Financial Mathematics 32000

Lecture 1

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FINM 33000 and FINM 32000

This is the second course in the sequence.

- Fall 2024: Option pricing
- Spring 2025: Numerical methods [for option pricing]
- "Option pricing" is meant in a broad sense: the pricing and hedging of options and other financial *derivative contracts*
- A *derivative security* or *derivative contract* is a financial instrument whose payoff is defined in terms of an *underlying* (e.g.: A security such as a stock/bond. An index. An interest rate.)

Why do we need numerical methods

Because we do not have simple exact formulas

- when the *dynamics* of the underlying risks are too complicated.
  For example, when working outside the class of Gaussian models (such as BM and GBM), simple exact formulas are less common.
  Why isn't GBM enough? Because it's not consistent with the **volatility skew**.
- when the *contract* to be priced/hedged is too complicated.
  For example, path-dependent options are typically harder to price than European-style options. Path-dependent options include **American**-style options, which can be exercised before expiry.

can be exercised at t which is smaller than T (expiry)

delta_t is 1/252
because we have 252 trading days (n)

Realized Volatility

Realized variance of S, sampled at interval Δt, from time 0 to time T can be defined, using log-returns by letting $t_n = n\Delta t$ and $T = t_N$ and

$$RVar = \frac{1}{T}\sum_{n=0}^{N-1}\left(\log\frac{S_{t_{n+1}}}{S_{t_n}}\right)^2$$

T = 252 x 1/252

Alternatively could use simple returns, letting $\Delta S = S_{t_{n+1}} - S_{t_n}$ and

$$RVar = \frac{1}{T}\sum_{n=0}^{N-1}\left(\frac{\Delta S}{S_{t_n}}\right)^2$$

Both RVar are percentage return,
squared, and take the average
and annualizing it by multiplying by 1/252

Alternatively, subtract the sample mean. **Realized volatility** of S is

$$RVol = \sqrt{RVar}$$

If $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$, then $RVol \to \sigma$ as $\Delta t \to 0$.

Black Scholes

Define the function C^{BS} for $X > 0, K > 0, \sigma > 0, t < T$ by

$$C^{BS}(X, t, K, T, R_{grow}, R_{disc}, \sigma) := e^{-R_{disc}(T-t)}\left[FN(d_1) - KN(d_2)\right],$$

and

Forward Price

$$F := Xe^{R_{grow}(T-t)}$$

d_1,2
how many standard deviations are you in or out the money?
If +1 , it's 1 stdev in the money
If -1 , it's 1 stdev out the money
If 0 , it's at the money

and

$$d_{1,2} := d_{+,-} := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}.$$

Recall: if $\mathrm{d}X_t = R_{grow}X_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t$ where W is $\mathbb{P}$-BM. then

Discounted Payoff

$$e^{-R_{disc}(T-t)}\mathbb{E}_t(X_T - K)^+ = C^{BS}(X_t, t, K, T, R_{grow}, R_{disc}, \sigma)$$

so C^{BS} gives the time-t price of a T-expiry K-strike call on S.

Implied Volatility

Given a time-t price C_t of a European call option (K, T) on a no-div stock S, the time-t **implied volatility** is the σ such that

$$C_t = C^{BS}(S_t, t, K, T, r, r, \sigma)$$

where C^{BS} is the Black-Scholes formula and r is the interest rate.

- This exists and is unique, if

Vega is always positive

- The bigger the dollar price C_t, the bigger the implied vol σ_{imp}
- Gives is another way quoting an option price. Instead of saying the option is trading at \$`x.xx`, can say it's trading at `yy`% vol.

35% vol = vol of 0.35

Implied Volatility

For **general** underlying Y_T, let F_t be time-t forward price. Recall:

- Contract with payoff $Y_T - F_t$ has time-t price 0. Thus $F_t = \mathbb{E}_t Y_T$
- $F_t - K = (C_t - P_t)e^{r(T-t)}$ where P_t is time-t price of (K, T) European put. Thus $F_t = K_*$ if K_* is the strike where $C_t = P_t$.

Implied volatility of a European (K, T) call on X is the σ such that

forward price is martingale
0 drift

$$C_t = C^{BS}(F_t, t, K, T, 0, r, \sigma)$$

Today's expectation of next week's payoff is the same as tomorrow's expectation of next week's payoff

Implied volatility of a European (K, T) **put** on X is the σ such that

$$P_t = P^{BS}(F_t, t, K, T, 0, r, \sigma)$$

This is incorrect, look at Mark
KN(-d2) - F(N(-d1)

where $P^{BS} = e^{-R_{disc}(T-t)}\left[FN(d_1) - KN(d_2)\right]$ is B-S put formula.

- In frictionless markets, call implied vol = put implied vol

Interpretations

Interpretations of time-t implied volatility

- Intuitively, implied vol is in some sense the market's forward-looking expectation of "realized volatility" from time t until T, along paths that go near K.
- A language / a metric / a scale in which to quote option prices. Instead of quoting in dollars, can quote in vol points. (Use of this language does not mean we actually believe the Black-Scholes assumptions!) Analogy: quoting a bond price as a yield-to-maturity does not mean we actually believe that interest rates are constant.

Volatility smile/skew

If the underlying truly has GBM dynamics with volatility σ then

$$C(K,T) = C^{BS}(K,T,\sigma) \qquad \text{for all } (K,T)$$

Hence

$$\sigma_{imp}(K,T) = \sigma \qquad \text{for all } (K,T)$$

However, empirically,

- Plotting σ_{imp} against K, you typically see not a flat line, but rather a volatility *smile* or a volatility *skew*.
- Also, σ_{imp} varies wrt T; implied volatility has a *term structure*.
- The function $\sigma_{imp}(K,T)$ is called the implied vol *surface* or *skew*.

Volatility surface: SPX, by strike

SPX Index | 90) Asset ▾ | 91) Actions ▾ | 92) Views ▾ | 93) Settings ▾ | Volatility Surface

S&P 500 INDEX 4720.29 USD | Bloomberg | Mid | As of 12-Jan-2022 | 15:35

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

Strike | Listed | 10) Edit | ☑ Fwd | ☑ Strikes

Exp Date	ImpFwd	4670.0	4675.0	4680.0	4685.0	4690.0	4695.0	4700.0	4705.0	4710.0	4715.0	4720.0	4725.0	4730.0	4735.0	4740.0	4745.0	4750.0	4755.0	4760.0	4765.0	4770.0	4775.0
		4670.0	4675.0	4680.0	4685.0	4690.0	4695.0	4700.0	4705.0	4710.0	4715.0	4720.0	4725.0	4730.0	4735.0	4740.0	4745.0	4750.0	4755.0	4760.0	4765.0	4770.0	4775.0
14 Jan 2022	4719.93	18.92	18.66	18.40	18.14	17.88	17.61	17.35	17.07	16.79	16.49	16.18	15.89	15.67	15.53	15.41	15.29	15.16	15.04	14.92	14.82	14.73	14.66
18 Jan 2022	4719.83	13.84	13.68	13.52	13.36	13.19	13.01	12.84	12.65	12.47	12.27	12.07	11.88	11.70	11.57	11.46	11.35	11.24	11.14	11.04	10.95	10.87	10.80
19 Jan 2022	4719.86	14.40	14.24	14.07	13.90	13.72	13.54	13.36	13.17	12.98	12.78	12.58	12.37	12.18	12.01	11.87	11.74	11.62	11.50	11.38	11.27	11.17	11.08
21 Jan 2022	4719.69	14.83	14.64	14.46	14.27	14.08	13.89	13.70	13.51	13.32	13.14	12.95	12.77	12.60	12.43	12.27	12.12	11.97	11.83	11.70	11.58	11.47	11.37
24 Jan 2022	4719.61	14.62	14.45	14.29	14.12	13.95	13.78	13.61	13.44	13.28	13.11	12.95	12.78	12.62	12.47	12.31	12.16	12.02	11.88	11.75	11.62	11.50	11.38
26 Jan 2022	4719.58	15.66	15.49	15.33	15.16	14.99	14.83	14.66	14.49	14.33	14.16	13.99	13.83	13.67	13.51	13.35	13.20	13.04	12.90	12.75	12.61	12.48	12.35
28 Jan 2022	4719.11	16.16	16.00	15.84	15.67	15.51	15.34	15.17	15.01	14.84	14.67	14.51	14.34	14.18	14.01	13.85	13.70	13.54	13.39	13.24	13.09	12.95	12.81
31 Jan 2022	4718.92	15.55	15.40	15.25	15.09	14.94	14.78	14.63	14.47	14.32	14.16	14.00	13.85	13.70	13.54	13.39	13.24	13.10	12.95	12.81	12.67	12.54	12.41
2 Feb 2022	4718.98	15.76	15.61	15.46	15.31	15.15	15.00	14.85	14.70	14.54	14.39	14.24	14.09	13.93	13.79	13.64	13.49	13.35	13.20	13.06	12.93	12.79	12.66
4 Feb 2022	4718.25	15.92	15.78	15.63	15.48	15.33	15.18	15.04	14.89	14.74	14.59	14.44	14.29	14.15	14.00	13.86	13.72	13.57	13.44	13.30	13.16	13.03	12.90
7 Feb 2022	4718.32	15.46	15.32	15.18	15.04	14.90	14.76	14.62	14.48	14.34	14.20	14.06	13.92	13.78	13.64	13.50	13.37	13.23	13.10	12.97	12.84	12.72	12.59
9 Feb 2022	4717.54	15.59	15.45	15.32	15.18	15.04	14.90	14.76	14.62	14.48	14.34	14.20	14.07	13.93	13.79	13.66	13.52	13.39	13.26	13.13	13.01	12.88	12.76
11 Feb 2022	4717.17	15.79	15.66	15.52	15.39	15.25	15.12	14.98	14.85	14.71	14.57	14.44	14.31	14.17	14.04	13.91	13.78	13.65	13.52	13.39	13.27	13.14	13.02
18 Feb 2022	4715.36	15.70	15.58	15.45	15.33	15.20	15.07	14.95	14.82	14.69	14.57	14.44	14.32	14.19	14.07	13.95	13.82	13.70	13.58	13.46	13.35	13.23	13.12
25 Feb 2022	4714.54	15.80	15.68	15.57	15.45	15.33	15.22	15.10	14.99	14.87	14.75	14.64	14.52	14.41	14.29	14.18	14.07	13.95	13.84	13.73	13.62	13.52	13.41
28 Feb 2022	4714.09	15.63	15.51	15.40	15.29	15.17	15.06	14.95	14.83	14.72	14.61	14.50	14.38	14.27	14.16	14.05	13.94	13.83	13.72	13.61	13.51	13.40	13.30
18 Mar 2022	4711.30	16.36	16.26	16.16	16.06	15.96	15.86	15.76	15.66	15.56	15.46	15.36	15.27	15.17	15.07	14.97	14.87	14.77	14.68	14.58	14.49	14.39	14.30
31 Mar 2022	4710.35	16.73	16.63	16.54	16.45	16.36	16.26	16.17	16.08	15.98	15.89	15.80	15.71	15.61	15.52	15.43	15.34	15.24	15.15	15.06	14.97	14.88	14.79
14 Apr 2022	4709.51	17.03	16.94	16.85	16.77	16.68	16.59	16.50	16.42	16.33	16.24	16.15	16.07	15.98	15.89	15.81	15.72	15.63	15.55	15.46	15.38	15.30	15.21
29 Apr 2022	4709.62	17.39	17.31	17.23	17.15	17.07	16.99	16.91	16.83	16.75	16.67	16.59	16.51	16.43	16.35	16.27	16.19	16.11	16.03	15.95	15.87	15.79	15.72
20 May 2022	4706.54	17.83	17.75	17.68	17.61	17.53	17.46	17.39	17.31	17.24	17.16	17.09	17.02	16.94	16.87	16.80	16.72	16.65	16.58	16.50	16.43	16.36	16.29
31 May 2022	4705.83	17.94	17.86	17.79	17.72	17.65	17.58	17.51	17.44	17.37	17.29	17.22	17.15	17.08	17.01	16.94	16.87	16.80	16.73	16.66	16.59	16.52	16.45
17 Jun 2022	4703.73	18.31	18.24	18.17	18.11	18.04	17.97	17.90	17.83	17.77	17.70	17.63	17.56	17.50	17.43	17.36	17.30	17.23	17.16	17.10	17.03	16.96	16.90
30 Jun 2022	4703.82	18.46	18.40	18.33	18.27	18.20	18.14	18.07	18.00	17.94	17.87	17.81	17.74	17.68	17.61	17.55	17.48	17.42	17.35	17.28	17.22	17.16	17.09
15 Jul 2022	4703.38	18.62	18.56	18.49	18.43	18.36	18.30	18.23	18.17	18.10	18.04	17.97	17.91	17.84	17.78	17.72	17.65	17.59	17.53	17.46	17.40	17.34	17.28
19 Aug 2022	4701.68	19.08	19.02	18.96	18.90	18.84	18.78	18.72	18.66	18.60	18.54	18.48	18.42	18.36	18.30	18.24	18.18	18.12	18.06	18.00	17.94	17.89	17.83
16 Sep 2022	4699.69	19.34	19.28	19.23	19.17	19.11	19.05	19.00	18.94	18.88	18.83	18.77	18.71	18.66	18.60	18.54	18.49	18.43	18.37	18.32	18.26	18.21	18.15
30 Sep 2022	4700.41	19.52	19.47	19.41	19.36	19.30	19.25	19.19	19.14	19.08	19.03	18.98	18.92	18.87	18.81	18.76	18.70	18.65	18.59	18.54	18.49	18.43	18.38
21 Oct 2022	4700.74	19.67	19.62	19.57	19.51	19.46	19.40	19.35	19.30	19.24	19.19	19.14	19.08	19.03	18.97	18.92	18.87	18.81	18.76	18.71	18.66	18.60	18.55

Option Pricing (OVME) | Legend | Zoom | 100%

99) Quick Pricer

SN 417340 EST GMT-5:00 G721-6557-166 12-Jan-2022 15:56:46

Volatility surface: SPX, by moneyness

SPX Index | 90) Asset ▾ | 91) Actions ▾ | 92) Views ▾ | 93) Settings ▾ | Volatility Surface

S&P 500 INDEX 4719.19 USD Bloomberg Mid As of 12-Jan-2022 15:40

1) Vol Table | 2) 3D Surface | 3) Term | 4) Skew | 5) Dividends | 6) Prices

Moneyness | Listed | 10) Edit | ☑ Fwd | ☑ Strikes

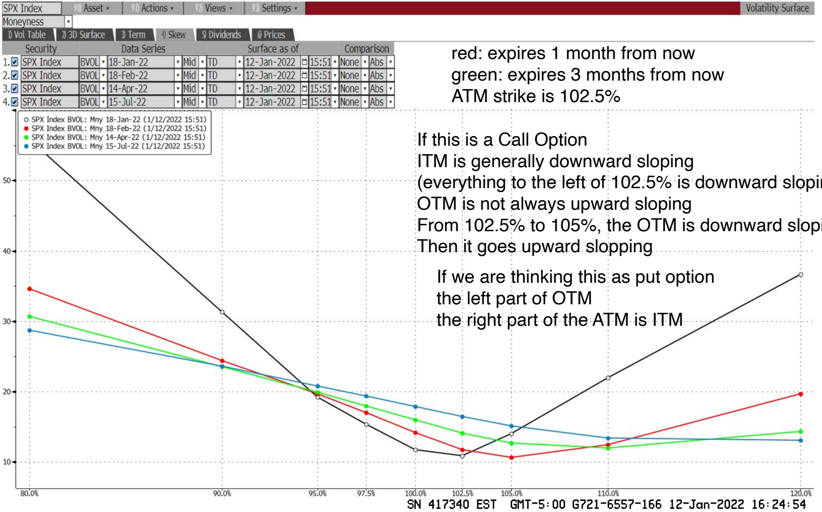
Exp Date	ImpFwd	80.0%	90.0%	95.0%	97.5%	100.0%	102.5%	105.0%	110.0%	120.0%
		3775.4	4247.3	4483.2	4601.2	4719.2	4837.2	4955.1	5191.1	5663.0
14 Jan 2022	4718.94	76.98	45.47	27.95	21.98	16.30	15.63	22.34	32.87	51.93
18 Jan 2022	4718.84	56.42	31.81	19.91	15.90	12.16	11.10	14.47	22.08	36.64
19 Jan 2022	4718.80	54.06	30.48	20.04	16.48	12.65	11.10	14.13	21.08	34.82
21 Jan 2022	4718.59	50.26	29.54	20.77	17.01	13.02	11.03	13.49	20.00	32.02
24 Jan 2022	4718.52	46.25	27.57	20.05	16.67	13.00	10.69	12.28	18.03	29.06
26 Jan 2022	4718.49	44.31	27.95	21.01	17.73	14.04	11.31	12.04	17.31	27.60
28 Jan 2022	4718.04	42.99	28.07	21.47	18.21	14.55	11.60	11.86	16.63	26.43
31 Jan 2022	4717.85	40.80	26.61	20.51	17.47	14.04	11.20	11.19	15.39	24.87
2 Feb 2022	4717.91	39.91	26.55	20.65	17.65	14.27	11.40	11.12	15.07	24.06
4 Feb 2022	4717.17	39.32	26.44	20.75	17.79	14.47	11.62	11.07	14.50	23.30
7 Feb 2022	4717.25	37.68	25.47	20.09	17.25	14.08	11.35	10.71	13.78	22.31
9 Feb 2022	4716.46	37.05	25.38	20.17	17.37	14.23	11.51	10.74	13.53	21.76
11 Feb 2022	4716.10	36.55	25.33	20.29	17.53	14.46	11.77	10.81	13.27	21.23
18 Feb 2022	4714.26	34.84	24.65	19.93	17.33	14.45	11.91	10.80	12.53	19.73
25 Feb 2022	4713.46	33.92	24.32	19.82	17.33	14.66	12.26	11.03	12.15	18.56
28 Feb 2022	4713.02	33.22	23.91	19.51	17.11	14.51	12.15	10.93	11.88	18.12
18 Mar 2022	4710.20	32.16	23.89	19.84	17.67	15.37	13.23	11.92	11.94	16.13
31 Mar 2022	4709.28	31.35	23.79	19.99	17.95	15.81	13.75	12.34	11.96	15.15
14 Apr 2022	4708.41	30.83	23.70	20.09	18.18	16.17	14.26	12.83	12.09	14.34
29 Apr 2022	4708.51	30.33	23.69	20.28	18.47	16.59	14.78	13.34	12.27	13.77
20 May 2022	4705.41	29.80	23.69	20.50	18.81	17.09	15.41	14.01	12.65	13.34
31 May 2022	4704.69	29.46	23.59	20.51	18.89	17.22	15.61	14.23	12.77	13.26
17 Jun 2022	4702.57	29.27	23.68	20.75	19.21	17.63	16.09	14.74	13.12	13.37
30 Jun 2022	4702.67	29.05	23.68	20.84	19.34	17.80	16.29	14.95	13.26	13.26
15 Jul 2022	4702.22	28.83	23.74	21.01	19.51	17.98	16.52	15.25	13.52	13.05
19 Aug 2022	4700.52	28.57	23.86	21.30	19.90	18.48	17.12	15.89	14.10	13.22
16 Sep 2022	4698.54	28.32	23.87	21.43	20.11	18.77	17.47	16.28	14.47	13.32
30 Sep 2022	4699.26	28.27	23.93	21.55	20.26	18.94	17.66	16.48	14.64	13.42
21 Oct 2022	4699.58	28.18	23.89	21.58	20.38	19.17	17.94	16.76	14.84	13.60

Option Pricing (OVME) Legend Zoom 100%

99) Quick Pricer

SN 417340 EST GMT-5:00 G721-6557-166 12-Jan-2022 15:58:14

Volatility skews: SPX, by moneyness, 4 expirations

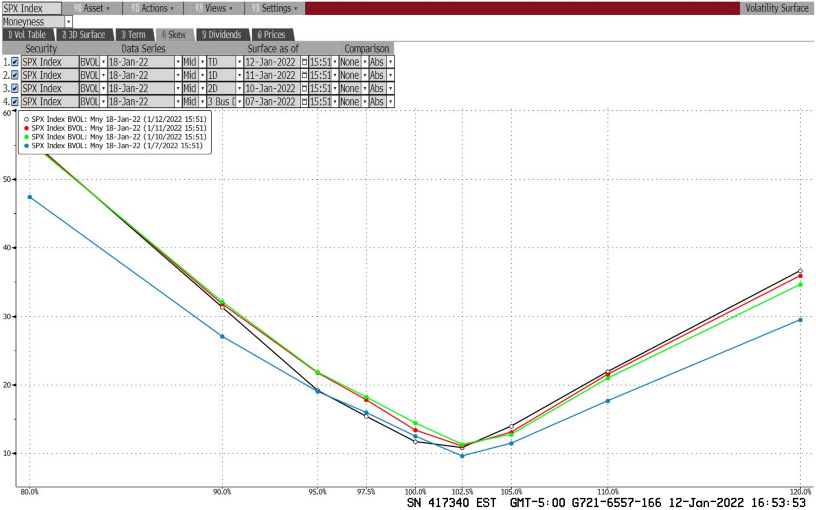


red: expires 1 month from now
green: expires 3 months from now
ATM strike is 102.5%

If this is a Call Option
ITM is generally downward sloping
(everything to the left of 102.5% is downward sloping)
OTM is not always upward sloping
From 102.5% to 105%, the OTM is downward sloping,
Then it goes upward slopping

If we are thinking this as put option
the left part of OTM
the right part of the ATM is ITM

Volatility skews: SPX, by moneyness, 4 observations



Volatility smile/skew: EBAY

EBAY US $ * ⬇ 74.61 -1.59 B 1s Equity OCM

DELAY 12:26 Vol 5,752,860 Op 76.6 Q Hi 76.68 Q Lo 74.07 Q ValTrd 433.120m

OPTION MONITOR 1 COMP Center: 74.12 1 <GO> to Edit Spreadsheet

BID d	ASK d	IVBD	IVAS	IVMD	OPIN d		BID d	ASK d	IVBD	IVAS	IVMD	OPIN d
Bid Price	Ask Price	Imp Volat Bid	Imp Volat Ask	Imp Volat Mid	Open Intrst	EBAY <-CALLS PUTS->	Bid Price	Ask Price	Imp Volat Bid	Imp Volat Ask	Imp Volat Mid	Open Intrst
74.610	74.620					QXB JUL4	74.610	74.620				
44.60	44.80	N.A.	69.07	N.A.	335	1) 30 16)		.05	N.A.	74.86	69.02	478
39.60	39.80	N.A.	70.07	54.16	305	2) 35 17)		.05	N.A.	62.87	57.97	453
34.70	34.80	N.A.	57.73	36.43	378	3) 40 18)		.05	N.A.	52.53	48.44	1,169
29.70	29.90	N.A.	52.91	46.90	355	4) 45 19)	.05	.10	43.44	47.80	45.89	735
24.80	25.00	37.23	46.40	42.65	514	5) 50 20)	.15	.20	41.55	43.68	42.61	1,559
20.00	20.20	36.56	41.46	39.18	889	6) 55 21)	.30	.35	38.88	40.05	39.50	2,205
						XBA JUL4						
15.40	15.50	34.85	37.76	36.34	1,931	7) 60 22)	.65	.75	35.56	37.06	36.31	4,757
11.10	11.30	33.40	34.36	33.88	3,851	8) 65 23)	1.40	1.45	33.46	34.45	33.95	9,285
7.40	7.60	31.33	32.79	32.06	3,313	9) 70 24)	2.65	2.75	31.84	32.56	32.20	4,905
4.50	4.60	29.70	31.03	30.37	7,000	10) 75 25)	4.70	4.80	30.15	30.81	30.48	2,415
2.45	2.55	29.16	29.88	29.52	2,166	11) 80 26)	7.60	7.80	28.98	30.39	29.68	619
1.20	1.30	28.78	29.25	29.02	1,429	12) 85 27)	11.40	11.60	27.94	29.89	28.92	209
.55	.65	28.82	29.50	29.16	1,115	13) 90 28)	15.70	15.90	27.27	30.24	28.80	
.20	.30	27.69	30.02	28.92	16	14) 95 29)	20.40	20.70	N.A.	32.77	29.60	
.10	.15	28.66	30.60	29.69	305	15) 100 30)	25.30	25.50	N.A.	32.83	30.88	

Volatility smile/skew: GOOG

ATM strike is $555

GOOG US $ ↑555.40 +.50 Q555.20/555.37Q 1x1
At 12:19 d Vol 1,878,854 O 559.62Q H 562.00P L 552.95P Val 1.048B

GOOG US Equity | 95) Templates | 96) Actions | 97) Expiry | Option Monitor: Option Monitor

GOOGLE INC-C ↑555.4001 .5001 .0901% 555.20 / 555.3701 Hi 562.00 Lo 552.95 Volm 1878854 HV .00

☐Calc Mode Center 555.00 Strikes 19 Exch US Composit 92) Next Earnings(EM) 04/16/14 C

81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure

Calls								Strike	Puts							
Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt		Ticker	Bid	Ask	Last	IVM	DM	Volm	OInt
19 Apr 14 (10d); CSize 100; IDiv .77; R .12; IFwd 556.55								19	19 Apr 14 (10d); CSize 100; IDiv .77; R .12; IFwd 556.55							
1) GOOG 4 C510	48.20	50.80	50.00	58.97	.84	2	34	510.00	58) GOOG 4 P510	3.20	3.60	3.08	55.30	-.14	69	203
2) GOOG 4 C515	43.70	46.80	43.00y	57.94	.82		37	515.00	59) GOOG 4 P515	4.10	4.40	4.24	55.20	-.17	28	304
3) GOOG 4 C520	39.90	40.60	44.40	53.44	.81	11	35	520.00	60) GOOG 4 P520	4.90	5.30	4.90	54.18	-.20	13	485
4) GOOG 4 C525	36.20	37.70	40.00	55.38	.76	1	58	525.00	61) GOOG 4 P525	6.00	6.40	6.14	53.70	-.23	27	163
5) GOOG 4 C530	32.40	33.50	34.00	54.21	.73	1	76	530.00	62) GOOG 4 P530	7.10	7.70	7.05	52.96	-.27	14	183
6) GOOG 4 C535	28.90	29.80	30.56	53.31	.69	45	73	535.00	63) GOOG 4 P535	8.70	9.10	7.80	52.84	-.30	45	348
7) GOOG 4 C540	25.60	26.50	27.52	53.30	.65	12	214	540.00	64) GOOG 4 P540	10.40	10.80	9.91	52.60	-.35	26	249
8) GOOG 4 C545	22.50	23.30	23.70	52.81	.61	18	168	545.00	65) GOOG 4 P545	12.10	12.70	12.30	51.82	-.39	17	148
9) GOOG 4 C550	19.60	20.20	20.00	52.13	.56	168	358	550.00	66) GOOG 4 P550	14.30	14.80	13.80	51.63	-.44	84	155
10) GOOG 4 C555	17.00	17.50	17.95	51.84	.52	192	256	555.00	67) GOOG 4 P555	16.60	17.10	16.50	51.18	-.48	41	159
11) GOOG 4 C560	14.60	15.00	15.41	51.29	.47	472	694	560.00	68) GOOG 4 P560	19.20	19.70	18.61	51.09	-.53	101	113
12) GOOG 4 C565	12.40	13.00	13.14	51.28	.43	115	321	565.00	69) GOOG 4 P565	21.80	22.50	20.50	50.64	-.58	1	29
13) GOOG 4 C570	10.50	10.70	10.51	50.74	.38	154	652	570.00	70) GOOG 4 P570	24.90	25.70	23.82	50.26	-.62	1	51
14) GOOG 4 C575	8.80	9.20	9.20	50.26	.34	244	317	575.00	71) GOOG 4 P575	28.00	28.80	31.60y	49.87	-.67		8
15) GOOG 4 C580	7.30	7.80	7.50	50.28	.30	124	431	580.00	72) GOOG 4 P580	31.40	32.30	29.10	49.19	-.71	11	47
16) GOOG 4 C585	5.90	6.30	6.43	50.02	.25	28	211	585.00	73) GOOG 4 P585	34.80	36.00	38.00y	48.78	-.75		11
17) GOOG 4 C590	4.80	5.30	5.07	50.20	.22	71	333	590.00	74) GOOG 4 P590	37.70	40.20	36.23	46.97	-.80	8	21
18) GOOG 4 C595	3.90	4.40	4.15	50.05	.19	91	136	595.00	75) GOOG 4 P595	41.60	44.10	40.26	44.85	-.84	8	13
19) GOOG 4 C600	3.20	3.60	3.39	49.94	.16	80	837	600.00	76) GOOG 4 P600	45.90	48.30	63.95y	44.80	-.87		35
17 May 14 (38d); CSize 100; IDiv .23; R .16; IFwd 556.62								19	17 May 14 (38d); CSize 100; IDiv .23; R .16; IFwd 556.62							
20) GOOG 5 C510	52.10	54.80	51.30y	36.10	.79		15	510.00	77) GOOG 5 P510	6.70	7.20	5.80	33.61	-.20	6	323

93) Default color legend Zoom 90%

DM is Delta (Delta Midpoint),
positive for calls, negative for ITM
Close to 0.5 around ATM
Delta goes to 0 for deeper OTM (both for puts and calls)
Difference between DM calls and DM put is 1

ATM Strike: 550

Volatility surface: GOOG

GOOG US $ ↓556.80 +1.90 P556.65/556.84Q 2x1
At 12:32 d Vol 1,926,365 O 559.62Q H 562.00P L 552.95P Val 1.075B

GOOG US Equity | 95) Templates | 96) Actions | 97) Expiry | Option Monitor: Option Monitor

GOOGLE INC-C ↓556.80 1.90 .3424% 556.6501 / 556.8401 Hi 562.00 Lo 552.95 Volm 1926365 HV .00

Calc Mode Center 555.00 Strikes 19 Exch US Composit 92) Next Earnings(EM) 04/16/14 C

81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure

Expiry	19 Apr 2014				17 May 2014				21 Jun 2014				20 Sep 2014				17 Jan 2015			
Calls/Puts	Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts	
Strike	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM
505.00	.91	51.46	-.11	56.24	.83	33.50	-.17	34.07	.80	28.63	-.20	28.79	.74	26.46	-.26	26.02	.71	25.49	-.29	25.08
510.00	.88	50.89	-.14	55.57	.81	33.13	-.20	34.25	.77	28.55	-.23	28.69	.72	26.38	-.28	26.03	.70	25.20	-.30	24.56
515.00	.85	51.33	-.16	55.34	.79	32.84	-.22	33.86	.75	28.26	-.25	28.51	.70	26.34	-.29	25.96	.68	25.16	-.32	24.68
520.00	.82	51.42	-.19	54.24	.76	33.23	-.24	33.63	.73	28.38	-.27	28.43	.68	26.45	-.31	25.86	.66	25.12	-.33	24.60
525.00	.79	51.53	-.22	54.26	.73	33.14	-.27	33.50	.70	28.74	-.30	28.37	.66	26.29	-.33	25.68	.65	25.21	-.35	24.43
530.00	.75	51.68	-.26	53.59	.70	33.56	-.30	33.37	.67	28.54	-.32	28.27	.64	26.18	-.35	25.61	.63	24.98	-.37	24.60
535.00	.71	52.68	-.29	53.02	.67	33.28	-.33	33.16	.65	28.54	-.35	28.08	.62	26.24	-.37	25.49	.62	25.14	-.38	24.31
540.00	.67	52.50	-.33	52.64	.63	33.20	-.37	32.79	.62	28.27	-.38	27.94	.60	26.08	-.39	25.46	.60	25.08	-.40	24.43
545.00	.62	52.27	-.38	52.45	.60	32.95	-.40	32.84	.59	28.20	-.41	27.78	.58	26.14	-.42	25.45	.58	25.15	-.42	24.24
550.00	.58	51.82	-.42	51.92	.57	32.76	-.43	32.58	.56	28.04	-.44	27.75	.56	25.98	-.44	25.37	.57	24.96	-.43	24.37
555.00	.53	51.92	-.47	51.67	.53	32.61	-.47	32.44	.53	27.99	-.46	27.64	.54	26.03	-.46	25.25	.55	25.07	-.45	24.23
560.00	.48	50.99	-.52	51.18	.50	32.35	-.50	32.40	.51	27.99	-.49	27.55	.52	25.97	-.48	25.30	.53	25.01	-.47	24.27
565.00	.44	50.88	-.56	50.90	.46	32.34	-.54	32.05	.48	27.74	-.52	27.41	.50	25.95	-.50	25.28	.52	25.00	-.48	24.20
570.00	.39	50.26	-.61	50.53	.43	32.02	-.57	32.02	.45	27.70	-.55	27.48	.48	25.86	-.52	25.22	.50	24.83	-.50	24.31
575.00	.35	49.95	-.65	50.23	.39	31.85	-.61	31.75	.42	27.58	-.58	27.31	.46	25.80	-.54	25.13	.49	24.90	-.52	24.28
580.00	.30	49.76	-.69	50.05	.36	31.95	-.64	31.76	.39	27.49	-.61	27.20	.44	25.71	-.56	25.17	.47	24.86	-.53	24.03
585.00	.26	49.38	-.73	50.06	.33	31.86	-.67	31.66	.36	27.40	-.64	27.10	.42	25.63	-.58	25.28	.45	24.83	-.55	24.02
590.00	.23	49.41	-.76	51.33	.30	31.71	-.70	31.47	.34	27.28	-.66	27.28	.40	25.76	-.60	25.23	.44	24.78	-.57	23.95
595.00	.19	49.62	-.79	51.70	.27	31.30	-.73	31.21	.31	27.35	-.69	27.23	.38	25.89	-.62	25.18	.42	24.39	-.58	23.86
600.00	.16	49.15	-.82	51.71	.24	31.34	-.76	31.59	.29	27.29	-.71	27.10	.37	25.86	-.64	25.20	.41	24.59	-.60	23.87
605.00	.14	49.64	-.85	51.82	.22	31.06	-.78	31.40	.27	27.28	-.74	26.76	.35	25.87	-.66	25.24	.39	24.74	-.61	23.91
610.00	.12	49.72	-.88	50.09	.19	31.02	-.80	31.89	.24	27.21	-.76	27.17	.33	25.94	-.68	25.11	.38	24.65	-.63	23.79

93) Default color legend Zoom 85%

Volatility surface: AMZN

AMZN US $ ↓326.48 -.59 P326.36/326.51P 2x3
At 13:08 d Vol 2,908,935 O 328.47Q H 328.49D L 322.50Q Val 947.826M

AMZN US Equity 95) Templates 96) Actions 97) Expiry Option Monitor: Option Monitor

AMAZON.COM INC ↓326.48 -.59 -.1804% 326.360 / 326.51 Hi 328.49 Lo 322.50 Volm 2908935 HV 24.76

Calc Mode Center 326.66 Strikes 5 Exch US Composit 92) Next Earnings(EM) 04/25/14 E

81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure

Expiry	19 Apr 2014				17 May 2014				21 Jun 2014				19 Jul 2014				18 Oct 2014			
Calls/Puts	Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts		Calls		Puts	
Strike	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM	DM	IVM
280.00	.98	55.66	-.02	53.40	.86	48.25	-.14	46.67	.83	40.35	-.16	38.61	.81	36.77	-.18	35.63	.77	34.48	-.22	33.08
285.00	.98	47.38	-.03	50.74	.83	48.31	-.16	45.71	.81	39.40	-.19	38.04	.79	36.61	-.20	35.10	.75	34.06	-.24	32.75
290.00	.94	53.36	-.05	48.05	.81	47.17	-.19	44.94	.78	39.05	-.21	37.48	.76	36.34	-.23	34.44	.73	33.43	-.27	32.50
295.00	.93	47.70	-.06	45.69	.78	45.17	-.22	44.18	.75	38.71	-.24	37.02	.74	35.31	-.26	34.08	.71	32.87	-.29	32.11
300.00	.89	47.88	-.09	43.23	.75	44.56	-.25	43.42	.72	38.21	-.27	36.51	.71	34.87	-.29	34.00	.68	32.81	-.31	31.80
305.00	.85	45.62	-.13	41.09	.71	43.74	-.29	42.74	.69	37.40	-.31	36.07	.68	34.48	-.32	33.54	.66	32.30	-.34	31.68
310.00	.82	39.98	-.18	39.65	.67	43.80	-.33	42.16	.65	37.05	-.34	35.56	.65	33.86	-.35	33.14	.63	32.05	-.37	31.33
315.00	.74	38.38	-.26	38.31	.63	42.99	-.37	41.56	.61	36.59	-.38	35.35	.62	33.54	-.38	32.92	.61	32.05	-.39	31.12
320.00	.65	37.84	-.35	37.21	.58	42.54	-.42	41.19	.58	36.16	-.42	34.67	.58	33.56	-.42	32.66	.58	31.64	-.42	30.93
325.00	.54	36.78	-.46	36.31	.54	41.87	-.46	40.53	.54	35.78	-.46	34.46	.54	33.18	-.46	32.43	.55	31.49	-.45	30.69
330.00	.43	35.57	-.57	34.99	.49	41.35	-.51	40.00	.50	35.45	-.50	33.89	.51	33.00	-.49	32.16	.53	31.53	-.47	30.65
335.00	.32	34.82	-.69	34.19	.44	40.81	-.56	39.45	.46	35.01	-.54	33.67	.47	32.67	-.53	31.98	.50	31.29	-.50	30.59
340.00	.22	34.59	-.78	33.80	.40	40.58	-.60	39.57	.42	34.77	-.58	33.26	.44	32.63	-.57	31.62	.47	31.21	-.53	30.29
345.00	.15	34.47	-.85	34.14	.35	40.18	-.65	38.82	.39	34.63	-.61	33.73	.40	32.40	-.60	31.50	.45	31.09	-.56	30.05
350.00	.09	34.92	-.91	33.55	.31	40.07	-.70	38.37	.35	34.56	-.65	33.51	.37	32.30	-.63	31.43	.42	30.80	-.58	30.03
355.00	.06	35.90	-.97	28.83	.27	39.71	-.73	38.86	.31	34.42	-.70	32.51	.34	32.17	-.67	31.27	.40	30.93	-.61	30.04
360.00	.04	37.29	-.99	29.45	.24	39.64	-.77	38.67	.28	34.01	-.72	33.19	.31	32.09	-.70	31.04	.37	30.75	-.63	29.83
365.00	.03	38.73			.20	39.53	-.80	38.60	.25	33.81	-.75	32.76	.28	31.82	-.73	30.85	.35	30.49	-.66	29.60
370.00	.02	40.89	-1.0		.17	39.55	-.84	37.03	.22	33.96	-.79	31.97	.25	31.75	-.76	30.72	.32	30.37	-.68	29.49

93) Default color legend Zoom 100%

the imp vol has a hump at May instead of decreasing from April (for google, it is solely decreasing over time)

On days around earnings call, there's lots of vol, so an increase in vol

Volatility smile: possible explanations

Why does the smile exist?

- The market prices options using a risk-neutral distribution of log-returns with *fatter tails* than the Normal (*excess kurtosis*).

What contributes to those fat tails?

- Clustering of volatility.
- Jumps introduce extreme outcomes into return distributions, especially short-term.

Volatility skew in equity markets

Why's the smile skewed? (Why fatter tails on *left*?) Physical causes:

- Negative correlation between price and instantaneous volatility.

  Empirically, instantaneous volatility increases as price decreases.

  This fattens the left tail of the distribution (negative skewness).
- Possibility of big down-jumps (crashes) also fattens the left tail.
- "The bull walks up the stairs, the bear jumps out the window."

Text

Combined with risk preferences:

The world is net long SPX
So people fear more downside —> high demand for puts

- "Supply and demand": Fear of downside leads to demand for protection/insurance, driving up the prices (hence implied vols) of OTM puts. And willingness to sell part of the upside leads (covered-) call writers to supply OTM calls, driving prices down.

people are willing to sell a chunk of potential upside
only collect +15% strike, collect premium for call options for +20% spot

Volatility skew: possible explanations

Why is the smile skewed? Another way to think about it:

- Implied volatility at strike K depends on the option price, which depends on expected future volatility – specifically, on expected future volatility *along paths near* K. Why? Future volatility along paths away from K does not help the option holder, because linear payoffs (e.g. zero, forward) are insensitive to vol. Only the *convex* part of the payoff brings benefits from volatility.
- Future volatility along paths near downside strikes is likely higher than future volatility along paths near upside strikes.
  So by the above logic, downside-strike options should trade higher above their intrinsic lower bound than upside-strike options do.

Volatility skew: implications

Presence of the volatility skew is inconsistent with GBM dynamics.

- Black-Scholes not adequate for pricing contracts such as forward-starting options, VIX options, etc, which are not just combinations of vanilla options. Consistent dynamic model needed.
- For simple purposes, using Black-Scholes may be adequate (even if logically inconsistent):
  - Interpolating/extrapolating call or put prices at liquid strikes, to price a call or put at an illiquid strike
  - Detecting strikes at a given expiry that seem under/overpriced

These purposes may just require fitting an implied vol curve.

Pricing an American

- An *American* option on S with strike K and expiry T pays

$$\text{Put: } (K - S_\tau)^+ \qquad \text{Call: } (S_\tau - K)^+$$

at the time of exercise τ, where $\tau \leq T$ is a *stopping time* chosen by the holder. Not a stopping time: $\tau = \operatorname{argmin}_{t \in [0,T]} S_t$

- By definition the *stopping time* τ can be random, but for each t it must satisfy $\{\tau = t\} \in \mathcal{F}_t$; thus the decision on whether to exercise at time t must depend only on information that has been revealed up to and including time t.
- The ability to exercise early makes the contract path-dependent; its payout no longer depends only on the underlying at a particular time T.

Pricing an American: call options

In general, pricing early-exercisable contracts requires numerics.

An exception: if $r > 0$, then it is *never optimal to exercise early an American call on a non-dividend-paying stock.*

Model-independent reason: (with all expiries $= T$; all strikes $= K$; discount bond Z), the time-t (where $t < T$) values of

$$\text{American call} \geq \text{European call} \geq \text{Forward} = S_t - KZ_t > S_t - K$$

American is worth $>$ the exercise payoff $S_t - K$. So *don't exercise.*

- But what if you think S is going down? Shorting the stock beats exercising your call. By shorting you still get S_t, but you *delay* paying K, and if stock dips, you may pay *less* than K to cover.

cover short position

Therefore, *American call price = European call price*

Pricing an American call: arbitrage argument

Arbitrage argument to show that American = European call:

Let $r \geq 0$ and no dividends. If American > European then go short the American and long the European for an immediate profit.

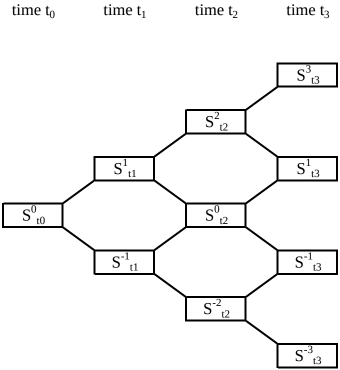
- If the counterparty never exercises, then no further liabilities.
- If the counterparty exercises, then you are short a share and long a value of K dollars in the bank account, which becomes $\geq K$ by expiry. Covering the short at expiration requires $\leq K$ dollars, due to the European.

Since American > European leads to arbitrage, we must have *American call price = European call price.*

And *early exercise of the American cannot be strictly optimal.*

Binomial tree

Consider an N-period model (example: $N = 3$) with a bank account $B_t = e^{rt}$ and a stock S with dynamics

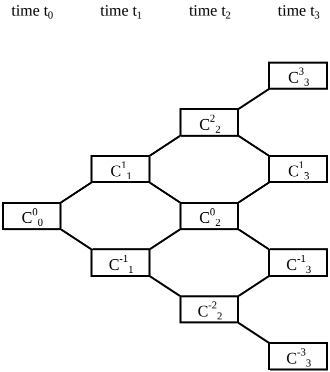


Superscripts are not powers. Subscripts $t_n = n\Delta t$ for $n = 0, \ldots, N$.

Binomial tree

Want to find time-0 price C_0^0 of some option.

To simplify notation, we will write C_n^j instead of $C_{t_n}^j$.



Consider three contract types: European, barrier, and American.

Pricing a European

If the expiry date is at time $T = t_N$ and the payoff function is $f(S)$, then for all j, the option price is

$$C_N^j = f(S_{t_N}^j).$$

Now induct backwards. If C_{n+1}^j has been computed for all j, then apply the one-period no-arbitrage results in the one-period subtree rooted at each (j, n) to find

$$C_n^j = e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}],$$

where the up-probability is

$$p_n^j := \frac{S_{t_n}^j e^{r\Delta t} - S_{t_{n+1}}^{j-1}}{S_{t_{n+1}}^{j+1} - S_{t_{n+1}}^{j-1}}.$$

Pricing an up-and-out barrier option

An *up-and-out put* on S with strike K, expiry T, and barrier H pays

$$(K - S_T)^+ \mathbf{1}_{\max_{t \in \mathcal{T}} S_t < H}.$$

where $\mathcal{T} \subseteq [0, T]$ are the observation times, specified in the contract. Continuous monitoring: $\mathcal{T} = [0, T]$. Discrete monitoring: $\mathcal{T}$ is some finite set. The option price is a function of *three* state variables:

$$C(S^j_{t_n},\ t_n,\ \mathbf{1}_{\text{knockout prior to time } t_n}).$$

where "knockout prior to time t_n" is the event that there exists $t < t_n$ with $t \in \mathcal{T}$ and $S_t \geq H$. Computationally: $C(S, t, 1)$ does not need to be tracked; it's always zero. So what we track in the tree is $C^j_n := C(S^j_{t_n}, t_n, 0)$, the option price *in the case of no prior knockout.*

Pricing an up-and-out barrier option

Let $t_N = T$. At terminal nodes, $C_N^j = (K - S_{t_N}^j)^+ \mathbf{1}_{S_{t_N}^j < H \text{ or } t_N \notin \mathcal{T}}$.

Inducting backwards, if C_{n+1}^j has been computed for all j, then at node (j, n) we have two cases:

$$t_n \in \mathcal{T} \text{ and } S_{t_n}^j \geq H \Rightarrow \quad C_n^j = 0 \tag{1}$$

$$t_n \notin \mathcal{T} \text{ or } S_{t_n}^j < H \Rightarrow \quad C_n^j = e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}] \tag{2}$$

So everywhere at or beyond the barrier at monitoring times, set the price to 0; elsewhere, take the usual discounted average of the next-time-step values.

(And in building a tree, you try to have the dates important in the contract – such as expiry and barrier monitoring dates – be among the dates represented in the tree.)

Pricing an American put

Notation: $x \wedge y := \min(x, y)$.

On a time interval $[t_1, t_2 \wedge \tau]$, consider a portfolio with price process V, that includes a static position in an American option, to be exercised at time $\tau \geq t_1$. We say it is an *arbitrage* if $V_{t_1} = 0$ and

$$\mathsf{P}(V_{t_2 \wedge \tau} < 0) = 0 \text{ and } \mathsf{P}(V_{t_2 \wedge \tau} > 0) > 0$$

for *some* stopping time $\tau \geq t_1$, if the portfolio is long the option;

for *all* stopping times $\tau \geq t_1$, if the portfolio is short the option.

Pricing an American

At expiry $T = t_N$, we have $C_N^j = (K - S_{t_N}^j)^+$. Inducting backwards, if C_{n+1}^j is the no-arb price that has been computed for each j then

$$C_n^j = \max((K - S_{t_n}^j)^+,\ e^{-r\Delta t}[p_n^j C_{n+1}^{j+1} + (1 - p_n^j) C_{n+1}^{j-1}])$$

This is because the holder at node (j, n) can either exercise and receive $(K - S_{t_n}^j)^+$, or hold on to the option which tomorrow will have no-arbitrage price of C_{n+1}^{j+1} (if up) or C_{n+1}^{j-1} (if down), which implies

- If $C_n^j < \max$, then there is arbitrage: go long the option.
- If $C_n^j > \max$, then there is arbitrage: short the option; if holder exercises, you close out with a profit; if not, then use the funds to buy a portfolio that superreplicates the time-t_{n+1} option value.

The time-0 option price is C_0^0.

Pricing an American: another formulation

Math fact: For any adapted process Z_{t_n}, define $Y_{t_N} = Z_{t_N}$ and

$$Y_{t_n} = \max(Z_{t_n},\ \mathbb{E}_{t_n} Y_{t_{n+1}}), \qquad n = N-1, N-2, \ldots, 0.$$

(E.g.: $Z_t = e^{-rt}(K - S_t)^+$ and $Y_t = e^{-rt}\times$ time-t value of American).

Then the **optimality principle of dynamic programming** says that

$$Y_0 = \max\{\mathbb{E} Z_\tau :\ \tau \text{ is a stopping time, } 0 \le \tau \le T\}$$

and $Y_0 = \mathbb{E} Z_{\tau^*}$ where $\tau^* := \min\{t_n :\ Y_{t_n} = Z_{t_n}\}$.

This leads to another formulation of the American option price:

$$C_0 = \max\{\mathbb{E} e^{-r\tau}(K - S_\tau)^+ :\ \tau \text{ is a stopping time, } 0 \le \tau \le T\}$$

(which is also valid in continuous time for continuous processes S).

Another application of dynamic programming

Interview (?!) question:

Take a 52 card deck with 26 red and 26 black cards, in random order.

Reveal cards sequentially without replacement.

With each black card you get $+1$ dollar.

With each red card you get -1 dollar.

You can stop playing the game at any time.

What's the optimal stopping time, and what's your expected profit from the optimal strategy?

(Easier question: 6 cards – 3 red, 3 black)

Another application of dynamic programming

Trees as approximations of diffusions

- Approximate the risk-neutral dynamics of a diffusion process (such as GBM) using a discrete model with a finite number of branches from each state.
- The expectations (hence option prices) that we find in the tree approximate the expectations (hence option prices) under the diffusion process.
- For example, let us use a tree to approximate the diffusion

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

Use binomial tree to approximate diffusion dynamics

- Diffusion is

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

where W is $\mathbb{P}$-BM. So $X := \log S$ has dynamics

$$\mathrm{d}X_t = \nu \mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\nu := R_{grow} - \sigma^2/2$.

- Approximate dynamics of $X := \log S$ using a binomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x_u) = p$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x_d) = 1 - p$$

where Δx_u and Δx_d and p will be nonrandom parameters.

Match the means and variances

- Let $\Delta t := T/N$ where N is number of time steps.
- Now choose $p, \Delta x_u, \Delta x_d$ such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu \mathrm{d}s + \int_t^{t+\Delta t} \sigma \mathrm{d}W_s = \nu\Delta t + \sigma\Delta W.$$

So

$$\nu\Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p\Delta x_u + (1-p)(-\Delta x_d)$$

$$\sigma^2\Delta t = \mathrm{Var}_t(X_{t+\Delta t} - X_t) = p(\Delta x_u)^2 + (1-p)(-\Delta x_d)^2 - (\nu\Delta t)^2$$

Two equations and three unknowns: p, Δx_u, Δx_d. Can impose another condition (such as $\Delta x_u = \Delta x_d$ or $p = 1/2$) and solve.

Then price the option in a tree with these parameters.

Not flexible enough

- However, often we want to specify Δx_u and Δx_d in advance.

  For example, we may want $\Delta x_u = \Delta x_d = \Delta x$, so that the *same* price levels are represented at every time point in the tree.

  Maybe, furthermore, we want to specify the size of Δx to optimize convergence or to choose *which* price levels are represented.

- This leaves only one free parameter p.

  Not enough to match both the mean and the variance.

Solution: trinomial trees.

Use trinomial tree to approximate diffusion dynamics

- Diffusion is

$$\mathrm{d}S_t = R_{grow} S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

  where W is $\mathbb{P}$-BM. So $X := \log S$ has dynamics

$$\mathrm{d}X_t = \nu \mathrm{d}t + \sigma \mathrm{d}W_t$$

  where $\nu := R_{grow} - \sigma^2/2$.

- Approximate dynamics of $X := \log S$ using a trinomial tree.

$$\mathbb{P}(X_{t+\Delta t} = X_t + \Delta x) = p_u$$

$$\mathbb{P}(X_{t+\Delta t} = X_t) = p_m$$

$$\mathbb{P}(X_{t+\Delta t} = X_t - \Delta x) = p_d$$

Match the means and variances

- Let $\Delta t := T/N$ where N is number of time steps.

  Choose $\Delta x \approx \sigma\sqrt{3\Delta t}$ for accuracy reasons.

  Can modify these suggestions to make sure certain price levels or times are represented in the tree.

- Now choose p_u, p_m, p_d such that diffusion and tree agree on mean and variance of the increments. In the diffusion,

$$X_{t+\Delta t} - X_t = \int_t^{t+\Delta t} \nu \mathrm{d}s + \int_t^{t+\Delta t} \sigma \mathrm{d}W_s = \nu\Delta t + \sigma\Delta W.$$

  So

$$\nu\Delta t = \mathbb{E}_t(X_{t+\Delta t} - X_t) = p_u\Delta x + p_m 0 + p_d(-\Delta x)$$

$$\sigma^2\Delta t = \mathrm{Var}_t(X_{t+\Delta t} - X_t) = p_u(\Delta x)^2 + p_m(0)^2 + p_d(-\Delta x)^2 - (\nu\Delta t)^2$$

$$p_u + p_m + p_d = 1$$

Solve for probabilities

Solve system of three equations in three unknowns.

$$p_u = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2} + \frac{\nu\Delta t}{\Delta x}\right]$$
$$p_m = 1 - \frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2}$$
$$p_d = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \nu^2(\Delta t)^2}{(\Delta x)^2} - \frac{\nu\Delta t}{\Delta x}\right]$$

Intuition: For fixed Δt and Δx,

- The bigger the σ, the more probability mass in the wings, the less in the middle.
- The bigger the ν, the more mass in the up-branch, the less in the down-branch.

Option pricing

To find option prices, induct backwards in tree with time points

$$t_n = n\Delta t \qquad n = 0, \ldots, N,$$

and log-stock-price points

$$x_j = \log S_0 + j\Delta x \qquad j = -N, \ldots, N.$$

Option price in tree for node at time t_n and log-price x_j is

$$C_n^j = e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^j + p_d C_{n+1}^{j-1}].$$

With large enough N, the option price in the tree $\approx$ the option price for diffusion, because in the $N \to \infty$ (or equivalently $\Delta t \to 0$) limit, the tree's option price $\to$ option price under diffusion dynamics.

For Europeans, the proof is by a form of the CLT.

Can't replicate general payoffs if 3 states 2 assets

- Recall that in the 3-state model, general options cannot be replicated using stock and bank acct. There is not a unique arbitrage-free price for the option. So how can we talk about finding "the" price of the option?
- Answer: The trinomial tree is not the model of the market. The model is GBM in continuous time. In that model, replication using $\{B, S\}$ is possible. So unique price exists.
  The tree is a *computational device* that we use in order to approximate the risk-neutral expectations (hence the prices) of the continuous-time model.

Option Pricing: Knock-outs and Americans

Pricing of knock-outs

$$C_n^j = e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^{j} + p_d C_{n+1}^{j-1}]\mathbf{1}_{\text{no knock-out at time } t_n}$$

Pricing of American put

$$C_n^j = \max((K - S_{t_n}^j)^+,\ e^{-r\Delta t}[p_u C_{n+1}^{j+1} + p_m C_{n+1}^{j} + p_d C_{n+1}^{j-1}])$$

Formulas are similar to binomial tree.