FINM 33000 – Homework 4: Matheus Raka Pradnyatama

Problem 1 a)

$$\Theta_t^{\text{rep}} := \begin{cases} (0 \text{ share of asset},\ 0 \text{ bonds}) & \text{if } S_t \leq K \\ (1 \text{ share of asset},\ -K \text{ bonds}) & \text{if } S_t > K \end{cases}$$

According to the proposed replicating portfolio, if $S_t > K$, we replicate the payoff by going long 1 share of asset, and going short K units of bonds.

In order to buy 1 share of asset, we need cash in the amount of S_t
We can finance this by selling (going short) K units of bonds. However, if $S_t > K$, then we don't have enough cash from selling K units of bonds to buy 1 share of asset.

Therefore, we need to borrow some cash to have 1 share of asset and $-K$ units of bonds, for the situation if $S_t > K$. This means that the time-0 value of the replicating portfolio is not zero, because we are borrowing cash to finance our proposed strategy.

Problem 1) b)

Let's say that a is the number of times that we have an upward move (+1) and $(12 - a)$ is the number of times that we have a downward move (-1) in the value of S.

At $t = 12$, the value of S is:
$S_{12} = S_0 + (1) * a + (-1)(12 - a) = 100 + a - 12 + a = 88 + 2a$

At $t = 12$, the payoff of the call option will be:
$\max(S_{12} - K, 0) = \max(88 + 2a - 105, 0) = \max(2a - 17, 0)$

For the call option to have a positive payoff, $(2a - 17) > 0$
Since $a \leq 12$, the only possible values of a is 9, 10, 11, and 12.
To calculate the total payoff, we need to sum the payoffs if a is 9, 10, 11, and 12.

Since a has binary possible outcomes (+1 and -1) and that each increment of S is independent from all other increments, we can model a using a binomial distribution.
$P\{X = k\} = \binom{n}{k} * p^k * (1 - p)^{n-k} = \binom{12}{a} * 0.4^a * (0.6)^{12-a}$

The payoff for a specific value of a will be:

$Expected\ Payoff = P\{X = a\} * (2a - 17)$

$Expected\ Payoff = \left\{\binom{12}{a} * 0.4^a * (0.6)^{12-a}\right\} * (2a - 17)$

Since we need to take into account all possible a, we need to sum all the possible expected payoffs. This will be the time-0 value of the call.

$$\sum_{a=9}^{12} \binom{12}{a} * 0.4^a * (0.6)^{12-a} * (2a - 17)$$

Problem 2 a)

If S is a martingale,

$$E(S_t) = E(S_0)$$

We know that $S_0 = 2.16$

$$E(S_t) = E(S_0) = 2.16$$

Let's denote:

S_{up} as the price of S when we reach a take-profit level of \$3

S_{down} as the price of S when we reach a stop-loss level of \$2

p is the probability that S will reach \$3 and I can exit the trade with a positive profit

$E(S_t) = p * S_{up} + (1 - p) * S_{down} = p * 3 + (1 - p) * 2 = 3p + 2 - 2p = p + 2$

$$E(S_t) = E(S_0)$$
$$p + 2 = 2.16$$
$$p = 2.16 - 2 = 0.16 = 16\%$$

The probability that I will exit the trade with a positive profit is 16%

This aligns with the intuition that since it's further to reach \$3 from \$2.16, the probability is smaller, i.e. 16%, compared to the probability that I will reach \$2, which is 84%.

Problem 2 b) $M_t = A^{S_t}$

For M to be a martingale, $E_t(M_{t+1} - M_t) = 0$

$$E(M_{t+1} | F_t) = M_t \quad (1)$$

We know that at $(t+1)$, S_t either becomes $(S_t + 0.01)$ with probability u or $(S_t - 0.01)$ with probability $(1-u)$

$$E(M_{t+1} | F_t) = u \cdot (A^{S_t + 0.01}) + (1-u) A^{S_t - 0.01}$$

From (1), $E(M_{t+1} | F_t) = M_t = A^{S_t}$, therefore,

$$A^{S_t} = u \cdot A^{S_t} \cdot (A^{0.01}) + A^{S_t - 0.01} - u \cdot A^{S_t - 0.01}$$

$$A^{S_t} = u \cdot A^{S_t} \cdot (A^{0.01}) + \frac{A^{S_t}}{A^{0.01}} - u \cdot \frac{A^{S_t}}{A^{0.01}}$$

$$1 = u \cdot A^{0.01} + \frac{(1-u)}{A^{0.01}}$$

$$A^{0.01} = u \cdot A^{0.02} + 1 - u, \text{ say } A^{0.01} = x$$

$$x = ux^2 + 1 - u$$

$$ux^2 - x + (1-u) = 0$$

$$A^{0.01} = x = \frac{1 \pm \sqrt{1 - 4u(1-u)}}{2u} = \frac{1 \pm \sqrt{1 - 4u + 4u^2}}{2u} = \frac{1 \pm (1-2u)}{2u}$$

$$A^{0.01} = \begin{cases} \frac{2-2u}{2u} = \frac{1-u}{u} & \rightarrow \quad A^{0.01} = \frac{1-u}{u} \\ \frac{1-1+2u}{2u} = 1 & \rightarrow \text{not possible, since } 0 < A < 1 \end{cases}$$

So, $A = \left(\frac{1-u}{u}\right)^{100}$ //

Problem 2) c)

Based on class notes Stochastic Calculus page 28

$$E[M_n] = E[M_0]$$

$$E[M_0] = E[A^{S_0}] = A^{S_0} = A^{2.16}$$

$$M_t = A^{S_t}$$

$$E[M_t] = E[M_0]$$

Let p be the probability that S reaches \$3.

$(1-p)$ = probability that S reaches \$2

$S_t = 3 \rightarrow$ $S_t = 2 \rightarrow$

$$p \cdot A^3 + (1-p) A^2 = A^{2.16}$$

$$A^3 \cdot p + A^2 - A^2 p = A^{2.16}$$

$$(A^3 - A^2)\, p = A^{2.16} - A^2$$

$$p = \frac{A^{2.16} - A^2}{A^3 - A^2} = \frac{A^2(A^{0.16} - 1)}{A^2(A-1)}$$

$$p = \frac{(A^{0.16} - 1)}{A-1} \quad , \text{ where } A = \left(\frac{1-u}{u}\right)^{100} \quad //$$

probability that I will exit the trade with a profit.

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