FINM 32000: Homework 5

Due Thursday May 8, 2025 at 11:59pm

Problem 1

Let r be the constant interest rate. Let $0 < T_1 < T_2$.

(a) Let F_t be the time-t forward price for T_2-delivery of some arbitrary underlying S, not necessarily tradeable. Recall from FINM 33000, that a *forward price* is *not the same thing* as the *value of a forward contract*. By definition of the time-t *forward price* F_t for T_2-delivery:

a forward contract paying $S_{T_2} - F_t$ at time T_2 has time-t value 0.

Let f_t be the time-t value of a T_2-forward contract on the same underlying, but with some delivery price K (not necessarily equal to F_t).

Express f_t in terms of K and F_t and a discount factor.

Hint: consider a portfolio long one (K, T_2)-forward contract and short one (F_t, T_2)-forward contract. The portfolio has (in terms of f_t) what value at time t? The portfolio pays how much at expiration?

(b) If S is a *stock* paying no dividends, the forward price must be $F_t = S_t e^{r(T_2-t)}$; otherwise, arbitrage would exist.

> If, say, $F_t > S_t e^{r(T_2-t)}$, then arbitrage would exist: at time t, borrow S_t dollars, buy the stock, and short the forward (with delivery price F_t and time-t value 0). At time T_2, deliver the stock, and receive F_t, which is more than enough to cover your accumulated debt of $S_t e^{r(T_2-t)}$ dollars.

However, if S is the spot price of a barrel of crude oil (so, for all t, the time-t price for time-t delivery is S_t per barrel), then this argument fails. Explain briefly (one or two sentences, no math) why *this specific arbitrage* does not apply to crude oil, by specifically pinpointing, in the quote above, why we cannot simply replace “stock” with “crude oil”.

Hint: Consider practical complications.

So we need more assumptions to relate F_t and S_t (here and in (c,d,e,f,g), the S denotes spot crude oil, and F_t denotes the time-t forward price for T_2-delivery crude oil). One approach is to assume a specific model of the risk-neutral dynamics of S. For instance, let us assume that S satisfies

$$S_t = \exp(X_t)$$
$$\mathrm{d}X_t = \kappa(\alpha - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t.$$

where W is Brownian motion, under risk-neutral measure.

Then, since r is constant and $\mathbb{E}_t(e^{-r(T_2-t)}(S_{T_2} - F_t))$ must be 0, one can calculate

$$F_t = \mathbb{E}_t(S_{T_2}) = \exp\left[e^{-\kappa(T_2-t)} \log S_t + \left(1 - e^{-\kappa(T_2-t)}\right)\alpha + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T_2-t)}\right)\right],$$

where $\mathbb{E}_t$ is time-t conditional expectation. Suppose $\kappa = 0.472$, $\alpha = 4.4$, $\sigma = 0.368$, $r = 0.05$, and the time-0 spot price is $S_0 = 106.9$.

Let C be the time-0 price of a K-strike T_1-expiry European call on F. So this call pays $(F_{T_1} - K)^+$. Let the call option have strike $K = 103.2$ and expiration $T_1 = 0.5$. Let the forward have delivery date $T_2 = 0.75$. See the `ipynb` file.

(c) Estimate $C(S_0)$ using Monte Carlo simulation of S with 100 timesteps on $[0, T_1]$. Choose the number of paths large enough that the standard error [the sample standard deviation, divided by the square root of the number of paths] is less than 0.05. Report the standard error. Don't use any variance reduction technique.

(d) Estimate $\partial C/\partial S$ by using Monte Carlo simulation to calculate $(C(S_0 + 0.01) - C(S_0))/0.01$. For the $C(S_0+0.01)$ calculation, *reuse* the same normal random variables which you generated for the $C(S_0)$ calculation. (Do not re-*generate* random variables to compute $C(S_0 + 0.01)$)

(e) Calculate analytically $\partial f_0/\partial S$, where f_0 is the time-0 value of a position long one forward contract on a barrel of crude oil, with delivery date T_2 and some fixed delivery price K.

(f) Suppose you want to hedge a position short one call (so your hedge portfolio should replicate a position long one call), by continuously rebalancing a position in T_2-delivery forward contracts. Your hedge portfolio at time 0 should be long how many forward contracts? Your final answer should be a number.

The delivery price K of the forward contracts is irrelevant to the answer here; it would affect only how many units of the bank account to carry in the portfolio (which I am not asking you to compute).

(g) Consider the following "purchase agreement" contract. The holder of this contract receives time-T_2 delivery of θ barrels of crude oil, and pays, at time T_2, a delivery price of K dollars per barrel. The θ is chosen at time T_1 by the holder of the purchase agreement, subject to the restriction that $4000 \leq \theta \leq 5000$; in particular, $\theta = 0$ is not a valid choice, because the contract is a commitment to purchase at least 4000 barrels. Using your answer to (c), without running any new simulations, find the time-0 value of this contract.

Here K, T_1, T_2 have the same values as on the previous page.

Hint: Assume the holder acts optimally; thus θ is either 4000 or 5000, depending on F_{T_1}.

Problem 2

Each unit of the bank account has price $B_t = e^{rt}$ for all $t \geq 0$.

(a) Let S_t be the time-t price of a stock that continuously pays a constant proportional dividend yield q. This means that each 1 share of S at time 0 will grow, via dividend reinvestment, to e^{qt} shares of S at each time $t \geq 0$. In other words, the stock should not be regarded as a tradeable asset. Rather, units of the "bundle" should be regarded as tradeable, where 1 unit of the bundle is defined to be

$$e^{qt} \text{ shares of stock, at all times } t \geq 0$$

Aside from the above information, do not assume any specific dynamics for S.

Using replication, find the time-0 value of a forward contract which pays $S_T - K$ at time T. How many shares of S and units of B does the replicating portfolio hold at time 0?

What is the time-0 *forward price* for time-T delivery of S_T? (This is not the same thing as the *value of a forward contract*.)

(b) Let S_t be the time-t price of a stock that pays a fixed dollar dividend D discretely at time T_0, where $0 < T_0 < T$. Assume S does not pay any other dividends between time 0 and T.

So we will consider as tradeable the following bundle. One unit of the bundle consists of:

$$\begin{array}{rl} \text{1 share} & \text{at all times } t < T_0 \\ \text{1 share plus } De^{-rT_0} \text{ units of bank account} & \text{at all times } t \geq T_0 \end{array}$$

Like the case of the continuous proportional dividend yield in (a), the bundle here in (b) absorbs the dividend payments. Unlike (a), the bundle in (b) allocates the dividend into bank account units, instead of into more stock shares.

We are still not assuming any specific dynamics for S.

Using replication, find the time-0 value of a forward contract which pays $S_T - K$ at time T. How many shares of S and units of B does the replicating portfolio hold at time 0?

What is the time-0 *forward price* for time-T delivery of S_T? (This is not the same thing as the *value of a forward contract*.)