Mathew Rulla Pradnyatama HW5

Ito's Rule: $df(X_t) = \frac{df}{dx} \cdot d(X_t) + \frac{1}{2} \frac{d^2f}{dx^2} (dX_t)^2$

Problem 1

a) $f(x) = e^{x^2-1}$

$dX_t = dW_t$

$X_t = W_t$

L4.25: $(dW_t)^2 = dt$

$\frac{df}{dx} = 2x \cdot e^{x^2-1}$

$f'(X_t) = 2X_t \cdot e^{X_t^2-1}$

$f''(X_t) = 2e^{X_t^2-1} + 2X_t \cdot (2X_t) e^{X_t^2-1}$

$= (2+4X_t^2) e^{X_t^2-1}$

$dZ_t = df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$

$= 2W_t e^{W_t^2-1} \cdot dW_t + \frac{1}{2} \cdot (2+4W_t^2) e^{W_t^2-1} \cdot (dW_t)^2$

$dZ_t = (1+2W_t^2) e^{W_t^2-1} \cdot dt + 2W_t e^{W_t^2-1} dW_t$ //

Problem 1

b) $f(X_t) = e^{X_t} = f'(X_t) = f''(X_t)$

$$dZ_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= e^{W_t^2-1} \cdot (dt + 2W_t\,dW_t) + \frac{1}{2} e^{W_t^2-1}(dt + 2W_t\,dW_t)^2$$

$$(dt + 2W_t\,dW_t)^2 = (dt)^2 + 4W_t\,dW_t\,dt + 4W_t^2\,dt = 0 + 0 + 4W_t^2\,dt$$

L4.21: $(dt)^2 = 0$

$(dW_t)(dt) = 0$

$$dZ_t = e^{W_t^2-1}dt + 2W_t e^{W_t^2-1}dW_t + \frac{1}{2} e^{W_t^2-1}(4W_t^2\,dt)$$

$$dZ_t = (1+2W_t^2)e^{W_t^2-1}\,dt + 2W_t e^{W_t^2-1}\,dW_t$$

//

c) L4.12: X_t is a martingale iff $\mu_t = 0$ for all $t > 0$

$dX_t = \mu_t\,dt + \sigma_t\,dW_t$

drift term, μ_t must be 0.

here, drift term is $(1+2W_t^2)e^{W_t^2-1}$ which is not 0 for all $t>0$, with probability 1.

$\rightarrow$ Z is not a martingale

//

Problem 2

a) $d(e^{kt})d(X_t) = d(e^{kt})d(X_t) = \{ke^{kt}dt\}\{k(\theta - X_t)dt + \sigma dW_t\}$

$$= k^2 e^{kt}(\theta - X_t)\underbrace{(dt)^2}_{0} + k\sigma^{kt}\cdot\sigma\cdot\underbrace{dW_t\cdot dt}_{0} \quad (4.21)$$

$d(e^{kt})d(X_t) = 0$

$4.33: d(X_tY_t) = Y_t\,d(X_t) + X_t\,d(Y_t) + (dX_t)(dY_t)$

$$d(e^{kt}X_t) = e^{kt}d(X_t) + X_t\,d(e^{kt}) + \underbrace{d(e^{kt})\cdot d(X_t)}_{0}$$

$$= e^{kt}(k(\theta - X_t)dt + \sigma dW_t) + X_t\cdot ke^{kt}dt$$

$$= e^{kt}(k\theta dt - k\cdot X_t dt + \sigma dW_t) + X_t\cdot k\,e^{kt}dt$$

$$= e^{kt}k\cdot\theta dt - \cancel{e^{kt}k\cdot X_t dt} + e^{kt}\sigma dW_t + \cancel{X_t\cdot ke^{kt}dt}$$

$$d(e^{kt}X_t) = k\cdot\theta\cdot e^{kt}dt + \sigma e^{kt}dW_t$$

if $Z_t = e^{kt}\cdot X_t$,

$$Z_t = Z_0 + \int_0^T k\cdot\theta e^{k\cdot t}dt + \int_0^T \sigma\cdot e^{kt}\cdot dW_t$$

$$e^{kT}X_T = e^{k\cdot 0}X_0 + \int_0^T k\theta e^{kt}dt + \int_0^T \sigma\cdot e^{kt}dW_t$$

$$e^{kT}X_T = \underbrace{X_0}_{\text{Constant}} + \underbrace{\int_0^T k\cdot\theta e^{kt}dt}_{\text{Riemann integral w.r.t } dt} + \underbrace{\int_0^T \sigma\cdot e^{kt}dW_t}_{\text{Itô integral w.r.t. } d} \quad //$$

Problem 2

b) $\int_0^T k\theta e^{kt}dt = k\cdot\theta\int e^{u}\cdot\frac{1}{k}du = \theta\int e^{u}du = \theta e^{u}\Big|_{t=0}^{t=T} = \theta e^{kt}\Big|_{t=0}^{t=T}$

$u = kt$

$\frac{du}{dt} = k$

$dt = \frac{du}{k}$

$= \theta(e^{kT} - e^{k\cdot 0}) = \theta(e^{k\cdot T} - 1)$

$$e^{kT}X_T = X_0 + \theta(e^{k\cdot T} - 1) + \int_0^T \sigma\cdot e^{kt}dW_t$$

$$X_T = \frac{X_0}{e^{kT}} + \theta\left(1 - \frac{1}{e^{kT}}\right) + \int_0^T \sigma\cdot e^{kt-kT}dW_t$$

$$E(X_t) = X_0 E(e^{kT}) + \theta E(1 - e^{-kT}) + E\left(\int_0^T \sigma e^{k(t-T)}dW_t\right) \quad \nearrow 0$$

$\nearrow$ this is Itô integral

4.11, expected value of any Itô integral is 0

$$E(X_t) = X_0 e^{kT} + \theta(1 - e^{-kT}) \;/\!/$$

Problem 2

b) Variance of X_t

$$X_t = \underbrace{\frac{X_0}{e^{kT}} + \theta(1 - e^{-kT})}_{\text{Constant}} + \int_0^T \sigma \cdot e^{k(t-T)} \, dW_t$$

Since the first 2 terms are constant, $Var(X_t)$ depends on

$\int_0^T \sigma \cdot e^{k(t-T)} dW_t = \int_0^T \beta_t \, dW_t$, where $\beta_t = \sigma \cdot e^{k(t-T)}$

$$\int_0^T \beta_t \, dW_t \sim \text{Normal}\left(0, \int_0^T \beta_t^2 \, dt\right)$$

$$Var(X_t) = Var\left(\int_0^T \beta_t \, dW_t\right) = \int_0^T \beta_t^2 \, dt = \int_0^T \sigma^2 \cdot e^{2k(t-T)} dt$$

$u = t - T$

$du = dt$

$$Var(X_t) = \sigma^2 \cdot \int e^{2k \cdot u} du = (\sigma^2) \cdot \frac{1}{2k} \cdot e^{2k \cdot u} \Big|_{t=0}^{t=T}$$

$$= \frac{\sigma^2}{2k} \cdot e^{2k(t-T)} \Big|_{t=0}^{t=T} = \frac{\sigma^2}{2k} \cdot \left(e^{2k(T-T)} - e^{2k(0-T)}\right)$$

$$Var(X_t) = \frac{\sigma^2}{2k}\left(1 - e^{-2kT}\right)$$

//

5

Problem 3

a) Long-term-mean is around 10 → $\theta = 10$

Small variability → $k = 8$ //

b) long-term mean is around 15 → $\theta = 15$

big variability → $k = 3$ //

c) long-term mean is around 15 → $\theta = 15$

Small variability → $k = 8$ //

d) long-term mean is around 10 → $\theta = 10$

big variability → $k = 3$ //

e) long-term mean is around 5 → $\theta = 5$

big variability → $k = 3$ //

f) long-term mean is around 5 → $\theta = 5$

Small variability → $k = 8$ //

$k = 8$ → higher mean reversion rate → smaller variability

$k = 3$ → smaller mean reversion rate → bigger variability.