Financial Mathematics 33000

Lecture 5

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2024 October 30-31

Arbitrage in continuous time

Black-Scholes model

B-S formula via replication

Delta, Gamma, Theta

Arbitrage

- Let prices be $\mathcal{F}_t$-adapted Itô processes $(X_t^1, \ldots, X_t^N) =: \mathbf{X}_t$.
- A *portfolio/trading strategy* is an $\mathcal{F}_t$-adapted vector process $\mathbf{\Theta}_t := (\theta_t^1, \ldots, \theta_t^N)$ of quantities held in each asset $1, \ldots, N$.
- Say that the trading strategy is *self-financing* if its value $V_t := \mathbf{\Theta}_t \cdot \mathbf{X}_t$ satisfies (with probability 1) for all t

$$\mathrm{d}V_t = \mathbf{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t, \qquad \text{equivalently } V_t = V_0 + \int_0^t \mathbf{\Theta}_u \cdot \mathrm{d}\mathbf{X}_u$$

- *Arbitrage* is a [admissible] self-financing trading strategy $\mathbf{\Theta}_t$ with

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

or

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1.$$

Replication and hedging

- Definition: a trading strategy $\mathbf{\Theta}$ *replicates* a time-T payoff Y_T if it is self-financing, and its value $V_T = Y_T$ (with probability 1).
- Law of one price: At any time $t < T$, the no-arbitrage price of an asset paying Y_T must = price of replicating portfolio (if it exists)
- To *hedge* a payoff usually means: to [try to] replicate the *negative* of the payoff (or the portion of the payoff attributable to some particular source of risk). For example, to hedge a position that is short one option usually means to [try to] replicate a position that is *long* the option. I say "try to" because "hedge" can mean an approximation to replication – such as superreplication, or broadly speaking, any strategy to reduce some notion of risk.

Motivation for GBM to model a stock price

BM is a natural starting point for model-building

- BM is the $\Delta t \to 0$ limit, in distribution, of a random walk (with zero-mean IID steps, scaled to have variance Δt).

But some problems with W_t or $\alpha t + \beta W_t$ as a model for a stock price:

- BM can go negative, and so can scaled BM with drift.
- If $\mathrm{d}S_t = \alpha \mathrm{d}t + \beta \mathrm{d}W_t$ then each $S_{t+\Delta t} - S_t$ is independent of $\mathcal{F}_t$. A 10+ dollar move is equally likely, whether S_t is at 20 or 100.

For a *GBM* S, the drift and diffusion are *proportional to* S.

- S stays positive.
- *Log return* $\log \frac{S_{t+\Delta t}}{S_t}$ (and simple return $\frac{S_{t+\Delta t} - S_t}{S_t}$) is indep of $\mathcal{F}_t$. A 10+ *percent* move is equally likely, whether S_t is at 20 or 100.

Black-Scholes model

In continuous time, consider two basic assets:

- Money-market or bank account: each unit has price $B_t = e^{rt}$.

  Equivalently, it has dynamics

  dBt = r x e^(rt) dt
  dBt = r x Bt dt

  $$\mathrm{d}B_t = rB_t\mathrm{d}t \qquad B_0 = 1$$

  purely drift term

- Non-dividend-paying stock: share price S has GBM dynamics

  $$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \qquad S_0 > 0$$

  where instantaneous *volatility* $\sigma > 0$ and W is BM wrt physical probability. Think of σ as $\sqrt{\text{Variance of log-return, per unit time}}$

Arbitrage in continuous time

Black-Scholes model

B-S formula via replication

Delta, Gamma, Theta

Replication

Find: time-t price C_t of call which pays $C_T = (S_T - K)^+$ at time T.

- Lecture 6 will use the *martingale/risk-neutral pricing* approach:

  By Fundamental Thm, take risk-neutral $\mathbb{E}$ of discounted payoff.

- Lecture 5 will price options using *replication*, two ways:

  First: an intuitive derivation, by replicating B using C and S

  Then: a careful proof, by replicating C using S and B

Fischer Black, Myron Scholes, Robert Merton

Plan of intuitive derivation: Replicate B using (C, S)

- Construct risk-free (= zero $\mathrm{d}W$ term) portfolio of (C, S).
- If self-financing, then the portfolio value's drift must be proportional, at rate r, or else there is arbitrage of portfolio vs B.
- On the other hand, if $C_t = C(S_t, t)$ for some smooth function C, then Itô says that the portfolio value's drift can be expressed in terms of C's partial derivatives.
- Therefore $C(S, t)$ satisfies a PDE.
- Solve this PDE to obtain formula for C.

Construct a risk-free portfolio

- Use (1 option, $-a_t$ share), choosing a_t to cancel the option risk.

  Portfolio value is

  $$V_t = C_t - a_t S_t.$$

- So some authors claim that

  $$\mathrm{d}V_t = \mathrm{d}C_t - a_t \mathrm{d}S_t.$$

  But the product rule says that

  $$\mathrm{d}(a_t S_t) = a_t \mathrm{d}S_t + S_t \mathrm{d}a_t + (\mathrm{d}a_t)(\mathrm{d}S_t),$$

  so it's not true that $\mathrm{d}(a_t S_t) = a_t \mathrm{d}S_t$. Ignoring this point ...

Construct a risk-free portfolio

- Assume $C_t = C(S_t, t)$ where C is some smooth function. By Itô ,

$$\mathrm{d}V_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S_t)^2 - a_t\mathrm{d}S_t$$

where C and its partials are evaluated at (S_t, t).

- Now make these cancel by choosing $a_t := \frac{\partial C}{\partial S}(S_t, t)$. Then

$$\mathrm{d}V_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S_t)^2 = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2\right)\mathrm{d}t$$

- On the other hand, V_t is the value of a risk-free portfolio, so

$$\mathrm{d}V_t = rV_t\mathrm{d}t = r\left(C_t - S_t\frac{\partial C}{\partial S}\right)\mathrm{d}t$$

- Comparing right-hand sides,

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 = rC$$

The Black-Scholes PDE and formula

- So want C to solve a PDE for $(S,t) \in [0,\infty) \times (0,T)$

$$\boxed{\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC}$$

with terminal condition $C(S,T) = (S-K)^+$ given by the payoff.

- Solution: the *Black-Scholes formula*. For $t < T$,

$$\boxed{C^{BS}(S,t) := e^{-r(T-t)}(FN(d_1) - KN(d_2))}$$

where N is the standard normal CDF, and $F := Se^{r(T-t)}$ and

$$\boxed{d_{1,2} := d_{+,-} := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}}$$

and $C^{BS}(S,T) := (S-K)^+ = \lim_{t\to T} C^{BS}(S,t)$.

Can directly check: C^{BS} solves PDE. (How to *find* C^{BS}? Later.)

How not to do stochastic calculus

What about the claim that $\mathrm{d}(C_t - a_t S_t) = \mathrm{d}C_t - a_t \mathrm{d}S_t$?

Bogus justifications:

- The share holdings a_t are "instantaneously constant."

  Nonsense. In fact a_t is changing (and, moreover, changing so fast that we needed to introduce Itô calculus).

- Portfolio of (1 option, $-a_t$ shares) is "self-financing"

  It's not. In fact there's no way to vary this portfolio's share holdings without outside funding. (The option position does not provide any funding, because it is fixed at 1 unit).

The intuitive derivation is helpful (and can be improved); it shows role of delta hedge! But it's not a proof. Let's actually give a proof now.

Black-Scholes formula: Careful proof

- Plan: replicate 1 option using a portfolio of (S, B).
- Let $C^{BS}(S,t)$ be the B-S formula.

  We are not assuming that $C^{BS}(S_t,t)$ is the option price;

  that will be the *conclusion*.
- At all $t \leq T$ let's hold quantities

  $$a_t := \frac{\partial C^{BS}}{\partial S}(S_t,t) \text{ shares}, \quad b_t := \frac{C^{BS}(S_t,t) - a_t S_t}{B_t} \text{ bank acct units}$$

  which vary continuously in time. Portfolio value is then

  $$V_t = a_t S_t + b_t B_t = a_t S_t + (C^{BS}(S_t,t) - a_t S_t) = C^{BS}(S_t,t)$$

Black-Scholes formula: Careful proof

- In particular, the final portfolio value is $C^{BS}(S_T, T) = (S_T - K)^+$
- And the portfolio self-finances, because

$$
\begin{aligned}
\mathrm{d}V_t = \mathrm{d}C^{BS}(S_t, t) &= \left(\frac{\partial C^{BS}}{\partial t} + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial S^2}\sigma^2 S_t^2\right)\mathrm{d}t + \frac{\partial C^{BS}}{\partial S}\mathrm{d}S_t \\
&= r\left(C^{BS} - S_t\frac{\partial C^{BS}}{\partial S}\right)\mathrm{d}t + \frac{\partial C^{BS}}{\partial S}\mathrm{d}S_t \\
&= a_t\mathrm{d}S_t + rb_tB_t\mathrm{d}t = a_t\mathrm{d}S_t + b_t\mathrm{d}B_t
\end{aligned}
$$

  because C^{BS} solves the PDE.

- So the portfolio replicates the option.

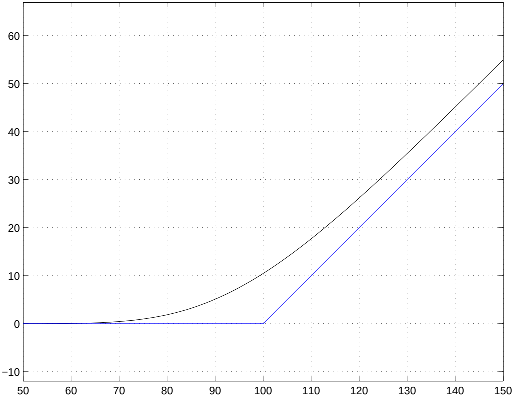
  Conclusion: at any time $t < T$, the unique no-arb price of the option equals the portfolio value, which is $C^{BS}(S_t, t)$.

Call price vs S

Let $K = 100$, $T - t = 1$, $\sigma = 0.2$, $r = 0.05$.

Call price $C^{BS}(S_t)$ and intrinsic value $:= (S_t - K)^+$

plotted against S_t



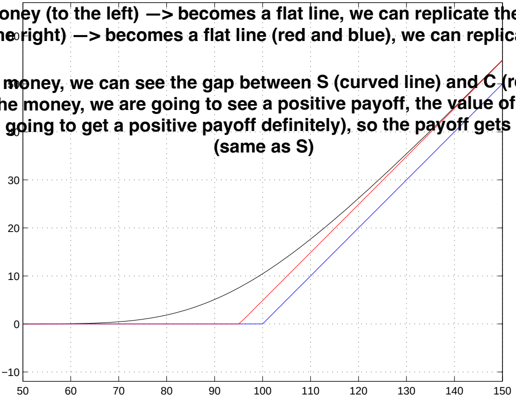
Call price vs S

Let $K = 100$, $T - t = 1$, $\sigma = 0.2$, $r = 0.05$.

Call price $C^{BS}(S_t)$ and intrinsic value $:= (S_t - K)^+$ and lower bound, plotted against S_t

as you go out of the money (to the left) —> becomes a flat line, we can replicate the option
you go in the money (to the right) —> becomes a flat line (red and blue), we can replicate the option

at the money, we can see the gap between S (curved line) and C (red line)
but as we go in the money, we are going to see a positive payoff, the value of holding an option declines (you're going to get a positive payoff definitely), so the payoff gets closer and closer (same as S)



Replication and linearity

Recall: in one-period binomial model, we replicated by holding $(c_u - c_d)/(s_u - s_d)$ shares, matching the slope of the payoff function.

In one-period three-state model, we could not replicate with a static portfolio of {bond, stock}, unless the option payoff is linear in S.

To achieve replication, we could introduce additional hedging assets, or we could go to a *multi-period* model.

Replication and linearity in continuous time

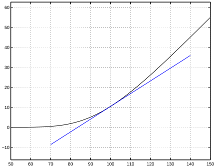
With extra nodes, the option value becomes “locally” linear in S.

At time t, match the slope, wrt $S_{t+\Delta t}$, of the possible values of $C_{t+\Delta t}$.

Slope changes in time, but that’s ok; just rebalance the portfolio.

Continuous time:

At time t match the slope, wrt $S_{t+\mathrm{d}t}$, of the possible values of $C_{t+\mathrm{d}t}$



Sensitivities or "Greeks": Delta, Gamma, Theta

Definition:

Suppose an asset or portfolio has time-t value $C_t = C(S_t, t)$.

- Its *delta*, at time-t, is $\frac{\partial C}{\partial S}(S_t, t)$.
- Its *gamma*, at time-t, is $\frac{\partial^2 C}{\partial S^2}(S_t, t)$.
- Its *theta*, at time-t, is $\frac{\partial C}{\partial t}(S_t, t)$.
- These definitions do not assume that C is a call price, and do not assume the Black-Scholes model.

In the remaining L5 slides, to get specific formulas, we do assume Black Scholes (L5.7).

Delta

For a call, in the B-S model, at time t,

$$\text{Delta} := \frac{\partial C^{BS}}{\partial S} = N(d_1) + S_t N'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = \boxed{N(d_1)}$$

recalling that $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Interpretations of delta:

- slope of option price C^{BS}, plotted as a function of S
- how much the option price moves, relative to S movements
- number of shares of S needed to replicate one option

This allows us to view the B-S call price

$$C^{BS}(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

as the value of the replicating portfolio, which consists of the value in the shares, and the value in the bank account.

Gamma

For a call, in the B-S model, at time t,

$$\text{Gamma} := \frac{\partial^2 C^{BS}}{\partial S^2} = \frac{\partial}{\partial S} N(d_1) = N'(d_1)\frac{\partial d_1}{\partial S} = \boxed{\frac{N'(d_1)}{S_t \sigma \sqrt{T-t}}}$$

Interpretations:

- convexity of C^{BS} wrt S
- how much the Delta moves, relative to S movements
- how much rebalancing the replicating portfolio needs to do, as S moves

Delta and gamma are also defined for portfolios. For N assets having time-t deltas $\mathbf{\Delta}_t \in \mathbb{R}^N$ and gammas $\mathbf{\Gamma}_t \in \mathbb{R}^N$, the portfolio $\mathbf{A}_t \in \mathbb{R}^N$ has time-t delta $\mathbf{A}_t \cdot \mathbf{\Delta}_t$ and gamma $\mathbf{A}_t \cdot \mathbf{\Gamma}_t$.

Call price

Call price $C^{BS}(S_t)$ and lower bound, plotted against S_t,

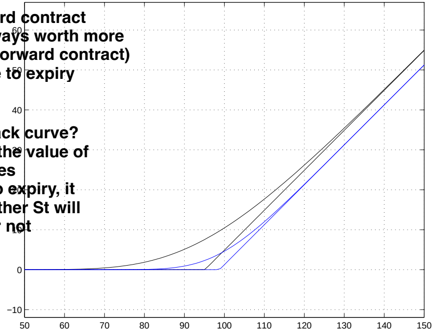
for $T-t=1$, and $T-t=0.25$.

1 year to expiry

3 months to expiry

this is call vs forward contract
call option (curve) is always worth more
than the intrinsic value (forward contract)
regardless of time to expiry

why blue curve below black curve?
as we go closer to expiry, the value of
the option declines
Why? as we get closer to expiry, it
becomes more clear whether St will
be higher than K or not

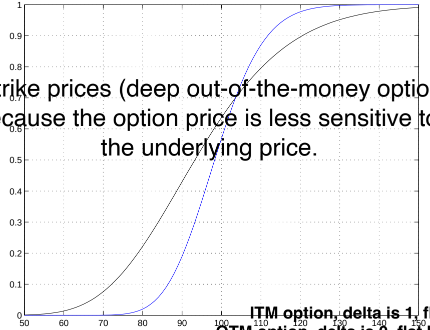


For ITM, we compare ITM call vs ITM forward
For OTM, we compare OTM call vs a zero contract (payoff is always 0) which is just a straight x-axis

higher K, option is less likely to be ITM
call price goes down
delta goes to 0

Call delta

Call delta $= N(d_1)$, plotted against S_t, for $T - t = 1$, $T - t = 0.25$.



For higher strike prices (deep out-of-the-money options), delta is closer to 0 because the option price is less sensitive to changes in the underlying price.

ITM option, delta is 1, flat line at top of curve
OTM option, delta is 0, flat line at bottom of the curve

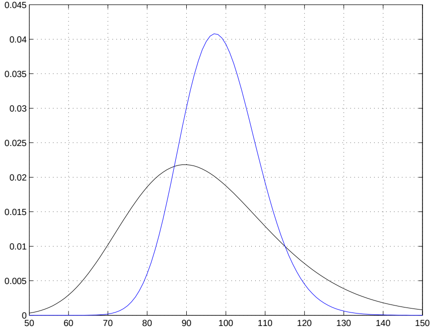
Stirike price is 100

Delta of a call is strictly between 0 and 1.

This plot how delta change at different time before expiry
blue line is if the time to expiry is 3 months
black line is if the time to expiry is 1 year
with less time to expiry, there is a higher slope (blue line)

Call gamma

Call gamma plotted against S_t, for $T - t = 1$, and $T - t = 0.25$.



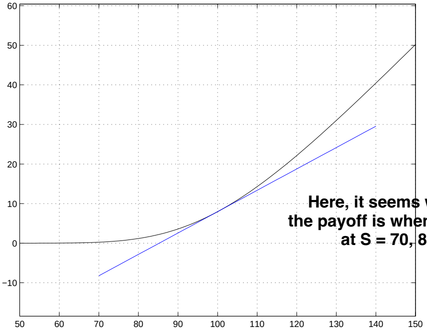
peak of gamma is higher for shorter time to expiry (blue line)

Gamma of a call is positive.

Gamma increases significantly as we get closer to strike price (100) and then decline

Discrete rebalancing

At time t, go long 1 call, short $\partial C/\partial S$ shares. Allocate the proceeds into the bank. Don't immediately rebalance. Let $r = 0$ and $S_t = 100$.



this is at time t

Here, it seems we always have positive payoff the payoff is whenever black line is above blue line at S = 70, 80 and S = 120, 130, 140, ...

Black curve (call value) minus blue line (shares + bank) = profit due to move in S. Always net positive profit?

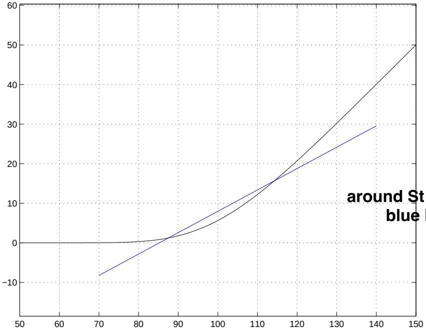
Discrete rebalancing

At time t, go long 1 call, short $\partial C/\partial S$ shares. Allocate the proceeds into the bank. Don't immediately rebalance. Let $r = 0$ and $S_t = 100$.

this is at time t + delta t

if delta hedged position moves a lot, we have positive payoff (far from 100, opposite ends)

if delta hedged position moves only a bit, we have negative payoff (around 100)



around St = 100, we are losing money blue line is above black line

Black curve (call value) minus blue line (shares + bank) = profit due to move in S. Always net positive profit? No, because of **time decay**

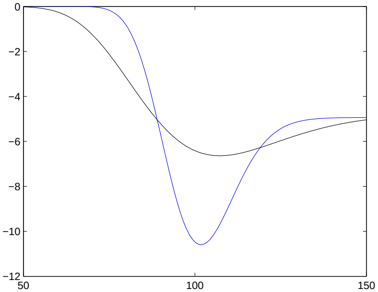
change in the call option as the time to the pricing date change

Call theta

For a call, in the B-S model, at time t,

$$\text{Theta} = \frac{\partial C^{BS}}{\partial t} = \frac{-S_t N'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Call's theta, plotted against S_t, for $T-t=1$ and $T-t=0.25$.



Greeks related to each other

BS PDE links theta, gamma, delta, and option price

theta is dc/dt
$$\boxed{\Theta + rS\Delta + \frac{1}{2}\Gamma\sigma^2 S^2 = rC}$$

(“Option” can have general time-T payoff, not necessarily call/put.)

In particular, if $r = 0$ then

$$\Theta = -\frac{1}{2}\Gamma\sigma^2 S^2$$

Discrete rebalancing and gamma

A discretely delta-hedged position that is long gamma

(meaning gamma > 0. For example: long call, short shares):

- has net profit if $|\Delta S|$ is large enough to overcome time decay

if delta hedged position moves a lot, we have positive payoff (far from 100, opposite ends)

- has net loss if $|\Delta S|$ is too small, relative to time decay

if delta hedged position moves only a bit, we have negative payoff (around 100)

A discretely delta-hedged position that is short gamma

(meaning gamma < 0. For example: short call, long shares):

- has net loss if $|\Delta S|$ is too large, relative to time decay
- has net profit if $|\Delta S|$ is small enough, relative to time decay

So such positions are sensitive to "realized volatility".

Dynamics of hedge

Delta-neutrality means delta = 0

- You have an option position, and want to trade shares to maintain delta-neutrality (delta=0).
- For which kind of options position – long gamma or short gamma – do you buy S on dips, and sell S on rallies?

This is 'gamma scalping'

Long Gamma:
If S goes up (rally), delta goes up
To maintain delta = 0, stock has to go down
as we want stock to go down, we are short the stock —> we sell the stock (during rallies)

Long Gamma:
If S goes down (dip), delta goes down
To maintain delta = 0, stock has to go up
as we want stock to go up, we are long the stock —> we buy the stock (during dip)

Short gamma position = Gamma < 0:
If S goes up, delta goes down, to maintain a delta-neutrality, must buy stock (buy when market rallies)
Each stock has a delta of +1. By adding more stock to your portfolio, we are offsetting the decline in your portfolio's delta which was caused by the market rally.

Short gamma position = Gamma < 0:
If S goes down, delta goes up, to maintain a delta-neutrality, must sell stock (sell during market dips)
by shorting more stocks, delta of portfolio will decrease and will eventually go to 0

Implied Volatility

Given a time-t price C for a given call option (K, T) on an underlying S_t assuming interest rate r, the **implied volatility** is the σ such that

$$C = C^{BS}(S_t, t, K, T, r, \sigma)$$

where C^{BS} is the Black-Scholes formula.

- This exists and is unique (if C is within arbitrage bounds).
- The bigger the dollar price C, the bigger the implied vol σ_I
- Gives us another way quoting an option price. Instead of saying the option is trading at $`x.xx`, can say it's trading at `yy`% vol.
- We will say much more about implied volatility next quarter

Realized Volatility

Realized variance of S, sampled at interval Δt, from time 0 to time T can be defined, using log-returns by letting $t_n = n\Delta t$ and $T = t_N$ and

$$RVar := \frac{1}{T}\sum_{n=0}^{N-1}\left(\log\frac{S_{t_{n+1}}}{S_{t_n}}\right)^2$$

Alternatively could subtract the sample mean from each return.

Alternatively could use simple returns, letting $\Delta S = S_{t_{n+1}} - S_{t_n}$ and

$$RVar := \frac{1}{T}\sum_{n=0}^{N-1}\left(\frac{\Delta S}{S_{t_n}}\right)^2$$

Realized volatility of S is $\boxed{RVol := \sqrt{RVar}}$.

If S follows GBM with instantaneous volatility σ, then $RVol \to \sigma$ as $\Delta t \to 0$. Thus realized volatility may be used as an estimate of σ.

PnL from Gamma Scalping

Let $r = 0$. You buy a call for $C(S_0, 0)$ where $C(S,t) := C^{BS}(S,t,\sigma_I)$.

Delta-hedge it at intervals Δt. In what cases would you profit/lose?

Taylor: $C(S + \Delta S, t + \Delta t) \approx C + (\Delta S)\frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + (\Delta t)\frac{\partial C}{\partial t}$

So your profit from t to $t + \Delta t$ is approximately

$$\begin{aligned}\frac{1}{2}\Gamma \times (\Delta S)^2 + \Theta \times \Delta t &= \frac{1}{2}\Gamma S^2 \left(\frac{\Delta S}{S}\right)^2 - \frac{1}{2}\Gamma \sigma_I^2 S^2 \Delta t \\ &= \frac{1}{2}\Gamma S^2 \left(\left(\frac{\Delta S}{S}\right)^2 - \sigma_I^2 \Delta t\right)\end{aligned}$$

Total profit from time 0 to T is

$$\sum_{n=0}^{N-1} \frac{1}{2}\Gamma_{t_n} S_{t_n}^2 \left(\left(\frac{\Delta S}{S_{t_n}}\right)^2 - \sigma_I^2 \Delta t\right)$$

Ignoring the ΓS^2, this would imply that you profit if $\boxed{RVol > \sigma_I}$.

Conclusion

Working under Black-Scholes dynamics,

- Today we priced options using *replication*,

  and we examined the behavior of the replicating portfolio.

- Next time we will price options using *martingale methods*: Apply Fundamental Thm, and take expectation of discounted payoff.