Option Pricing – Homework 6
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Problem 1.a)

At time T_2:
The long position has a payoff: $S_{T_2} - K$
For the short position, we must pay: $-(S_{T_2} - F_t)$
Net payoff:

$$S_{T_2} - K - \left(S_{T_2} - F_t\right) = S_{T_2} - K - S_{T_2} + F_t = F_t - K$$

Therefore, accounting for the discount factor from time T_2 to t:

$$f_t = e^{-r*(T_2-t)} * (F_t - K)$$

Problem 1.b)

The strategy asks us to buy and hold the stock until time T_2.
A stock, which is an agreement of ownership for a company, can be stored virtually in a virtual account (e.g. Robinhood account) or in a physical paper document until the future time T_2, with no storage cost.

However, crude oil has some storage cost. We must have a sufficient physical space to store a barrel of crude oil and ensure that the storage has the right temperature and air exposure so that the oil doesn't go bad (as it is a chemical substance). Since there is some storage cost, F_t might not be enough to compensate us for all the expenses related to the storage cost and that the arbitrage strategy can fail in generating a profit.

Problem 1.e)

Isolating S_t term out from F_t

$$F_t = \exp\left\{e^{-k*(T_2-t)} * \ln(S_t) + \left(1 - e^{-k*(T_2-t)}\right)\alpha + \frac{\sigma^2}{4k} * \left(1 - e^{-2k*(T_2-t)}\right)\right\}$$

$$F_t = \exp\left\{e^{-k*(T_2-t)} * \ln(S_t)\right\} * \exp\left\{\left(1 - e^{-k*(T_2-t)}\right)\alpha + \frac{\sigma^2}{4k} * \left(1 - e^{-2k*(T_2-t)}\right)\right\}$$

$$F_t = S_t^{e^{-k*(T_2-t)}} * \exp\left\{\left(1 - e^{-k*(T_2-t)}\right)\alpha + \frac{\sigma^2}{4k} * \left(1 - e^{-2k*(T_2-t)}\right)\right\}$$

$$f_t = e^{-r*(T_2-t)} * \left(S_{T_2} - F_t\right)$$

$$f_t = e^{-r*(T_2-t)} * \left(S_{T_2} - S_t^{e^{-k*(T_2-t)}} * \exp\left\{\left(1 - e^{-k*(T_2-t)}\right)\alpha + \frac{\sigma^2}{4k} * \left(1 - e^{-2k*(T_2-t)}\right)\right\}\right)$$

For $t = 0$,

$$f_0 = e^{-r*T_2} * \left(S_{T_2} - S_0^{e^{-k*T_2}} * \exp\left\{(1 - e^{-k*T_2})\alpha + \frac{\sigma^2}{4k} * (1 - e^{-2k*T_2})\right\}\right)$$

$$\frac{df_0}{dS} = e^{-r*T_2} * \left(1 - (e^{-k*T_2}) * S_0^{(e^{-k*T_2})-1} * \exp\left\{(1 - e^{-k*T_2})\alpha + \frac{\sigma^2}{4k} * (1 - e^{-2k*T_2})\right\}\right)$$

Exact solution is computed in the Jupyter Notebook.

Problem 1.f)

We have a short position for one call option. We want to hedge this.
Say we have a long position for n forward contracts.
Sensitivity of the long position should match the sensitivity of the short position.

$$n * Forward\ Sensitivity = Call\ Sensitivity$$

$$n * \frac{\partial f_0}{\partial S} = \frac{\partial C}{\partial S} = \Delta$$

$$n = \frac{\Delta}{\frac{\partial f_0}{\partial S}}$$

$$Hedge\ Ratio = \frac{\Delta}{\frac{\partial f_0}{\partial S}}$$

Exact solution is computed in the Jupyter Notebook.

Problem 1.g)

F_t is the time-t forward price for T_2-delivery

At time-T_2:

- The holder will receive delivery of θ barrel of crude oil, each with price S_{T_2}
- The holder will pay delivery price of K
- Payoff: $\theta * (S_{T_2} - K)$

At time-T_1:

- The holder will see the forward price: F_{T_1}, for T_2-delivery
- If the forward price is bigger than the delivery price, the holder will want to lock in the delivery price. He will want to buy more at the cheaper delivery price:
    - If $F_{T_1} > K$, then $\theta = 5{,}000$
- If the forward price is cheaper than the delivery price, the holder will want to buy less at the expensive delivery price:
    - If $F_{T_1} < K$, then $\theta = 4{,}000$

<u>Computing the Time-0 Value</u>

$$E\left[S_{T_2} \middle| \mathcal{F}_{T_1}\right] = F_{T_1}$$

$$E\left[S_{T_2} - K \middle| \mathcal{F}_{T_1}\right] = F_{T_1} - K$$

$$\theta * E\left[S_{T_2} - K \middle| \mathcal{F}_{T_1}\right] = \theta * \left(F_{T_1} - K\right)$$

Using the discount factor, the time-0 value of the contract will be:

$$V_0 = E\left[e^{-rT_1} * \ \theta * \left(F_{T_1} - K\right)\right]$$

Use the mean to get the expected time-0 value of the payoff

Problem 2. a)

If S_t is a stock that continuously pays a constant proportional dividend yield q,

The time-t forward price for time-T delivery of S_T is:

$$F_t = S_t * e^{(r-q)*(T-t)}$$

Therefore, **the time-0 forward price** for time-T delivery of S_T,

$$F_0 = S_0 e^{(r-q)T}$$

Time-0 Value of Forward Contract, f_0

$$f_0 = e^{-rT}(F_0 - K)$$

$$f_0 = e^{-rT}\left(S_0 e^{(r-q)T} - K\right)$$

Using replication:

For the stock:

1 share of stock at time-0 → e^{qt} share of stock at time-t

a share of stock at time-0 → $a * e^{qt}$ share of stock at time-t

For the bank account:

b unit of Bank at time-0 → b unit of Bank at time-t

$$V_t = a * e^{qt} * S_t + b * B_t$$
$$V_t = a * e^{qt} * S_t + b * e^{rt}$$
$$V_T = a * e^{qT} * S_T + b * e^{rT}$$

We want to replicate: $S_T - K$

$$V_T = S_T - K$$
$$a * e^{qT} * S_T + b * e^{rT} = S_T - K$$

$$a * e^{qT} * S_T = S_T$$
$$a * e^{qT} = 1$$
$$a = e^{-qT}$$

$$b * e^{rT} = -K$$
$$b = -\frac{K}{e^{rT}}$$
$$b = -K * e^{-rT}$$

At time-0, the replicating portfolio holds:

e^{-qT} shares of S and $(-Ke^{-rT})$ units of B.

Problem 2. b) At time T_0, from the dividend payment, we will get:

$$x * D$$

We will then allocate the dividend into the bank account, so we need to multiply it with B_t.

At time t, the value of the replicating portfolio is:

$$V_t = x * S_t + (y + x * D * e^{-rT_0}) * B_t$$

$y * B_t$ is the original bank account balance
Now with an additional $x * D * e^{-rT_0} * B_t$ from the dividend

$$V_t = x * S_t + (y + x * D * e^{-rT_0}) * e^{rt}$$
$$V_T = x * S_T + \left(x * D * e^{r(T-T_0)}\right) + y * e^{rT}$$

Set the replicating portfolio to $(S_T - K)$:

$$V_T = S_T - K$$
$$x * S_T + \left(x * D * e^{r(T-T_0)}\right) + y * e^{rT} = S_T - K$$

$$x * S_T = S_T$$
$$x = 1$$

$$\left(x * D * e^{r(T-T_0)}\right) + y * e^{rT} = -K$$
$$De^{r(T-T_0)} + y * e^{rT} = -K$$
$$y * e^{rT} = -K - De^{r(T-T_0)}$$
$$y = -K * e^{-rT} - De^{r(T-T_0)}e^{-rT}$$

$$y = -Ke^{-rT} - De^{-rT_0}$$

At time-0, the replicating portfolio holds:
1 share of S
$(-Ke^{-rT} - De^{-rT_0})$ units of B

Time-0 Value of Forward Contract, f_0

$$f_0 = x * S_0 + y * B_0 = x * S_0 + y * e^{r*0} = x * S_0 + y$$

$$f_0 = S_0 - Ke^{-rT} - De^{-rT_0}$$

Setting $f_0 = 0$,

$$S_0 - K * e^{-rT} - De^{-rT_0} = 0$$
$$S_0 - De^{-rT_0} = K * e^{-rT}$$
$$K * e^{-rT} = S_0 - De^{-rT_0}$$
$$K = e^{rT}(S_0 - De^{-rT_0})$$

Time- 0 Forward Price, F_0

$$F_0 = K = e^{rT}(S_0 - De^{-rT_0})$$

$$F_0 = e^{rT}(S_0 - De^{-rT_0})$$