Tutorial on Wave Digital Filters

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CCRMA DSP Seminar
January 25, 2008

Outline

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Wave digital filters model circuits used for filtering.
Overview

- Fettweis (1986), Wave Digital Filters: Theory and Practice.
- Wave Digital Filters (WDF) mimic structure of classical filter networks.
  - Low sensitivity to component variation.
- Use wave variable representation to break delay free loop.
- WDF adaptors have low sensitivity to coefficient quantization.
  - Direct form with second order section biquads are also robust
  - Transfer function abstracts relationship between component and filter state
  - WDF provides direct one-to-one mapping from physical component to filter state variable

Wave digital filters model circuits used for filtering

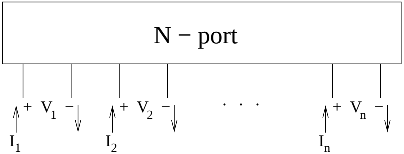
Applications

- Modeling physical systems with equivalent circuits.
  - Piano hammer mass spring interaction
  - Generally an ODE solver
  - Element-wise discretization and connection strategy
  - Real time model of loudspeaker driver with nonlinearity
  - Multidimensional WDF solves PDEs
- Ideal for interfacing with digital waveguides (DWG).

Outline

Classical Network Theory

N-port linear system



- Describe a circuit in terms of voltages (across) and current (thru) variables
- General N-port network described by V and I of each port
- Impedance or admittance matrix relates V and I
- $$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \underbrace{\begin{pmatrix} Z_{11} & Z_{12} & \dots & Z_{1N} \\ Z_{21} & \ddots & & Z_{2N} \\ \vdots & & & \vdots \\ Z_{N1} & \dots & & Z_{NN} \end{pmatrix}}_{\mathbf{Z}} \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{pmatrix}$$

Classical Network Theory

Element-wise discretization for digital computation

- For example, use Bilinear transform $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$
- Capacitor: $Z(s) = \frac{1}{sC}$

$$Z(z^{-1}) = \frac{T}{2C}\frac{1+z^{-1}}{1-z^{-1}} = \frac{V(z^{-1})}{I(z^{-1})}$$

$$v[n] = \frac{T}{2C}(i[n] + i[n-1]) + v[n-1]$$

- v[n] depends instantaneously on i[n] with $R_0 = \frac{T}{2C}$
- This causes problems when trying to make a signal processing algorithm
- Can also solve for solution using a matrix inverse (what SPICE does).

Classical Network Theory

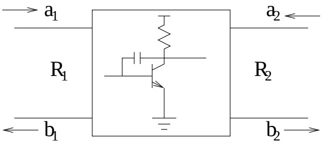
Wave variable substitution and scattering

$$A = V + RI \qquad V = \frac{A+B}{2}$$
$$B = V - RI \qquad I = \frac{A-B}{2R}$$

- Variable substitution from V and I to incident and reflected waves, A and B
- An N-port gives an $N \times N$ scattering matrix
- Allows use of scattering concept of waves

Classical Network Theory

Two port is commonly used in microwave electronics to characterize amplifiers



- Input port (1) and output port (2)

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

- Represent as scattering matrix and wave variables

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

- Scattering matrix **S** determines reflected wave b_n as a linear combination of N incident waves $a_1, \ldots, a_n$
- Guts of the circuit abstracted away into **S** or **Z** matrix

Outline

Wave Digital Elements

Basic one port elements

- Work with voltage wave variables b and a. Substitute into Kirchhoff circuit equations and solve for b as a function of as.
- Wave reflectance between two impedances is well known

$$\rho = \frac{b}{a} = \frac{R_2 - R_1}{R_2 + R_1}$$

- Define a port impedance R_p
- Input wave comes from port and reflects off the element's impedances.
  - Resistor $Z_R = R$, $\rho_R(s) = \frac{1-R_p/R}{1+R_p/R}$
  - Capacitor $Z_C = \frac{1}{sC}$, $\rho_C(s) = \frac{1-R_pCs}{1+R_pCs}$
  - Inductor $Z_L = sL$, $\rho_L(s) = \frac{s-R_p/L}{s+R_pL}$

Wave Digital Elements

Discretize the capacitor by bilinear transform

- Plug in bilinear transform

$$\frac{b_n}{a_n} = \frac{1 - R_p C \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{1 + R_p C \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$(1 + R_p C \tfrac{2}{T})b[n] + (1 - R_p C \tfrac{2}{T})b[n-1] = (1 - R_p C \tfrac{2}{T})a[n] + (1 + R_p C \tfrac{2}{T})a[n-1]$$

- Choose R_p to eliminate depedence of $b[n]$ on $a[n]$, e.g., $R_p = \frac{T}{2C}$, resulting in:

$$b[n] = a[n-1]$$

- Note that chosen R_p exactly the instantaneous resistance of the capacitor when discretized by the bilinear transform

Wave Digital Elements

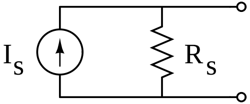
T-ports and I-ports

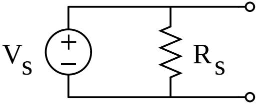
- *T-port.* Port resistance can be chosen to perfectly match element resistance to eliminate instantaneous reflection and avoid delay-free loop.
- *I-port.* If port is not matched, $b[n]$ depends on $a[n]$ instantaneously. R_p can be chosen as any positive value.
  - Short circuit
    $b[n] = -a[n]$
  - Open circuit
    $b[n] = a[n]$
  - Voltage source of voltage V
    $b[n] = -a[n] + 2V$
  - Current source of current I
    $b[n] = a[n] - 2R_p I$

Wave Digital Elements

One port summary for voltage waves

Element	Port Resistance	Reflected wave
Resistor	$R_p = R$	$b[n] = 0$
Capacitor	$R_p = \frac{T}{2C}$	$b[n] = a[n-1]$
Inductor	$R_p = \frac{2L}{T}$	$b[n] = -a[n-1]$
Short circuit	R_p	$b[n] = -a[n]$
Open circuit	R_p	$b[n] = a[n]$
Voltage source V_s	R_p	$b[n] = -a[n] + 2V_s$
Current source I_s	R_p	$b[n] = a[n] + 2R_pI_s$
Terminated V_s	$R_p = R_s$	$b[n] = V_s$
Terminated I_s	$R_p = R_s$	$b[n] = R_pI_s$

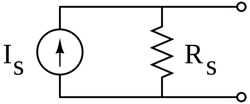


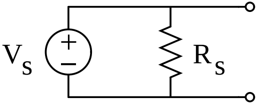


Wave Digital Elements

One port summary for current waves. Signs are flipped for some reflectances.

Element	Port Resistance	Reflected wave
Resistor	$R_p = R$	$b[n] = 0$
Capacitor	$R_p = \frac{T}{2C}$	$b[n] = -a[n-1]$
Inductor	$R_p = \frac{2L}{T}$	$b[n] = a[n-1]$
Short circuit	R_p	$b[n] = a[n]$
Open circuit	R_p	$b[n] = -a[n]$
Voltage source V_s	R_p	$b[n] = a[n] + 2V_s$
Current source I_s	R_p	$b[n] = -a[n] + 2R_pI_s$
Terminated V_s	$R_p = R_s$	$b[n] = V_s$
Terminated I_s	$R_p = R_s$	$b[n] = R_pI_s$





Wave Digital Elements

Two ports

- Series
- Parallel
- Transformer
- Unit element

Outline

Adaptors

Adaptors perform the signal processing calculations

- Treat connection of N circuit elements as an N-port
- Derive scattering junction from Kirchhoff's circuit laws and port impedances determined by the attached element
- Scattering matrix is $N \times N$
- Parallel and series connections can be simplified to linear complexity
- Signal flow diagrams to reduce number of multiply or add units
    - Dependent port - one coefficient can be implied,
    - Reflection free port - match impedance to eliminate reflection

Two-Port Parallel Adaptor

$$b_1 = a_2 + \gamma(a_2 - a_1)$$

$$b_2 = a_1 + \gamma(a_2 - a_1)$$

$$\gamma = (R_1 - R_2)/(R_1 + R_2)$$

N-Port Parallel Adaptor

- G_ν are the port conductances

$$G_n = G_1 + G_2 + \cdots + G_{n-1}, G_\nu = 1/R_\nu$$

- Find scattering parameters

$$\gamma_\nu = \frac{G_\nu}{G_n}, \nu = 1 \text{ to } n - 1$$

- Note γ sum to 2

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = 2$$

- Use intermediate variable to find reflected waves

$$a_o = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_n a_n$$
$$b_\nu = a_o - a_\nu$$

Parallel Adaptor

with port n reflection free (RFP)

- G_ν are the port conductances
- R_n is set equal to equivalent resistance looking at all the other ports (their Rs in parallel) to make port n RFP

$$G_n = G_1 + G_2 + \cdots + G_{n-1}, G_\nu = 1/R_\nu$$

$$\gamma_n = 1$$

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} = 1$$

$$\gamma_\nu = \frac{G_\nu}{G_n}, \nu = 1 \text{ to } n-1$$

$$b_n = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1}, \quad \text{RFP}$$

$$b_\nu = b_n + a_n - a_\nu$$

Series Adaptor

with port n dependent

- R_ν are the port resitances.
- Find scattering parameters

$$\gamma_\nu = \frac{2R_\nu}{R_1 + R_2 + \cdots + R_n}$$

- Note scattering parameters sum to 2

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = 2$$

- Use intermediate variable to find reflected waves

$$\begin{aligned} a_o &= a_1 + a_2 + \cdots + a_n \\ b_\nu &= a_\nu - \gamma_\nu a_o \end{aligned}$$

Series Adaptor

with port n reflection free (RFP)

- R_ν are the port resitances
- R_n is set equal to equivalent resistance looking at all the other ports (their Rs in series) to make port n RFP

$$
\begin{aligned}
R_n &= R_1 + R_2 + \cdots + R_{n-1} \\
\gamma_n &= 1 \\
\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} &= 1 \\
\gamma_\nu &= \frac{R_\nu}{R_n}, \nu = 1 \text{ to } n-1 \\
b_n &= -(a_1 + a_2 + \cdots + a_{n-1}), \quad \text{RFP} \\
b_\nu &= a_\nu - \gamma_\nu(a_n - b_n)
\end{aligned}
$$

Adaptors

Dependent ports

- Adaptors have property of low coefficient sensitivity, e.g., coefficients can be rounded or quantized.
- Dependent ports take advantage of property that γs sum to two.
- Use this fact along with quantization to ensure that adaptor is (pseudo-)passive.

Connection Strategy

Parameter updates

- Parameter updates propagate from leaf through its parents to the root
- Each adaptor's RFP must be recalculated when a child's port resistance changes
- Parameter update is more complicated than solving the Kirchhoff's equations directly, where the parameters are just values in the resistance matrix.

Connection Strategy

Avoid delay-free loops with adaptors connected as a tree

- Sarti et. al., Binary Connection Tree - implement WDF with three-port adaptors
- Karjalainen, BlockCompiler - describe WDF in text, produces efficient C code
- Scheduling to compute scattering
  - Directed tree with RFP of each node connected to the parent
  - Label each node (a, b, c, ...)
  - Label downward going signals *d* by node and port number
  - Label upward going signals *u* by node
  - Start from leaves, calculate all *u* going up the tree
  - Then start from root, calculate all *d* going down the tree

Connection Strategy

Automatic generation of WDF tree structure

- SPQR tree algorithms find biconnected and triconnected graphs (Fränken, Ochs, and Ochs, 2005. Generation of Wave Digital Structures for Networks Containing Multiport Elements.)
- Q nodes are one ports
- S and P nodes are Series and Parallel adaptors
- R nodes are triconnected elements
  - Implemented with similarity transform of $N \times N$ scattering matrix into two-port adaptors (Meerkötter and Fränken, Digital Realization of Connection Networks by Voltage-Wave Two-Port Adaptors”
  - Includes bridge connections and higher order connections
- Implemented in WDInt package for Matlab (http://www-nth.uni-paderborn.de/wdint/index.html)

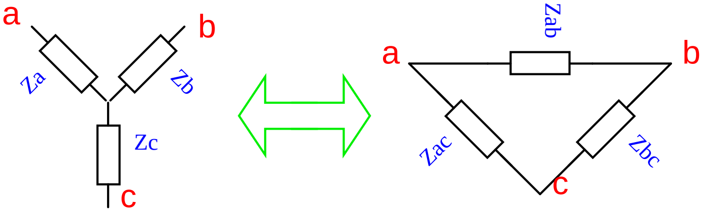
WDF Interconnections

More about the R nodes

- For example, bridge connection
- higher order triconnections also common
- N-port scattering junction
    - can be reduced to implementation by two port adaptors using similarity transform - keeps robust properties for quantization
    - reduces operations for filtering vs scatter matrix
- In general parameter update is complicated

R-Nodes

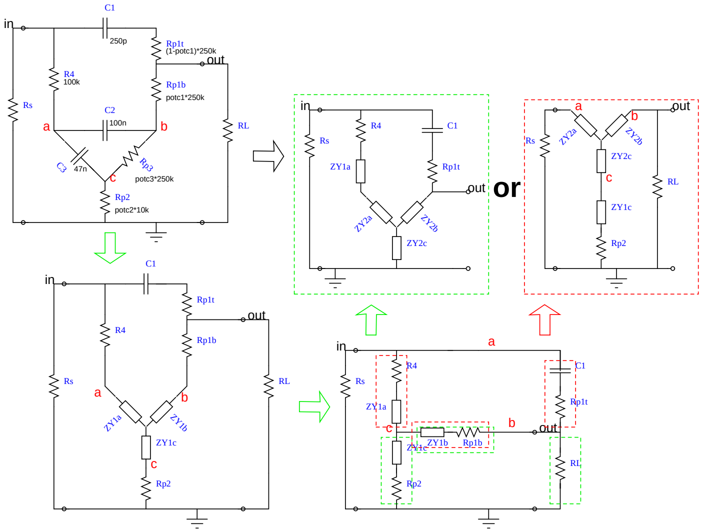
Other strategies: $Y - \Delta$ $\Delta - Y$ transformations



- Circuit analysis technique to replace triconnected impedances with equivalents that can be connected in series or parallel.
- Must discretize general impedances, no longer correspondence between prototype circuit element and WDF element

R-Nodes

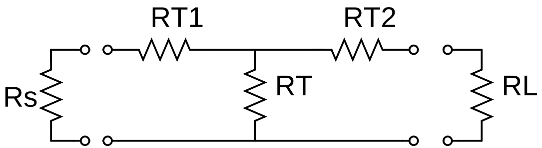
Example: Guitar amplifier tone stack with input and output loading



Tone Stack

Formulate blocks compatabile with scattering

- Observe that tone stack is a specific two-port
- Direct implementation of a 2-port scattering matrix
- Or convert into an equivalent circuit with impedances and use adaptors
- Tabulate the scattering parameters or impedances as they vary with parameter changes



Outline

Nonlinearity
Literature

- Meerkötter and Scholz (1989), Digital Simulation of Nonlinear Circuits by Wave Digital Filter Principles.
- Sarti and De Poli (1999), Toward Nonlinear Wave Digital Filters.
- Karjalainen and Pakarinen (2006), Wave Digital Simulation of Vacuum-Tube Amplifier
- Petrausch and Rabenstein (2004), Wave Digital Filters with Multiple Nonlinearities
- Either conceive as nonlinear resistor or dependent source
- Introduces an I-port, may lead to delay-free loops
- DFL must be solved as a system of equations in wave variables

Nonlinear Conductance

Meerkötter and Scholz (1989).

- Current is a nonlinear function of voltage, $i = i(v)$
- In wave variables

$$a = f(v) = v + R_p i(v)$$
$$b = g(v) = v - R_p i(v)$$

- Substituting wave variables into Kirchhoff variable definition of nonlinear resistance and solving for $b(a)$

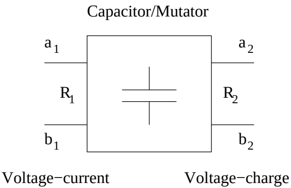
$$b = b(a) = g(f^{-1}(a))$$

- f^{-1} must exist
- Port resistance R_p can be chosen arbitrarily within constraints that $f(v)$ be invertible
- Instantaneous dependence exists regardless of R_p

Extension to nonlinear reactances

Nonlinear Capacitor. Sarti and De Poli (1999).

Use a "mutator" to integrate the Kirchhoff variable so that the nonlinear reactance can be defined in terms of wave variables.



- Define wave variable such that port resistance can be an impedance.

$$\begin{aligned} A(s) &= V(s) + R(s)I(s) \\ B(s) &= V(s) - R(s)I(s) \end{aligned}$$

- Recall that standard wave definitions for a one port such as a capacitor are

$$\begin{aligned} A_1 &= V_1 + R_1 I_1 \\ B_1 &= V_1 - R_1 I_1 \end{aligned}$$

Extension to nonlinear reactances

Nonlinear Capacitor. Sarti and De Poli (1999).

For the capacitor with the usual single resistive port, define a second port across the capacitor with its port impedance $R(s) = \frac{1}{sC}$

$$\begin{aligned} A_2(s) &= V_2(s) + \frac{1}{sC} I_2(s) \\ B_2(s) &= V_2(s) - \frac{1}{sC} I_2(s) \end{aligned}$$

This looks like the usual wave variable definitions if $I(s)$ is replaced by its integral $Q(s)$, charge, and port impedance $R_2 = 1/C$.

$$\begin{aligned} A_2(s) &= V_2(s) + \frac{1}{C} Q(s) \\ B_2(s) &= V_2(s) - \frac{1}{C} Q(s) \end{aligned}$$

A_2 and B_2 can be substituted into the definition of a generic nonlinear capacitance $\boxed{Q = f(V) = C(V) \cdot V}$ to find the nonlinear “reflection” as it is done for the nonlinear resistor. R_2 can be chosen rather arbitrarily as before.

Extension to nonlinear reactances

Voltage-current to voltage-charge wave conversion (Felderhoff 1996).

Compute scattering relations between the resistive and the integrated port using two relations: consistency of voltage for the two ports $V_1 = V_2$, and $I_1 + I_2 = 0$ across junction .

$$V = \frac{A_1+B_1}{2} = \frac{A_2+B_2}{2} \tag{1}$$

$$I_1 = \frac{A_1-B_1}{2R_1} = -\frac{A_2-B_2}{2R(s)} = -\frac{A_2-B_2}{2\frac{1}{sC}} \tag{2}$$

bilinear transform $s \rightarrow z$

$$\frac{A_1-B_1}{R_1} = -\frac{A_2-B_2}{\frac{1}{C}\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}}$$

$$(1+z^{-1})(A_1-B_1) = -\frac{2}{T}CR_1(1-z^{-1})(A_2-B_2)$$

$$(A_1+z^{-1}A_1-B_1-z^{-1}B_1) = -\frac{2}{T}CR_1(A_2-z^{-1}A_2-B_2+z^{-1}B_2)$$

$$a_1[n]+a_1[n-1]-b_1[n]-b_1[n-1] = -\frac{2}{T}CR_1(a_2[n]-a_2[n-1]-b_2[n]+b_2[n-1])$$

substitute $b_2 = a_1 + b_1 - a_2$ using (1)

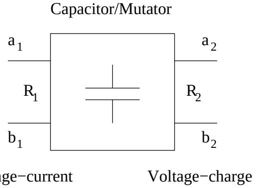
$$\begin{aligned} a_1[n]+a_1[n-1]-b_1[n]-b_1[n-1] &= -\frac{2}{T}CR_1(a_2[n]-a_2[n-1] \\ &\quad -(a_1[n]+b_1[n]-a_2[n])+a_1[n-1]+b_1[n-1]-a_2[n-1]) \end{aligned}$$

Extension to nonlinear reactances

Voltage-current to voltage-charge wave conversion.

Set $\boxed{R_1 = T/(2C)}$ to eliminate reflection at port 1.

$$a_1[n-1] - b_1[n] = -a_2[n] + a_2[n-1]$$



Resulting scattering junction (or mutator according to Sarti and De Poli) converts between voltage-current and voltage-charge waves:

$$b_2 = a_1 + (a_1[n-1] - a_2[n-1])$$

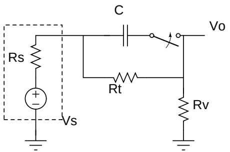
$$b_1 = a_2 + (a_1[n-1] - a_2[n-1])$$

Port resistance for new mutated waves corresponding to voltage and charge is $R_2 = \frac{1}{C}$.
Port resistance for usual waves corresponding to voltage and current is $R_1 = \frac{T}{2C} = \frac{T}{2} R_2$.

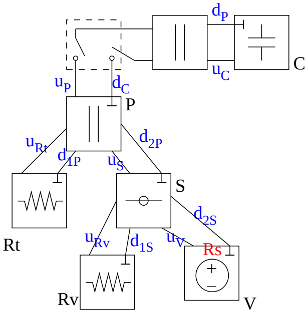
Outline

Parametrized Linear Circuit Example

Volume pot with bright switch



Output is voltage over R_v, $V_o = \frac{(d_{1S}+u_{Rt})}{2}$ $R_V = R_{pot}(\text{vol})$, $R_t = R_{pot}(1-\text{vol})$. Changes in vol require recomputation of γ's starting from bottom of tree. Use RFP to allow open circuit when C is disconnected.



$$u_{Rv} = u_{Rt} = 0, \qquad u_V = V$$

$$u_S = -(u_{Rv} + u_V)$$

$$u_P = \gamma_{1P} u_{Rt} + \gamma_{2P} u_S$$

$$u_C = d_P[n-1]$$

$$d_P = u_P + \gamma_P(u_C - u_P)$$

$$d_C = u_C + \gamma_P(u_C - u_P), \quad \text{or} \quad d_C = u_P$$

$$d_{1P} = u_P + d_C - u_{Rt}, \quad \text{don't care}$$

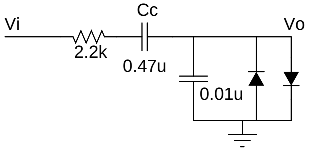
$$d_{2P} = u_P + d_C - u_S$$

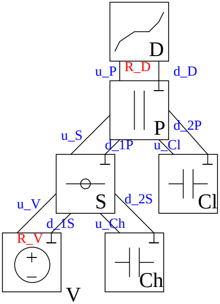
$$d_{1S} = u_{Rv} - \gamma_{1S}(d_{2P} - u_S), \quad \text{output value}$$

$$d_{2S} = u_V - \gamma_{2S}(d_{2P} - u_S), \quad \text{don't care}$$

Nonlinear Circuit Example

Diode clipper





- Diode clipper circuit found in guitar distortion pedals
- Treat diodes together as single nonlinear one-port

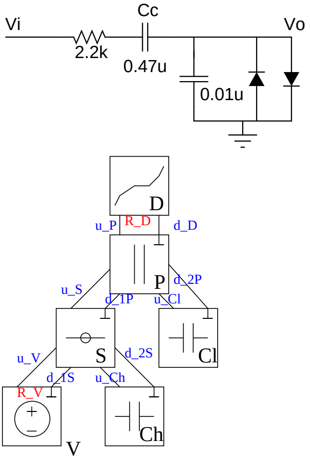
$I(V) = 2I_s \sinh(V/V_d)$

Solve for $b(a)$, $\frac{a-b}{2R_p} = 2I_s \sinh\left(\frac{a+b}{2V_d}\right)$

- Isolate nonlinearity at root of tree
- Incorporate resistor into voltage source

Nonlinear Circuit Example

Diode clipper: Computational algorithm



$$u_V = V, \qquad u_{Ch} = d_{2S}[n-1]$$

$$u_S = -(u_V + u_{Ch}), \qquad u_{Cl} = d_{2P}[n-1])$$

$$u_P = \gamma_{1P} u_S + \gamma_{2P} u_{Cl}$$

$$d_D = f(u_P), \quad \text{nonlinear function}$$

$$d_{1P} = u_P + d_D - u_S$$

$$d_{2P} = u_P + d_D - u_{Cl}$$

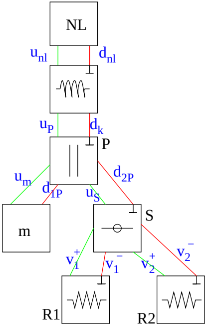
$$d_{1S} = u_V - \gamma_{1S}(d_{1P} - u_S), \quad \text{don't care}$$

$$d_{2S} = u_{Ch} - \gamma_{2S}(d_{1P} - u_S)$$

Nonlinear Musical Acoustics Example

WDF hammer and nonlinear felt

- Draw mass/spring/waveguide system in terms of equivalent circuits
- Waveguides look like resistors to the lumped hammer. Waves enter lumped junction directly.
- WDF result in tree like structures with adaptors/scattering junctions at the nodes, and elements at the leaves.
- The root of the tree allowed to have instantaneous reflections
- Nonlinearity gives instantaneous reflection, WDF handles only 1 nonlinearity naturally.
- Compression ($d = \frac{u_{nl} - d_{nl}}{2R_{nl}}$ from next slide) must ≥ 0, otherwise hammer is not in contact.



To left and right waveguides

Nonlinear Musical Acoustics Example

Computations

v_1^+, v_2^- from waveguides

$$u_S = -(v_1^+ + v_2^-)$$

$$u_m = d_{1P}[n-1]$$

$$u_P = \gamma_{1P} u_m + \gamma_{2P} u_S$$

$$u_{nl} = u_P + (u_P[n-1] - d_{nl}[n-1])$$

$$d_{nl} : \text{ solve } \left\{ \frac{u_{nl}+d_{nl}}{2} = k \left(\frac{u_{nl}-d_{nl}}{2R_{nl}} \right)^{\gamma} \right\}$$

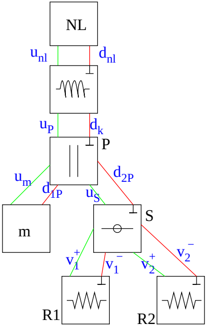
$$d_k = d_{nl} + (u_P[n-1] - d_{nl}[n-1])$$

$$d_{1P} = u_P + d_k - u_m$$

$$d_{2P} = u_P + d_k - u_S$$

$$v_1^- = v_1^+ - \gamma_{1S}(d_{2P} - u_S)$$

$$v_2^+ = v_2^- - \gamma_{2S}(d_{2P} - u_S)$$



To left and right waveguides

Outline

Observations

- Scattering formulation works well with DWG - DWG looks like resistor in WDF
- For a standalone simulation of nonlinear circuits, may not be the best choice
- More difficult for parameter update than direct solving
- Root node is special - easy to implement nonlinearity or parameter changes
- May be able to design circuits without bridge connections that have equivalent transfer function

Summary

- Wave digital formulation uses matched (reflection-free) ports to eliminate reflections and avoid delay-free loops
- Elements are connected in tree structure with reflection-free ports connected to the parent node.
- Useful for building up a model component-wise.

Outline

Two-port adaptor reflectance and transmittance

N-port parallel adaptor

N-port series adaptor

Capacitor

Inductor

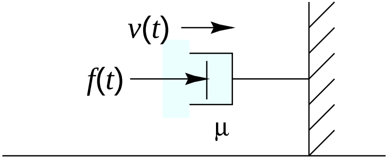
Outline

Mechanical Impedance Analogues from Music 420

The following mechanical examples are taken from:
"Lumped Elements, One-Ports, and Passive Impedances", by Julius O. Smith III, (From Lecture Overheads, Music 420).
http://ccrma.stanford.edu/ jos/OnePorts/
`http://ccrma.stanford.edu/~jos/OnePorts/`

Dashpot

Ideal dashpot characterized by a constant impedance μ



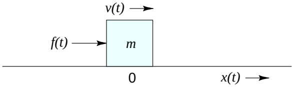
- Dynamic friction law

  $f(t) \approx \mu v(t)$ "Ohm'sLaw"(Force $=$ Friction_coefficient$\times$Velocity)
- Impedance

$$R_\mu(s) \triangleq \mu \geq 0$$

- Dashpot = *gain* for force input and velocity output
- Electrical analogue: *Resistor* $R = \mu$
- More generally, losses due to friction are
  - *frequency dependent*
  - *hysteretic*

Mass



Ideal mass of *m* kilograms sliding on a frictionless surface

- Newton's 2nd Law

$$f(t) = ma(t) \triangleq m\dot{v}(t) \triangleq m\ddot{x}(t) (\text{Force} = \text{Mass} \times \text{Acceleration})$$

- *Differentiation Theorem*

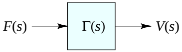
$$F(s) = m[sV(s) - v(0)] = msV(s)$$

for Laplace Transform when $v(0) = 0$.

- *Impedance*

$$R_m(s) \triangleq \frac{F(s)}{V(s)} = ms$$

Mass



"Black Box" Description

- *Admittance*

$$\Gamma_m(s) \triangleq \frac{1}{R_m(s)} = \frac{1}{ms}$$

- *Impulse Response* (unit-momentum input)

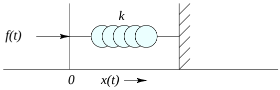
$$\gamma_m(t) \triangleq \mathcal{L}^{-1}\left\{\Gamma_m(s)\right\} = \frac{1}{m}u(t)$$

- *Frequency Response*

$$\Gamma_m(j\omega) = \frac{1}{mj\omega}$$

- Mass admittance = *Integrator* (for force input, velocity output)
- Electrical analogue: *Inductor* $L = m$.

Spring (Hooke's Law)



Ideal spring

- Hooke's law

$$f(t) = kx(t) \triangleq k \int_0^t v(\tau)d\tau (\text{Force} = \text{Stiffness} \times \text{Displacement})$$

- *Impedance*

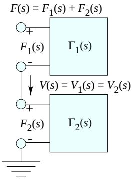
$$R_k(s) \triangleq \frac{F(s)}{V(s)} = \frac{k}{s}$$

- *Frequency Response*

$$\Gamma_k(j\omega) = \frac{j\omega}{k}$$

- Spring = *differentiator* (force input, velocity output)
- Velocity $v(t)$ = "compression velocity"
- Electrical analogue: *Capacitor* $C = 1/k$ (1/stiffness = "compliance")

Series Connection of One-Ports



- Series Impedances *Sum*:

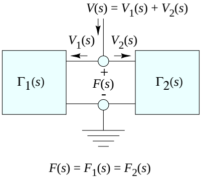
$$R(s) = R_1(s) + R_2(s)$$

- Admittance:

$$\Gamma(s) = \frac{1}{\frac{1}{\Gamma_1(s)} + \frac{1}{\Gamma_2(s)}} = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}$$

- Physical Reasoning:
  - *Common Velocity* $\Rightarrow$ *Series* connection
  - *Summing Forces* $\Rightarrow$ *Series* connection

Parallel Combination of One-Ports



- Parallel Admittances *Sum*

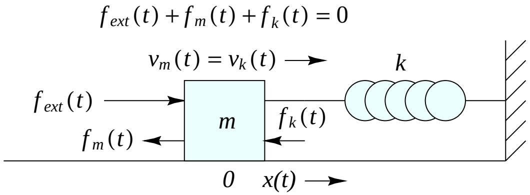
$$\Gamma(s) = \Gamma_1(s) + \Gamma_2(s)$$

- Impedance:

$$R(s) = \frac{1}{\frac{1}{R_1(s)} + \frac{1}{R_2(s)}} = \frac{R_1 R_2}{R_1 + R_2}$$

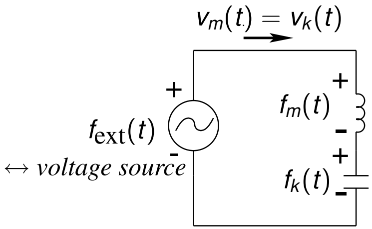
  or, for EEs, $R = R_1 || R_2$
- Physical Reasoning:
  - *Common Force* $\Rightarrow$ *Parallel* connection
  - *Summing Velocities* $\Rightarrow$ *Parallel* connection

Mass-Spring-Wall (Series)



Physical Diagram:

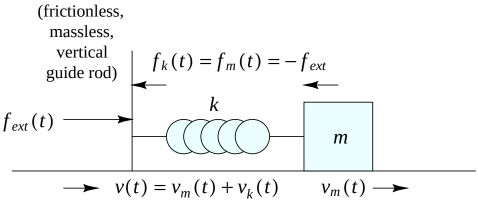
Electrical Equivalent Circuit:



Mass $m \leftrightarrow$ Inductance $L = m$ (impedance $R_m(s) = ms$)

Spring $k \leftrightarrow$ Capacitance $C = \frac{1}{k}$ (impedance $R_k(s) = \frac{k}{s}$)

Spring-Mass (Parallel)



Physical Diagram:

Electrical Equivalent Circuit: