FINM 33000: HW 7 Solutions

November 21, 2024

Problem 1

(a) Due to symmetry, the standard normal cdf N and density $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ satisfy

$$1 - N(x) = N(-x) \qquad \text{and} \qquad N'(x) = N'(-x)$$

for all x, which we will use below. By the hint,

$$\begin{aligned}
\mathbb{E}(Y_T - K)^+ &= \mathbb{E}(Y_T - M_t)\mathbf{1}_{Y_T > K} + \mathbb{E}(M_t - K)\mathbf{1}_{Y_T > K} \\
&= (\sigma\sqrt{T-t})\mathbb{E}X_T \mathbf{1}_{X_T > \frac{K-M_t}{\sigma\sqrt{T-t}}} + (M_t - K)\mathbb{E}\mathbf{1}_{X_T > \frac{K-M_t}{\sigma\sqrt{T-t}}} \\
&= (\sigma\sqrt{T-t})N'\Big(\frac{K - M_t}{\sigma\sqrt{T-t}}\Big) + (M_t - K)N\Big(\frac{M_t - K}{\sigma\sqrt{T-t}}\Big)
\end{aligned}$$

Therefore the call price is

$$e^{-r(T-t)}\mathbb{E}(Y_T - K)^+ = e^{-r(T-t)}\Big[(\sigma\sqrt{T-t})N'\Big(\frac{M_t - K}{\sigma\sqrt{T-t}}\Big) + (M_t - K)N\Big(\frac{M_t - K}{\sigma\sqrt{T-t}}\Big)\Big]$$

(b) The option price is $C_t = C(Y_t, t)$, where the pricing function from (a), for $r = 0$, is

$$C(Y, t) = (\sigma\sqrt{T-t})N'\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big) + (Y - K)N\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big)$$

Differentiate wrt Y to obtain the time-t delta $\Delta(Y_t, t)$ where

$$\Delta(Y, t) = N''\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big) + \frac{Y - K}{\sigma\sqrt{T-t}}N'\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big) + N\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big) = \boxed{N\Big(\frac{Y - K}{\sigma\sqrt{T-t}}\Big)}$$

where the cancellation is because $N''(x) = -xN'(x)$.

(c) From (b) we have the price $C(K, t) = \boxed{\dfrac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}}$.

Take the t-derivative of price to obtain the theta $\boxed{\dfrac{-\sigma}{2\sqrt{2\pi(T-t)}}}$

Take the σ-derivative of price to obtain the vega $\boxed{\dfrac{\sqrt{T-t}}{\sqrt{2\pi}}}$

Problem 2

(a) $0.96 - 0.44 = 0.52$

(b) Put call parity: $34 - 216 + 250 \times 0.96 = 58$

(c) $34 + 250 \times 0.44 = 144$

(d) $216e^{rT} = 216/0.96 = 225$

(e) $0.44/0.96 = 11/24$

(f) $N(d_1) - 1 = S_0 N(d_1)/S_0 - 1 = 144/216 - 1 = -1/3$