Financial Mathematics 33000

Lecture 4

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2024 October 23-24

Brownian motion and stochastic integration

Itô's rule/lemma/formula

Filtrations in continuous time

- Represent the arrival of information by a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Each $\mathcal{F}_t$ represents what has been determined at or before time t.
- If we want to specify a filtration in continuous time, we can't just list the infinitely many info sets. Instead we could designate some process(es) – such as Brownian motion – that drive the risk in the market, and use the filtration generated by these risk sources.
- Write $\{\mathcal{F}_t^Z\}_{t\geq 0}$ for the filtration *generated by* process(es) Z. This means $\mathcal{F}_t^Z$ contains all info about the history of Z through time t.
- Model asset prices as processes adapted to the filtration, so they are "functions of" Z and its history. Require trading strategies to be adapted to the filtration, so they don't "look into the future".

Brownian motion: Motivation

Let $\Delta t > 0$. Consider random walk $U_1^{\Delta t}, U_2^{\Delta t}, \ldots$ started at $U_0 = 0$, with step sizes $X_n \sim \text{Normal}(0, \Delta t)$.

Embed this in continuous time by letting $V_t := V_t^{\Delta t} = U_{\lfloor t/\Delta t \rfloor}^{\Delta t}$.

Then for $t = n\Delta t$,

variance is delta t
sum of all variances = n x delta t

$$V_t = X_1 + \cdots + X_n \sim \text{Normal}(0, t)$$

because the sum of independent normal random variables is also normal. Likewise, if $s = m\Delta t < n\Delta t = t$ then

$$V_t - V_s \sim \text{Normal}(0, t - s)$$

Brownian motion W_t is, in some sense, the limit of the random walks $V_t^{\Delta t}$ as $\Delta t \to 0$ (thus $n \to \infty$ for each fixed t).

Brownian motion: Definition

A *Brownian motion* or *Wiener process* is a stochastic process W with

- $W_0 = 0$

independent of all history of brownian motion up to and including at time s

- W has *independent increments*: $W_t - W_s$ is independent of $\mathcal{F}_s^W$ for $0 \le s < t$. This implies $\{\Delta W_{t_0}, \ldots, \Delta W_{t_{N-1}}\}$ are independent, where $\Delta W_{t_n} := W_{t_{n+1}} - W_{t_n}$ and $0 \le t_0 < t_1 < t_2 < \cdots < t_N$.
- W has *Gaussian* (*normal*) increments: if $0 \le s < t$ then

$$W_t - W_s \sim \text{Normal}(0,\ t - s)$$

~N(0, delta t)

So $\Delta W_t := W_{t+\Delta t} - W_t$ is $\sqrt{\Delta t}$ times a $N(0, 1)$ random variable. $N(m, v)$ denotes Normal(mean m, variance v) distribution.

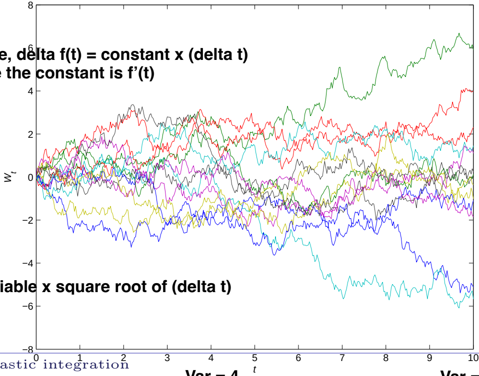
- W has continuous *sample paths* (*trajectories*): W_t is continuous in t, with probability 1.

Brownian motion: Properties

- W exists and is, in some sense, the limit of a symmetric random walk, as step size and time interval approach zero.
- W_t is a martingale (with respect to $\mathcal{F}_t^W$).
- W_t is nowhere differentiable in t, with probability 1.

Var(W_10) = 10
Var(W_10 - W_4)) = 10-4 = 6

if f is differentiable, delta f(t) = constant x (delta t)
where the constant is f'(t)



delta w(t) ~ N(0,1) variable x square root of (delta t)

Var = 4
Var = 10

Itô processes: motivation via discrete sums

Use Brownian motion as source of risk to drive Itô processes,

which can model asset prices, interest rates, etc.

Construct Itô process: Divide time interval $[0, T]$ into N periods,

of length $\Delta t := T/N$. For $n = 0, \ldots, N$,

let $t_n := n\Delta t$ be the nth time point.

Let X start at X_0 and evolve via

$X_{t_{n+1}} = X_{t_n} + \mu_{t_n}\Delta t + \sigma_{t_n}\Delta W_{t_n}$

Interpretation: new price = old price,

plus a *drift* coefficient times Δt,

plus a *diffusion* coefficient times the random shock

$$\Delta W_{t_n} := W_{t_{n+1}} - W_{t_n} \sim N(0, \Delta t)$$

Itô processes: motivation via discrete sums

Summing all the increments,

$$X_T = X_0 + \sum_{n=0}^{N-1} \mu_{t_n} \Delta t + \sum_{n=0}^{N-1} \sigma_{t_n} \Delta W_{t_n}$$

The continuous-time analogue of this sum is

$$X_0 + \int_0^T \mu_t \mathrm{d}t + \int_0^T \sigma_t \mathrm{d}W_t$$

which we will define by taking $N \to \infty$ limits of the discrete sums.

And then we will abbreviate this by writing

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

Change in X = Drift $\times$ (time increment) + diffusion $\times$ (change in W)

Itô integrals: definition

Let μ_t and σ_t satisfy integrability conditions (which this course will not require you to know), be adapted to $\mathcal{F}_t^W$, and be continuous in t (or, more generally, have one-sided continuity and two-sided limits).

Then, for each path of μ, we define the Riemann integral

$$\int_0^T \mu_t \mathrm{d}t := \lim_{N\to\infty} \sum_{n=0}^{N-1} \mu_{t_n} \Delta t$$

and define the *Itô integral* of σ with respect to W by:

$$\int_0^T \sigma_t \mathrm{d}W_t := \lim_{N\to\infty} \sum_{n=0}^{N-1} \sigma_{t_n} \Delta W_{t_n}$$

Likewise define $\int_\tau^T$ by taking $\Delta t := (T-\tau)/N$ and $t_n := \tau + n\Delta t$.

Note: This limit does not necessarily exist in a pathwise sense. It *does* exist *in* L^2 and *in probability*, which you are not expected to know.

Itô integrals: properties

Some properties:

- For σ constant, $\int_0^T \sigma \mathrm{d}W = \lim \sum \sigma \Delta W_{t_n} = \sigma(W_T - W_0) = \sigma W_T$.

  For constant μ, σ, scaled BM + drift can be written various ways:

  $$\boxed{X_t = X_0 + \mu t + \sigma W_t} = X_0 + \int_0^t \mu \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s \iff \boxed{\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t}$$

- Itô integrals are linear in the integrand: for constants a and b and processes ρ and σ,

  $$\int_0^T (a\rho_s + b\sigma_s)\mathrm{d}W_s = a\int_0^T \rho_s \mathrm{d}W_s + b\int_0^T \sigma_s \mathrm{d}W_s$$

- Itô integrals are time-additive: for $0 \le \tau \le T$,

  $$\int_0^T \sigma_s \mathrm{d}W_s = \int_0^\tau \sigma_s \mathrm{d}W_s + \int_\tau^T \sigma_s \mathrm{d}W_s$$

Itô integrals are martingales

Let $\mathrm{d}X_t = \sigma_t \mathrm{d}W_t$ or equivalently

$$X_t := X_0 + \int_0^t \sigma_s \mathrm{d}W_s.$$

Then $\boxed{X \text{ is a martingale}}$: for all $t < T$,

$$\mathbb{E}_t X_T = X_t.$$

Equivalently, $\mathbb{E}_t(X_T - X_t) = \mathbb{E}_t \int_t^T \sigma_s \mathrm{d}W_s = 0$. Idea of proof:

$$\mathbb{E}_t \mathrm{d}X_t = \mathbb{E}_t(\sigma_t \mathrm{d}W_t) = \sigma_t \mathbb{E}_t(W_{t+\mathrm{d}t} - W_t) = 0$$

A corollary: $\mathbb{E}$ of any Itô integral is zero.

Itô processes

Define an *Itô process* to be a stochastic process X of the form

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s$$

which is the sum of an initial value, a Riemann integral (the *drift* term), and an Itô integral (the *diffusion* term). Shorthand notation:

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

If X_t is an Itô process then

- X_t is continuous in t (because W is. Irrelevant whether μ, σ are).
- X_t is adapted to $\mathcal{F}_t^W$
- X_t is a martingale iff $\mu_t = 0$ for all $t > 0$, with probability 1.

  (Not sufficient: $\mathbb{E}\mu_t = 0$)

Stochastic differential equations

- Recall that in an Itô process

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

the μ_t and σ_t can depend on the entire history of W up to time t.

Solutions of Itô *stochastic differential equations* (SDE) are a subclass of Itô processes. In an SDE, the μ and σ have the form $\mu_t = \mu(X_t, t)$ and $\sigma_t = \sigma(X_t, t)$ for some functions $\mu(x, t)$ and $\sigma(x, t)$.

- Usually, specify the μ and σ functions, and define X to satisfy

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t, \qquad X_0 = \text{constant}.$$

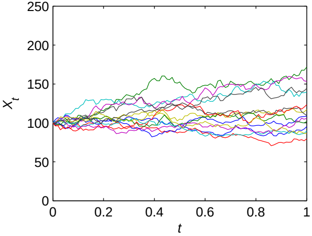
Existence and uniqueness of a *solution* X can be guaranteed by Lipschitz-type technical conditions on μ and σ.

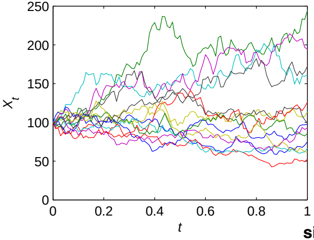
Geometric Brownian motion: $\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$

Let $X_0 = 100$. Trajectories for $\mu = -0.15, +0.15$ and $\sigma = 0.20, 0.40$:

sigma is 0.2

miu is +0.15, more positive values from 100 has an up trend lots of them finish above $100





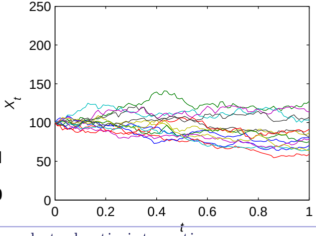
sigma is 0.4

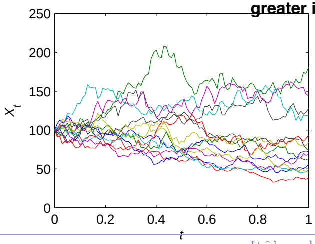
miu is +0.15

sigma is 0.4 greater variability greater instantaneous vol

sigma is 0.2

miu is -0.15, more negative values from 100 has a down trend lots of them finish below $100





sigma is 0.4

miu is -0.15

Motivation for GBM to model a stock price

BM is a natural starting point for model-building

- BM is the $\Delta t \to 0$ limit, in distribution, of a random walk (with zero-mean IID steps, scaled to have variance Δt).

But some problems with W_t or $\mu t + \sigma W_t$ as a model for a stock price:

- BM can go negative, and so can scaled BM with drift.
- If $\mathrm{d}S_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$ then each $S_{t+1} - S_t$ is independent of $\mathcal{F}_t$.

  A 10+ dollar move is equally likely, whether S_t is at 20 or 100.

For a *GBM* S, the drift and diffusion are *proportional to* S.

- S stays positive. **we can see in the 4 graphs, the stock price will never be negative**
- Each *log return* $\log(S_{t+1}/S_t)$ is independent of $\mathcal{F}_t$.

  A 10+ *percent* move is equally likely, whether S_t is at 20 or 100.

T = N x delta(t)
N = number of steps
delta(t) is the length of each step

Realized Volatility

Realized variance of S, sampled at interval Δt, from time 0 to time T can be defined, using log-returns by letting $t_n = n\Delta t$ and $T = t_N$ and

$$RVar := \frac{1}{T} \sum_{n=0}^{N-1} \left(\log \frac{S_{t_{n+1}}}{S_{t_n}} \right)^2$$

Alternatively could subtract the sample mean from each return.

Alternatively could use simple returns, letting $\Delta S = S_{t_{n+1}} - S_{t_n}$ and

$$RVar := \frac{1}{T} \sum_{n=0}^{N-1} \left(\frac{\Delta S}{S_{t_n}} \right)^2$$

Realized volatility of S is $\boxed{RVol := \sqrt{RVar}}$

If S follows GBM with instantaneous volatility σ, then $RVol \to \sigma$ as $\Delta t \to 0$. Thus realized volatilty may be used as an estimate of σ.

Integral with respect to an Itô process

- Let $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$. Define the integral of a ($\mathcal{F}_t^W$-adapted, sufficiently integrable) process θ with respect to X, as follows:

$$\int_0^t \theta_s \mathrm{d}X_s := \int_0^t \theta_s \mu_s \mathrm{d}s + \int_0^t \theta_s \sigma_s \mathrm{d}W_s$$

Shorthand:

$$\theta_t \mathrm{d}X_t = \theta_t \mu_t \mathrm{d}t + \theta_t \sigma_t \mathrm{d}W_t$$

- For vectors $\mathbf{\Theta}_t = (\theta_t^1, \ldots, \theta_t^J)$ and $\mathbf{X}_t = (X_t^1, \ldots, X_t^J)$, define

$$\int_0^t \mathbf{\Theta}_s \cdot \mathrm{d}\mathbf{X}_s := \sum_{j=1}^J \int_0^t \theta_s^j \mathrm{d}X_s^j$$

Shorthand:

$$\mathbf{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t = \sum_j \theta_t^j \mathrm{d}X_t^j$$

Example of application of stochastic integration

- Model the source of risk using a Brownian motion W.
- Model a stock price process via, for example,

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

for constants μ, σ, X_0. Equivalently,

$$X_t = X_0 + \int_0^t \mu X_s \mathrm{d}s + \int_0^t \sigma X_s \mathrm{d}W_s$$

- Then d $PNL_t = \theta_t \,\mathrm{d}X_t$ with $PNL_0 = 0$, or equivalently

$$PNL_t = \int_0^t \theta_s \mathrm{d}X_s$$

is a $P\&L$ (profit or loss) due to trading X_t according to a self-financing strategy that holds θ_s units at each time $s \in [0, t]$.

Brownian motion and stochastic integration

Itô's rule/lemma/formula

Itô's rule/lemma/formula

- Itô's rule expresses the change in $f(X_t)$ wrt t

  in terms of: f' and f'' and the change in X wrt t.

  Thus it is the *chain rule of stochastic calculus*.
- Itô's rule expresses $f(X)$ in terms of integrals of f' and f''.

  Thus it is the *fundamental theorem of stochastic calculus*.

Kiyosi Itô (1915-2008)

Itô's rule/formula/lemma: Statement

Given an Itô process X_t with dynamics $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$, find dynamics of $f(X_t)$ or $f(X_t, t)$, where f is a real-valued function.

- Example: X_t is some underlying, $f(X_t, t)$ is value of a derivative asset. Know dynamics of X_t. Want to learn dynamics of $f(X_t, t)$.

Itô's rule: If f is sufficiently smooth, then $f(X_t)$ is an Itô process and

$$\boxed{\mathrm{d}f(X_t) = \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2}$$

where the partials of f are evaluated at X_t, and $(\mathrm{d}X_t)^2$ is given by the "multiplication" rules

$$\boxed{(\mathrm{d}t)^2 = 0, \quad (\mathrm{d}W_t)(\mathrm{d}t) = 0, \quad (\mathrm{d}W_t)^2 = \mathrm{d}t.}$$

which imply $(\mathrm{d}X_t)^2 = (\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t)^2 = \sigma_t^2 (\mathrm{d}W_t)^2 = \sigma_t^2 \mathrm{d}t$.

Itô's rule: Restatement

So Itô's rule

$$\boxed{\mathrm{d}f(X_t) = \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2}$$

can be more explicitly restated as

don't memorize this
but learn how do derive this from above equation

$$\mathrm{d}f(X_t) = \left(\mu_t \frac{\partial f}{\partial x}(X_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}(X_t)\right)\mathrm{d}t + \sigma_t \frac{\partial f}{\partial x}(X_t)\mathrm{d}W_t$$

look at notes

Or in integrated form,

$$f(X_t) = f(X_0) + \int_0^t \left(\mu_s \frac{\partial f}{\partial x}(X_s) + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}(X_s)\right)\mathrm{d}s + \int_0^t \sigma_s \frac{\partial f}{\partial x}(X_s)\mathrm{d}W_s$$

Itô's rule: Idea of proof

It's just a second order Taylor expansion

$$\mathrm{d}f(X_t) = \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2$$

- If X were a differentiable function of t then $\mathrm{d}X = X'(t)\mathrm{d}t$, and $(\mathrm{d}X)^2 = [X'(t)]^2(\mathrm{d}t)^2$, negligible relative to $\mathrm{d}t$, so drop the $(\mathrm{d}t)^2$. (Thus we obtain the chain rule of ordinary calculus.)
- But if $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$,

  then $(\mathrm{d}X_t)^2$ has terms involving $\mathrm{d}W_t$, which acts like $(\mathrm{d}t)^{1/2}$.

$$\begin{array}{rlll} (\mathrm{d}t)^2 & & \ll \mathrm{d}t & \text{so drop it} \\ (\mathrm{d}t)(\mathrm{d}W_t) & = (\mathrm{d}t)^{3/2} & \ll \mathrm{d}t & \text{so drop it} \\ (\mathrm{d}W_t)^2 & = \mathrm{d}t & & \text{cannot drop} \end{array}$$

Why drop terms smaller than $\mathrm{d}t$

Intuitively, does

$$(\mathrm{d}t)^p$$

vanish?

- Answer: With $\Delta t = T/N$,

$$\begin{aligned}\int_0^T (\mathrm{d}t)^p &= \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta t)^p \\ &= \lim_{N\to\infty} N(\Delta t)^p = T \lim_{N\to\infty} (\Delta t)^{p-1}\end{aligned}$$

If $p = 1$, then this does not vanish. If $p > 1$, then this vanishes.

Why is $(\mathrm{d}W_t)^2 = \mathrm{d}t$

We know $\mathbb{E}(\Delta W)^2 = \Delta t$. Why can we delete $\mathbb{E}$ from $\mathbb{E}(\mathrm{d}W)^2 = \mathrm{d}t$?

Intuitive idea:

$$\int_0^T (\mathrm{d}W_t)^2 = \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta W_{t_n})^2$$

Let's show that this limit is T; then $\int_0^T (\mathrm{d}W_t)^2 = \int_0^T \mathrm{d}t$, as claimed.

- Expectation of the $\sum_{n=0}^{N-1}$ (which we'll write as $\sum$) is

$$\mathbb{E}\sum = \sum \mathbb{E}(\Delta W)^2 = \sum \Delta t = T$$

- Variance of the $\sum$ is

$$\begin{aligned}\operatorname{Var}\sum = \sum \operatorname{Var}(\Delta W)^2 &= \sum \left(\mathbb{E}(\Delta W)^4 - [\mathbb{E}(\Delta W)^2]^2\right) \\ &= \sum \left(3(\Delta t)^2 - (\Delta t)^2\right) \\ &= 2(\Delta t)^2 N = 2(\Delta t)T \to 0 \text{ as } N \to \infty.\end{aligned}$$

Itô's rule for $f : \mathbb{R}^2 \to \mathbb{R}$

Let X_t and Y_t be Itô processes and f be sufficiently smooth.

Then $f(X_t, Y_t)$ is an Itô process and

$$\mathrm{d}f(X_t, Y_t) = \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{\partial f}{\partial y}\mathrm{d}Y_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\mathrm{d}Y_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(\mathrm{d}X_t)(\mathrm{d}Y_t)$$

with the same "multiplication" rules as before. (For now, X and Y depend on the same W. Later, when we allow multiple Brownian motions, we will need one more multiplication rule.)

drift = 1, multiplied with dt
diffusion = 0, multiplied with dW
so here, time is an Ito's process
Y_t = t

- Important special case: $Y_t = t$. Then $\mathrm{d}Y_t = 1\mathrm{d}t + 0\mathrm{d}W_t = \mathrm{d}t$, so

$$\boxed{\mathrm{d}f(X_t, t) = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2}$$

change in option price = dt * Theta + dX * Delta + 0.5 * (dX)^2 * Gamma

Example 0: If f(t) is smooth, then df(t) = f'(t) dt

Itô's rule: Example 1

Let $Z_t := W_t^2$. Use Itô to write Z as sum of drift and diffusion terms.

Solution: $Z_t = f(X_t)$ where $f(x) := x^2$ (so $f'(x) = 2x$ and $f''(x) = 2$) and $X_t := W_t$ (so $\mathrm{d}X_t = 0\mathrm{d}t + 1\mathrm{d}W_t = \mathrm{d}W_t$). Hence

$$\mathrm{d}Z_t = \mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2 = 2W_t\mathrm{d}W_t + \frac{1}{2}2(\mathrm{d}W_t)^2$$
$$= \mathrm{d}t + 2W_t\mathrm{d}W_t$$

Or could write in integrated form:

$$Z_t = Z_0 + \int_0^t \mathrm{d}s + \int_0^t 2W_s\mathrm{d}W_s$$

$$W_t^2 = t + \int_0^t 2W_s\mathrm{d}W_s$$

a brownian motion, is not smooth
smooth = continuously differentiable

Compare: If w smooth, then $[w(t)]^2 = \int_0^t 2w(s)w'(s)\mathrm{d}s$

small w, this is not a brownian motion

Itô's rule: Example 2

Let $Z_t := W_t^3$. Use Itô to write Z as sum of drift and diffusion terms.

Solution: $Z_t = f(X_t)$ where $f(x) := x^3$ (so $f'(x) = 3x^2$, $f''(x) = 6x$) and $X_t := W_t$ (so $\mathrm{d}X_t = 0\mathrm{d}t + 1\mathrm{d}W_t = \mathrm{d}W_t$). Hence

$$\begin{aligned} \mathrm{d}Z_t &= 3W_t^2 \mathrm{d}W_t + \frac{1}{2} 6W_t (\mathrm{d}W_t)^2 \\ &= 3W_t \mathrm{d}t + 3W_t^2 \mathrm{d}W_t \end{aligned}$$

Or could write in integrated form:

$$\begin{aligned} Z_t &= Z_0 + \int_0^t 3W_s \mathrm{d}s + \int_0^t 3W_s^2 \mathrm{d}W_s \\ W_t^3 &= \int_0^t 3W_s \mathrm{d}s + \int_0^t 3W_s^2 \mathrm{d}W_s \end{aligned}$$

Sanity check:

Itô's rule: Example 3

Geometric Brownian motion S is defined by $S_0 > 0$ and the dynamics

$$\boxed{\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t}$$

where μ and *volatility* σ are constant. (For now assume such S exists and is positive.) Black-Scholes assumed GBM for stock prices.

- What are the dynamics of $\log S_t$? Solution:

  Apply Itô's rule with $f(x) := \log x$, $f'(x) = 1/x$, $f''(x) = -1/x^2$:

$$\begin{aligned}\mathrm{d}\log S_t = \mathrm{d}f(S_t) = f'(S_t)\mathrm{d}S_t + \frac{1}{2}f''(S_t)(\mathrm{d}S_t)^2 &= \frac{1}{S_t}\mathrm{d}S_t + \frac{1}{2}\frac{-1}{S_t^2}(\mathrm{d}S_t)^2 \\ &= \frac{1}{S_t}(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 (\mathrm{d}W_t)^2 \\ &= (\mu - \sigma^2/2)\mathrm{d}t + \sigma \mathrm{d}W_t\end{aligned}$$

Itô's rule: Example 3

Equivalently, $\log S_t = \log S_0 + \int_0^t (\mu - \sigma^2/2)\mathrm{d}u + \int_0^t \sigma \mathrm{d}W_u$, so

$$\boxed{\log S_t = \log S_0 + (\mu - \sigma^2/2)t + \sigma W_t}$$

- Distribution of S_t? Solution:

$$\boxed{\log S_t \sim N(\log S_0 + (\mu - \sigma^2/2)t,\ \sigma^2 t)}$$

so S_t has *lognormal* distribution. Its log (and $\log(S_t/S_0)$, the *log return*) are normal with variance $\sigma^2 t$ and standard deviation $\sigma\sqrt{t}$.

- Explicit expression for S_t in terms of W_t:

$$\boxed{S_t = e^{\log S_t} = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}}$$

which helps us to understand volatility

What does volatility 64% mean:

- It means standard deviation of $\log(S_t/S_0) = 0.64$, where $t = 1\text{y}$.
- If $t = 3\text{m}$, then the standard deviation of $\log(S_t/S_0) =$ **32%**
- If $t = 1\text{d}$, then the standard deviation of $\log(S_t/S_0) =$ **4%**

Rule of 16
for conventional market, 1 year = 252 days
for crypto market, 1 year = 365 days because crypto trades 24/7

Upper bound??

- Can standard deviation be $> 100\%$? **yes**
- Can volatility be $> 100\%$?

the std and vol of both simple and log returns can be bigger than 100%
log returns can be (-inf, inf)
std and vol can be bigger than 100% and smaller than -100%

How to estimate volatility from daily price data?

- Take sample standard deviation of daily log returns $\log \frac{S_{t_{n+1}}}{S_{t_n}}$, and annualize, by multiplying by $\sqrt{252}$ if using 252 trading days/year

Itô's rule: Example 3

- Compute $\mathbb{E}S_t$? Solution 1: If $X \sim N(m, v)$ then $\mathbb{E}e^X = e^{m+v/2}$ so

$$\boxed{\mathbb{E}S_t = S_0 e^{\mu t}}$$

Solution 2: Take expectations of both sides of

$$\begin{aligned} S_t &= S_0 + \int_0^t \mu S_u \mathrm{d}u + \int_0^t \sigma S_u \mathrm{d}W_u \\ \Rightarrow \mathbb{E}S_t &= S_0 + \int_0^t \mu \mathbb{E}S_u \mathrm{d}u + 0 \\ m(t) &= S_0 + \int_0^t \mu m(u) \mathrm{d}u \end{aligned}$$

where $m(t) := \mathbb{E}S_t$. Differentiate both sides wrt t, to get

$$m'(t) = \mu m(t) \qquad m(0) = S_0$$

Solution to this ODE: $m(t) = S_0 e^{\mu t}$

Itô's rule: Example 4

Let X_t, Y_t be Itô processes. Find the dynamics of X_tY_t.

Soln: Let $f(x,y) = xy \Rightarrow \frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial x \partial y} = 1$.

Then by Itô's rule,

$$\mathrm{d}(X_tY_t) = \mathrm{d}f(X_t, Y_t) = \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{\partial f}{\partial y}\mathrm{d}Y_t + \frac{\partial^2 f}{\partial x \partial y}(\mathrm{d}X_t)(\mathrm{d}Y_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\mathrm{d}Y_t)^2 = \boxed{Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t + (\mathrm{d}X_t)(\mathrm{d}Y_t)}$$

Intuition:

$$\mathrm{d}(XY) = (X + \mathrm{d}X)(Y + \mathrm{d}Y) - XY = X\mathrm{d}Y + Y\mathrm{d}X + (\mathrm{d}X)(\mathrm{d}Y)$$

Ordinary calculus says drop third term if X, Y are differentiable in t.

Itô calculus says keep the third term if X, Y are Itô processes.

The distinction is that $(\mathrm{d}t)(\mathrm{d}t) \ll \mathrm{d}t$ but $(\mathrm{d}W)(\mathrm{d}W) = \mathrm{d}t$.

Physical or risk-neutral probabilities?

- Question: Are the probabilities and expectations in L4 referring to physical measure or risk-neutral measure?
- Answer: Any probability measure. L4 is purely math.

  If the assumptions are with respect to physical measure, then the conclusions will be with respect to physical measure.

  If the assumptions are with respect to risk-neutral measure, then the conclusions will be with respect to risk-neutral measure.
- Analogy:

  If I say that $\mathbf{2} + \mathbf{3} = \mathbf{5}$, am I referring to 5 apples or 5 oranges?

  if i'm talking about 2 apples and 3 apples, i'm referring to 5 apples

  if i'm talking about 2 oranges and 3 oranges, i'm referring to 5 oranges

Key points

- Stochastic integration is useful in continuous-time finance:
  - Model prices as drift + integral of volatility wrt Brownian motion
  - Model profit (loss) as the stochastic integral of quantity wrt price
- GBM model: volatility (as % of price) is constant
  - Resulting log-return distributions are normal.
  - Resulting prices are lognormal (and positive).
  - Options on GBM can be priced/hedged using Black-Scholes.
- Itô's rule is the Fundamental Theorem of stochastic calculus
  - Given how S_t evolves in time, Ito shows how functions of S evolve
  - Example: S_t is a stock price, and $f(S_t, t)$ is option price process