Financial Mathematics 32000

Lecture 8

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Fourier transform

Let $f : \mathbb{R} \to \mathbb{R}$ be integrable, meaning $\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x < \infty$.

The *Fourier transform* of f is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(x) e^{izx} \mathrm{d}x$$

Theorem: If $\hat{f}$ is integrable then the *inversion* formula holds:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(z) e^{-izx} \mathrm{d}z$$

(at every x where f is continuous)

Intuition: This represents $f(x)$ as a combination of functions e^{-izx} having various “frequencies” z. The transform $\hat{f}(z)$ gives the weighting of the e^{-izx} function. More precisely, ...

Intuition of inversion formula

For arbitrary $L > 0$, on the interval $x \in [-\pi L, \pi L]$, the functions $e^{-ikx/L}$, for integer k, are orthonormal wrt the inner product

$$\langle a, b\rangle := \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} a(x)\overline{b(x)}\mathrm{d}x$$

General f may be represented on $[-\pi L, \pi L]$, using the "basis" functions $e^{-ikx/L}$. Specifically, the kth function has coefficient

$$\langle f, e^{-ikx/L}\rangle = \frac{1}{2\pi L}\int_{-\pi L}^{\pi L} f(x)e^{ikx/L}\mathrm{d}x = \frac{1}{2\pi L}\widehat{f_L}(k/L).$$

where $f_L := f$ on $[-\pi L, \pi L]$, and 0 elsewhere. For $x \in [-\pi L, \pi L]$,

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi L}\widehat{f_L}(k/L)e^{-ikx/L} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(z)e^{-izx}\mathrm{d}z$$

as $L \to \infty$.

Characteristic function

The characteristic function (CF) of any random variable X is the function $F_X : \mathbb{R} \to \mathbb{C}$ defined by

$$F_X(z) := \mathbb{E}e^{izX}$$

If X has a density f, then $F_X(z) = \int_{-\infty}^{\infty} f(x)e^{izx}\mathrm{d}x = \hat{f}(z)$, so the CF of X is the same thing as the Fourier transform of its density. We can also talk about the CF of a distribution, which means the same thing as the CF of a variable X with that distribution.

A characteristic function uniquely identifies a distribution.

Indeed, in the case that F_X is integrable, applying the inversion formula to F_X gives the density of X.

Characteristic function

Just as a distribution can be specified by giving a CDF or a density, it can be specified by giving a CF.

Why may we like to work with CF instead of a density or a CDF?

- Sometimes the CF is much simpler than the density.
- The following facts help us prove limit theorems (such as CLT): The CF of the *sum* of *independent* random variables is the *product* of the CFs. And the pointwise convergence of CFs to a continuous limit F is equivalent to the convergence of the corresponding distributions to a limiting distribution with CF F.
- Quantities of the form $\mathbb{E}g(X)$ can often be expressed easily in terms of the CF of X. This is useful for derivatives pricing.

Deriving a CF

Examples:

- Let X be Normal(a, b^2).

  Then the CF of X is

$$F_X(z) = \mathbb{E}e^{izX} = e^{\mathbb{E}izX + (1/2)\mathrm{Var}(izX)} = e^{iaz - b^2z^2/2}$$

- Let U be Uniform$[0, 1]$.

  Then the CF of U is

$$F_U(z) = \mathbb{E}e^{izU} = \int_0^1 1 \times e^{izu}\mathrm{d}u = \frac{e^{izu}}{iz}\Big|_0^1 = \frac{e^{iz} - 1}{iz}$$

  for $z \neq 0$. And $F_U(0) = 1$.

Using the CF to compute moments

The nth moment of X is the expectation $\mathbb{E}X^n$. To compute:

Take n derivatives of $F_X(z) = \mathbb{E}e^{izX}$ wrt z:

$$F_X^{(n)}(z) = \mathbb{E}((iX)^n e^{izX})$$

Evaluate at $z = 0$ to get $\mathbb{E}X^n = (-i)^n F_X^{(n)}(0)$.

- Example: If X is Normal$(0,1)$ then $F_X(z) = e^{-z^2/2}$ so

$$\begin{aligned} F_X'(z) = -ze^{-z^2/2} \quad &\Rightarrow \quad \mathbb{E}X = (-i)(0) = 0 \\ F_X''(z) = (-1+z^2)e^{-z^2/2} \quad &\Rightarrow \quad \mathbb{E}X^2 = (-i)^2(-1) = 1 \end{aligned}$$

To obtain moments of e^X: Let us extend the domain of F_X or $\hat{f}$ (which was $\mathbb{R}$) to a strip in $\mathbb{C}$ defined by $\{z: \ \mathbb{E}|e^{izX}| < \infty\}$.

Then $\mathbb{E}\big[(e^X)^n\big] = F_X(-in)$

Using the CF to compute the CDF

To obtain the CDF of X at any point of continuity k:

$$\mathbb{E}\mathbf{1}_{X<k} = \mathbb{P}(X<k) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{F_X(z)}{iz}e^{-izk}\right]\mathrm{d}z$$

Sanity check: the k-derivative of the RHS is

$$\begin{aligned}-\frac{1}{\pi}\int_0^\infty \mathrm{Re}[-F_X(z)e^{-izk}]\mathrm{d}z &= \frac{1}{2\pi}\int_0^\infty F_X(z)e^{-izk} + F_X(-z)e^{izk}\mathrm{d}z \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty F_X(z)e^{-izk}\mathrm{d}z = f(k),\end{aligned}$$

which recovers the density, as it should.

We have used the fact that

$$\overline{F_X(z)} = \overline{\mathbb{E}e^{izX}} = \mathbb{E}e^{-i\overline{z}X} = F_X(-\overline{z})$$

Using the CF to compute asset-or-nothing call price

Assume $\mathbb{E}e^X < \infty$. Define the measure $\mathbb{P}^*$ by the property that

$$\mathbb{E}^* Y = \mathbb{E}\left[\frac{e^X}{\mathbb{E}e^X} Y\right]$$

for all random variables Y such that the $\mathbb{E}$ exists. (If e^X is a time-T share price, then $\mathbb{P}^*$ is share measure.) The $\mathbb{P}^*$-CF of X is

$$F^*(z) = \mathbb{E}^* e^{izX} = \mathbb{E}\left[\frac{e^X}{\mathbb{E}e^X} e^{izX}\right] = \frac{\mathbb{E}e^{(iz+1)X}}{\mathbb{E}e^X} = F_X(z-i)/F_X(-i).$$

Therefore, for $k \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}(e^X \mathbf{1}_{X>k}) &= \mathbb{E}e^X \mathbb{E}\left[\frac{e^X}{\mathbb{E}e^X} \mathbf{1}_{X>k}\right] = \mathbb{E}e^X \mathbb{P}^*(X>k) \\
&= F_X(-i)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{F_X(z-i)/F_X(-i)}{iz} e^{-izk}\right] \mathrm{d}z\right) \\
&= \frac{F_X(-i)}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{F_X(z-i)}{iz} e^{-izk}\right] \mathrm{d}z.
\end{aligned}$$

Using the CF to compute call prices

The time-0 price of a call on e^X struck at K is

$$e^{-rT}\left[\mathbb{E}(e^X \mathbf{1}_{e^X>K}) - K\mathbb{P}(e^X>K)\right]$$

Let $k := \log K$. The first term, by L8.18, equals

$$\mathbb{E}(e^X \mathbf{1}_{X>k}) = \frac{F_X(-i)}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{F_X(z-i)}{iz}e^{-izk}\right]\mathrm{d}z$$

The second term, by L8.9, equals

$$K\mathbb{P}(X>k) = K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{F_X(z)}{iz}e^{-izk}\right]\mathrm{d}z\right)$$

Typical application: $X = \log S_T$.

This was the approach in Heston (93).

Using the CF to compute $\mathbb{E}g(X)$

Parseval/Plancherel Theorem: If f and g are integrable and square integrable ($\int_{-\infty}^{\infty} f^2 < \infty$ and $\int_{-\infty}^{\infty} g^2 < \infty$) then

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(z)\overline{\hat{g}(z)}\mathrm{d}z = \left\langle \frac{\hat{f}}{\sqrt{2\pi}}, \frac{\hat{g}}{\sqrt{2\pi}} \right\rangle$$

Idea: The transformation $f \mapsto \hat{f}/\sqrt{2\pi}$ is *unitary*: it preserves inner products. Analogous to unitary mappings in $\mathbb{R}^n$: reflections/rotations that preserve lengths and angles (dot products) of vectors.

Application: Let $f :=$ density of $\log S_T$, and $g(\log S_T)$ be some payoff.

Example: for call option, $g(x) = (e^x - K)^+$. We want to compute

$$\mathbb{E}g(\log S_T) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = (\text{we hope})\frac{1}{2\pi}\int \hat{f}(z)\overline{\hat{g}(z)}\mathrm{d}z$$

But $\int (e^x - K)^+ \mathrm{d}x = \infty$, so the call payoff g is not integrable.

Using the CF to compute $\mathbb{E}g(X)$

Solution: let $f_\alpha(x) := e^{\alpha x} f(x)$ and $g_{-\alpha}(x) := e^{-\alpha x} g(x)$, where

$$1 < \alpha < \sup\{p : \mathbb{E}S_T^p < \infty\}.$$

Then $fg = f_\alpha g_{-\alpha}$ and

$$\hat{f}_\alpha(z) = \int_{-\infty}^{\infty} e^{\alpha x} f(x) e^{izx} \mathrm{d}x = \int_{-\infty}^{\infty} f(x) e^{ix(z-\alpha i)} \mathrm{d}x = \hat{f}(z - \alpha i)$$

and likewise $\hat{g}_{-\alpha}(z) = \hat{g}(z + \alpha i)$. Then

$$\begin{aligned}
\mathbb{E}g(\log S_T) &= \int_{-\infty}^{\infty} f_\alpha(z) \overline{g_{-\alpha}(z)} \mathrm{d}z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(z - \alpha i) \overline{\hat{g}(z + \alpha i)} \mathrm{d}z \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(z - \alpha i) \hat{g}(-z + \alpha i) \mathrm{d}z = \frac{1}{2\pi} \int_{-\infty - \alpha i}^{\infty - \alpha i} \hat{f}(z) \hat{g}(-z) \mathrm{d}z.
\end{aligned}$$

(Note that $\hat{g}$ here is evaluated at points $u + \alpha i$ where $u \in \mathbb{R}$.)

Example: compute call price

In the case that $g(x) := (e^x - K)^+$, the payoff transform is, at any point $z = u + \alpha i$ where $u \in \mathbb{R}$ and $\alpha > 1$,

$$\hat{g}(z) = \int_{-\infty}^{\infty} (e^x - K)^+ e^{ixz} \mathrm{d}x = \int_{\log K}^{\infty} (e^{x(iz+1)} - Ke^{ixz}) \mathrm{d}x$$

$$= \left(\frac{e^{(1+iz)x}}{1+iz} - K\frac{e^{izx}}{iz} \right) \Bigg|_{x=\log K}^{x=\infty} = K\frac{e^{iz\log K}}{iz} - \frac{e^{(1+iz)\log K}}{1+iz} = \frac{e^{(1+iz)\log K}}{z(i-z)}$$

(the $x = \infty$ terms vanished because $1 + iz$ and iz have real parts < 0).

$$\text{Hence} \quad \mathbb{E}(e^X - K)^+ = \frac{1}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} F_X(z) \frac{e^{(1-iz)\log K}}{-z(z+i)} \mathrm{d}z.$$

Equivalently, via $z \mapsto z - i$, this is sometimes rewritten for $\alpha > 0$ as

$$\frac{1}{2\pi} \int_{-\infty-\alpha i}^{\infty-\alpha i} F_X(z-i) \frac{e^{-iz\log K}}{z(i-z)} \mathrm{d}z$$

Computational benefits (vs. L8.11): One integral. Ability to choose α.

Numerical integration

Can rewrite this as an integral over positive real line. With $k = \log K$,

$$\begin{aligned}
\mathbb{E}(e^X - K)^+ &= \frac{1}{\pi}\int_{0-\alpha i}^{\infty-\alpha i} \mathrm{Re}\left(F_X(z-i)\frac{e^{-izk}}{z(i-z)}\right)\mathrm{d}z \\
&= \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left(F_X(x-(\alpha+1)i)\frac{e^{-i(x-\alpha i)k}}{(x-\alpha i)((\alpha+1)i-x)}\right)\mathrm{d}x \\
&= \frac{1}{e^{\alpha k}}\int_0^{\infty} \mathrm{Re}(h(x)e^{-ixk})\mathrm{d}x
\end{aligned}$$

where $h(x) = \frac{F_X(x-(\alpha+1)i)}{\pi(x-\alpha i)((\alpha+1)i-x)}$. Can evaluate integral using

- Fixed integration points, e.g. "midpoint"/"rectangular" rule

  $\Delta_x e^{-\alpha k}\sum_{n=1}^{N}\mathrm{Re}\big(h(x_n)e^{-ix_n k}\big)$ where $x_n = (n-1/2)\Delta_x$

  Can calculate this by simple summation, or by FFT.

- Adaptive integration algorithms, e.g. Python `scipy.integrate`

FFT

FFT algorithm: Input $\mathsf{v} \in \mathbb{C}^N$. Output $\mathsf{FFT}[\mathsf{v}] \in \mathbb{C}^N$ where

$$\mathsf{FFT}[\mathsf{v}]_m = \sum_{n=1}^{N} e^{-i(2\pi/N)(n-1)(m-1)} \mathsf{v}_n, \qquad m = 1, 2, \ldots, N$$

Python `numpy.fft` computes FFT (with 0-based indexing). Can price calls at log-strikes $k_m = k_1 + (m-1)\Delta_k$ for $m = 1, \ldots N$, because $x_n k_m = (n-1)(m-1)\Delta_x \Delta_k + (n-1/2)k_1 \Delta_x + (m-1)\Delta_x \Delta_k / 2$.
If $\Delta_x \Delta_k = 2\pi/N$ then running *one* FFT solves for *all* $(m = 1, \ldots, N)$ log-strikes k_m, by calculating the midpoint rule using FFT as

$$\Delta_x e^{-\alpha k_m} \mathrm{Re} \sum_{n=1}^{N} h(x_n) e^{-i x_n k_m} = \Delta_x e^{-\alpha k_m} \mathrm{Re}(e^{-i\pi(m-1)/N} \mathsf{FFT}[\mathsf{v}]_m)$$

where $\mathsf{v}_n := h(x_n) e^{-i k_1 x_n}$.

Numerical integration in Fourier space

Deriving a CF: Heston model

B-S implied vol in Heston model

Interview questions

Conclusions

The Heston (1993) stochastic volatility model

Let W^S and W^V be $\mathbb{P}$-BM with correlation ρ. Let

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{V_t}S_t\mathrm{d}W_t^S$$

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}W_t^V$$

Interpretation of the parameters $\kappa, \theta, \eta > 0$:

$$\begin{aligned} \kappa = &\text{ rate of mean-reversion} \\ \theta = &\text{ long-term mean} \\ \eta = &\text{ volatility of volatility} \end{aligned}$$

Then $X_t := \log S_t$ has dynamics

$$\mathrm{d}X_t = (r - \frac{1}{2}V_t)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^S$$

The Heston CF

We want to find the CF of X_T, in order to price options on S_T.

Let's find indeed, for all $t < T$, the time-t conditional CF

$$M_t := M(X_t, V_t, t; z) := \mathbb{E}_t e^{izX_T}$$

For each z, this M_t is a *martingale*, because if $s < t$ then

$M_s = \mathbb{E}_s e^{izX_T} = \mathbb{E}_s[\mathbb{E}_t e^{izX_T}] = \mathbb{E}_s M_t$.

So we *set its drift equal to zero*. By Itô $\mathrm{d}M_t =$

$$\frac{\partial M}{\partial t}\mathrm{d}t+\frac{\partial M}{\partial X}\mathrm{d}X+\frac{\partial M}{\partial V}\mathrm{d}V+\frac{1}{2}\frac{\partial^2 M}{\partial X^2}(\mathrm{d}X)^2+\frac{1}{2}\frac{\partial^2 M}{\partial V^2}(\mathrm{d}V)^2+\frac{\partial^2 M}{\partial V \partial X}(\mathrm{d}V)(\mathrm{d}X)$$

So we have a (backward Kolmogorov) PDE

$$\frac{\partial M}{\partial t}+\frac{\partial M}{\partial X}(r-\frac{1}{2}V)+\frac{\partial M}{\partial V}\kappa(\theta-V)+\frac{1}{2}\frac{\partial^2 M}{\partial X^2}V+\frac{1}{2}\frac{\partial^2 M}{\partial V^2}\eta^2 V+\frac{\partial^2 M}{\partial V \partial X}\rho\eta V = 0$$

Solve for A and B

Let's guess that

$$M(X_t, V_t, t; z) = \mathbb{E}_t e^{izX_T} = e^{A(t;z)+izX_t+B(t;z)V_t}$$

for some A and B, which may depend on the model's parameters.

The guess is that the CF is exponential-*affine* wrt (X, V). *Affine* means a constant plus a linear transformation. The PDE becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t}+\frac{\mathrm{d}B}{\mathrm{d}t}V+iz(r-\frac{1}{2}V)+B\kappa(\theta-V)+\frac{1}{2}(iz)^2V+\frac{1}{2}\eta^2B^2V+izB\rho\eta V=0$$

We want this to vanish for all V. So $A(T) = B(T) = 0$ and

$$\frac{\mathrm{d}A}{\mathrm{d}t}+irz+\kappa\theta B=0$$

$$\frac{\mathrm{d}B}{\mathrm{d}t}-\frac{iz}{2}-\frac{z^2}{2}+(i\rho\eta z-\kappa)B+\frac{\eta^2}{2}B^2=0$$

Solve for A and B

So

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -\frac{\eta^2}{2}(B - c_1)(B - c_2)$$

where c_1, c_2 depend on ρ, η, κ, z. This is a *Riccati* ODE, with solution

$$B(t) = c_1 c_2 \frac{1 - e^{(T-t)(c_2-c_1)\eta^2/2}}{c_1 - c_2 e^{(T-t)(c_2-c_1)\eta^2/2}}.$$

The other ODE is $\mathrm{d}A/\mathrm{d}t = -irz - \kappa\theta B$ hence

$$A(t) = irz(T - t) + \kappa\theta \int_t^T B(u)\mathrm{d}u$$

which also has an explicit solution.

The Heston CF

Conclusion: The [time-t conditional] Heston CF is

$$F_X(z) = e^{A+izX_t+BV_t}.$$

$$A := irz(T-t) + \frac{\kappa\theta}{\eta^2}\left[(\kappa_* - \gamma)(T-t) - 2\log\left(1 + \frac{\kappa_* - \gamma}{2\gamma}(1 - e^{-\gamma(T-t)})\right)\right]$$

$$B := \frac{-(zi + z^2)(1 - e^{-\gamma(T-t)})}{2\gamma e^{-\gamma(T-t)} + (\gamma + \kappa_*)(1 - e^{-\gamma(T-t)})}$$

$$\kappa_* := \kappa - i\rho\eta z$$

$$\gamma := \sqrt{\kappa_*^2 + \eta^2(zi + z^2)}$$

Use the principal branch of the complex log with this formulation.

Plug $F_X(z)$ into L8.11 or L8.15 to get call prices.

Heston model

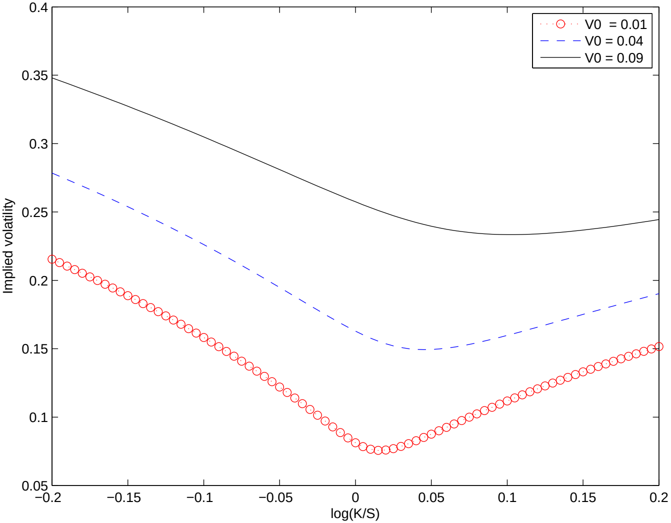
Let $T = 0.25$.

Let $r = 0$, $\kappa = 1$, $\theta = 0.04$, $\eta = 1.0$, $\rho = -0.5$, $V_0 = 0.04$.

Use Fourier method to compute Heston-model option prices, and express those prices as Black-Scholes implied volatilities.

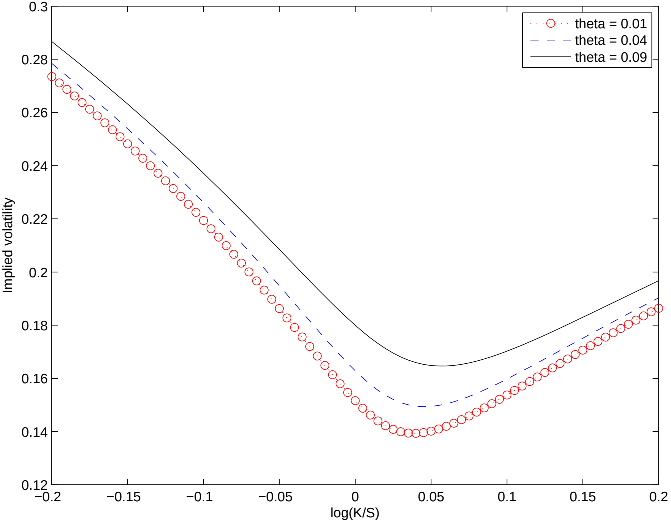
Plot the implied volatility skews for various parameter values.

Effect of V_0

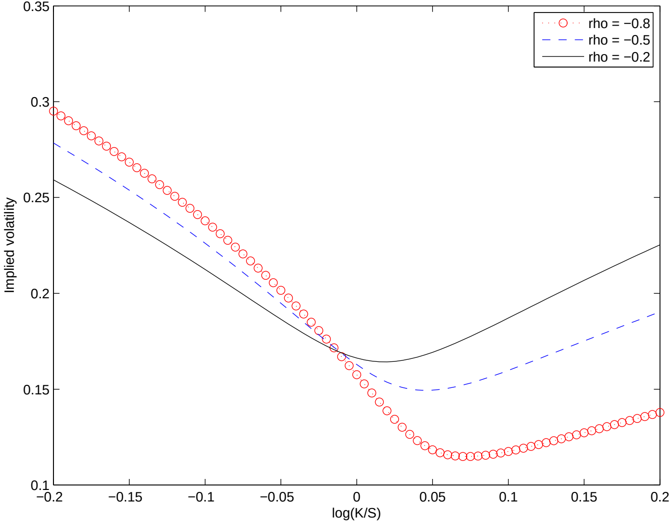


0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
Implied volatility
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2
log(K/S)
V0 = 0.01
V0 = 0.04
V0 = 0.09

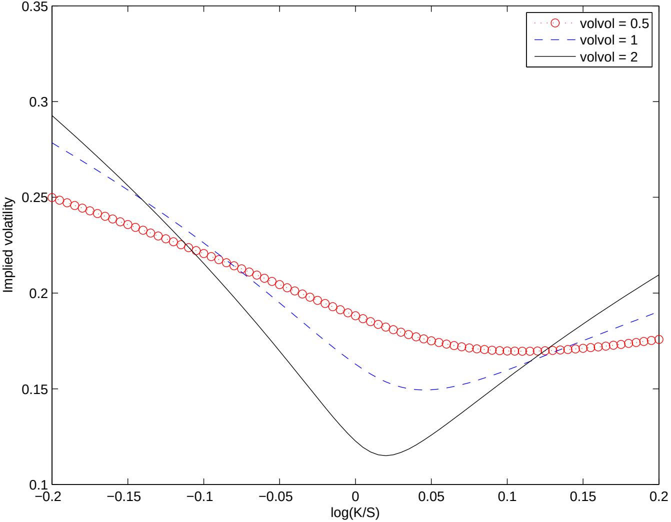
Effect of θ



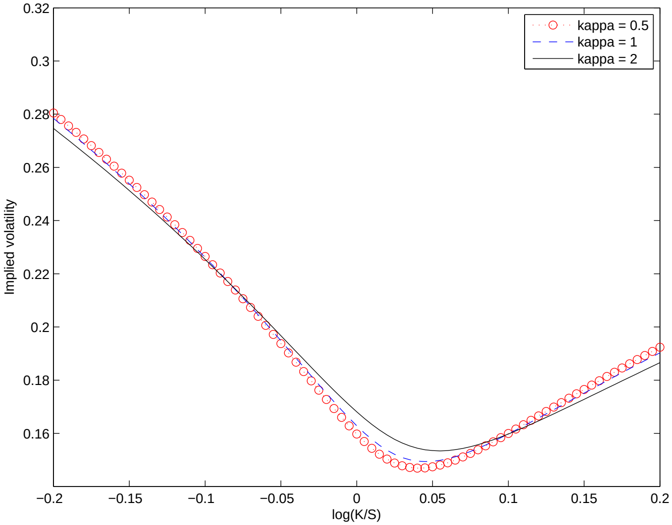
Effect of ρ



Effect of vol-of-vol η



Effect of κ



Numerical integration in Fourier space

Deriving a CF: Heston model

B-S implied vol in Heston model

Interview questions

Conclusions

Moment generating functions

- (Goldman) What is a moment generating function?

  Answer: The moment generating function of X is

  $$M_X(u) = \mathbb{E}e^{uX}$$

  at all $u \in \mathbb{R}$ where this is finite. Similar to characteristic function, but CF always exists, while MGF may be infinite for all $u \neq 0$.

- (Goldman) Why is its first derivative the mean?

  Answer: $M'_X(u) = \mathbb{E}Xe^{uX}$ so $M'_X(0) = \mathbb{E}X$.

- (FHLBC) How to calculate $\mathbb{E}X^2$ given the MGF of X?

  Answer: $M''_X(u) = \mathbb{E}X^2e^{uX}$ so $M'_X(0) = \mathbb{E}X^2$.

Heston

- (Murex) What is the Heston model?
- (Security Benefit) What are the parameters of the Heston model?
- (UBS) For Heston model, what will the volatility surface look like if the volatility of volatility increases?

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Numerical integration (quadrature)

- Given g the density of X, and payoff $f(X)$, numerical integration calculates the expectation $\int f(x)g(x)\mathrm{d}x$.
- Midpoint quadrature (let $h = fg$, restrict to domain of length L, divided into N intervals of length $\Delta x = L/N$ and midpoints x_n. Numerical result: $\sum_n h(x_n)\Delta x$), is equivalent to doing first-order Taylor approximation of h at each midpoint.

  Max error in approximating h on each interval is $O(\Delta x)^2$. Error in approximating $\int h$ on each interval is $O(\Delta x)^3$. Total error in approximating $\int h$ on N intervals is $N \times O(\Delta x)^3 = O(\Delta x)^2$.
- Thus one-dimensional numerical integration of payoff×density is a preferred approach, provided that the density is available.

Numerical integration in Fourier space

- Given the CF of X, another way to calculate expectations of payoffs $f(X)$ is by numerical integration in Fourier space. Instead of integrating Payoff $\times$ Density, the Fourier transform approach integrates Payoff transform $\times$ Density transform (CF)
- One-dimensional numerical integration of payoff transform $\times$ CF is also a preferred approach – provided that the CF is available.
- Often useful for pricing vanilla contracts quickly – for example in *calibration* applications.

Trees and FD

Trees

- Trees can be regarded as *explicit* finite difference methods.
  But FD have greater flexibility, because FD can also be done by implicit, C-N, etc.

Finite Differences

- Explicit: Simple. Equivalent to trinomial tree. But only **first**-order accurate in Δt, and have stability restrictions.
- Implicit and C-N: Unconditionally stable and (in C-N case) **second**-order accurate in Δt. But requires solution of linear system at each time step.

Monte Carlo

- Easy to code (and parallelize) even for complex contract/dynamic
- Estimates have random noise, which goes to zero as $O(1/\sqrt{M})$.
- Advantages on multidimensional problems (Multi-asset contracts. Or multi-factor dynamics. Or some path-dependent contracts.) For a 1-dimensional problem, FD methods typically more efficient than MC. But FD computational burden grows exponentially as the number of dimensions grows. ("Curse of Dimensionality").
- FD / quadrature: If error is constant$/N^2$ and work is const $\times N^D$ then work to achieve ε error is const $\times$ (constant$/\sqrt{\varepsilon})^D$. MC: standard error $\sqrt{\operatorname{Var} Y}/\sqrt{M}$. If work is constant $\times DM$ then work to achieve ε standard error is constant $\times D \operatorname{Var}(Y)/\varepsilon^2$, which is "worse" than FD if $D \leq 3$.

Conclusions

Fastest to slowest execution:

- Explicit formula, such as Black-Scholes
- 1-dimensional numerical integration
- Low-dimensional PDE solution, or numerical integration, or trees
- Monte Carlo
- High-dimensional PDE solution, or numerical integration, or trees

But remember:

- Rapid coding may be more important than rapid execution.
- Finance rewards those who see relevant relationships/similarities between A and B (which may denote assets/risks/situations). For pricing purposes, this $\rightarrow$ simplifications/approximations/bounds.