wave Variables

We have been using our usual traveling-wave decomposition of force and velocity waves:

$$\begin{aligned} f(t) &= f^+(t) + f^-(t) = R_0 v^+(t) - R_0 v^-(t) \\ v(t) &= v^+(t) + v^-(t) = \frac{f^+(t)}{R_0} - \frac{f^-(t)}{R_0} \end{aligned}$$

where R_0 is the wave impedance of the medium, or

$$\begin{bmatrix} f(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} R_0 & -R_0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v^+(t) \\ v^-(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{R_0} & -\frac{1}{R_0} \end{bmatrix} \begin{bmatrix} f^+(t) \\ f^-(t) \end{bmatrix}$$

Inverting these gives

$$\begin{bmatrix} v^+(t) \\ v^-(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1/R_0 & 1 \\ -1/R_0 & 1 \end{bmatrix} \begin{bmatrix} f(t) \\ v(t) \end{bmatrix}$$

$$\begin{bmatrix} f^+(t) \\ f^-(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & R_0 \\ 1 & -R_0 \end{bmatrix} \begin{bmatrix} f(t) \\ v(t) \end{bmatrix}$$

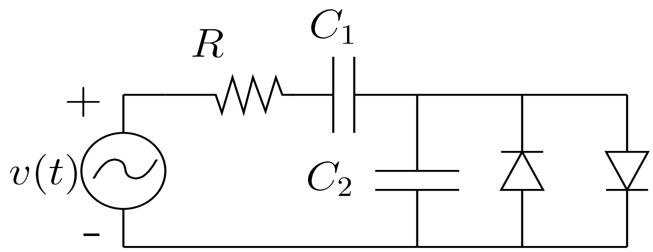
In the WDF literature, the second case is typically used, multiplied by 2, and replacing force and velocity by voltage and current:

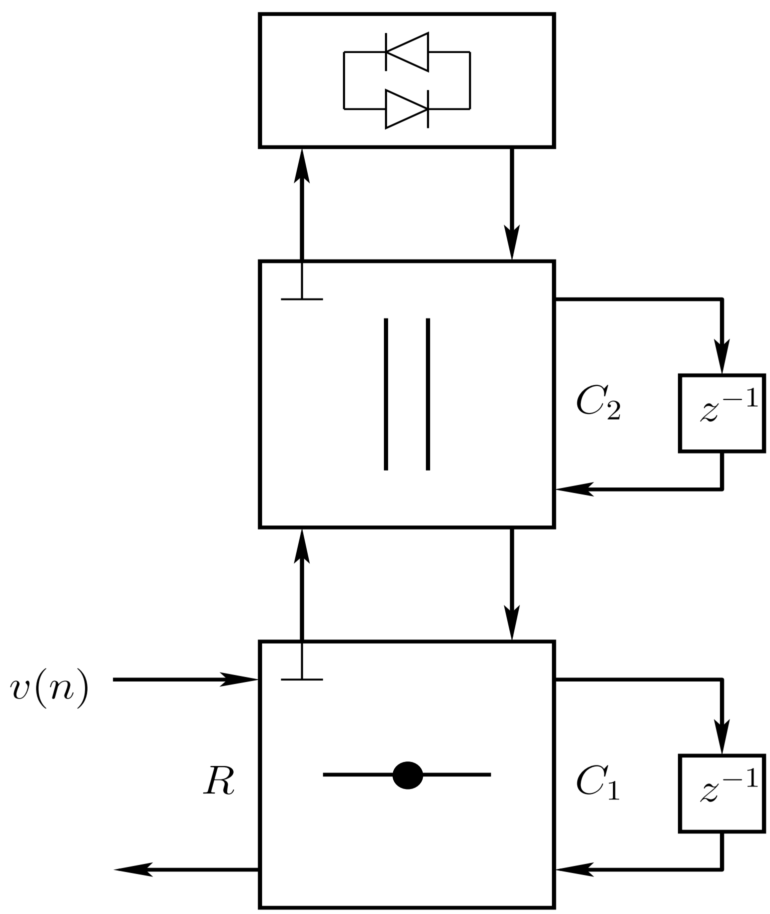
$$\begin{aligned} a(t) &= v(t) + R_0\, i(t) \\ b(t) &= v(t) - R_0\, i(t) \end{aligned}$$

where $v(t)$ is now *voltage* and $i(t)$ denotes *current*. Thus, $a(t) = 2v^+(t)$ and $b(t) = 2v^-(t)$ (doubled voltage traveling-wave components)

Binary Connection Tree

It has become common practice to organize WDF elements into a *Binary Connection Tree* (BCT):





Reflection-Free Ports

- The symbol $\perp$ on a WDF adaptor port denotes a *reflection-free port* (RFP)
- To make a port reflection-free, its wave-impedance must be the
  - *parallel combination* of the other port impedances for a parallel adaptor, or
  - *series combination* of the other port impedances for a series adaptor

  This choice of port impedance zeros the impedance step "seen" by waves in the RFP, thus suppressing instantaneous reflection from it
- All ports *outgoing* from the BCT root must be RFPs, for computability (no delay-free loops)
- Computations propagate (each sample) from the leaves of the tree (delay element outputs) up to the root, where there is a final reflection which then propagates back down to all of the reflection-free ports, thereby updating all of the delay elements (capacitor/spring and inductor/mass states)
- When an element value changes (typically a resistor), RFPs must be *recalculated up to the root*.

Reflection-Free Port Coefficients

For an N-port adaptor, with port wave-impedances R_i, $i = 1, 2, \ldots, N$, let's arbitrarily designate port N as the *reflection-free port* (the one on top). It is convenient to define the port *conductances* $G_i \triangleq 1/R_i$. To suppress reflection on port N, we need, for a *parallel adaptor*,

$$\begin{aligned} R_N &= R_1 \parallel R_2 \parallel \cdots \parallel R_{N-1} \Leftrightarrow \\ G_N &= G_1 + G_2 + \cdots + G_{N-1} \end{aligned}$$

and, for a *series adaptor*,

$$R_N = R_1 + R_2 + \cdots + R_{N-1}.$$

Recall the *alpha parameters* for an N-port series scattering junction, derived from the physical constraints that the velocities be equal and the forces sum to zero at the (series) junction:

$$\alpha_i \triangleq \frac{2R_i}{R_1 + R_2 + \cdots + R_N} = \boxed{\frac{R_i}{R_N}}$$

when port N is reflection free.

Since $\sum_{i=1}^{N} \alpha_i = 2$, we have $\boxed{\alpha_N = 1}$ and $\boxed{\displaystyle\sum_{i=1}^{N-1} \alpha_i = 1}$.

Example

See page 42 of David Yeh's WDF Tutorial[1]

Shockley diode equation ("diode law")

$$I(t) = I_s \cdot \left(e^{\frac{V_d}{nV_T}} - 1 \right)$$

where

$$\begin{aligned} I &= \text{diode current} \\ I_s &= \text{diode reverse leakage current} \\ V_d &= \text{voltage across the diode} \\ n &= \text{ideality factor (1 for ideal, up to 2 or more otherwise)} \\ V_T &= \text{thermal voltage } kT/q \\ k &= \text{Boltzmann constant} \\ q &= \text{electron charge} \\ T &= \text{temperature} \end{aligned}$$

[1] https://ccrma.stanford.edu/~dtyeh/papers/wdftutorial.pdf

Topology Issues

- Classical WDFs are composed of *parallel* and *series* connections of elements
- A Binary Connection Tree (BCT) can represent any such parallel/series network
- R-Nodes
  - Some circuits, such as the "bridged T" circuit, cannot be represented using parallel/series connections of elements
  - These circuits are modeled using more general *scattering matrices*
  - Such circuits are called R-*Nodes* in the overall WDF network graph
  - R-Nodes connect naturally to BCT graphs, since all signals are compatible traveling-wave components
  - An open issue is how to minimize the computational complexity of R-node scattering matrices

SPQR Decomposition

Every graph can be decomposed into Series (S), Parallel (P), and R ("Rigid") type subgraphs (Q is the degenerate case consisting of only one graph edge)

- S and P handled by standard WDF methods (BCT)
- R node characterized by its scattering matrix
- Modified Nodal Analysis (MNA) may be used to find the R-node scattering matrix (see Werner et al. reference below)

WDF State Space Interpretation

Digital filters can be expressed in state-space form as

$$\underline{x}(n+1) = A\,\underline{x}(n) + B\,\underline{u}(n)$$

by simply enumerating all delay elements as state variables $\underline{x}^T(n) = [x_1(n), x_2(n), \ldots, x_N(n)]$, and finding the state transition matrix A by inspection. Any inputs are collected in $\underline{u}(n)$ and determine the B matrix.

- For WDFs, the A matrix is a *scattering matrix*
- The A matrix is *orthogonal* (lossless) for reactive elements (masses, springs)
- The state variables are all *sampled traveling waves*
- Physical state variables (bilinear transformed) are obtainable by *summing* (capacitors, springs) or *subtracting* (inductors, masses) the input and output of the unit delays:

$$y_k(n) = x_k(n) \pm x_k(n-1)$$

- In comparison to other state-space models, WDF state-space form has top numerical properties due to its lossless scattering formulation

Nonlinear Wave Digital Filters

A WDF network tree can have a multiport *instantaneous nonlinearity* at its root:

- A typical instantaneous nonlinearity is a *nonlinear resistor* $R(v)$ (such as a diode) or a *dependent source* (as used in transistor models, etc.)
- Because the resistance of a nonlinear resistor depends on the voltage across it, there is no way to avoid an instantaneous reflection in general (no fixed port-impedance can match it for all input conditions)
- The nonlinearity is placed at the *root* of the BCT
  A delay-free path is "computable" only there (we get one per tree)
- Each sample, computations propagate up the tree to the root, reflecting instantaneously, then back down to all the reflection-free ports
- The nonlinear reflectance can be pre-computed and stored for fast interpolated table look-up in real time (no iterations)
- If the nonlinearity cannot be placed at the root of the WDF BCT (e.g., because there are two or more

nonlinearities in the circuit) the delay-free-path may be solved iteratively using Newton's method et al.

- Alternatively, all nonlinearities can be placed at the root of the WDF tree and connect to the BCT through an R-Node. References:
  1. "Wave Digital Filter Adaptors for Arbitrary Topologies and Multiport Linear Elements"[2]
  2. "Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities"[3]

     Kurt Werner et al.
     Int. Conf. Digital Audio Effects (DAFx-15)
     Trondheim, Norway, 2015

Dynamic Nonlinearities

Nonlinearities can be *instantaneous* or *dynamic* (having *memory*)

- A dynamic nonlinearity can sometimes be converted into an instantaneous nonlinearity:
- Convert to the physical units in which the nonlinearity is instantaneous

[2] http://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_53.pdf
[3] https://www.ntnu.edu/documents/1001201110/1266017954/DAFx-15_submission_54.pdf

Choice of WDF Topology

Summarizing points above,

- Generally try to make a Binary Connection Tree (BCT) using only three-port adaptors
- At the *root* of the tree, include all
  - nonlinearities
  - non-adaptable elements such as switches
- When everything is linear and adaptable, place a time-varying element at the root, to minimize update propagation when that element changes
- When multiple elements are at the root, or when topology is not merely series + parallel connections, there will generally be at least one R node

Free WDF Software

Real Time Wave Digital Filter Software (DAFx-2016):

- GitHub: RT-WDF
- DAFx16 Paper

Overview and Demo of Various Wave Digital Filter Software (DAFx-2015, KeyNote 2, Part 2):

- Video (YouTube)
- Slides (PDF)

WDF References

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4. A. Sarti and G. De Sanctis, “Systematic methods for the implementation of nonlinear wave-digital structures,” IEEE Trans. Circuits and Systems I: Regular Papers, vol. 56, no. 2, pp. 460–472, Feb. 2009.

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6. T. Felderhoff, “A new wave description for nonlinear elements,” in IEEE Int. Symposium on Circuits and Systems, vol. 3, Sep. 1996, pp. 221–224.

7. A. Sarti and G. De Poli, “Toward nonlinear wave digital filters,” IEEE Trans. Signal Process., vol. 47, no. 6, pp. 1654–1668, June 1999.

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