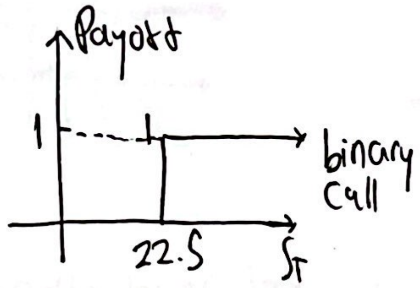
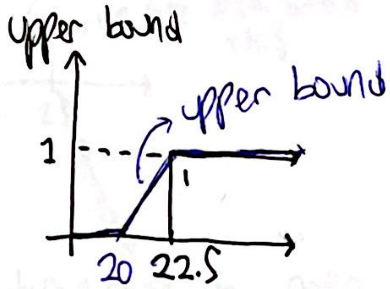
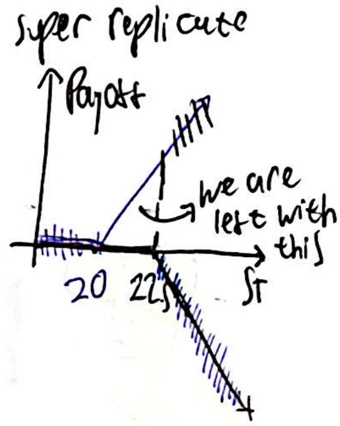
# Option Pricing - Matheus Pradnyatama HW2

## Problem 1

a)




For the upper bound, we can super replicate by going long $C(20)$ and short $C(22.5)$.

$$\frac{1}{K_2-K_1} = \frac{1}{22.5-20} = \frac{1}{2.5} = \frac{2}{5}$$

For the payoff to be 1 (replicating Binary Call), adjust by $\frac{2}{5}$

$$\left\{ \frac{2}{5} \text{ unit of } C(20),\ -\frac{2}{5} \text{ unit of } C(22.5) \right\}$$

For $S_T > 22.5$,

$$V_T = \frac{2}{5}(C_T(20)) - \frac{2}{5}(C_T(22.5)) = \frac{2}{5}\{S_T - 20 - S_T + 22.5\} = \frac{2}{5}(2.5) = 1$$

match the payoff of the binary call

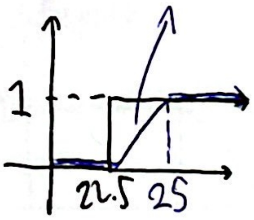
upper bound on $BC_0(22.5)$ is

$$= \frac{2}{5}C_0(20) - \frac{2}{5}C_0(22.5) = \frac{2}{5}(6.15-4.15) = 0.8$$

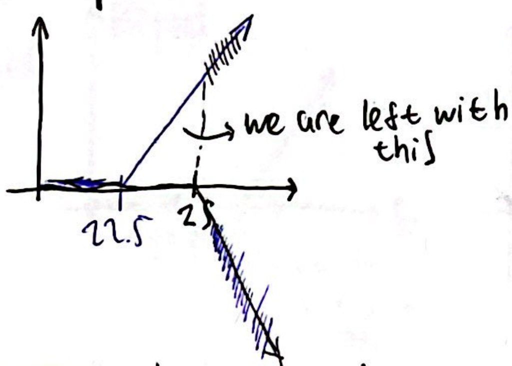
$BC$ = Binary Call

$\hookrightarrow BC_0(22.5) \leq 0.8 \rightarrow$ upper bound //

1) a) Lower Bound




Sub-replicate




For the lower band, we can subreplicate by going long $C(22.5)$ and short $C(25)$, adjust by $\frac{2}{5}$ to make payoff 1

$\left\{ \frac{2}{5} \text{ unit of } C(22.5),\ -\frac{2}{5} \text{ unit of } C(25) \right\}$

For $S_T > 22.5$,

$$V_T = \frac{2}{5} C_T(22.5) - \frac{2}{5} C_T(25) = \frac{2}{5}\{S_T - 22.5 - S_T + 25\} = \frac{2}{5}(2.5) = 1$$

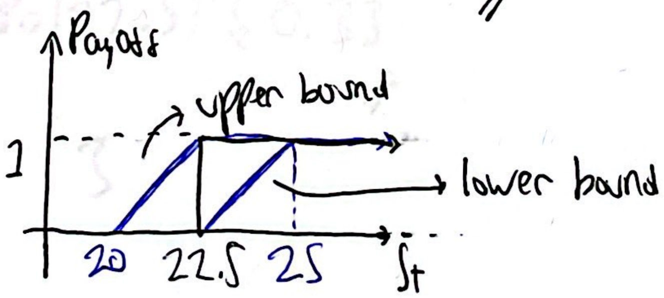
$$S_T \leqslant 22.5,\ V_T = \frac{2}{5} \cdot 0 - \frac{2}{5} \cdot 0 = 0$$

Lower bound on $BC_0(22.5)$ is

$$= \frac{2}{5} C_0(22.5) - \frac{2}{5} C_0(25) = \frac{2}{5}\{4.15 - 2.6\} = 0.62$$

Lower band: $0.62 \leqslant BC_0(22.5)$ //

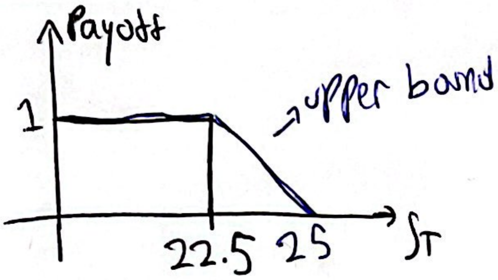
Final graph:

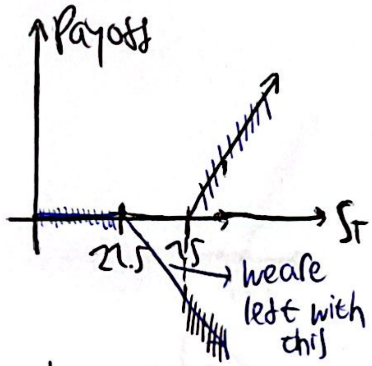


## Problem 1

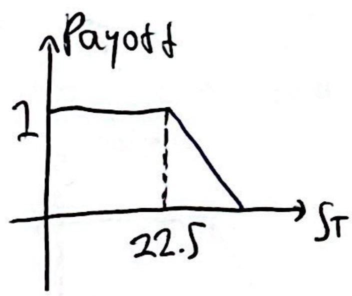
Superreplicate

b)




long C(25) and short C(22.5)




add 1 unit of discount bond

Supperreplicate portfolio: $\left\{ \frac{2}{5} \text{ unit of } C(25), -\frac{2}{5} \text{ unit of } C(22.5), 1 \text{ unit of discount bond} \right\}$

Payoff when $S_T < 22.5$

$$V_T = \frac{2}{5}(C_T(25)) - \frac{2}{5}(C_T(22.5)) + 1 = \frac{2}{5}(0) - \frac{2}{5}(0) + 1 = 1$$

upper bound on Binary Put ($BP_0(22.5)$):

$$= \frac{2}{5}C_0(25) - \frac{2}{5}C_0(22.5) + 0.95 = \frac{2}{5}(2.6 - 4.15) + 0.95 = 0.33$$

upper bound: $BP_0(22.5) \leq 0.33$ //

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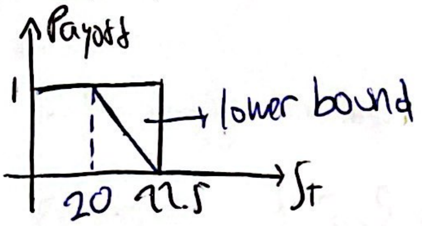
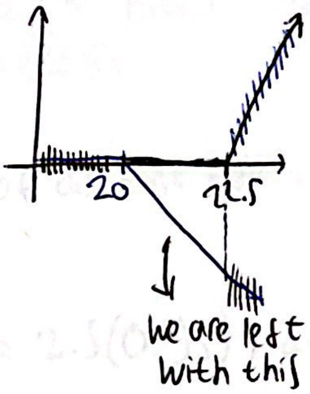
## Problem 1

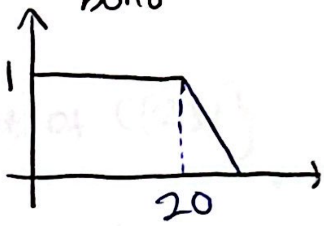
b) (continued)




Subreplicate




add 1 unit of discount bond




we can subreplicate by having:

$\left\{-\frac{2}{5} \text{ unit of } C(20), \frac{2}{5} \text{ unit of } C(22.5), 1 \text{ unit of discount bond}\right\}$

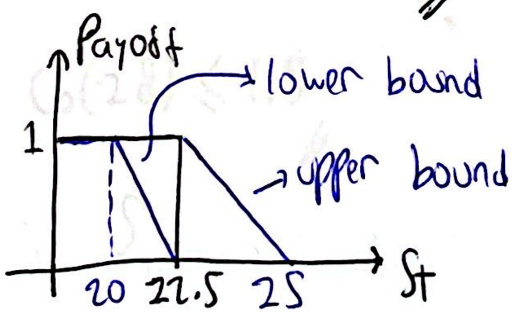
Lower bound on Binary Put ($BP_0(22.5)$):

$$= -\frac{2}{5} \times C_0(20) + \frac{2}{5} C_0(22.5) + 0.95$$

$$= -\frac{2}{5}(6.15 - 4.15) + 0.95 = 0.15$$

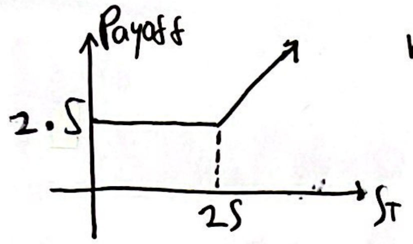
Lower bound: $0.15 \leq BP_0(22.5)$

Final graph:



# Problem 1

c)




we can replicate using 2.5 unit of discount bond, and 1 unit of $C(25)$

Portfolio: $\{2.5$ unit of discount bond, 1 unit of $C(25)\}$

time-0 price:

$$C_0 := 2.5 z_0 + (1) C_0(25) = 2.5(0.95) + 2.6 = 4.975$$

//

d)


using equation from 4.2d

$$0 \leq C_0(k_1) - C_0(k_2) \leq (k_2 - k_1) z_0$$

$k_1 < k_2$ — $27.5 < 28$, $28 < 30$

$$C_0(k_2) \leq C_0(k_1) \leq (k_2 - k_1) z_0 + C_0(k_2)$$

$$C_0(30) \leq C_0(28) \leq (30-28)(0.95) + C_0(30)$$

$$0.8 \leq C_0(28) \leq (2)(0.95) + (0.8)$$

$$0.8 \leq C_0(28) \leq 2.7$$

$$C_0(28) \leq C_0(27.5) \leq (28-27.5)(0.95) + C_0(28)$$

$$C_0(28) \leq 1.5$$

upper bound: $C_0(28) \leq 1.5$

//

S

## Problem 2

a) $\underline{S > k_*}$

$\int_0^{k_*} f''(k)(k-S)^+ dk = 0$, we won't exercise the put, payoff $= 0$

$$\int_{k_*}^{\infty} f''(k)(S-k)^+ dk = \int_{k_*}^{S} f''(k)(S-k)\,dk = (S-k)\,f'(k)\Big|_{k_*}^{S} - \int_{k_*}^{S} f'(k)(-dk)$$

$u = S-k$
$du = -dk$
$dv = f''(k)$
$v = f'(k)$

$$= (S-k)f'(k)\Big|_{k_*}^{S} + f(k)\Big|_{k}^{S}$$

$$= \underbrace{(S-S)f'(S)}_{0} - (S-k_*)f'(k_*) + f(S) - f(k_*)$$

$$= f(S) - f(k_*) - (S-k_*)f'(k_*)$$

$$f(S) = f(k_*) + f'(k_*)(S-k_*) + \int_0^{k_*} f''(k)(k-S)^+ dk + \int_{k_*}^{\infty} f''(k)(S-k)^+ dk \quad (1)$$

$$f(S) = \cancel{f(k_*)} + \cancel{f'(k_*)(S-k_*)} + 0 + f(S) - \cancel{f(k_*)} - \cancel{(S-k_*)f'(k_*)}$$

$f(S) = f(S)$ , for $S > k_*$, equation (1) is true.

$\underline{S = k_*}$

$$f(S) = f(k_*) + f'(k_*)(S-k_*) + \int_0^{k_*} f''(k)(k-S)^+ dk + \int_{k_*}^{\infty} f''(k)(S-k)^+ dk$$

$$= f(S) + f'(S)(S-S) + \int_0^{S} f''(k)(k-S)^+ dk + \int_S^{\infty} f'(k)(S-S)^+ dk$$

$f(S) = f(S) + 0 + 0 + 0$

$f(S) = f(S)$ // $\therefore$ For $S = k_*$, equation (1) is true also

$\underline{S < k_*}$

$\int_{k_*}^{\infty} f''(k)(S-k)^+ dk = 0$, we won't exercise the call, payoff $= 0$

$$\int_0^{k_*} f''(k)(k-S)^+ dk = \int_S^{k_*} f''(k)(k-S)\,dk = (k-S)\,f'(k)\Big|_S^{k_*} - \int_S^{k_*} f'(k)\,dk$$

$u = k-S$
$du = dk$
$dv = f''(k)$
$v = f'(k)$

$$= (k-S)f'(k)\Big|_S^{k_*} - f(k)\Big|_S^{k_*}$$

$$= (k_*-S)f'(k_*) + 0 - f(k_*) + f(S)$$

$\underline{s < k_*}$

$$f(s) = f(k_*) + f'(k_*)(s - k_*) + \int_0^{k_*} f''(k)(k-s)^+ dk + \int_{k_*}^{\infty} f''(k)(s-k)^+ dk \quad (1)$$

$$= \cancel{f(k_*)} + \cancel{f'(k_*)(s - k_*)} + \cancel{(k_* - s) f'(k_*)} - \cancel{f(k_*)} + f(s) + 0$$

$f(s) = f(s)$, $\therefore$ For $s < k_*$, equation (1) is true also

We have proven that for any $k_* > 0$ and any $s > 0$,

$$f(s) = f(k_*) + f'(k_*)(s - k_*) + \int_0^{k_*} f''(k)(k-s)^+ dk + \int_{k_*}^{\infty} f''(k)(s-k)^+ dk \quad //$$

# Problem 2

b) Put option: $\int_0^{K*} f''(K)(K-S)^+ dK$

using the Riemann Sum: $\int_0^{K*} f''(K)(K-S)^+ dK \approx \sum_{K=0}^{K*} f''(K)(K-S)^+ \Delta K$

the strikes are multiples of $5 \rightarrow \Delta K = 5$

$K_* = 1960 \rightarrow \sum_{K=0}^{1960} f''(K)(K-S)^+(5)$

Since $(K-S)^+$ is the payoff of the put, the quantity is $f''(K)(5)$

$f(S_T) = -2\log(S_T)$

$f'(S_T) = -\frac{2}{S_T}$

$f''(S_T) = -\frac{2}{S_T^2}(-1) = \frac{2}{S_T^2}$

$f''(K) = f''(1950) = \frac{2}{1950^2}$, strike 1950

Number of puts with strike 1950 to hold

$= \frac{2}{(1950)^2} \cdot 5 = 2.63 \times 10^{-6}$ //