Financial Mathematics 33000

Lecture 7

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Other Dynamics: Normal

Other Dynamics: Forward prices

Other Dynamics: Dividends

Analytical or Computational

Bachelier model

Suppose underlying X has risk-neutral dynamics

$$dX_t = R_{grow} X_t dt + \sigma \, dW_t$$

Then a K-strike T-expiry call on X has time-t price

- $e^{-r(T-t)}\mathbb{E}_t(X_T - K)^+$ which can be evaluated by HW 7.1
- Equivalently, $C(X_t, t)$ where $C(X, t)$ satisfies the PDE

$$\frac{\partial C}{\partial t} + R_{grow} \, x \frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} = rC$$

with terminal condition $C(X, T) = (X - K)^+$

CME usage of Bachelier model

CME Group | Advisory Notice

TO: Clearing Member Firms
Chief Financial Officers
Back Office Managers

FROM: CME Clearing

ADVISORY #: 20-152

SUBJECT: CME Clearing Plan to Address the Potential of a Negative Underlying in Certain Energy Options Contracts

DATE: April 8th, 2020

The purpose of this advisory is to assure CME clearing firms and end clients that if major energy prices continue to fall towards zero in the coming months, CME Clearing has a tested plan to support the possibility of a negative options underlying and enable markets to continue to function normally. That plan is as follows:

- If WTI Crude Oil futures prices settle, in any month, to a price between $8.00/bbl and $11.00/bbl, CME Clearing MAY switch its pricing and margining options models from the existing models to the Bachelier model, currently utilized in numerous spread options products where negative underlying prices and strike levels are a regular occurrence. If

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Forward prices

The time-t *forward price* for time-T delivery of some underlying X_T (not necessarily a tradeable asset) is defined to be the particular level of K^* such that the forward contract on X_T with delivery price K^* and delivery date T has time-t value 0. F_t

- Forward price $\neq$ Value of forward contract

  The time-t value of a K-strike forward contract is $(F_t - K)Z_t$, because $X_T - K = (X_T - F_t) + (F_t - K)$ is replicated by 1 unit of the F_t-strike forward contract $+(F_t - K)$ units of the bond.

- Put-call parity without assuming the underlying X is tradeable:

$$\boxed{C_t - P_t = (F_t - K)Z_t}$$

  if the call, put, forward contracts all have strike K and expiry T.

Forward prices, assuming constant r

- If X is a no-dividend stock, then by solving $X_t - K^* e^{-r(T-t)} = 0$,

$$\boxed{F_t = X_t e^{r(T-t)}}$$

- For general X, by solving $e^{-r(T-t)}\mathbb{E}_t(X_T - K^*) = 0$,

$$\boxed{F_t = \mathbb{E}_t X_T}$$

  and F is a martingale with final level $F_T = X_T$. Therefore, if F follows GBM, can price options on F_T (and thus, options on X_T), using C^{BS} formula with underlying $= F_t$ and $R_{grow} = 0$.

- F_t also equals the time-t *futures price* for time-T delivery of X. (A futures contract is not the same thing as a forward contract. But futures prices = forward prices, if interest rates non-random.)

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Analytical or Computational

What is a tradeable asset?

Common language standpoint: something you can buy/sell.

Mathematical standpoint: We have already defined tradeable assets.

- A tradeable asset is just a member of the vector $\mathbf{X}$ of adapted stochastic asset price processes.
- Each trading/portfolio strategy $\boldsymbol{\Theta}$ in the assets $\mathbf{X}$ is allowed to change at any times in some designated set of trading times. At all t, the portfolio's time-t value is defined to be $V_t := \boldsymbol{\Theta}_t \cdot \mathbf{X}_t$.
- The strategy $\boldsymbol{\Theta}$ in assets $\mathbf{X}$ is defined to be self-financing if

$$\mathrm{d}V_t = \boldsymbol{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t$$

as you recall. These definitions already incorporate "tradeability".

What is a tradeable asset?

Embedded within these mathematical definitions are requirements that can be labelled as [frictionless] "tradeability":

- The ability to buy and hold arbitrary quantities at prices $\mathbf{X}$. This includes negative quantities.
- The definition that declares "self-financing (no deposits / no withdrawals)" to be equivalent to "value changes are fully attributable to asset price changes" assumes that portfolio values
  - Are not allowed to change due to transaction costs.
  - Are not allowed to change due to dividends or storage costs.

What is a tradeable asset?

Financial modeling standpoint:

- A tradeable asset is an object which satisfies [to whatever extent the financial modeler demands] the "tradeability" requirements implicit in our mathematical definitions about trading strategies.

In particular, a tradeable asset X has the following properties (or at least can be modelled as having the following properties):

- Is available to be bought or sold frictionlessly at all designated times t, at price X_t
- Can be held in arbitrary quantities (including negative), without receiving any dividends, nor incurring any costs.

Examples

X is *not* the price process of a tradeable asset for $t \in [0, T]$ if:

(ignoring trivial/exceptional circumstances)

- X_t is the time-t price of a dividend paying stock
- X_t is the forward price of a stock (dividend-paying or not)
- $X_t = S_t^2$ where S_t is a stock that follows B-S dynamics
- $X_t = (S_t - K)^+$ where S_t is a stock that follows B-S dynamics
- X_t is an interest rate
- X_t is the S&P 500 index
- X_t is the time-t temperature in this room

Examples

X *is* the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t price of a non-dividend paying stock
- X_t is the time-t price of a dividend-paying stock, *together* with all of its re-invested dividend payments since time 0.

X could be the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t value of contract which pays (only) at time T the quantity V_T, where V_T is *any* random variable (not necessarily an asset price) revealed at time $T > t$.

  Ex: X_t is the time-t value of an option on a time-T interest rate.

  Ex: X_t is the time-t value of a contract paying S_T^2 at time T.

Examples

X could be the price process of a tradeable asset for $t \in [0, T]$ if

- X_t is the time-t value of contract which pays (only) at time T the quantity V_T, where V_T is *any* random variable revealed at time T.
  Ex: X_t is the time-t value of an option on a time-T interest rate.
  Ex: X_t is the time-t value of a contract paying S_T^2 at time T.

Why "could be"?

- Maybe X is not available in the particular market, and cannot be synthesized from what is available. (But note that in derivatives pricing the typical question is: If the contract that pays V_T *would* be made available, what price would it have? So we typically treat as tradeable the derivative that we propose to introduce.)

Contrast two examples

Let S the price process of a tradeable asset.

Can the process X_t be the price process of a tradeable asset ...

- If $X_T = S_T^2$?

  Yes (if that contract is available, or if you make it available).

  But you cannot dictate that $X_t = S_t^2$ at earlier times t.

- If $X_t = S_t^2$ for all $t \in [0, T]$?

  No. Because arbitrage would arise (except in trivial cases).

Including dividend-paying stocks in our portfolios

Two approaches:

- Enlarge the mathematical theory to allow assets to have dividend/consumption streams. Need to (re)define concepts, such as "self-financing portfolio".
- Keep the mathematical theory as it is. But to apply it to some object, you must bundle the object *together* with whatever dividend/consumption stream it generates. This *bundle/package* can be considered a tradeable asset.

We take the second approach.

Stock paying continuous dividends

Let's price an option which pays $f(S_T)$ at time T, where S is the price of a stock that pays dividends to stockholders at a constant yield q.

- This means that if Q_t denotes the total dollar amount of dividend paid during $[0, t]$ by one share, then $\mathrm{d}Q_t = qS_t\mathrm{d}t$.

So S is not tradeable, but it makes sense to consider as tradeable:

- A contract which pays S_T at time T.
- Equivalently, this is a zero-delivery-price forward contract on S_T.
- Equivalently, this is a bundle/package, starting at time 0 with $A_0 := e^{-qT}$ shares, pooled with all reinvested dividends. Thus

$$\mathrm{d}A_t = A_t\mathrm{d}Q_t/S_t = A_t(qS_t\mathrm{d}t)/S_t = qA_t\mathrm{d}t$$

So at time t the bundle has $A_t = e^{-q(T-t)}$ shares. Note $A_T = 1$.

Dynamics

- Let X_t be the value of this bundle/package. We have

$$X_t = e^{-q(T-t)} S_t.$$

  Note that $X_T = S_T$; so the payoff $f(S_T)$ is identical to $f(X_T)$.

- Now assume Black-Scholes dynamics for S:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

  where W is P-BM. Then $\log X_t = -q(T-t) + \log S_t$, hence

$$\mathrm{d} \log X_t = q\mathrm{d}t + \mathrm{d} \log S_t = (\mu + q - \sigma^2/2)\mathrm{d}t + \sigma \mathrm{d}W_t$$

  hence

$$\mathrm{d}X_t = (\mu + q) X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t.$$

The replication approach

By the same replication arguments that we have already seen (L5), the option price must be $\tilde{C}(X_t, t)$ where $\tilde{C}$ satisfies the B-S PDE

$$\frac{\partial \tilde{C}}{\partial t} + rX\frac{\partial \tilde{C}}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \tilde{C}}{\partial X^2} = r\tilde{C}$$

with $\tilde{C}(X,T) = f(X)$. Let $C(S,t) := \tilde{C}(e^{-q(T-t)}S,\ t)$. Then

$$\begin{aligned}
\frac{\partial C}{\partial S} &= e^{-q(T-t)}\frac{\partial \tilde{C}}{\partial X} \\
\frac{\partial^2 C}{\partial S^2} &= e^{-2q(T-t)}\frac{\partial^2 \tilde{C}}{\partial X^2} \\
\frac{\partial C}{\partial t} &= \frac{\partial \tilde{C}}{\partial t} + qe^{-q(T-t)}S\frac{\partial \tilde{C}}{\partial X}
\end{aligned}$$

so $X\frac{\partial \tilde{C}}{\partial X} = S\frac{\partial C}{\partial S}$, and $X^2\frac{\partial^2 \tilde{C}}{\partial X^2} = S^2\frac{\partial^2 C}{\partial S^2}$, and $\frac{\partial \tilde{C}}{\partial t} = \frac{\partial C}{\partial t} - qS\frac{\partial C}{\partial S}$, where the LHS of these three equations are evaluated at $X = e^{-q(T-t)}S$.

Solution

Hence

$$\frac{\partial C}{\partial t} + (r-q)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

with terminal condition $C(S,T) = f(S)$.

- Suppose we have a call, $f(S) := (S-K)^+$.

  The PDE solution is

  $$C(S,t) = C^{BS}(S,t,K,T,r-q,r,\sigma) = e^{-r(T-t)}[FN(d_1) - KN(d_2)],$$

  where $F := Se^{(r-q)(T-t)}$ and

  $$d_{1,2} := d_\pm := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}.$$

The risk-neutral pricing / martingale approach

The bundle X has physical dynamics

$$\mathrm{d}X_t = (\mu + q)X_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t.$$

therefore risk-neutral dynamics

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t\mathrm{d}\tilde{W}_t.$$

So stock price $S_t = e^{q(T-t)}X_t$ has risk-neutral dynamics

$$\mathrm{d}S_t = (r - q)S_t\mathrm{d}t + \sigma S_t\mathrm{d}\tilde{W}_t.$$

because $\mathrm{d}\log S_t = -q\mathrm{d}t + \mathrm{d}\log X_t$. Therefore, the call option price

$$C_t = e^{-r(T-t)}\mathbb{E}_t(S_T - K)^+ = C^{BS}(S_t, t, K, T, r - q, r, \sigma)$$

Reasonable to assume continuously-paid div yields?

Depends on the underlying.

- FX: Yes usually.

  When the underlying is a foreign currency, the "dividend yield" is the interest rate paid by the foreign-currency-denominated bank account. Can model these interest payments as a continuous dividend.

- Single stocks: No.

  Dividend payments are discrete, not continuous. If a stock has ex-dividend dates on the 15th of Feb, May, Aug, Nov, with a dividend yield of 4% annual, then taking $q = 0.04$ is wrong if the pricing date = Mar 1 and expiry = Apr 20. No dividends!

Reasonable to assume continuously-paid div yields?

- Single stocks: No.

  Moreover, if we want to account for early exercise (single-stock options are usually "American-style"), then dividend timing matters (e.g., to determine whether to exercise an American call, necessary to check the day immediately before ex-div date).

- Stock index: Maybe.

  ~ 400 companies, asynchronously paying typically 4 divs/year, could be modeled as a continuous (but time-varying) yield. But rather than forecasting q to compute $F_t = S_t e^{(r-q)(T-t)}$, consider estimating $F_t = \mathbb{E}_t S_T$ directly, using put-call parity, or futures.

How to model discrete dividends?

A simple approach:

- European option on S_T is a European option on forward price F_T because $F_T = S_T$
- Model the forward price as GBM with drift 0.
- Solve for time-0 option price in terms of F_0.
  Can rewrite solution in terms of S_0.

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Analytical or Computational

Pricing and hedging

The ingredients of a derivatives pricing/hedging problem/solution:

- **Contract** to be priced/hedged
- **Dynamics** of underlying
- **Solution** approach
  - Replication or Expectation
  - Analytical or Computational

A look ahead

This quarter has focused on analytic solutions. But we do not have simple exact formulas

- when the dynamics of the risk factors are too complicated.

  For example, simple exact formulas are less common:

  ... when working with higher-dimensional models,

  ... or when outside the class of Gaussian models (BM/GBM)

- when the contract to be priced/hedged is too complicated.

  For example, due to early-exercise or path dependency.

Then we use numerical methods (trees, finite differences, Monte Carlo, Fourier, and reinforcement learning) to evaluate $\mathbb{E}$ or solve PDE.

Spring quarter!