FINM 33000: Homework 5

Due Thursday, November 7, 2024 at 11:59pm

Problem 1

Let W be a Brownian motion and let

$$Z_t = \exp(W_t^2 - 1), \qquad \text{for } t \geq 0.$$

Write Z_t in terms of drift and diffusion components. You may give your answer using differential notation, i.e. of the form

$$\mathrm{d}Z_t = \underline{\hspace{2em}}\mathrm{d}t + \underline{\hspace{2em}}\mathrm{d}W_t,$$

or using integral notation. Give two solutions (a,b):

(a) Let $X_t = W_t$, so $\mathrm{d}X_t = \mathrm{d}W_t$. Then apply Itô to $f(x) = e^{x^2-1}$.

(b) Let $X_t = W_t^2 - 1$, so $\mathrm{d}X_t = \mathrm{d}t + 2W_t\mathrm{d}W_t$. Then apply Itô to $f(x) = e^x$.

(c) Is Z a martingale?

Comment on part (b)

In (b), where X is defined by $X_t = W_t^2 - 1$, how did we know that $\mathrm{d}X_t = \mathrm{d}t + 2W_t\mathrm{d}W_t$.

One way is to use Ito's rule on the function $g(w) = w^2 - 1$, to obtain
$\mathrm{d}X_t = \mathrm{d}g(W_t) = g'(W_t)\mathrm{d}W_t + \frac{1}{2}g''(W_t)(\mathrm{d}W_t)^2 = \mathrm{d}t + 2W_t\mathrm{d}W_t$, as claimed.

Alternatively, a second way is

$$\begin{aligned}
\mathrm{d}(W_t^2 - 1) &= \mathrm{d}(W_t^2) - \mathrm{d}(1) \text{ by Fact 1 below} \\
&= \mathrm{d}(W_t^2) \text{ by Fact 2 below} \\
&= \mathrm{d}t + 2W_t\mathrm{d}W_t \text{ by L4.27}
\end{aligned}$$

Fact 1: By L4.17 together with L4.10, or alternatvely by L4.26, we have for any constants a, b, and any Ito processes F_t, G_t,

$$\boxed{\mathrm{d}(aF_t + bG_t) = a\ \mathrm{d}F_t + b\ \mathrm{d}G_t}$$

Fact 2: According to L4.10, for any constants a, b, c, we have

$$\boxed{\mathrm{d}(at + bW_t + c) = a\mathrm{d}t + b\mathrm{d}W_t}$$

and (by taking $a = b = 0$) in particular, $\mathrm{d}c = 0$ for any constant c.

Problem 2

Let W be a Brownian motion. Define X by

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where κ, θ, σ, and X_0 are all constants.

(a) Write $e^{\kappa T} X_T$ as the sum of a constant, a Riemann integral with respect to $\mathrm{d}t$, and an Itô integral with respect to $\mathrm{d}W_t$, such that both integrands may depend on t, but not on X.

Hint: As a first step, calculate $\mathrm{d}(e^{\kappa t} X_t) = \ldots$

(b) Find explicit formulas for the mean and variance of X_T.

In part (b) you may use the following fact (without providing a proof):

If β_t is a *nonrandom* piecewise continuous function of t, then

$$\boxed{\int_0^T \beta_t \mathrm{d}W_t \qquad \text{has distribution:} \qquad \text{Normal}\Big(\text{mean } 0,\ \text{variance} \int_0^T \beta_t^2 \mathrm{d}t\Big)}$$

essentially because the sum of independent normals is normal; more specifically because

$$\sum_{n=0}^{N-1} \beta_{t_n} \Delta W_{t_n} \qquad \text{has distribution:} \qquad \text{Normal}\Big(\text{mean } 0,\ \text{variance} \sum_{n=0}^{N-1} \beta_{t_n}^2 \Delta t\Big)$$

for all positive integer N, where $\Delta t := T/N$ and $t_n := n\Delta t$ and $\Delta W_{t_n} := W_{t_{n+1}} - W_{t_n}$.

Problem 3

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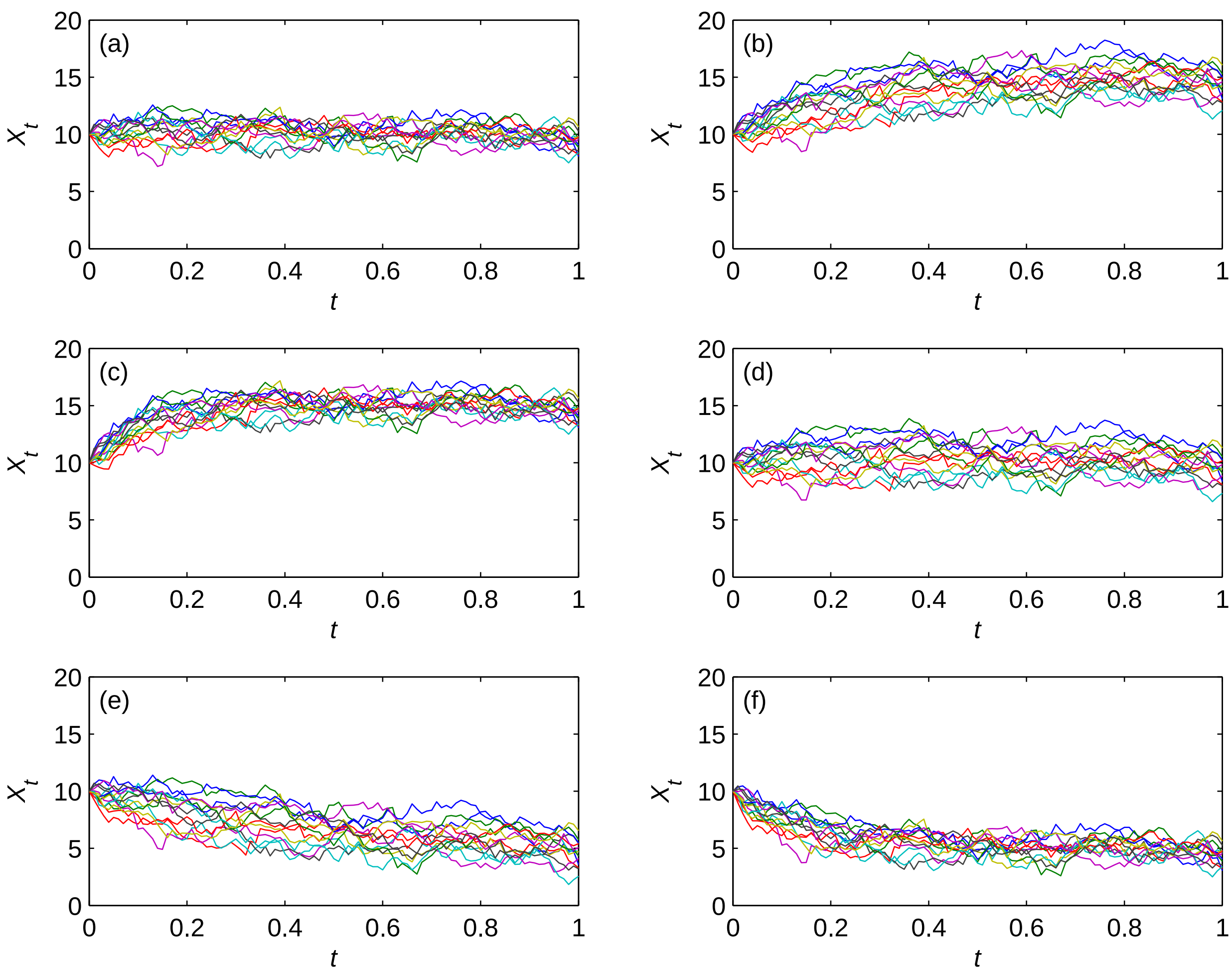
Problem 3

Here are 6 examples of the dynamics in Problem 2. All have the form

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t$$

where $\sigma = 4$ and $X_0 = 10$ in all cases. For each of the $2 \times 3 = 6$ combinations of choices $\kappa \in \{3, 8\}$ and $\theta \in \{5, 10, 15\}$, I plotted some sample paths.



For each process (a,b,c,d,e,f), state the κ and the θ that I used to generate that process. No justification necessary.

Hint: In each case the process is *mean-reverting*. Intuitively, θ is the mean reversion "level" or the "long-term mean"; and κ is the mean reversion "rate" or "speed".