FINM 33000: HW 6 Solutions

November 21, 2024

Problem 1

(a) The plan's payoff is (# of shares you acquire) $\times$ (market value of each share) $-$ 3000, which is

$$\frac{3000}{\min(30,\ 0.75 \times S_1)} \times S_1 - 3000 = \max\left(\frac{3000}{30},\ \frac{3000}{0.75 \times S_1}\right) \times S_1 - 3000$$

d1=0.3373

d2 = 0.1173

$$= \max\big(100S_1,\ 4000\big) - 3000 = 100 \max\big(S_1 - 40,\ 0\big) + 1000$$

the same as 100 calls struck at 40, plus 1000 dollars. Each call has time-0 value 4.4811 by the Black-Scholes formula, so the total value of the plan is $100 \times 4.4811 + 1000e^{-0.05} = 1399.34$

(b) Go short 100 one-year 40-strike calls. This position does not need to be rebalanced.

(c) Calls are unavailable, but we can dynamically *replicate*, using stock, the position short 100 one-year 40-strike calls. Black-Scholes delta at time 0 is $N(d_1) = 0.6243$, so we should be short $100 \times 0.6243 = 62.43$ shares. This position must be rebalanced.

Problem 2

Note: we won't need to use the assumption $\beta > 1$. Our calculations will be valid for *general* leverage ratios β, including $\beta < 0$ (which would model an "inverse ETF").

(a) Let b_t be the number of units of the bank account. The total portfolio value is

$$L_t = \beta \frac{L_t}{S_t} S_t + b_t B_t$$

Therefore

$$b_t = (1-\beta)\frac{L_t}{B_t}.$$

(b) By definition of self-financing,

$$\mathrm{d}L_t = \beta \frac{L_t}{S_t} \mathrm{d}S_t + b_t \mathrm{d}B_t$$

Plugging in $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$ and $\mathrm{d}B_t = rB_t\mathrm{d}t$, and the b_t solution from (a),

$$\begin{aligned}\mathrm{d}L_t &= \beta L_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t) + (1-\beta) r L_t \mathrm{d}t \\ &= (\beta\mu + (1-\beta)r) L_t \mathrm{d}t + \beta\sigma L_t \mathrm{d}W_t\end{aligned}$$

which is GBM with drift $\beta\mu + (1-\beta)r$ and volatility $\beta\sigma$. Intuition: drift μ and volatility σ both get multiplied by β. Moreover, pay interest if $\beta > 1$ (or earn interest if $\beta < 1$)

(Follow-up question: what are the *risk-neutral* drift and volatility of L?)

Problem 3

(a) Call price C_t equals put price P_t plus value of a forward with the same strike and same expiry as the call and put:

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

(b) Using put-call parity, the B-S call formula, and the fact that $N(x) + N(-x) = 1$,

$$\begin{aligned}
P_t &= C_t - S_t + Ke^{-r(T-t)} \\
&= S_t N(d_1) - Ke^{-r(T-t)} N(d_2) - S_t + Ke^{-r(T-t)} \\
&= S_t(N(d_1) - 1) - Ke^{-r(T-t)}(N(d_2) - 1) \\
&= S_t(-N(-d_1)) - Ke^{-r(T-t)}(-N(-d_2)) \\
&= Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1)
\end{aligned}$$

(c) By put-call parity, delta of the put = delta of the call, minus delta of the forward contract.

The call's delta is $N(d_1)$.

The forward contract's delta is the S-derivative of $S - Ke^{-r(T-t)}$, which is 1.

(d) By put-call parity, gamma of put = gamma of call, minus gamma of forward contract.

The call has gamma $N'(d_1)/(S_t\sigma\sqrt{T-t})$.

The forward contract's gamma is the second S-derivative of $S - Ke^{-r(T-t)}$, which is 0.

So the put has the same gamma $N'(d_1)/(S_t\sigma\sqrt{T-t})$.