Matthews Raka Pradnyatama HW7

Problem 1a)

$$E_t(Y_T - K)^+ = E\left(\sigma\sqrt{T-t}\, X_T \mathbb{1}_{X_T > c} + (m_T - K)\mathbb{1}_{X_T > c}\right)$$

where $c = \dfrac{K - m_t}{\sigma\sqrt{T-t}}$

$$E(\sigma\sqrt{T-t}\, X_T \mathbb{1}_{X_T>c}) = \sigma\sqrt{T-t}\, E(X_T \mathbb{1}_{X_T > c}) \quad \nearrow L6.17$$

$$= \sigma\sqrt{T-t} \cdot \int_c^{\infty} \frac{1}{\sqrt{2\pi}} x \cdot e^{-x^2/2}\, dx$$

$$= \frac{\sigma\sqrt{T-t}}{2\pi} \cdot e^{-c^2/2}$$

L6.17: $E(\mathbb{1}_{Y>K}) = P(Y>K) = N\left(\frac{m-K}{\sqrt{v}}\right)$

$$\underset{\substack{\downarrow\ \downarrow \\ m\ \ v}}{X_T \sim N(0,1)} \rightarrow E(\mathbb{1}_{X_T > c}) = P(X_T > c) = N\left(\frac{m-c}{\sqrt{v}}\right) = N\left(\frac{0-c}{1}\right).$$

$$= N(-c) = 1 - N(c)$$

$$\underset{\substack{\downarrow \\ \text{time-}t\text{ price of the option}}}{C_t} = \left\{ \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-c^2/2} + (m_T - K)\cdot(1 - N(c)) \right\} \underset{\substack{\downarrow \\ \text{discount factor}}}{e^{-r(T-t)}} \quad //$$

where $c = \dfrac{K - m_t}{\sigma\sqrt{T-t}}$

Problem 1 b) $r=0 \to e^{-r(T-t)} = e^0 = 1$

delta: $\frac{\partial C_t}{\partial Y_t}$, 1st deriv of option w.r.t. Y_t

$$\boxed{N'(x) = \frac{1}{2\pi} e^{-x^2/2}}$$

$Y_t = M_t \to C = \frac{K}{\sigma\sqrt{T-t}} - \frac{Y_t}{\sigma\sqrt{T-t}}$

$$\frac{\partial C}{\partial Y_t} = -\frac{1}{\sigma\sqrt{T-t}}$$

$$\frac{d e^{-C^2/2}}{\partial Y_t} = e^{-C^2/2}\left(-\frac{2C}{2}\right)\frac{dC}{dY_t} = -C \cdot e^{-C^2/2}\left(-\frac{1}{\sigma\sqrt{T-t}}\right) = C \cdot \frac{e^{-C^2/2}}{\sigma\sqrt{T-t}}$$

$$= \frac{(K-Y_t)}{\sigma\sqrt{T-t}} \cdot \frac{e^{-C^2/2}}{\sigma\sqrt{T-t}} = \frac{(K-Y_t)e^{-C^2/2}}{\sigma^2(T-t)}$$

First term of $\Delta = \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} \cdot \frac{(K-Y_t)e^{-C^2/2}}{\sigma^2(T-t)} = \frac{(K-Y_t)}{\sigma\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-(K-Y_t)^2/2(\sigma\sqrt{T-t})^2}$

$$= \frac{K-Y_t}{\sigma\sqrt{T-t}} \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\left(\frac{K-Y_t}{\sigma\sqrt{T-t}}\right)^2/2}\right) = \frac{K-Y_t}{\sigma\sqrt{T-t}} \cdot N'\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

2nd term of $\Delta = \frac{d}{dY_t}\left\{(Y_t-K) \cdot \left(N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)\right)\right\}$

$$= N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right) + (Y_t-K) \cdot \frac{1}{\sigma\sqrt{T-t}} \cdot N'\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

$$\Delta = \left\{\frac{(K-Y_t)}{\sigma\sqrt{T-t}} + \frac{(Y_t-K)}{\sigma\sqrt{T-t}}\right\} N'\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right) + N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right)$$

$$\Delta = N\left(\frac{Y_t-K}{\sigma\sqrt{T-t}}\right) \quad \sim \text{delta}$$

Problem 1) b) (continued)

$$\text{gamma} = \frac{\partial \Delta}{\partial Y_t} = \frac{1}{\sigma\sqrt{T-t}} \cdot N\left(\frac{Y_t - K}{\sigma\sqrt{T-t}}\right)$$

//

Problem 1 c)

$m_t = y_t = k$

$$\text{Price} = C_t = \left\{ \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-c^2/2} + \overset{\nearrow 0}{(k-k)}(1-N(c)) \right\} e^0$$

$$C_t = \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-c^2/2} = \frac{\sigma(T-t)^{1/2}}{\sqrt{2\pi}} e^{-c^2/2} = \frac{\sigma(T-t)^{1/2}}{\sqrt{2\pi}}$$

$$c = \frac{k-m_t}{\sigma\sqrt{T-t}} = \frac{k-k}{\sigma\sqrt{T-t}} = 0$$

$$\text{theta} = \frac{dC_t}{dt} = \frac{\sigma}{\sqrt{2\pi}} \cdot \frac{1}{2}(T-t)^{-1/2} \cdot \frac{d(-t)}{dt} = \frac{-\sigma}{2\sqrt{2\pi}}(T-t)^{-1/2} \quad //$$

$$\text{vega} = \frac{dC_t}{d\sigma} = \frac{(T-t)^{1/2}}{2\pi} \quad //$$

P4

Problem 2) a)

in a no-arbitrage situation, the price of a binary put and binary call option equals the PV of 1 unit of bond, discounted at the risk-free rate.

$$C_B(K) + P_B(K) = e^{-r.T}$$

$$P_B(K) = 0.96 - 0.44 = 0.52 \;//$$ → time-0 price of binary put

2) b) Put-call parity: $C_0(K) = P_0(K) + S_0 - K \cdot Z_0$

$$C_0(K) = P_0(K) + S_0 - K \cdot e^{-r.T}$$

$$P_0(K) = C_0(K) - S_0 + K \cdot e^{-r.T} = 34 - 216 + (250)(0.96)$$

$$P_0(K) = 58 \;//$$ → time-0 price of put on S

(L5.13)

2) c) price of call: $C^{BS} = e^{-r(T-t)}\left(S_t \cdot e^{r.(T-t)} \cdot N(d_1) - K \cdot N(d_2)\right)$

$$C^{BS} = S_t \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

at $t=0$, $C^{BS} = S_0 N(d_1) - K \cdot e^{-r.T} \cdot N(d_2)$

Value of binary call that pays $1_{S_T > K}$, at $t=0$, (L6.2d)

$$C_B = e^{-r(T-t)} N(d_2) = e^{-r(T)} \cdot N(d_2)$$

$$0.44 = 0.96 \cdot N(d_2) \rightarrow N(d_2) = \frac{0.44}{0.96}$$

S

Value of asset-or-nothing call on S, at $t=0$

$$S_0 N(d_1) = C^{BS} + K \cdot e^{-r \cdot T} N(d_2) = 34 + 250(0.96)\left(\frac{0.44}{0.96}\right)$$

$$S_0 \cdot N(d_1) = 144$$

//

2) d) risk neutral measure, $E\left(\frac{S_T}{B_T}\right) = \frac{S_0}{B_0} = \frac{S_0}{1} = S_0$

$L_{1}:19$: $\frac{1}{B_T} = e^{-r \cdot T}$

$$E_0(e^{-r \cdot T} \cdot S_T) = S_0$$

$$e^{-r \cdot T} E_0(S_T) = S_0$$

$$E_0(S_T) = S_0 \cdot e^{r \cdot T} = (216)(1/0.96) = 225$$

//

2) e) risk-neutral prob. of $S_T > K$ $\rightarrow$ L6.28

$$= P(S_T > K) = N(d_2) = \frac{C_B}{e^{-r \cdot T}} = \frac{0.44}{0.96} = 0.4583$$

//

2)f) L6.28: $S_t N(d_1)$ = value of vanilla call replicator's share holdings

$N(d_1)$ → number of units of shares being hold in the replicated call

$N(d_1) = \frac{dC}{dS}$ → delta of call

Since here, we are replicating a put option, we need to find delta of put option

Put call parity: $C = P + S - K \cdot Z$

$$\frac{\partial C}{\partial S} = \frac{\partial P}{\partial S} + 1$$

$$\Delta C = \Delta P + 1 \sim \Delta P = \Delta C - 1$$

from 2)c), $S_0 \cdot N(d_1) = 144$

$$N(d_1) = \frac{144}{216} = \frac{2}{3} = \Delta C$$

$$\Delta P = \frac{2}{3} - 1 = -\frac{1}{3} = -0.3333$$

at time-0, we need to short 0.3333 units of S, to replicate the put option //