Financial Mathematics 32000

Lecture 2

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Analytic approximation techniques

Analytic approximation techniques/heuristics help us to

- Analyze the accuracy of numerical methods
- Perform sanity checks on the output of numerical methods

Two of those analytic heuristics are:

- Taylor approximation
- Normal approximation

Taylor approximation

Postpone error analysis until we do finite difference methods.

A key tool in that error analysis will be Taylor approximation:

If f has $n+1$ continuous derivatives in a neighborhood of x_0 then

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + O(x-x_0)^{n+1} \end{aligned}$$

as $x \to x_0$.

We may use Taylor approximation for two different purposes:

- To approximate prices and sensitivities – example on next page.
- To analyze the error in tree or finite difference calculation of prices and sensitivities

Interview question: price ATM option

No calculators allowed. Your interviewer says to you:

> *Spot is* 100*. No dividends. What's the price of a European-style* 1*-year at-the-money-forward (ATMF) vanilla option with* 20% *implied volatility?*

(ATMF at time 0 means $K = F_0$. Recall $F_0 = S_0 e^{rT}$ if no divs.)

Answer: Black-Scholes call price is

$$S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where

log(1) = 0

$$d_{1,2} := d_{+,-} := \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$$

ATM option prices are almost linear in vol

ATMF option price is

$$S_0 N(d_1) - Ke^{-rT} N(d_2) = S_0 (N(\sigma\sqrt{T}/2) - N(-\sigma\sqrt{T}/2))$$

For small $|x|$,

$$N(x) = N(0) + N'(0)x + \frac{1}{2}N''(0)x^2 + O(x^3) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x + 0 + O(x^3).$$

So option price is approximately

$$S_0\left(\frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}} - \frac{1}{2} + \frac{\sigma\sqrt{T}/2}{\sqrt{2\pi}}\right) = \frac{S_0\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \times S_0\sigma\sqrt{T}.$$

Your answer: 8 dollars

(True answer: 7.97 dollars)

Follow-up question: ATM delta

Same assumptions. What's the delta of the ATMF call?

Differentiating the approximate option price $S_0\sigma\sqrt{T}/\sqrt{2\pi}$ with respect to S_0, we have a delta of

$$\frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.08$$

Or do we? I thought the delta of an ATMF call should be close to 0.5.

Normal approximation

Pretend that S_T is normally distributed (rather than lognormally).

In particular, pretend that the *simple* return on the forward

$$\frac{F_T}{F_0} - 1 = \frac{S_T}{F_0} - 1 \quad \sim \quad \text{Normal}(0, \sigma\sqrt{T})$$

ATMF option on no-div stock has price

$$e^{-rT}\mathbb{E}(S_T - F_0)^+ = \frac{e^{-rT}}{2}\mathbb{E}|S_T - F_0| = \frac{F_0 e^{-rT}}{2}\mathbb{E}\left|\frac{S_T}{F_0} - 1\right| = \frac{S_0}{2}\mathbb{E}\left|\frac{S_T}{F_0} - 1\right|$$

K=F_0

$$= \frac{1}{\sqrt{2\pi}} S_0 \sigma \sqrt{T}$$

because the mean absolute deviation of a normal random variable is $\sqrt{\frac{2}{\pi}} \times$ its standard deviation (by calculating $\int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x$)

UNIT 0: Analytic approximation

UNIT 1: Trees

Pricing options; Europeans, barriers, Americans

Approximating diffusions: GBM and Local Volatility

Appendix: Lookback option

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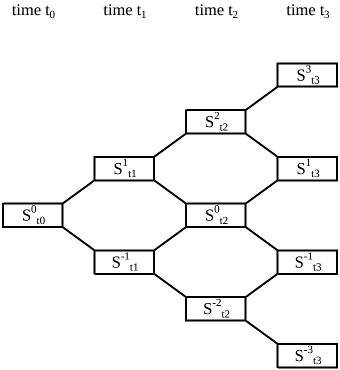
Binomial tree

Consider an N-period model (example: $N = 3$) with a bank account $B_t = e^{rt}$ and a stock S with dynamics

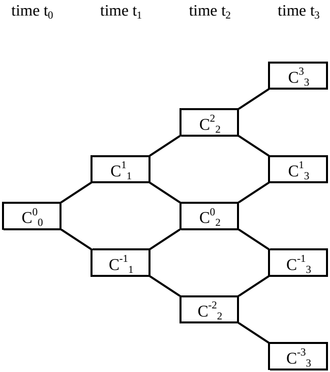


Superscripts are not powers. Subscripts $t_n = n\Delta t$ for $n = 0, \ldots, N$.

Binomial tree

Want to find time-0 price C_0^0 of some option.

To simplify notation, we will write C_n^j instead of $C_{t_n}^j$.



Consider three contract types: European, barrier, and American.

Pricing a European

If the expiry date is at time $T = t_N$ and the payoff function is $f(S)$, then for all j, the option price is

$$C_N^j = f(S_{t_N}^j).$$

Now induct backwards. If C_{n+1}^j has been computed for all j, then apply the one-period no-arbitrage results in the one-period subtree rooted at each (j, n) to find

$$C_n^j = e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}],$$

where the up-probability is

$$p_n^j := \frac{S_{t_n}^j e^{r\Delta t} - S_{t_{n+1}}^{j-1}}{S_{t_{n+1}}^{j+1} - S_{t_{n+1}}^{j-1}}.$$

Pricing an up-and-out barrier option

An *up-and-out put* on S with strike K, expiry T, and barrier H pays

$$(K - S_T)^+ \mathbf{1}_{\max_{t \in \mathcal{T}} S_t < H}.$$

where $\mathcal{T} \subseteq [0, T]$ are the observation times, specified in the contract. Continuous monitoring: $\mathcal{T} = [0, T]$. Discrete monitoring: $\mathcal{T}$ is some finite set. The option price is a function of *three* state variables:

$$C(S^j_{t_n},\ t_n,\ \mathbf{1}_{\text{knockout prior to time } t_n}).$$

where "knockout prior to time t_n" is the event that there exists $t < t_n$ with $t \in \mathcal{T}$ and $S_t \geq H$. Computationally: $C(S, t, 1)$ does not need to be tracked; it's always zero. So what we track in the tree is $C^j_n := C(S^j_{t_n}, t_n, 0)$, the option price *in the case of no prior knockout.*

Pricing an up-and-out barrier option

Let $t_N = T$. At terminal nodes, $C_N^j = (K - S_{t_N}^j)^+ \mathbf{1}_{S_{t_N}^j < H \text{ or } t_N \notin \mathcal{T}}$.

Inducting backwards, if C_{n+1}^j has been computed for all j, then at

We are assuming that there's no knockout prior to n

node (j, n) we have two cases:

today is a monitoring day We breach the barrier today

$$t_n \in \mathcal{T} \text{ and } S_{t_n}^j \geq H \Rightarrow \quad C_n^j = 0 \tag{1}$$

C today = (0,0), two scenarios of why C is 0 today:
First 0 is because today is not a monitoring day
Second 0 is because we breached the barrier today

$$t_n \notin \mathcal{T} \text{ or } S_{t_n}^j < H \Rightarrow \quad C_n^j = e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}] \tag{2}$$

today is not a monitoring day We didn't breach the barrier today

So everywhere at or beyond the barrier at monitoring times, set the price to 0; elsewhere, take the usual discounted average of the next-time-step values.

(And in building a tree, you try to have the dates important in the contract – such as expiry and barrier monitoring dates – be among the dates represented in the tree.)

Pricing an American put

Notation: $x \wedge y := \min(x, y)$.

On a time interval $[t_1, t_2 \wedge \tau]$, consider a portfolio with price process V, that includes a static position in an American option, to be exercised at time $\tau \geq t_1$. We say it is an *arbitrage* if $V_{t_1} = 0$ and

No cost

$$\mathsf{P}(V_{t_2 \wedge \tau} < 0) = 0 \text{ and } \mathsf{P}(V_{t_2 \wedge \tau} > 0) > 0$$

Type 1 arbitrage: No possibility of a loss

for *some* stopping time $\tau \geq t_1$, if the portfolio is long the option;

for *all* stopping times $\tau \geq t_1$, if the portfolio is short the option.

Pricing an American

At expiry $T = t_N$, we have $C_N^j = (K - S_{t_N}^j)^+$. Inducting backwards, if C_{n+1}^j is the no-arb price that has been computed for each j then

$$C_n^j = \max((K - S_{t_n}^j)^+,\ e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}])$$

This is because the holder at node (j, n) can either exercise and receive $(K - S_{t_n}^j)^+$, or hold on to the option which tomorrow will have no-arbitrage price of C_{n+1}^{j+1} (if up) or C_{n+1}^{j-1} (if down), which implies

- If $C_n^j <$ max, then there is arbitrage: go long the option.
- If $C_n^j >$ max, then there is arbitrage: short the option; if holder exercises, you close out with a profit; if not, then use the funds to buy a portfolio that superreplicates the time-t_{n+1} option value.

The time-0 option price is C_0^0.

Pricing an American: another formulation

Math fact: For any adapted process Z_{t_n}, define $Y_{t_N} = Z_{t_N}$ and

$$Y_{t_n} = \max(Z_{t_n},\ \mathbb{E}_{t_n} Y_{t_{n+1}}), \qquad n = N-1, N-2, \ldots, 0.$$

(E.g.: $Z_t = e^{-rt}(K - S_t)^+$ and $Y_t = e^{-rt} \times$ time-t value of American).

Then the **optimality principle of dynamic programming** says that

$$Y_0 = \max\{\mathbb{E} Z_\tau : \ \tau \text{ is a stopping time, } 0 \le \tau \le T\}$$

and $Y_0 = \mathbb{E} Z_{\tau^*}$ where $\tau^* := \min\{t_n : \ Y_{t_n} = Z_{t_n}\}$.

This leads to another formulation of the American option price:

$$C_0 = \max\{\mathbb{E} e^{-r\tau}(K - S_\tau)^+ : \ \tau \text{ is a stopping time, } 0 \le \tau \le T\}$$

(which is also valid in continuous time for continuous processes S).

Another application of dynamic programming

Interview (?!) question:

Take a 52 card deck with 26 red and 26 black cards, in random order.

Reveal cards sequentially without replacement.

With each black card you get $+1$ dollar.

With each red card you get -1 dollar.

You can stop playing the game at any time.

What's the optimal stopping time, and what's your expected profit from the optimal strategy?

(Easier question: 6 cards – 3 red, 3 black)

Another application of dynamic programming

UNIT 0: Analytic approximation

UNIT 1: Trees

Pricing options; Europeans, barriers, Americans

Approximating diffusions: GBM and Local Volatility

Appendix: Lookback option

Trees as approximations of diffusions

- Approximate the risk-neutral dynamics of a diffusion process (such as GBM) using a discrete model with a finite number of branches from each state.
- The expectations (hence option prices) that we find in the tree approximate the expectations (hence option prices) under the diffusion process.
- For example, let us use a tree to approximate the diffusion

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

Use binomial tree to approximate diffusion dynamics

- Diffusion is

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where W is $\mathbb{P}$-BM. So $X := \log S$ has dynamics

$$\mathrm{d}X_t = \nu \mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\nu := R_{grow} - \sigma^2/2$.

- Approximate dynamics of $X := \log S$ using a binomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x_u) = p$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x_d) = 1 - p$$

where Δx_u and Δx_d and p will be nonrandom parameters.

Match the means and variances

- Let $\Delta t := T/N$ where N is number of time steps.
- Now choose $p, \Delta x_u, \Delta x_d$ such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu \mathrm{d}s + \int_t^{t+\Delta t} \sigma \mathrm{d}W_s = \nu\Delta t + \sigma\Delta W.$$

So

$$\nu\Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p\Delta x_u + (1-p)(-\Delta x_d)$$

$$\sigma^2\Delta t = \mathrm{Var}_t(X_{t+\Delta t} - X_t) = p(\Delta x_u)^2 + (1-p)(-\Delta x_d)^2 - (\nu\Delta t)^2$$

Two equations and three unknowns: p, Δx_u, Δx_d. Can impose another condition (such as $\Delta x_u = \Delta x_d$ or $p = 1/2$) and solve.

Then price the option in a tree with these parameters.

Not flexible enough

- However, often we want to specify Δx_u and Δx_d in advance.

  For example, we may want $\Delta x_u = \Delta x_d = \Delta x$, so that the *same* price levels are represented at every time point in the tree.

  Maybe, furthermore, we want to specify the size of Δx to optimize convergence or to choose *which* price levels are represented.

- This leaves only one free parameter p.

  Not enough to match both the mean and the variance.

Solution: trinomial trees.

Use trinomial tree to approximate diffusion dynamics

- Diffusion is

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where W is $\mathbb{P}$-BM. So $X := \log S$ has dynamics

$$\mathrm{d}X_t = \nu \mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\nu := R_{grow} - \sigma^2/2$.

- Approximate dynamics of $X := \log S$ using a trinomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x) = p_u$$

$$\mathbb{P}(X_{t+\Delta t} = X_t) = p_m$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x) = p_d$$

Match the means and variances

- Let $\Delta t := T/N$ where N is number of time steps.

  Choose $\Delta x \approx \sigma\sqrt{3\Delta t}$ for accuracy reasons.

  Can modify these suggestions to make sure certain price levels or times are represented in the tree.

- Now choose p_u, p_m, p_d such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu \mathrm{d}s + \int_t^{t+\Delta t} \sigma \mathrm{d}W_s = \nu\Delta t + \sigma\Delta W.$$

  So

expectation of the log return

$$\nu\Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p_u\Delta x + p_m 0 + p_d(-\Delta x)$$

$$\sigma^2\Delta t = \mathrm{Var}_t(X_{t+\Delta t} - X_t) = p_u(\Delta x)^2 + p_m(0)^2 + p_d(-\Delta x)^2 - (\nu\Delta t)^2$$

variance of the log return

E(x^2) - (E(x))^2

$$p_u + p_m + p_d = 1$$

Solve for probabilities

Solve system of three equations in three unknowns.

$$p_u = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2} + \frac{\nu\Delta t}{\Delta x}\right]$$

$$p_m = 1 - \frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2}$$

$$p_d = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2} - \frac{\nu\Delta t}{\Delta x}\right]$$

Intuition: For fixed Δt and Δx,

- The bigger the σ, the more probability mass in the wings, the less in the middle. bigger vol—> bigger possible price change
- The bigger the ν, the more mass in the up-branch, the less in the down-branch.

Option pricing

To find option prices, induct backwards in tree with time points

$$t_n = n\Delta t \qquad n = 0, \ldots, N,$$

and log-stock-price points

$$x_j = \log S_0 + j\Delta x \qquad j = -N, \ldots, N.$$

Option price in tree for node at time t_n and log-price x_j is

$$C_n^j = e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^{j} + p_d C_{n+1}^{j-1}].$$

With large enough N, the option price in the tree $\approx$ the option price for diffusion, because in the $N \to \infty$ (or equivalently $\Delta t \to 0$) limit, the tree's option price $\to$ option price under diffusion dynamics.

For Europeans, the proof is by a form of the CLT.

Can't replicate general payoffs if 3 states 2 assets

- Recall that in the 3-state model, general options cannot be replicated using stock and bank acct. There is not a unique arbitrage-free price for the option. So how can we talk about finding "the" price of the option?
- Answer: The trinomial tree is not the model of the market.
  The model is GBM in continuous time. In that model, replication using $\{B, S\}$ is possible. So unique price exists.
  The tree is a *computational device* that we use in order to approximate the risk-neutral expectations (hence the prices) of the continuous-time model.

Option Pricing: Knock-outs and Americans

Pricing of knock-outs

We only have payoff if there's no knock-out = no breach exceeding the barrier.
no breach —> ITM ; breach —> OTM

$$C_n^j = e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^{j} + p_d C_{n+1}^{j-1}]\mathbf{1}_{\text{no knock-out at time } t_n}$$

Pricing of American put

$$C_n^j = \max((K - S_{t_n}^j)^+,\ e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^{j} + p_d C_{n+1}^{j-1}])$$

Formulas are similar to binomial tree.

For american put and knock outs (barrier options), there is just one more scenario that we need to add
the middle part (middle scenario) is the newly added part here

Local volatility models

We want a model that is consistent with the observed non-constant implied vol skew. One approach: *local volatility* models specify the instantaneous volatility σ to be a function of (S_t, t).

- In the continuous-time setting,

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(S_t, t)S_t\mathrm{d}W_t$$

where W is $\mathbb{P}$-BM.

Here, the sigmas is different. Thus, the up, middle, and down probabilities are different in different nodes in the tree

- In the tree setting, we let σ depend on the node (S_t, t).

Note that the local volatility σ is not the same thing as σ_{imp}.

Analytic computation of option prices is difficult in the diffusion setting, even for European calls/puts (exception: $\sigma(S_t, t) = \sigma(t)$). But by approximating the diffusion in a tree, the computations are easy.

Option pricing in the tree setting

Suppose we are given the local volatility function σ.

- Let $\Delta t = T/N$ (unless this fails to place important dates in the tree).
- To choose Δx, let σ_{avg} be some "representative" or "average" σ in the tree, and let σ_{max} be an upper bound on the σ in the tree. Two guidelines: to make local discretization error small, we want

$$\Delta x \approx \sigma_{avg}\sqrt{3\Delta t}$$

but for stability reasons, we want

$$\Delta x \geq \sigma_{max}\sqrt{\Delta t}$$

So we can let $\Delta x = \max(\sigma_{avg}\sqrt{3\Delta t}, \sigma_{max}\sqrt{\Delta t})$

Option pricing in the tree setting

- Then at each node, use the $\sigma(S_t, t)$ prevailing at that particular node to generate the ν and the probabilities for the branches out of that node. Same formulas as L1:

$$p_{u,d} = \frac{1}{2}\left[\frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2} \pm \frac{\nu \Delta t}{\Delta x}\right], \quad p_m = 1 - \frac{\sigma^2 \Delta t + \nu^2 (\Delta t)^2}{(\Delta x)^2}$$

- Price options – including path-dependent and American-style options – as we did for GBM.

  The only change is that σ (hence ν, p_u, p_m, p_d) vary across nodes.

What if σ is not given

Then *calibrate* it to the prices of listed options.

The general idea of calibration:

- Observe prices of liquidly traded assets (e.g. listed options)
- Choose the model's parameters (e.g. the σ function) in such a way that the model generates theoretical prices that match closely the observed prices.

Then one can apply that model, with the calibrated parameters, to

- Compute hedges and risk sensitivities
- Price illiquid options
- Price and hedge complex deals (e.g. structured products and exotic options)

The implied vol needs to incorporate all the sigmas in the different nodes between time 0 and time T

Calibration of local volatility $\sigma(t)$

If σ is a non-random function of t and

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t$$

then $\mathrm{d}\log S_t = (r - \frac{1}{2}\sigma^2(t))\mathrm{d}t + \sigma(t)\mathrm{d}W_t$ so

$$\log S_T = \log S_0 + \left(r - \frac{\bar{\sigma}_T^2}{2}\right)T + \int_o^T \sigma(t)\mathrm{d}W_t$$

$$\sim \mathrm{Normal}\left(\log S_0 + (r - \frac{\bar{\sigma}_T^2}{2})T,\ \bar{\sigma}_T^2 T\right) \text{ where } \boxed{\bar{\sigma}_T := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)\mathrm{d}t}}$$

(intuition: a nonrandomly weighted sum of indep normals is normal)

So time-0 call prices $C(K,T) = C^{BS}(\bar{\sigma}_T)$, thus $\sigma_{imp}(K,T) = \bar{\sigma}_T$.

Calibration: If given $C(K,T)$ at various expiries T, then obtain σ_{imp} and use boxed equation to (not uniquely) find $\sigma(t)$.

Calibration of local volatility $\sigma(S, t)$ in trinom tree

It's harder when σ depends on (S, t). SKIP

Here's the rough idea. Assume $r = 0$ and a tree with S-levels s_j with equal spacing $\Delta S = s_{j+1} - s_j$. Let $\Delta K := \Delta S$.

Given: $C_0(K, \tau)$, the time-0 price of a strike-K expiry-τ call, for all (K, τ). Find at each node (s_j, t_n): Local volatility σ which generates risk-neutral probabilities consistent with call prices (and stock prices).

- The probability of reaching node (s_j, t_n) equals $(\Delta K)\mathrm{Fly}_0(s_j, t_n)$ where $\mathrm{Fly}_0(K, \tau)$ is the time-0 price of a *butterfly*:

$$\mathrm{Fly}_0(K, \tau) := \frac{C_0(K - \Delta K, \tau) - 2C_0(K, \tau) + C_0(K + \Delta K, \tau)}{(\Delta K)^2}$$

Calibration of local volatility in a trinomial tree

SKIP

- The probability of reaching node (s_j, t_n) *and then going up* equals $(\Delta t/\Delta K)\mathrm{Cal}_0(s_j, t_n)$, where $\mathrm{Cal}_0(K, \tau)$ is the time-0 price of a *calendar spread*:

$$\mathrm{Cal}_0(K, \tau) := \frac{C_0(K, \tau + \Delta t) - C_0(K, \tau)}{\Delta t}$$

Hence the conditional up-probability from node (s_j, t_n) is the ratio

$$p_u = \frac{\mathbb{P}(\text{reach } (s_j, t_n) \text{ then up})}{\mathbb{P}(\text{reach } (s_j, t_n))} = \frac{(\Delta t/\Delta K)\mathrm{Cal}_0}{(\Delta K)\mathrm{Fly}_0}$$

Calibration of local volatility in a trinomial tree

SKIP

On the other hand, the up-probability is also obtained by solving

$$\begin{aligned} p_u(\Delta K) + p_m(0) + p_d(-\Delta K) &= 0 \\ p_u(\Delta K)^2 + p_m(0) + p_d(-\Delta K)^2 &= \sigma^2 s_j^2(\Delta t) \end{aligned} \quad \Rightarrow p_u = \frac{\sigma^2 s_j^2 \Delta t}{2(\Delta K)^2}$$

where σ, p_u, p_m, p_d all depend on (s_j, t_n). Therefore

$$\frac{\sigma^2 s_j^2 \Delta t}{2(\Delta K)^2} = \frac{(\Delta t/\Delta K)\mathrm{Cal}_0}{(\Delta K)\mathrm{Fly}_0}$$

Conclusion: at node (s_j, t_n), the local volatility calibrated at time 0 is

$$\sigma(s_j, t_n) = \sqrt{\frac{2}{s_j^2} \times \frac{\mathrm{Cal}_0(s_j, t_n)}{\mathrm{Fly}_0(s_j, t_n)}}$$

Calendar Spread

Butterfly Spread

Conceptually, the local volatility $\sigma(s,t)$ can be inferred from prices of options with strikes near s and expiries near t.

Calibration of local volatility in a trinomial tree

Comments: SKIP

- We have derived a discrete version of the “Dupire equation.” We'll return to this formula in the PDE context.
- If you try to implement this formula directly, the calibrated σ will be sensitive to noise in the option price observations. In practice some regularization/smoothing procedure is advisable.
- Calibration is much easier in the case where σ depends only on t instead of jointly on (S, t).

SKIP

Another path-dependent option: Lookback

A fixed-strike lookback with strike K, start date 0, and expiry T pays

SKIP

$$\text{Call:} \qquad (\max_{t\in[0,T]\cap\mathcal{T}} S_t - K)^+$$

$$\text{Put:} \qquad (K - \min_{t\in[0,T]\cap\mathcal{T}} S_t)^+$$

where $\mathcal{T}$ is some set of monitoring times.

Pricing a fixed-strike lookback call

With risk-neutral dynamics SKIP

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

the level of S_t determines the time-t conditional distribution of

$$\max_{u\in[t,T]\cap\mathcal{T}} S_u.$$

But that's not enough information, if option is on $\max_{u\in[0,T]\cap\mathcal{T}} S_u$.

Need also to track the running maximum

$$M_t := \max_{u\in[0,t]} S_u.$$

So the time-t price of a fixed-strike lookback call is a function of *three* state variables. Due to path dependence, not just a function of (t, S_t).

$$C(t, S_t, M_t)$$

Pricing a fixed-strike lookback call in a tree

SKIP

- Recall: the knockout option, also path-dependent, has price

$$C(t, S_t, \mathbf{1}_{\text{knockout prior to time } t})$$

  Here the path-dependence is simple. The path-dependent state variable has only two states: 0/1 (live/dead).

  And the "1" state does not need to be tracked in the tree.

- Lookback is more complicated than the barrier. At each node (t, S_t) in the tree, need to store a table of values associating each possible M_t with $C(t, S_t, M_t)$.

Example

SKIP

Assumes r=0	
nu	-0.042
sigma	0.29
deltaX	0.25
deltaT	0.25

running max	option price: fixed strike lookback call

pu	0.15
pm	0.66
pd	0.19

K 100

j		n=1 time 0	n=2 time 0.25	n=3 time 0.5	n=4 time 0.75	n=5 time 1.00
j=4	271.83					271.83 171.83
j=3	211.70				211.70 120.60	211.70 111.70
j=2	164.87			164.87 77.71	164.87 71.81	211.70 111.70 164.87 64.87
j=1	128.40		128.40 42.47	128.40 38.40	164.87 64.87 128.40 33.80	164.87 64.87 128.40 28.40
j=0	100	100.00 13.84	100.00 10.96	128.40 29.20 100.00 7.79	128.40 28.40 100.00 4.21	164.87 64.87 128.40 28.40 100.00 0.00
j=-1	77.88		100.00 1.57	100.00 0.62	128.40 28.40 100.00 0.00	128.40 28.40 100.00 0.00
j=-2	60.65			100.00 0.00	100.00 0.00	128.40 28.40 100.00 0.00
j=-3	47.24				100.00 0.00	100.00 0.00
j=-4	36.79					100.00 0.00