Financial Mathematics 33000

Lecture 2

Roger Lee

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Binomial model specification

- Times 0 and T. No intermediate trading; all portfolios are static.
- Up and down state $\{\omega_u, \omega_d\}$ at time T, each with probability > 0.
- Bank account: Each unit has time-t value $B_t = e^{rt}$ for $t = 0, T$.
- Stock S: Let S_T take values $S_T(\omega_u) = s_u$ and $S_T(\omega_d) = s_d$ in the up and down state respectively, where $s_u > s_d$.
- "Option" contract C, paying $C_T(\omega_u) = c_u$ and $C_T(\omega_d) = c_d$.
- Nonrandom S_0, s_u, s_d, c_u, c_d, r

Exercises: Find an arbitrage

- Suppose that $S_0 e^{rT} \leq s_d$. Find an arbitrage.
- Suppose that $S_0 e^{rT} \geq s_u$. Find an arbitrage.

Option pricing via replication

Given $S_0, s_u, s_d, c_u, c_d, r$, find arbitrage-free time-0 option price C_0.

Solution: Construct portfolio (α, β) of (bank acct, stock) that replicates the option. Want $\mathsf{P}(\text{time-}T \text{ portfolio value} = C_T) = 1$.

$$\alpha e^{rT} + \beta s_u = c_u$$

beta = deltaC / deltaS
if beta = 0.12
for every dollar movement in stock price, the option moves by 0.12
You should hold 0.12 shares stock to make the stock and option moves together by 0.12
If you hold 0.12 stock, share price moves 1, option moves by 0.12 also as beta becomes 1

$$\alpha e^{rT} + \beta s_d = c_d$$

if beta = - 0.35
per one dollar increase in stock price, the option go down by 0.35
You should short 0.35 share of stock
if you short 0.35 stock, beta becomes 1

Solve for α and β:

This is delta hedging to ensure that the stock and option moves by the same amount

$$\beta = \frac{c_u - c_d}{s_u - s_d} \qquad \text{and} \qquad \alpha = e^{-rT}(c_d - \beta s_d)$$

By no-arb, time-0 option value = time-0 portfolio value. Conclude:

$$C_0 = \alpha + \beta S_0.$$

Value of replicating portfolio

Rewrite, collecting c_u and c_d terms:

$$C_0 = \alpha + \beta S_0 = e^{-rT}(c_d - \beta s_d + \beta S_0 e^{rT}) = e^{-rT}\left[c_d + \frac{c_u - c_d}{s_u - s_d}(S_0 e^{rT} - s_d)\right]$$

$$\text{Therefore } \boxed{C_0 = e^{-rT}(p_u c_u + p_d c_d)}$$

where

$$p_u := \frac{S_0 e^{rT} - s_d}{s_u - s_d}, \qquad p_d := \frac{s_u - S_0 e^{rT}}{s_u - s_d} = 1 - p_u$$

Two special cases of (c_u, c_d) are $(1,0)$ and $(0,1)$:

Special case 1:
in the up contract, it pays $1
in the down contract, it pays $0

- Let an "up-contract" U pay $U_T(\omega_u) = 1$ and $U_T(\omega_d) = 0$.

  Then time-0 up-contract value is $U_0 = e^{-rT} p_u$.

Special case 2:[
in the up contract, it pays $0
in the down contract, it pays $1

- Let a "down-contract" D pay $D_T(\omega_u) = 0$ and $D_T(\omega_d) = 1$.

  Then time-0 down-contract value is $D_0 = e^{-rT} p_d$.

Interpreting the pricing formula as a decomposition

Result $C_0 = e^{-rT}(p_u c_u + p_d c_d)$ can be understood as a decomposition.

- Example: A contract that pays 5 in the up state and 3 in the down state decomposes into 5 up-contracts plus 3 down-contracts.

So $C_T = 5U_T + 3D_T$ hence $C_0 = 5U_0 + 3D_0 = e^{-rT}(5p_u + 3p_d)$. = E(C_T)

- More generally, payment C_T of (c_u, c_d) in (up,down) states decomposes as

$$C_T = c_u U_T + c_d D_T$$

which has time-0 value

$$C_0 = c_u U_0 + c_d D_0 = c_u \times e^{-rT} p_u + c_d \times e^{-rT} p_d$$

Interpreting the pricing formula as an expectation

Result $C_0 = e^{-rT}(p_u c_u + p_d c_d)$ can be understood also as expectation:

$$C_0 = e^{-rT}\mathbb{E}C_T$$

or equivalently: $C_0/B_0 = \mathbb{E}(C_T/B_T)$ where $B_0 = 1$ and $B_T = e^{rT}$

What is the meaning of $\mathbb{E}$?

- $\mathbb{E}$ is expectation wrt the measure $\mathbb{P}$ that assigns probability p_u to up-move, and probability p_d to down-move.
- Note that p_d, p_u are > 0 and < 1, or else arbitrage exists. So $\mathbb{P}$ is indeed a probability measure.
- But $\mathbb{P}$ does not represent actual physical probabilities. No reason to think that p_u and p_d are actual probabilities.

Two probability measures

- P is called the *actual* or *physical* probability measure.

  It has *no direct relevance here*:

  Given the specification of this model, we do not care about the value of $\mathsf{P}(\text{up})$ for the purpose of pricing.

- $\mathbb{P}$ is called a *risk-neutral measure* or *martingale measure*. Some authors denote as "$\mathbb{Q}$". Important in derivatives pricing.

- Irrelevance of physical probabilities ?!

physical probabilities —> price of asset —> price of replicating assets

Two probability measures

	P measure	$\mathbb{Q}$ measure
Name	Actual / physical	Risk-neutral / martingale
Prices	$\neq e^{-rT} \times$ expected payoff	$= e^{-rT} \times$ expected payoff
Typical uses	Prediction/forecasting. Stat arb	Options/derivatives. Pure arb
Typical users	Buy-side (e.g. asset managers)	Sell-side (e.g. banks)
Other notation	$\mathbb{P}$ measure	$\mathbb{P}$ measure

(Oversimplification! Exceptions exist. And activities using both measures.)

One period, two states

The Fundamental Theorem

One-period, more discrete states

The Fundamental Theorem, again

The [first] Fundamental Theorem of Asset Pricing

$$\text{No arbitrage} \iff \begin{array}{l}\text{there exists a probability measure } \mathbb{P},\\ \text{equivalent to } \mathsf{P}, \text{such that the discounted prices}\\ \text{of all tradeable assets are martingales wrt } \mathbb{P}.\end{array}$$

Definitions:

- $\mathbb{P}$ *equivalent to* P means: for any event A, $\mathbb{P}(A) = 0$ iff $\mathsf{P}(A) = 0$.
- In this one-period model, M_t is a *martingale* means: $M_0 = \mathbb{E}M_T$.
  (Today's level equals today's expectation of tomorrow's level)
- *Discounted* price means price X divided by bank acct price: X/B

Thus, to say that the discounted price X/B is a martingale here means that $X_0/B_0 = \mathbb{E}(X_T/B_T)$; equivalently $X_0 = e^{-rT}\mathbb{E}X_T$.

Proof of Fundamental Theorem

Let's prove it in the case of the one-period binomial model, with an arbitrary number of assets, including a stock and a bank account. (True much more generally, but need technical assumptions)

- Proof that No arb $\Rightarrow$ existence of martingale measure $\mathbb{P}$:

  We proved this L2.5-L2.8. The measure $\mathbb{P}$ is, explicitly:

  $\mathbb{P}(\text{up}) = p_u$ and $\mathbb{P}(\text{down}) = 1 - p_u$, with p_u specified in L2.6.

  We need to check that $\mathbb{P}$ is a probability measure $(0 \leq p_u \leq 1)$, and indeed an *equivalent* probability measure $(0 < p_u < 1)$, which follows (how?) from no-arbitrage. With respect to that $\mathbb{P}$,

  $$X_0/B_0 = \mathbb{E}(X_T/B_T)$$

  for all assets X, by L2.8.

Proof of Fundamental Theorem

if there's no type 1 arb, there's no type 2 arb

- Proof that Existence of martingale measure $\mathbb{P} \Rightarrow$ No type-1 arb:

  Consider any static portfolio $\mathbf{\Theta}$ of assets $\mathbf{X}$. Each asset price $X_0^n = e^{-rT}\mathbb{E}X_T^n$. Multiply by quantity θ^n, then $\sum$ across assets:

  $$\mathbf{\Theta} \cdot \mathbf{X}_0 = e^{-rT}\mathbb{E}(\mathbf{\Theta} \cdot \mathbf{X}_T).$$

  So discounted *portfolio* value is also martingale: $V_0 = e^{-rT}\mathbb{E}V_T$.

  If $V_0 \neq 0$, then not arb; we're done. So take $V_0 = 0 \Rightarrow \mathbb{E}V_T = 0$.

  If $\mathbb{P}(V_T < 0) \neq 0$ then not arb; done. So take $\mathbb{P}(V_T < 0) = 0$.

  Then $\mathbb{P}(V_T > 0) = 0$, because a nonnegative, zero-expectation, random variable must vanish with probability 1. (Reason: if $\mathbb{P}(V_T > 0) > 0$, then $\mathbb{P}(V_T > \varepsilon) > 0$ for some $\varepsilon > 0$, hence $\mathbb{E}V_T \geq \varepsilon\mathbb{P}(V_T > \varepsilon) > 0$). Conclusion: $\mathbf{\Theta}$ is not a (type-1) arb. $\square$

Option pricing via the Fundamental Theorem

An alternative to pricing via replication is to use Fundamental Thm:

Basic asset prices ⇒ risk-neutral probabilities ⇒ option price

(1) Apply Fundamental Thm to S to infer risk-neutral probabilities:

$$S_0 = e^{-rT}\mathbb{E}S_T = e^{-rT}[p_u s_u + (1-p_u)s_d].$$

Solve to obtain

$$p_u = \frac{S_0 e^{rT} - s_d}{s_u - s_d}.$$

(2) Now use p_u to price the option:

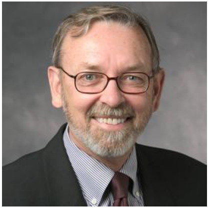
$$C_0 = e^{-rT}\mathbb{E}C_T = e^{-rT}[p_u c_u + (1-p_u)c_d]$$

Two techniques for derivatives pricing:

Build replicating portfolio, or Find $\mathbb{E}$(discounted payoff).

Fundamental Theorem of Asset Pricing

Rigorous justifications in general settings are achieved by 1981, by Michael Harrison, Stanley Pliska, and David Kreps.



One period, two states

The Fundamental Theorem

One-period, more discrete states

The Fundamental Theorem, again

Replication in two-state model

Recall: we replicated the option using $\beta := (c_u - c_d)/(s_u - s_d)$ shares.

- Match the slope by choosing the appropriate number of shares.

  Match the level using the appropriate number of bank acct units.
- Another view: For each asset, write its payoff as a vector of up-state and down-state payoffs. Replication possible because

$$\begin{pmatrix} c_u \\ c_d \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} e^{rT} \\ e^{rT} \end{pmatrix}, \begin{pmatrix} s_u \\ s_d \end{pmatrix} \right\}$$

A three-state model

- Times 0 and T. No intermediate trading; all portfolios are static.
- Up, middle, down state at time T, each with positive probability
- Bank account: Each unit has time-t value $B_t = e^{rt}$, for $t = 0, T$.
- Stock S: Let S_T take values $s_u > s_m > s_d$ in up, mid, down states respectively.
- Option C: Let C_T take values c_u, c_m, c_d in up, mid, down states.

Replication in three-state model

Example: Let $r = 0$, let $S_0 = 100, s_u = 130, s_m = 100, s_d = 80$.

Consider a 90-call: $c_u = 40, c_m = 10, c_d = 0$. Can we replicate it?

Answer:

Can replicate option on the upside by holding 1 share of S

Can replicate option on the downside by holding 0.5 shares of S

But can't simultaneously replicate both risks.

Replication and spanning

Another view: Write payoffs as vectors

Bank acct payoff $\begin{pmatrix} \text{up-payoff} \\ \text{mid-payoff} \\ \text{down-payoff} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Stock payoff $\begin{pmatrix} 130 \\ 100 \\ 80 \end{pmatrix}$

And the 90-call payoff is

$$\begin{pmatrix} 40 \\ 10 \\ 0 \end{pmatrix} \notin \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 130 \\ 100 \\ 80 \end{pmatrix} \right\}$$

Complete markets

Market is said to be *complete* if every random variable Y_T can be replicated, meaning there exists $\mathbf{\Theta}$ such that $\mathsf{P}(\mathbf{\Theta} \cdot \mathbf{X}_T = Y_T) = 1$, where $\mathbf{\Theta} := (\theta^1, \ldots, \theta^N)$ and $\mathbf{X}_T(\omega) := (X_T^1(\omega), \ldots, X_T^N(\omega))$.

Examples:

- The market $\{\text{bank acct}, \text{stock}\}$ in the 2-state model is complete.

  For every Y_T (every (c_u, c_d)) we were able to solve for $\Theta = (\alpha, \beta)$.

- The market $\{\text{bank acct}, \text{stock}\}$ in the 3-state model is *incomplete*.

  For instance $Y_T :=$ the 90-call payoff could not be replicated.

The martingale measures in this example

Probability measures on this space can be specified by probabilities (p_u, p_m, p_d). They form a martingale measure iff they are an equivalent probability measure such that A/B is MG for each asset A.

The first condition is that p_u, p_m, p_d are positive (> 0), and

$$p_u + p_m + p_d = 1.$$

The second condition is that $\mathbb{E}(S_T/B_T) = S_0/B_0$. Equivalently,

$$130p_u + 100p_m + 80p_d = 100.$$

This system has infinitely many solutions. Two examples:

here the solution is 14 —> consistent with absence of arb

here the solution is 13.5 —> consistent with absence of arb

$$(p_u, p_m, p_d) = (0.20, 0.50, 0.30) \quad \text{and} \quad (p_u, p_m, p_d) = (0.30, 0.25, 0.45)$$

40pu+10pm+0pd = 40 x 0.2 + 40x0.5 + 0 = 8+2

Martingale measure exists but is not unique.

The [first] Fundamental Theorem

The first fundamental theorem still holds in the multiple-state setting with an arbitrary number of assets (irrespective of completeness).

$$\boxed{\text{No arb} \iff \exists \text{ equivalent martingale measure } \mathbb{P}}$$

Proof of "$\Leftarrow$" is like in binomial model, but "$\Rightarrow$" is harder.

Let's just give some intuition ...

One period, two states

The Fundamental Theorem

One-period, more discrete states

The Fundamental Theorem, again

Why can't we price by taking the payoff's expectation using actual probabilities

Because people are not risk-neutral.

- Your \$10000 car has a actual 1% chance of being unrecoverably stolen this year. You may be willing to pay a lot more than \$100 to insure against this. Not because you are irrational, but because you are risk-averse.
- Consider a physically 50/50 coin flip worth \$1 million or nothing. You might rationally refuse to pay more than \$400K for this coin flip. Because each dollar in the bad state may be more precious than a dollar in the good state.

What is the risk-neutral probability of an event

It's the price of a one-unit payout contingent on the event.

Consider an event G. Consider an asset which pays:

1 bank acct unit if G occurs, otherwise 0.

Let p_G denote the time-0 price of this "G" asset, in units of B.

What can we say about p_G?

Answer: The following are consequences of no-arbitrage.

- If $\mathsf{P}(G) = 0$ then $p_G = 0$.
- If $0 < \mathsf{P}(G) < 1$ then $0 < p_G < 1$.
- If $\mathsf{P}(G) = 1$ then $p_G = 1$.

Likewise for an asset contingent on some event H.

What is the risk-neutral probability of an event

Consider disjoint events G and H. Consider an asset which pays:

$$1 \text{ bank acct unit if } G \cup H \text{ occurs, } 0 \text{ otherwise.}$$

Let $p_{G \cup H}$ denote the time-0 price of this asset, in units of B.

Then $p_{G \cup H} = ?$

Answer: Replicate this $G \cup H$ asset

by holding 1 unit of the G asset and 1 unit of the H asset.

- By law of one price, $p_{G \cup H} = p_G + p_H$.

These prices p satisfy the definition of a probability measure.

So define the risk-neutral probability of an event to be the *price* of an asset which pays: 1 bank acct unit if the event occurs, else 0.

Why can we price by taking $\mathbb{P}$-expectations

Assume J possible outcomes $\{\omega_1, \ldots, \omega_J\}$. Asset pays random Y units of B at time T. Thus payout is $Y(\omega_j)$ bank acct units, if jth outcome occurs. What's the time-0 price of the asset which pays Y?

Answer:

Replicate it by a portfolio of J assets: for each $j = 1, \ldots, J$, hold

$Y(\omega_j)$ units of the jth asset; each unit pays: 1 if ω_j occurs, else 0

Replicating portfolio's time-0 value, in units of the bank account, is

$$\sum_{j=1}^{J} \text{Quantity} \times \text{Price} = \sum_{j=1}^{J} Y(\omega_j) p_{\omega_j}$$

which is the expectation of Y wrt risk-neutral probabilities!

Why can we price by taking $\mathbb{P}$-expectations

Converting to dollars (from units of bank account),

- Let X_t be the time-t value in dollars of an asset which pays X_T dollars at time T.
- Then X_t/B_t is its time-t value, and X_T/B_T is the payout, expressed in units of the bank account.
- So, according the the previous page,

$$\frac{X_0}{B_0} = \mathbb{E}\frac{X_T}{B_T}$$

  where $\mathbb{E}$ denotes risk neutral expectation.
- (What's missing from this proof? Need to show it works even if the "basic" assets don't exist.)

Summary: why can we price using $\mathbb{P}$-expectations

Because the following actions result in identical calculations:

- Pricing:

  Take a payoff, decompose into a portfolio of 0/1 "Arrow-Debreu" or "binary" assets, and sum the quantity $\times$ price of each asset.

- Taking a $\mathbb{P}$-expectation:

  Take a random variable, decompose into its possible realizations, and sum the level $\times$ the $\mathbb{P}$-probability of each realization.

(All "prices" are relative to a designated asset, e.g. the bank account)

Summary of summary: risk-neutral pricing works because risk-neutral probabilities *are* prices (L2.28). So taking a risk-neutral expectation does the same calculation as pricing by replication (L2.29).

How are actual and risk-neutral probabilities related

- Risk-neutral probabilities $\mathbb{P}$ depend on actual probabilities P combined with the market's risk preferences
- The measures $\mathbb{P}$ and P are "equivalent" meaning that they agree on all events that have probability 0 (or probability 1).
- Again in the discrete setting with outcomes $\{\omega_1, \ldots, \omega_n\}$ each of nonzero probability, the relationship between the risk-neutral measure $\mathbb{P}$ and the actual measure P can be expressed by the "Radon-Nikodym derivative"

$$\frac{\mathbb{P}(\omega)}{\mathsf{P}(\omega)}$$

It's typically bigger in "bad" states ω, smaller in "good" states ω.

The second fundamental theorem of asset pricing

An arbitrage-free market is complete iff there exists a *unique* martingale measure.

Complete $\Rightarrow$ uniqueness: Assume one-period, J-states $\{\omega_1, \ldots, \omega_J\}$ with > 0 probabilities, N assets including a bank acct with value e^{rt}. For each $j = 1, \ldots, J$, define the *Arrow-Debreu* payoff

$$A_T^j(\omega) = \begin{cases} 1 & \text{for } \omega = \omega_j \\ 0 & \text{for } \omega \neq \omega_j \end{cases}$$

By completeness, A_T^j has a replicating portfolio $\boldsymbol{\Theta}^j$ hence a unique arbitrage-free time-0 price $\boldsymbol{\Theta}^j \cdot \mathbf{X}_0$. So for *any* martingale measure $\mathbb{P}^*$, we have $e^{-rT}\mathbb{E}^* A_T^j = \boldsymbol{\Theta}^j \cdot \mathbf{X}_0$, where $\mathbb{E}^*$ means expectation wrt $\mathbb{P}^*$. So $\mathbb{P}^*(\{\omega_j\}) = \boldsymbol{\Theta}^j \cdot \mathbf{X}_0 e^{rT}$ is *unique* MM probability of $\{\omega_j\}$.

Second fundamental theorem of asset pricing: Proof

Uniqueness $\Rightarrow$ complete: By no-arb, there exists a MM $\mathbb{P}$. Suppose market is incomplete. Let's construct a MM $\mathbb{P}^*$ different from $\mathbb{P}$.

Look for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_J)$ such that an equivalent MM is formed by

$$\mathbb{P}^*(\{\omega_j\}) := \mathbb{P}(\{\omega_j\}) + \varepsilon_j \qquad j = 1, \ldots, J.$$

Want ε such that $\mathbb{E}^*$ still prices all assets correctly. Equivalently, want

$$e^{-rT} \sum_j \varepsilon_j X_T^n(\omega_j) = 0 \qquad n = 1, \ldots, N.$$

Equivalently, want ε orthogonal to each $\mathbf{x}^n \in \mathbb{R}^J$ that represents X^n.

By incompleteness $\text{span}\{\mathbf{x}^1, \ldots, \mathbf{x}^N\} \neq \mathbb{R}^J$, so by Gram-Schmidt, there exists a unit vector $\mathbf{v}$ orthogonal to $\mathbf{x}^1, \ldots, \mathbf{x}^N$.

So $\mathbb{P}^*$ is another equivalent MM, where $\varepsilon := \frac{\min_j |\mathbb{P}(\{\omega_j\})|}{2}\mathbf{v}$. $\square$