FINM 33000: Homework 7

Due Thursday, December 5, 2024 at 11:59pm

Problem 1

Let r be the constant risk-free interest rate. Suppose that, under risk-neutral probabilities, the time-t-conditional distribution of the random variable Y_T is Normal with some mean M_t (known at time $t < T$) and variance $\sigma^2(T-t)$, where $\sigma > 0$ and T are constants.

(a) Find the time-t price of an option which pays $(Y_T - K)^+$ at time T, where K is a constant.

Hint: Let $X_T := \frac{Y_T - M_t}{\sigma\sqrt{T-t}}$. Then

$$\begin{aligned}(Y_T - K)^+ &= (Y_T - M_t)\mathbf{1}_{Y_T > K} + (M_t - K)\mathbf{1}_{Y_T > K} \\ &= (\sigma\sqrt{T-t})X_T\mathbf{1}_{X_T > \frac{K-M_t}{\sigma\sqrt{T-t}}} + (M_t - K)\mathbf{1}_{X_T > \frac{K-M_t}{\sigma\sqrt{T-t}}}\end{aligned}$$

In the first term, the fact that $\frac{1}{\sqrt{2\pi}}\int_c^\infty xe^{-x^2/2}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}e^{-c^2/2}$ (for any c) will help you find the expectation. In the second term, the expectation will involve N, the standard normal CDF. In both terms, M_t can be treated as a constant, because we are conditioning on the information available at time t.

(b) Let $r = 0$ and assume $M_t = Y_t$ (in other words, $\mathbb{E}_t Y_T = Y_t$, which is true if Y is a martingale).

For the option in (a), find its time-t delta and gamma, with respect to Y_t.

(c) Let $r = 0$ and assume $M_t = Y_t$. Assume the option in (a) is at-the-money at time t: $Y_t = K$. Find the option's time-t price. From that price, find its theta, and vega. Vega is defined as the partial derivative of the option pricing function with respect to σ.

Problem 2

(The recording of the November 27 office hours may be relevant to this problem.)

Suppose that the bank account B and a non-dividend-paying stock price S have the following dynamics under risk-neutral measure.

$$\begin{aligned}\mathrm{d}B_t &= rB_t\mathrm{d}t & B_0 &= 1 \\ \mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t & S_0 &= 216\end{aligned}$$

Let $T > 0$, and assume that $\exp(-rT) = 0.96$. Let $K = 250$ and assume that a T-expiry K-strike call on S has time-0 price 34, and a T-expiry K-strike binary call on S has time-0 price 0.44.

The expectation in (d) and probability in (e) are with respect to risk-neutral measure. Compute:

(a) The time-0 price of a T-expiry K-strike binary put on S.

(b) The time-0 price of a T-expiry K-strike put on S.

(c) The time-0 price of a T-expiry K-strike asset-or-nothing call on S.

(d) The time-0 expectation of S_T.

(e) $\mathbb{P}(S_T > K)$.

(f) A portfolio holds $\{B, S\}$ in quantities that vary continuously in time. It is self-financing and its time-T value is $(K - S_T)^+$. How many units of S does it hold at time 0?