FINM 33000 HW 2 Solutions

(1a) Recall the diagram of the payoff of a call spread with strikes (K_1, K_2), meaning that it is long a standard K_1-strike call and short a standard K_2-strike call, where $K_2 > K_1$. Scaling this call spread by $1/(K_2 - K_1)$ produces a payoff which is 1 for $S_T \geq K_2$ and 0 for $S_T \leq K_1$.

So the binary call payoff is bounded above by $1/2.5$ call spreads with strikes $(20, 22.5)$, which has time-0 price $1/2.5 \times (6.15 - 4.15) = 0.80$, and bounded below by $1/2.5$ call spreads with strikes $(22.5, 25)$ which has time-0 price $1/2.5 \times (4.15 - 2.60) = 0.62$.

So lower and upper bounds are 0.62 and 0.80 respectively.

(1b) Binary put + binary call = bond. Therefore the binary put's time-0 value is bounded below by $0.95 - 0.80 = 0.15$ and bounded above by $0.95 - 0.62 = 0.33$.

(1c) The payoff is $\max(2.5, S_T - 22.5) = 2.5 + \max(0, S_T - 25)$ which is replicated by 2.5 bonds and one 25-strike call, which has total time-0 price $2.5 \times 0.95 + 2.6 = 4.975$

(1d) Consider the super-replicating portfolio: 0.8 calls with strike 27.5, and 0.2 calls with strike 30. Its time-0 value is $1.5 \times 0.8 + 0.8 \times 0.2 = 1.36$, so upper bound is 1.36.

(2a) For $s \geq K_*$ the right-hand side is

$$\begin{aligned} f(K_*) + f'(K_*)(s - K_*) + \int_{K_*}^{\infty} f''(K)(s-K)^+ \mathrm{d}K &= f(K_*) + f'(K_*)(s - K_*) + \int_{K_*}^{s} f''(K)(s-K) \mathrm{d}K \\ &= f(K_*) + f'(K_*)(s - K_*) + (s-K)f'(K)\Big|_{K=K_*}^{K=s} + \int_{K_*}^{s} f'(K)\mathrm{d}K \\ &= f(K_*) + f'(K_*)(s - K_*) - (s - K_*)f'(K_*) + f(s) - f(K_*) \\ &= f(s) \end{aligned}$$

For $s < K_*$ the terms in blue get replaced by $\int_0^{K_*} f''(K)(K-s)^+ \mathrm{d}K = \int_s^{K_*} f''(K)(K-s)\mathrm{d}K = (K-s)f'(K)\Big|_{K=s}^{K=K_*} - \int_s^{K_*} f'(K)\mathrm{d}K = (K_* - s)f'(K_*) + f(s) - f(K_*)$, the same result as the blue.

(2b) Let $K_n = 5n$ where $n = 1, 2, 3, \ldots$. Let $f(s) = -2\log s$. Then

$$\int_0^{K_*} f''(K)(K-s)^+ \mathrm{d}K = \int_0^{K_*} \frac{2}{K^2}(K-s)^+ \mathrm{d}K \approx \sum_{K_n < K_*} \frac{2}{K_n^2}(K_n - s)^+ \Delta K$$

so the coefficient of $(1950 - s)^+$ is $\frac{-2}{1950^2} \times 5 \approx 2.63 \times 10^{-5}$, so go long 2.63×10^{-5} puts at strike 1950.

The CBOE uses this $2(\Delta K)/K^2$ weighting to approximate the $-2\log$ payoff in calculating the VIX.

`https://cdn.cboe.com/resources/futures/vixwhite.pdf`

(But why are they interested in $-2\log$ in the first place? That's another story ...)