FINM 3300 Matheus Raka Pradnyatama HW1

1) a) (-1 unit of B, 1 unit of B^*)

$V_0 = (-1)B_0 + (1)B_0^* = (-1)(1) + (1)(1) = 0$

This satisfies the type 1 arbitrage condition that $V_0 = 0$

$V_T = (-1)B_T + (1)B_T^* = (-1)e^{r \cdot T} + (1) \cdot e^{r^* T} = e^{r^* T} - e^{r \cdot T}$

Say, $r^* = 0.07$, $r = 0.05$

$V_1 = e^{r^*} - e^{r} = e^{0.07} - e^{0.05} > 0$

$V_2 = e^{2r^*} - e^{2r} = e^{2(0.07)} - e^{2(0.05)} > 0$

$\therefore V_T = e^{r^* T} - e^{r \cdot T} > 0$, for $r < r^*$

This satisfies the type 1 arbitrage condition that

$P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$

Portfolio: (-1 unit of B, 1 unit of B^*) is a type 1 arbitrage. //

1) b) Say: Long $10z$, Short $x \cdot S$, Long $y \cdot C$

$$V_0 = (10)(z_0) - x \cdot (S_0) + y(C_0)$$

$$V_0 = 10(0.9) - x \cdot 100 + y(0.5) = 0 \quad \text{so that } V_0 = 0$$

$$9 - 100x + 0.5y = 0$$

$$9 + 0.5y = 100x \quad ③$$

at time T: $z_T = 1$ (lecture 1 slide 19)

$$S_T \geq 0$$

$$C_T = \max\{S_T - K, 0\} = \max\{S_T - 110, 0\}$$

For $S_T \leq 110$, $V_T = 10 \cdot z_T - x \cdot S_T + y \cdot C_T = 10(1) - x \cdot S_T + 0$

$$V_T = 10 - x \cdot S_T$$

if $S_T = 110 \rightarrow V_T = 10 - 110x$ ①

For $S_T > 110$, $V_T = (10)(1) - x \cdot S_T + y(S_T - 110) = 10 - x \cdot S_T + y \cdot S_T - 110y$

$$V_T = 10 + (y - x)S_T - 110y \quad ②$$

We want $V_T \geq 0$: ① $V_T = 10 - 110x \geq 0$

$$110x \leq 10$$

$$x \leq \frac{1}{11} = 0.0909$$

② $V_T = 10 + (y - x)S_T - 110y \geq 0$

Since $S_T > 110$, for $V_T \geq 0$, $(y - x) \geq 0$

$$y \geq x$$

Say $x = \frac{1}{11}$, ③ $9 + 0.5y = 100(1/11)$

$$0.5y = \frac{100}{11} - 9$$

$$y = \frac{200}{11} - 18 = \frac{2}{11} = 0.1818 > \frac{1}{11}$$

Satisfies $y \geq x$

Try plugging $x = \frac{1}{11}, y = \frac{2}{11}$, for $S_T > 110$,

$V_T = 10(1) + \left(\frac{2}{11} - \frac{1}{11}\right) \cdot S_T - 110\left(\frac{2}{11}\right) = \frac{1}{11} S_T - 10 > 0$ for $S_T > 110$

If $V_T = 111$, $V_T = \frac{111}{11} - 10 = \frac{1}{11} > 0$

this satisfies the condition that $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$

Since $y + 0.5y = 100x$ is also satisfied, $V_0 = 0$

Static portfolio that is a type 1 arbitrage:

(10 units of Z, -0.0909 unit of S, 0.1818 unit of C)

Check $V_0 = 0$:

$V_0 = 10(Z_0) - 0.0909(S_0) + 0.1818(C_0)$

$V_0 = 10(0.9) - \frac{1}{11}(100) + \frac{2}{11}(0.5)$

$V_0 = 0$ → satisfies the type 1 arbitrage condition.

1) c) Suppose (1 unit of S, -1 unit of G, -1 unit of C)

V_0

- long 1 unit of S: $(1)(100) = 100$
- Short 1 unit of G: $(-1)(G_0) = (-1)(85) = -85$
- Short 1 unit of C: $(-1)C_0 = -20$

$V_0 = 100 - 85 - 20 = -5 \rightarrow V_0 < 0$ ①

at time T,

if $S_T < 110$, $V_T = (1)(S_T) - 1 \cdot G_T - 1 \cdot C_T = S_T - S_T - 0 = 0$
$V_T = 0$.

if $S_T = 110$, $V_T = (1)(S_T) - 1 \cdot G_T = S_T - 110 = 110 - 110 = 0$
$V_T = 0$

if $S_T > 110$, $V_T = (1)S_T - 1 \cdot G_T - 1 \cdot C_T = S_T - (110) - (S_T - 110)$
$V_T = 0$

this means for all S_T, $V_T = 0 \rightarrow P(V_T \geq 0) = 1$ ②

From ① and ②, we have satisfied the requirements that $V_0 < 0$, and $P(V_T \geq 0) = 1$

the portfolio: (1 unit of S, -1 unit of G, -1 unit of C) is a type 2 arbitrage. //

1) d) Suppose we are long a units of z, short 1 unit of $C(20)$, and long 1 unit of $C(25)$.

at time	0	T
z:	0.9	1
$C(20)$:	6.4	$\max(S_T-20,0)$
$C(22.5)$:	3.1	$\max(S_T-22.5,0)$
$C(25)$:	1	$\max(S_T-25,0)$

(a unit of z, -1 unit of $C(20)$, 1 unit of $C(25)$)

$V_0 = (a)(0.9)-(1)(6.4)+1\cdot(1) = 0.9a-5.4$

to make this a type 1 arbitrage: $V_0 = 0$

$0.9a-5.4=0$

$a=6$

Portfolio: {6 units of z, -1 units of $C(20)$, 1 unit of $C(25)$}

For $S_T<20$, $V_T = (6)(1)-1\cdot C_T(20)+1\cdot C_T(25)$

$V_T = 6-1(0)+1(0) = 6 > 0$

For $S_T=20$, $V_T=(6)(1)-1(0)+(1)(0)=6>0$

For $20<S_T<25$, $V_T = 6-1(S_T-20)+0 = 6-S_T+20$

$V_T = 26-S_T > 0$, for $20<S_T<25$

For $S_T=25$, $V_T = 6-1(25-20)+0 = 6-5 = 1>0$

For $S_T>25$, $V_T = 6-1(S_T-20)+1(S_T-25) = 6-\cancel{S_T}+20+\cancel{S_T}-25$

$V_T = 1>0$

we can see that for all S_T, $V_T>0$

This satisfies the conditions that $P(V_T\geq 0)=1$ and $P(V_T>0)>0$

$V_0 = (6)(0.9)-(1)(6.4)+1(1)=0$

$\rightarrow$ this satisfies the condition that $V_0=0$

the portfolio: {6 units of z, -1 unit of $C(20)$, 1 unit of $C(25)$}

is a type 1 arbitrage. //

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1)e) Suppose we are long Y and short a unit of X : $(1\ Y, -aX)$

$V_0 = Y_0 - aX_0 = 0$ → so that this is a type 1 arbitrage.

$$-10 - a \cdot 0.2 = 0$$
$$10 = -a \cdot 0.2$$
$$a = -50$$

this means we actually need to be long 50 units of X.

Proposed portfolio: (+1 unit of Y, +50 units of X)

$V_0 = (1)(-10) + 50(0.2) = -10 + 10 = 0$ → satisfies $V_0 = 0$

$V_T = (1)Y_T + 50(X_T) = (S_T - 100) + (50)\left(-2\log\left(\frac{S_T}{100}\right)\right)$

$V_T = S_T - 100 - 100\log\left(\frac{S_T}{100}\right) = S_T - 100 - 100\log(S_T) + 100\log(100)$

$S_T = 1 \rightarrow V_T = 362 > 0$

$S_T = 100 \rightarrow V_T = 0$

$S_T = 105 \rightarrow V_T = 0.121 > 0$

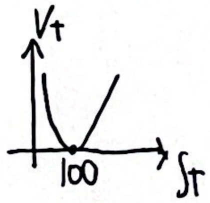
Say, $V_T = f(S_T) = S_T - 100 - 100\log(S_T) + 100\log(100)$

$$f'(S_T) = 1 - \frac{100}{S_T} = 0$$

$$1 = \frac{100}{S_T} \rightarrow S_T = 100 \rightarrow \text{critical point}$$

$$f''(S_T) = (-100)(-1)S_T^{-2} = \frac{100}{S_T^2}$$

$f''(100) = \frac{100}{100^2} > 0 \rightarrow f(S_T) = V_T$ has a local minima at $S_T = 100$



this satisfies $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$

therefore, portfolio: (1 unit of Y, 50 units of X) is a type 1 arbitrage.

Problem 2

	C_0	C_T if Trump wins	C_T if Biden wins
US. Trump	0.17	1	0
US. Biden	0.83	0	1
AZ. Biden	0.80	0	1
AZ. Trump	0.20	1	0
GA. Biden	0.56	0	1
GA. Trump	0.44	1	0
PA. Biden	0.84	0	1
PA. Trump	0.16	1	0

a) Suppose (−1 unit of US. Trump, 1 unit of PA. Trump)

$V_0 = (-1)(0.17) + (1)(0.16) = -0.01 < 0 \rightsquigarrow V_0 < 0$

at T, there are 3 scenarios:

winner of US election	winner of PA state	V_T
Trump	Trump	$V_T = (-1)(1) + (1)(1) = 0$
Biden	Biden	$V_T = (-1)(0) + (1)(0) = 0$
Biden	Trump	$V_T = (-1)(0) + (1)(1) = 1$

$\Big\} P(V_T \geq 0) = 1$

there's no scenario in which Trump wins the US election but loses in Pennsylvania.

Since $V_0 < 0$, and $P(V_T \geq 0) = 1$, the portfolio

(−1 unit of US. Trump, 1 unit of PA. Trump) is a type 2 arbitrage. //

b) Suppose (1 unit of US. Biden, 1 unit of PA. Trump, −1 unit of Bank account)

$V_0 = (1)(0.83) + (1)(0.16) - (1)(1) = -0.01 < 0$

$\hookrightarrow B_0 = B_T = 1$

at T, there are 3 scenarios

winner of US election	winner of PA state	V_T
Trump	Trump	$V_T = (1)(0) + (1)(1) - 1 \cdot 1 = 0$
Biden	Biden	$V_T = (1)(1) + (1)(0) - 1 \cdot 1 = 0$
Biden	Trump	$V_T = (1)(1) + (1)(1) - 1 \cdot 1 = 2 - 1 = 1$

$\} P(V_T \geqslant 0) = 1$

There's no scenario in which Trump wins the US election but loses in Pennsylvania.

Since $V_0 < 0$, and $P(V_T \geqslant 0) = 1$, the portfolio (1 unit of US. Biden, 1 unit of PA. Trump, −1 unit of Bank account) is a type 2 arbitrage. //

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