Financial Mathematics 33000

Lecture 1

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Introduction

General properties of arbitrage-free prices

General properties of forwards and options

FINM 33000 and FINM 32000

I teach this course and its sequel.

- Fall 2024: Option pricing theory
- Spring 2025: Computational option pricing
- "Option pricing" is meant in a broad sense: the pricing and hedging of options and other financial *derivative contracts*
- A *derivative security* or *derivative contract* is a financial instrument whose payoff is defined in terms of an *underlying* (e.g.: A asset such as a stock or bond. An index. An interest rate.)

The main idea

I quote from Tomas Björk:

- A financial derivative is defined *in terms of* some underlying asset which already exists on the market
- A derivative cannot therefore be priced arbitrarily *in relation to the underlying prices* if we want to *avoid mispricing between the derivative and the underlying price.*
- We thus want to price the derivative in a way that is *consistent* with the underlying prices given by the market.
- We are not trying to compute the price of the derivative in some "absolute" sense. The idea instead is to determine the price of the derivative *in terms of the market prices of the underlying assets.*

We assume frictionless markets

We will specify a set of basic tradeable assets and a set of times.

At any such time, each basic tradeable asset has a market price, and we can buy/sell/hold arbitrary quantities at that time, at that price.

In other words, assume *frictionless markets*. In particular,

- No transaction costs: no commissions, no fees, no taxes
- No bid-ask spread. No slippage. No market impact
- No default risk. No counterparty risk
- No margin constraints
- Can hold fractional quantities of assets
- Can sell what you do not own (*sell short* or *go short* or *short*), and hold a negative quantity (a *short* position).

Introduction

General properties of arbitrage-free prices

General properties of forwards and options

Assets

- The market has risks described by a probability measure P.
- It includes N tradeable *assets* with nonrandom time-0 prices

$$\mathbf{X}_0 := (X_0^1, \dots, X_0^N)$$

time-0 price of asset 1
time-0 price of asset N

and random time-T prices ("payoffs" or "payouts")

$$\mathbf{X}_T := (X_T^1, \dots, X_T^N)$$

time-T price of asset 1
time-T price of asset N

No distinction between payoff/payout X_T vs. final asset price X_T.

- In this section, no assumptions about which times $t \in (0, T)$ exist in our market. Our general analysis applies to a one-period model (which includes only 0 and T), continuous-time model (which includes all of $t \in [0, T]$), and any intermediate model.

Examples of tradeable assets

- A *zero-coupon bond* or *discount bond* with maturity T:

  Each unit pays at time T a fixed payoff, let us say $Z_T = 1$.
- A non-dividend-paying *stock*: Each unit has time-t price $S_t \geq 0$.

  Can think of stock as a claim on a time-T random payoff $S_T \geq 0$.
- A *bank account* or *money market acct*: Each unit has time-t price

  $$B_t := \exp\left(\int_0^t r_u \mathrm{d}u\right). \quad \text{If } r \text{ is constant}, B_t = e^{rt}$$

  for some (possibly random) r_u, called the time-u *instantaneous spot rate* of interest or *short rate*. Note: B solves the diff eq

  $$\frac{\mathrm{d}B_t}{\mathrm{d}t} = r_t B_t \qquad \text{with } B_0 = 1.$$

  Can think of bank account as having time-T payoff $\exp(\int_0^T r_t \mathrm{d}t)$.

Static portfolios

- A *static portfolio* is a vector of quantities

$$\boldsymbol{\Theta} := (\theta^1, \ldots, \theta^N)$$

where each θ is nonrandom and constant in time.

Each θ^n denotes the number of units of asset n, for $n = 1, \ldots, N$.

If $\theta^n > 0$ we say the portfolio is *long* asset n.

If $\theta^n < 0$ we say the portfolio is *short* asset n.

V_t = the number of unit of asset N times the price of each unit of asset N

- The time-t *value* of portfolio $\boldsymbol{\Theta}$ is

$$V_t := \boldsymbol{\Theta} \cdot \mathbf{X}_t = \theta^1 X_t^1 + \cdots + \theta^N X_t^N$$

If we are dealing with multiple portfolios, we may give V a superscript to indicate which portfolio.

Arbitrage: common-language definition

Arbitrage is a combination of transactions that tries to profit from price inconsistencies. Examples:

- A stock is being bid at Exchange A for a higher price than it is being offered at Exchange B. Buy it at site B, sell it at site A.
- Asset F is a combination of assets G and H, but is priced lower than the sum of the constituent prices. Buy F, sell G, sell H.
  G + H > F
- A combination of assets is underpriced or overpriced relative to a "fair" or "predicted" value from a statistical model.
  Buy or sell that combination.

In common language, "arbitrage" may involve risk of loss.

Arbitrage: mathematical definition

V_0 = 0 means that to acquire the portfolio, there's no cost

A static portfolio $\boldsymbol{\Theta}$ is a "type 1" *arbitrage* if its value V satisfies

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

No chance of loss
There's a prob 1 that the portfolio has a positive value

(Zero initial investment, and no risk of loss, some chance of gain.)

A static portfolio $\boldsymbol{\Theta}$ is a "type 2" *arbitrage* if its value V satisfies

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1$$

V_0 < 0 = instead of paying, you're getting a credit to acquire the portfolio

(Initially receive a credit ... which you will definitely not repay.)

A static portfolio $\boldsymbol{\Theta}$ is an *arbitrage* if it's either a type 1 or type 2 arbitrage.

Say V_0 = -7, this means you get $7 upfront and you're gonna put that in a bank —> receive interest
Payoff at the end = $7 + X

Arbitrage

- Prices which admit arbitrage are, in some sense, incorrect.
  Existence of arbitrage is a severe form of the inconsistency and mispricing that we want to avoid.
- Assume no arbitrage, unless otherwise indicated.
  Thus, when we try to price some asset, we are looking for an *arbitrage-free* price.
- Some authors define arbitrage without "type 2".
  The distinction between our definition (type 1 or type 2) and their definition (type 1 only) is essentially immaterial, because:
  If there exists an asset whose price is always nonnegative and not always zero, then type 1 arb exists whenever type 2 arb exists.

Examples

A portfolio is *not* an arbitrage if its value satisfies:

- $V_0 = 0$, and $\mathsf{P}(V_T = 50) = 0.99$, $\mathsf{P}(V_T = -5) = 0.01$.

  If there's any chance of loss, then it's not an arbitrage. **arb: P(V_T >= 0) = 1**

- $V_0 = 1$, and $V_T = 2$ with probability 1.

  By definition, $V_0 = 1$ implies the portfolio is not an arbitrage.

  Initial investment is required to buy this portfolio. **arb: V_0 = 0**

- $V_0 = -2$, and $V_T = -1$ with probability 1. **arb: P(V_T >= 0) = 1**

  This is not an arb because $V_T = -1$. Receiving 2 initially, then later paying only 1, does *not* necessarily lock in a gain. Because, without assumptions about interest rates, we don't know whether the initial 2 can be parked in an asset worth at least 1 at time T.

Example: FIFA world cup contracts

Prices on polymarket.com: November 30, 2022

Group G	
Brazil	97%
Switzerland	59%
Serbia	36%
Cameroon	4%

Each contract pays \$1 at time T if that team advances

Exactly two teams will advance. Assume zero interest rates: $B_t = 1$.

Find arbitrage.

Arbitrage-free prices satisfy consistency conditions

Suppose portfolio $\boldsymbol{\Theta}^a$ *superreplicates* portfolio $\boldsymbol{\Theta}^b$, which means that $\mathsf{P}(V_T^a \geq V_T^b) = 1$. Then $V_0^a \geq V_0^b$, otherwise arbitrage exists.

Proof.

If instead $V_0^a < V_0^b$, then construct portfolio $\boldsymbol{\Theta} := \boldsymbol{\Theta}^a - \boldsymbol{\Theta}^b$.

(In other words, go *long* $\boldsymbol{\Theta}^a$ and *short* $\boldsymbol{\Theta}^b$.)

Its time-0 value is $V_0 = \boldsymbol{\Theta} \cdot \mathbf{X}_0 = \boldsymbol{\Theta}^a \cdot \mathbf{X}_0 - \boldsymbol{\Theta}^b \cdot \mathbf{X}_0 = V_0^a - V_0^b < 0$.

Its time-T value is $V_T = V_T^a - V_T^b \geq 0$ with probability 1.

Hence $\boldsymbol{\Theta}$ is an arbitrage. □

In this proof, we used a general technique for constructing arbitrage:

- Go long what is cheap (undervalued), and short what is rich (overvalued). In other words: buy low, sell high.

The law of one price

Likewise, if $\mathbf{\Theta}^a$ *subreplicates* $\mathbf{\Theta}^b$, meaning $\mathsf{P}(V_T^a \leq V_T^b) = 1$, then $V_0^a \leq V_0^b$. By combining the two inequalities, therefore,

$$\boxed{\text{If } \mathsf{P}(V_T^a = V_T^b) = 1, \text{ then } V_0^a = V_0^b.}$$

In other words, if $\mathbf{\Theta}^a$ *replicates* $\mathbf{\Theta}^b$, then $V_0^a = V_0^b$.

- This is the *law of one price*. Any two static portfolios with identical future payouts must have identical current prices.
- "You can summarize the essence of quantitative finance," according to Emanuel Derman, as follows:
  "If you want to know the value of a security, use the price of another security [or *portfolio* of securities] that's as similar to it as possible."

Price vs Value vs Payoff

- Time-t *price* = how much it costs to buy/sell something at time t.
  (Exceptions: "forward price", "futures price")
- Time-t *value* = how much it *should* cost to buy/sell something

Meaning of "should" depends on the context. In this course, the only notion of "should" is that arbitrage should not exist. So for us, "value" is what it costs to buy/sell something, in the absence of arbitrage. But since we have a standing assumption of no-arbitrage, we really have no distinction between price and value, *unless* we are in a situation where arbitrage exists (e.g. HW: "find an arbitrage").

- Payoff = Payout = how much a contract pays
  = Value of the contract *at expiration* (assuming single payment)

Introduction

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Discount bond: valuation

For a discount bond Z maturing at T, and a bank account B, we have

$$Z_0 = \frac{1}{B_T} = e^{-\int_0^T r_t \mathrm{d}t} \qquad \text{if interest rate } r_t \text{ is non-random}$$

$$= e^{-rT} \qquad \text{if } r \text{ is constant}$$

Proof.

A portfolio consisting of $1/B_T$ units of the bank account has time-T value $(1/B_T) \times B_T = 1$, which is identical to $Z_T = 1$.

- In particular, if r is constant, then 1 unit of bank is identical to e^{rT} bonds, and 1 bond is identical to e^{-rT} units of the bank acct.

So the portfolios must have equal time-0 values: $Z_0 = (1/B_T) \times 1$. □

Forward contract: definition

Consider a random variable S_T whose value is revealed at time T.

- A *forward contract* on S_T with maturity / delivery date T and nonrandom delivery price K obligates the holder to, at time T, pay K and receive S_T (dollars if "cash" settled. If "physical" settlement, you get an asset, whose time-T price we denote S_T.)
- So the forward contract has payoff $S_T - K$. Payoff diagram:
- Forward contract is an example of a *derivative* – a security whose payout is contractually related to some *underlying* variable.

Forward contract: valuation

Consider a forward contract on a non-dividend-paying stock S, with delivery date T and any delivery price K.

Then the time-0 value of the forward contract is $S_0 - KZ_0$.

Proof.

The portfolio

$$\boldsymbol{\Theta} = (1 \text{ share}, \; -K \text{ bonds})$$

has time-T value $V_T = \boldsymbol{\Theta}_T \cdot \mathbf{X}_T = 1 \times S_T - K \times 1 = S_T - K$.

The forward contract also has time-T value $S_T - K$.

So the time-0 value of the forward contract must equal the time-0 value of the replicating portfolio, which is

$$V_0 = \boldsymbol{\Theta}_0 \cdot \mathbf{X}_0 = (1, -K) \cdot (S_0, Z_0) = 1 \times S_0 - K \times Z_0.$$

Forward price

The *forward price* F_0 which sets at time 0 for delivery at time T is the delivery price such that the forward contract has zero value at time 0.

- A "forward price" and the "value of a forward contract" are not the same thing.
- A forward contract on a no-dividend stock S has time-0 value

$$S_0 - KZ_0.$$

Choice of K that makes value zero is S_0/Z_0.

$$\boxed{\text{Thus } F_0 = S_0/Z_0\text{. If } r \text{ is constant, then } F_0 = S_0 e^{rT}.}$$

This does not depend on the dynamics of S.

Forward price example

- If $r = 0.02$ and the share price today is $S_0 = 600$, and you and I want to enter costlessly today into a contract for time-1 delivery of S in exchange for a delivery price to be paid at time-1, the only arbitrage-free way to set that delivery price is $600 \times e^{0.02} \approx 612$.

Even if bullish, it'd be wrong for me to agree to pay, say, 650.

- Your portfolio $(-650 \text{ bond}, 1 \text{ share}, -1 \text{ forward contract})$ is an arbitrage because $V_0 = -650e^{-0.02} + 600 < 0$ and $V_T = 0$.
- Another arb: $(-600 \text{ bank}, 1 \text{ share}, -1 \text{ forward contract})$ because $V_0 = 0$ and $V_1 = -600e^{0.02} + S_1 - (S_1 - 650) > 0$.

  In other words, you sell me the contract, borrow 600 to buy the share today. At time 1, deliver the share, collect 650, repay 612.

Affine payoff

More generally, consider the following "affine" (or "linear") contract on a non-dividend-paying stock S. The contract pays, by definition,

$$a + bS_T$$

where a and b are constants. Then its time-0 value is

a*Z_0 + b*S_0

because it is replicated by

(a units of bond, b units of S)

Call option: definition

A (European-style) *call* option with strike K and expiry T on an underlying process S, gives the holder the right, but not obligation, at time T, to pay K and receive S_T (dollars, or asset worth S_T dollars). So call has payoff $(S_T - K)^+$, where $x^+ := \max(x, 0)$. Payoff diagram:

At time $t \leq T$, the call option is said to be:

in the money if $S_t > K$, *out of the money* if $S_t < K$.

at the money if $S_t = K$, *near the money* if $S_t \approx K$

Stock is $40, you sell a call with strike price $44
This will allow you to sell the stock $44, guarantees an income of $4 + receiving the premium of the call option

Uses of call options

Why would you use a call option? Examples:

- Suppose you are bullish on the underlying.
  Buying the call costs $x\%$ of the stock price, while, potentially, participating in $y\%$ of the gains (\$) in stock price, where $x < y$.
  So, compared to buying stock, buying a call can limit your downside, and/or increase your leverage.
- Suppose you own the underlying. Selling a call ("call writing") trades away some upside, in exchange for current income.
- Suppose you think the options market is overpricing the call.
  Profit by selling the call for more than what it costs to replicate.
  Or profit by selling the call outright, if you have directional views.

Call option: bounds wrt underlying

The time-0 price C_0 of a call on a no-dividend stock S satisfies

$$\boxed{(S_0 - KZ_0)^+ \leq C_0 \leq S_0}$$

call price is bounded by upper and lower bounds

Proof.

See payoff diagram:

- Call payoff dominates payoff of forward with delivery price K.
- Call payoff dominates a zero payoff.
- Call payoff is dominated by the stock.

Hence $C_0 \geq S_0 - KZ_0$ and $C_0 \geq 0$ and $C_0 \leq S_0$ □

Call option: bounds wrt other calls

The time-0 call prices $C_0(K_1)$ and $C_0(K_2)$, for strikes $K_1 < K_2$ (with same expiry, on same underlying) satisfy

$$\boxed{0 \leq C_0(K_1) - C_0(K_2) \leq (K_2 - K_1)Z_0}$$

Proof.

Consider a bull *call spread*, long the K_1 call, short the K_2 call.

The call spread payoff dominates the zero payoff, but is dominated by the payoff of $K_2 - K_1$ discount bonds. Payoff diagram:

So $C_0(K_1) - C_0(K_2)$ is bounded below by the time-0 value of zero, above by the time-0 value of $K_2 - K_1$ bonds. □

Put option: definition

A (European-style) *put* option with strike K and expiry T on an underlying process S, gives the holder the right, but not obligation, at time T to pay S_T (dollars, or asset worth S_T dollars) and receive K. So put has payoff $(K - S_T)^+$. Payoff diagram:

At time $t \leq T$, the put option is said to be:

in the money if $S_t < K$, *out of the money* if $S_t > K$.

at the money if $S_t = K$, *near the money* if $S_t \approx K$

Uses of put options

If you short a stock without put, your exposed to unlimited loss if stock drops significantly
If you buy a put, you can sell it at K

Why would you use a put option? Examples:

- Suppose you are bearish on the underlying.

  Buying a put limits your potential loss to the cost of the option.

  (Shorting stock exposes you to unlimited loss.)

- Suppose you own the underlying. Buying a put protects you against the underlying going below K. It's insurance.

- Suppose you think the options market is overpricing the put.

  Profit by selling the put for more than what it costs to replicate.

  Or sell the put outright, if you have a directional view.

Put-call parity

Let $P_0(K,T)$ and $C_0(K,T)$ be time-0 prices of a European put and call, with identical (K,T), on a no-dividend stock S. Let $Z_0(T)$ be the time-0 price of a T-maturity discount bond. Then

$$C_0(K,T) = P_0(K,T) + S_0 - KZ_0(T)$$

Proof.

Payoff diagram:

Payoffs are equal, hence prices at earlier date are equal. □

When was put-call parity discovered?

- It's in *Confusion de Confusiones* (1688) by José de la Vega

- Put-call parity is far older, and more fundamental, than any particular model e.g. Black-Scholes (1973)

Put option: bounds wrt underlying, and wrt other puts

The time-0 price of a put on a non-dividend-paying stock S satisfies

$$\boxed{(KZ_0 - S_0)^+ \leq P_0 \leq KZ_0.}$$

The time-0 put prices $P_0(K_1)$ and $P_0(K_1)$, for strikes $K_1 < K_2$ (with same expiry, on same underlying) satisfy

$$\boxed{0 \leq P_0(K_2) - P_0(K_1) \leq (K_2 - K_1)Z_0.}$$

Proof.

Compare payoffs. Or use put-call parity. $\square$

Put option: bounds wrt other puts, revisited

If $K_1 < K_2$ then $P_0(K_1) \leq P_0(K_2)$. Proof by comparing payoffs:

Better yet,

$$P_0(K_1) \leq$$

General payoffs

Using static positions in T-expiry bonds, forwards, calls, and puts on S, we can replicate or bound (superreplicate, subreplicate) general functions of S_T.

- Use bonds to adjust level.
- Use forwards (or perhaps S itself) to adjust slope.
- Use calls (or puts) to adjust convexity and concavity.

10 units of bonds
-2 units of stock
+1.5 units of K1 strike call
+4.5 units of K2 strike call
-3.7 units of K3 strike call
-0.3 units of K4 strike call

GOOG option quotes (source: nasdaq.com)

yellow boxes are in the money contracts

September 2023 | Composite | Calls & Puts | Near the Money | All (Types)

	Calls							Puts					
Exp. Date	Last	Change	Bid	Ask	Volume	Open Int.	Strike	Last	Change	Bid	Ask	Volume	Open Int.
September 29, 2023													
Sep 29	11.55	+1.17 ▲	12.35	12.80	1	37	119.00	0.03	-0.03 ▼	0.01	0.02	21	527
Sep 29	10.60	+1.15 ▲	11.60	11.75	31	117	120.00	0.02	-0.04 ▼	0.01	0.02	63	1556
Sep 29	8.62	--	10.60	10.70	--	168	121.00	0.02	-0.07 ▼	0.02	0.03	61	584
Sep 29	9.00	+1.40 ▲	9.60	9.70	5	121	122.00	0.03	-0.08 ▼	0.02	0.03	111	572
Sep 29	7.10	+0.44 ▲	8.60	8.70	28	90	123.00	0.04	-0.11 ▼	0.03	0.04	222	800
Sep 29	6.75	+1.15 ▲	7.60	7.75	23	87	124.00	0.04	-0.19 ▼	0.04	0.05	201	836
Sep 29	5.60	+0.79 ▲	6.65	6.75	7	94	125.00	0.05	-0.26 ▼	0.06	0.07	1866	2019
Sep 29	5.56	+1.60 ▲	5.70	5.80	29	83	126.00	0.10	-0.36 ▼	0.09	0.10	293	921
Sep 29	4.30	+1.14 ▲	4.75	4.85	53	196	127.00	0.15	-0.50 ▼	0.14	0.15	626	2837
Sep 29	3.76	+1.27 ▲	3.80	3.90	257	524	128.00	0.25	-0.68 ▼	0.23	0.24	1580	1504
Sep 29	2.92	+1.03 ▲	3.00	3.05	723	934	129.00	0.37	-0.89 ▼	0.37	0.39	4643	1861
Sep 29	2.16	+0.85 ▲	2.23	2.27	2425	1785	130.00	0.60	-1.11 ▼	0.60	0.61	3747	2490
Sep 29	1.61	+0.76 ▲	1.55	1.59	3639	1346	131.00	0.92	-1.40 ▼	0.91	0.93	1410	1473
Sep 29	1.00	+0.47 ▲	0.98	1.00	7072	2232	132.00	1.35	-1.65 ▼	1.34	1.37	500	1044
Sep 29	0.54	+0.24 ▲	0.56	0.58	5650	2003	133.00	2.05	-1.67 ▼	1.90	1.94	86	2980
Sep 29	0.30	+0.14 ▲	0.28	0.29	1724	1798	134.00	2.80	-1.85 ▼	2.62	2.67	107	847
Sep 29	0.13	+0.05 ▲	0.12	0.13	995	3160	135.00	3.65	-1.96 ▼	3.45	3.55	190	930
Sep 29	0.06	+0.01 ▲	0.05	0.06	601	9726	136.00	4.55	-2.35 ▼	4.40	4.50	32	784
Sep 29	0.03	-0.01 ▼	0.02	0.03	444	2538	137.00	5.38	-2.25 ▼	5.25	5.50	30	127
Sep 29	0.01	-0.02 ▼	0.01	0.02	386	3670	138.00	6.65	-2.17 ▼	6.20	6.50	1717	211
Sep 29	0.01	-0.01 ▼	--	0.01	16	1303	139.00	7.65	-2.15 ▼	7.30	7.55	1630	217
Sep 29	0.01	--	--	0.01	640	2618	140.00	9.45	-1.35 ▼	8.25	8.50	426	121
Sep 29	0.01	--	--	0.01	12	835	141.00	9.90	-2.70 ▼	9.20	9.55	8	--
Sep 29	0.01	--	--	0.01	10	1827	142.00	12.35	--	10.25	10.70	--	--
Sep 29	0.01	--	--	0.01	1	1044	143.00	12.65	+1.65 ▲	11.10	11.50	1	19
Sep 29	0.01	--	--	0.01	11	1697	144.00	14.20	+2.15 ▲	12.30	12.65	63	19

LAST TRADE: $131.36 (AS OF SEP 27, 2023 3:37 PM ET)

GOOG option quotes (source: nasdaq.com)

November 17, 2023

Nov 17	13.30	+0.27 ▲	14.55	14.70	18	4561	120.00	2.01	-0.54 ▼	2.01	2.04	139	9884
Nov 17	10.50	+1.20 ▲	10.75	10.90	74	5604	125.00	3.25	-0.75 ▼	3.20	3.25	382	6748
Nov 17	7.50	+1.07 ▲	7.55	7.60	2122	5480	130.00	4.95	-1.05 ▼	4.95	5.00	1542	5840
Nov 17	4.87	+0.79 ▲	4.90	4.95	3761	7288	135.00	8.30	-0.60 ▼	7.30	7.40	73	5933
Nov 17	2.95	+0.54 ▲	2.98	3.05	5111	15231	140.00	11.39	-0.57 ▼	10.40	10.50	4	3699

LAST TRADE: $131.35 (AS OF SEP 27, 2023 3:40 PM ET)