FINM 33000: Homework 4 Solutions

October 31, 2024

Problem 1

(a) If the strategy were *self-financing*, then the fact that its terminal value equals (with probability 1) the call payoff would indeed imply that the strategy replicates the call, and that the strategy's initial value equals the no-arbitrage initial value of the call.

But the strategy is not self-financing, for the following reason. With positive probability, it can happen that for some t, we have $S_{t-1} = K$ but $S_t = K + 1$. In this event, we have

$$\mathbf{\Theta}_{t-1} \cdot (S_t, 1) = (0, 0) \cdot (K + 1,\ 1) = 0,$$

which does not equal

$$\mathbf{\Theta}_t \cdot (S_t, 1) = (1, -K) \cdot (K + 1,\ 1) = 1,$$

so the portfolio requires one dollar of outside financing every time S increases from K to $K + 1$.

So the "replicating" strategy does not actually replicate, because it does not self-finance. (Note that the definition of arbitrage L3.22 and law of one price L3.23 require self-financing).

Thus we cannot conclude that the option value equals the portfolio value.

(b) Interest rates are zero, and the risk-neutral probabilities of up and down moves are each $1/2$, so the expected discounted payoff is

$$\begin{aligned}\mathbb{E}(S_{12} - 105)^+ &= 7\mathbb{P}(S_{12} = 112) + 5\mathbb{P}(S_{12} = 110) + 3\mathbb{P}(S_{12} = 108) + 1\mathbb{P}(S_{12} = 106) \\ &= \frac{1}{2^{12}}\left[7\underbrace{\binom{12}{0}}_{\mathbf{1}} + 5\underbrace{\binom{12}{1}}_{\mathbf{5x12}} + 3\underbrace{\binom{12}{2}}_{\mathbf{66}} + 1\underbrace{\binom{12}{3}}_{\mathbf{220}}\right].\end{aligned}$$

Question: suppose you are asked to find $\mathbb{E}S_{12}$, would you do it in a similar way?

Answer: you could say $\mathbb{E}S_{12} = 112\mathbb{P}(S_{12} = 112) + 110\mathbb{P}(S_{12} = 110) + \cdots + 88\mathbb{P}(S_{12} = 88) = \cdots$ but that would *not* be recommended. Instead, S discounted is a martingale, and (because $r = 0$) it follows that S is a martingale, so $\mathbb{E}S_{12} = S_0 = 100$.

Problem 2

Let's use cents rather than dollars. Let τ be the time that you exit the trade, and let p be the probability that you exit with a positive profit.

(a) By Optional Stopping theorem, $S_0 = \mathbb{E}S_\tau$ so

$$216 = S_0 = \mathbb{E}S_\tau = (1-p) \times 200 + p \times 300 = 200 + 100p.$$

Therefore $16 = 100p$ and $p = 0.16$.

(b) The condition for M to be a martingale is

$$0 = \mathbb{E}_t(M_{t+1} - M_t) = \mathbb{E}_t(A^{S_{t+1}} - A^{S_t}) = u \times A^{1+S_t} + (1-u) \times A^{-1+S_t} - A^{S_t}$$

Multiply by A^{1-S_t} to obtain

$$0 = uA^2 + (1-u) - A = (uA + u - 1)(A - 1)$$

The solution A with $0 < A < 1$ is

$$A = \frac{1-u}{u}$$

(c) By Optional Stopping theorem, $M_0 = \mathbb{E}M_\tau$, so

$$A^{216} = M_0 = \mathbb{E}M_\tau = (1-p) \times A^{200} + p \times A^{300} = A^{200} + p(A^{300} - A^{200})$$

so

$$p = \frac{1 - A^{16}}{1 - A^{100}}.$$

Note that if we express S in dollars rather than cents, then the answers to (b) and (c) would change:

$$A = \left(\frac{1-u}{u}\right)^{100}$$

and

$$p = \frac{1 - A^{0.16}}{1 - A}$$

(but the relationship between p and u would be unaffected).