FINM 33000: Homework 5 Solutions

November 7, 2024

Problem 1

(a) Let $X_t = W_t$ and $f(x) = e^{x^2-1}$.

Then $\mathrm{d}X_t = \mathrm{d}W_t$ and $f'(x) = 2xe^{x^2-1}$ and $f''(x) = (2+4x^2)e^{x^2-1}$. So

$$\mathrm{d}Z_t = 2X_t e^{X_t^2-1}\mathrm{d}X_t + \frac{1}{2}(2+4X_t^2)e^{X_t^2-1}(\mathrm{d}X_t)^2 = (1+2W_t^2)e^{W_t^2-1}\mathrm{d}t + 2W_t e^{W_t^2-1}\mathrm{d}W_t.$$

(b) Let $X_t = W_t^2 - 1$. Then $\mathrm{d}X_t = \mathrm{d}t + 2W_t\mathrm{d}W_t$. Let $f(x) = e^x$. Then $f'(x) = f''(x) = e^x$ and

$$\begin{aligned}\mathrm{d}Z_t &= e^{X_t}\mathrm{d}X_t + \frac{1}{2}e^{X_t}(\mathrm{d}X_t)^2 = e^{X_t}(\mathrm{d}t + 2W_t\mathrm{d}W_t) + \frac{1}{2}e^{X_t}4W_t^2\mathrm{d}t \\ &= e^{X_t}(1+2W_t^2)\mathrm{d}t + 2e^{X_t}W_t\mathrm{d}W_t = (1+2W_t^2)e^{W_t^2-1}\mathrm{d}t + 2W_t e^{W_t^2-1}\mathrm{d}W_t.\end{aligned}$$

(c) Since the drift does not always vanish, Z is not a martingale by L4.12.

Problem 2

(a) Solution 1: Apply multivariable Ito (L4.26) to $f(t,x) = e^{\kappa t}x$:

$$\mathrm{d}(e^{\kappa t}X_t) = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2 = \kappa e^{\kappa t}X_t\mathrm{d}t + e^{\kappa t}\mathrm{d}X_t + \frac{1}{2}\cdot 0$$

Solution 2: Apply the Ito product rule (L4.33) to the product of $e^{\kappa t}$ and X_t, and use the fact[1] that $\mathrm{d}e^{\kappa t} = \kappa e^{\kappa t}\mathrm{d}t$:

$$\mathrm{d}(e^{\kappa t}X_t) = e^{\kappa t}\mathrm{d}X_t + X_t\mathrm{d}e^{\kappa t} + \mathrm{d}X_t\mathrm{d}e^{\kappa t} = e^{\kappa t}\mathrm{d}X_t + \kappa e^{\kappa t}X_t\mathrm{d}t + 0$$

Using either Solution 1 or Solution 2 as a first step, therefore, the next step is

$$\mathrm{d}(e^{\kappa t}X_t) = e^{\kappa t}(\kappa(\theta - X_t)\mathrm{d}t + \sigma\mathrm{d}W_t) + \kappa e^{\kappa t}X_t\mathrm{d}t = \kappa\theta e^{\kappa t}\mathrm{d}t + \sigma e^{\kappa t}\mathrm{d}W_t$$

so

$$e^{\kappa T}X_T = X_0 + \int_0^T \kappa\theta e^{\kappa t}\mathrm{d}t + \int_0^T \sigma e^{\kappa t}\mathrm{d}W_t = X_0 + \theta(e^{\kappa T} - 1) + \int_0^T \sigma e^{\kappa t}\mathrm{d}W_t$$

(b) The integrand $\sigma e^{\kappa t}$ is nonrandom and $\int_0^T \sigma^2 e^{2\kappa t}\mathrm{d}t = \frac{\sigma^2}{2\kappa}(e^{2\kappa T} - 1)$, so by the hint,

$$e^{\kappa T}X_T \sim \text{Normal}\Big(\text{mean} = X_0 + \theta(e^{\kappa T} - 1),\ \text{variance} = \frac{\sigma^2}{2\kappa}(e^{2\kappa T} - 1)\Big)$$

$$\text{Therefore } X_T \sim \text{Normal}\Big(\text{mean} = \theta + (X_0 - \theta)e^{-\kappa T},\ \text{variance} = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T})\Big)$$

[1] If a (nonrandom) function $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable then $\mathrm{d}f(t) = f'(t)\mathrm{d}t$ by Ito's rule, or simply by the Fundamental Theorem of Calculus.

Problem 3

One way to solve is to compute, using Problem 2, the mean and variance of X_1 under each of the six combinations of parameters. Another way is to use the hint's interpretations of κ and θ.

(a) $\theta = 10, \kappa = 8$

(b) $\theta = 15, \kappa = 3$

(c) $\theta = 15, \kappa = 8$

(d) $\theta = 10, \kappa = 3$

(e) $\theta = 5, \kappa = 3$

(f) $\theta = 5, \kappa = 8$