Financial Mathematics 33000

Lecture 3

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Multi-period model

Stochastic processes

Multi-period models: Arbitrage and Fundamental Theorem

Now allow intermediate trading

Start with an example. Two periods, so three time points $t = 0, 1, 2$.

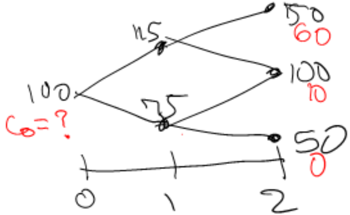
Four outcomes $\Omega = \{UU, UD, DU, DD\}$. Bank acct with $r = 0$.

Let $S_0 = 100$.

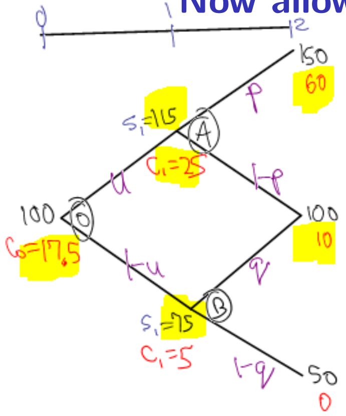
Let $S_1(UD) = S_1(UU) = 115$ and $S_1(DD) = S_1(DU) = 75$.

Let $S_2(UU) = 150$, $S_2(UD) = S_2(DU) = 100$, $S_2(DD) = 50$.

Replicate a 90-call with expiry $T = 2$? No way using a static portfolio of bank acct and stock. But suppose we allow trading at time $t = 1$.



Now allow intermediate trading



At (A):
Replication: hold (60-10)/(150-100)=1 share of stock, -90 units of bank.
Value = 1 x 115 - 90 x 1 = 25

Risk-neutral pricing:
115 = 150p+100(1-p) so p=0.3
so C = 0.3 x 60 + 0.7 x 10 = 25

At (B):

Replication: hold (10-0)/(100-50)=0.2 share of stock, and -10 units of bank
Value at B = 0.2 x 75 - 10 x 1 = 5

Risk-neutral pricing:
75 = 100q +50(1-q) so q=0.5
so C = 0.5x10+0.5x0 = 5

At (O):
Replication: hold (25-5)/(115-75)=0.5 units of stock, -32.5 units of bank.
Value at (O) is 0.5x100-32.5x1=17.5
[Skip risk-neutral approach.]

delta C / delta S will result in holding 1 share of stock = $150
we want $60 (replicate the call)
so we subtract 90 units of bank

delta C / delta S will result in holding 0.5 share of stock
= 0.5*115= 57.5,(up scenario) or
= 0.5*75= 37.5 (down scenario)
to have $25 in the up scenario or $5 in the down scenario (what we want to rreplicate is the call values)
so we subtract 32.5 units of bank
57.5 - 32.5 = $25
37.5 - 32.5 = $5

delta C / delta S will result in holding 0.2 share of stock = 0.2x$100 =$20
we want $10 (replicate the call)
so we subtract 10 units of bank

Multi-period model

Stochastic processes

Multi-period models: Arbitrage and Fundamental Theorem

Stochastic process

A stochastic process is a set of random variables, indexed by time

- Discrete time: the set of time points is countable, for example

$$X_0, X_1, X_2, X_3, \ldots$$

  or

$$X_{t_0}, X_{t_1}, X_{t_2}, X_{t_3}, \ldots$$

  where $t_0 < t_1 < t_2 < \cdots$

- Continuous time: the set of time points is an interval, for example

$$X_t,\ t \geq 0 \qquad \text{or} \qquad X_t,\ t \in [0, T]$$

Some statements that we give below will literally be true for finite sample spaces but ignore technicalities (integrability, measurability) in infinite case.

Random walk

A random walk (started at a nonrandom point S_0) is a stochastic process $S_0, S_1, S_2, \ldots$ such that

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \ldots,$ are independent and identically distributed random variables.

- A simple random walk: $S_0 = 0$ and $\mathbb{P}(X_n = 1) = p$ and $\mathbb{P}(X_n = -1) = 1 - p$, where $0 < p < 1$.
- Symmetric random walk: simple random walk with $p = 1/2$.

p is prob of downtick

$N \sim \text{Geometric}(p) \Rightarrow EN = \frac{1}{p}$

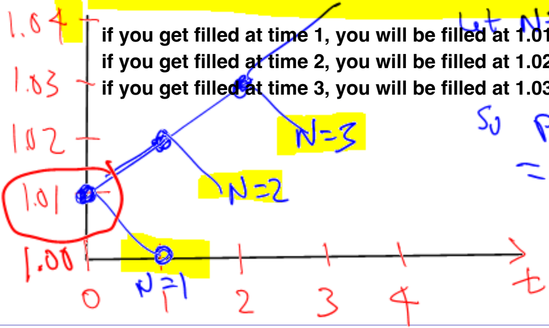
Interview question

E(N) = 1/(0.5) = 2

Here $p = \frac{1}{2}$.

A stock price S is currently 1.01. The stock can move only in steps of ± 0.01, and is a symmetric random walk. We will always put a bid (a limit order to buy) 0.01 below the current S, thus 1.00 for now. If S goes down we get filled at 1.00, if S goes to 1.02 the new bid is 1.01. What is the expectation of the price at which we finally buy the stock?

if you get filled at time 1, you will be filled at 1.01
if you get filled at time 2, you will be filled at 1.02
if you get filled at time 3, you will be filled at 1.03



Let N = # of steps that it takes for us to get filled.

So price we pay

$$= 0.99 + \frac{N}{100}$$

$$E = 0.99 + \frac{EN}{100} = 1.01$$

Filtration

In multi-period models, we want to represent the revelation of information as time passes.

- A *filtration* $\{\mathcal{F}_t : \ t \geq 0\}$ represents, for each t, all information revealed at or before time t.

Example: in the previous model,

- $\mathcal{F}_1$ is the information about whether the first step was U or D.
- $\mathcal{F}_2$ is the information about whether the first two steps were UU, UD, DU, or DD.

Filtrations

We want to represent the revelation of information as time passes.

- A *filtration* $\{\mathcal{F}_t : t \geq 0\}$ represents, for each t, all information revealed at or before time t.

Example: Flip a coin at times $1, 2, 3$.

- Sample space {`HHH`, `HHT`, `HTH`, `HTT`, `THH`, `THT`, `TTH`, `TTT`}
- $\mathcal{F}_1$ is the information available at time 1, namely whether the first flip was `H` or `T`.
- $\mathcal{F}_2$ is the information available at time 2, namely whether the first two flips were `HH`, `HT`, `TH`, or `TT`.
- $\mathcal{F}_3$ is the time-3 information, namely whether the first three flips were `HHH`, `HTH`, `THH`, `TTH`, `HHT`, `HTT`, `THT`, `TTT`

Filtrations

More precisely: at each time, the sample space is partitioned into "information sets". At that time, you know which info set you are in, but not which outcome will happen within the info set.

- The time-1 information sets are

  within an information set, we can't distinguish between the 4 possible outcomes

  $$\{\mathtt{HHH}, \mathtt{HTH}, \mathtt{HHT}, \mathtt{HTT}\}, \{\mathtt{THH}, \mathtt{TTH}, \mathtt{THT}, \mathtt{TTT}\}$$

  and $\mathcal{F}_1$ is the set of those information sets (and their unions).

- The time-2 *information sets* are

  within an information set, we can't distinguish between the 2 possible outcomes

  $$\{\mathtt{HHH}, \mathtt{HHT}\}, \{\mathtt{HTH}, \mathtt{HTT}\}, \{\mathtt{THH}, \mathtt{THT}\}, \{\mathtt{TTH}, \mathtt{TTT}\}$$

  and $\mathcal{F}_2$ is the set of those information sets (and their unions).

  at time 3, we can in granularity know the outcomes we have

- Time-3 info sets: $\{\mathtt{HHH}\}, \{\mathtt{HHT}\}, \{\mathtt{HTH}\}, \{\mathtt{HTT}\}, \{\mathtt{THH}\}, \{\mathtt{THT}\}, \{\mathtt{TTH}\}, \{\mathtt{TTT}\}$

Conditional Expectations

- You can take expectations conditional on the information available at a given time.

$$\mathbb{E}(X|\mathcal{F}_t) \quad \text{also written as} \quad \mathbb{E}_t(X)$$

  is defined to be the random variable whose value on each of the information sets A in $\mathcal{F}_t$ is $\mathbb{E}(X|A)$.

- For example, let X be the number of heads in the 3 flips. Then

$$\mathbb{E}_1(X) = \mathbb{E}(X|\mathcal{F}_1) = \begin{cases} 2 & \text{on } \{\mathtt{HHH}, \mathtt{HTH}, \mathtt{HHT}, \mathtt{HTT}\}, \\ 1 & \text{on } \{\mathtt{THH}, \mathtt{TTH}, \mathtt{THT}, \mathtt{TTT}\} \end{cases}$$

at time 1, you don't know whether you have 2H or 1H
possible outcomes are 2H and 1H

Conditional expectations

Again the notation

$$\mathbb{E}_t X := \mathbb{E}(X|\mathcal{F}_t)$$

means the conditional expectation of X, given the time-t information.

Some properties of conditional expectation:

For [integrable] random variables X, Y,

- "Taking out what's known":

  we can extract the constant, it already has certainty

  If X is measurable wrt $\mathcal{F}_t$ then $\mathbb{E}_t(XY) = X\mathbb{E}_t Y$.
- If X is independent of $\mathcal{F}_t$, then $\mathbb{E}_t X = \mathbb{E}X$.
- "Law of iterated expectations": If $s < t$ then $\mathbb{E}_s(\mathbb{E}_t X) = \mathbb{E}_s X$.

Let's assume that $\mathcal{F}_0$ is trivial. So $\mathbb{E}_0$ is the same thing as $\mathbb{E}$.

Adapted processes

A stochastic process Y is *adapted* to $\{\mathcal{F}_t\}$, if Y_t is $\mathcal{F}_t$-measurable for each t, meaning the value of Y_t is determined by the information in $\mathcal{F}_t$.

This means that Y_t is constant on each information set of $\mathcal{F}_t$.

For instance, in option pricing theory, we:

- Construct our models so that asset prices X_t are adapted to $\mathcal{F}_t$.

  Interpretation: At time t the market has revealed the price X_t.
- Define our trading strategies to require that the quantities θ_t be adapted to $\mathcal{F}_t$.

  Interpretation: Allow trading, but determined only by what has been revealed, not by future outcomes.

If E_t(M_T - M_t) > 0, we have an 'up' trend
If E_t(M_T - M_t) < 0, we have an 'down' trend

Martingales

We say M_t is a *martingale* with respect to a filtration $\{\mathcal{F}_t\}$

(if $\mathcal{F}_t$ unspecified, then assume filtration consisting of history of M)

if: M_t is adapted to $\{\mathcal{F}_t\}$, and for all t and all T with $0 \leq t < T$,

$$\boxed{\mathbb{E}_t M_T = M_t}$$

with probability 1. Other ways to say this:

- "Today's expectation of tomorrow's level is today's level"
- No "drift". No "trend": $\mathbb{E}_t(M_T - M_t) = 0$ $\quad \mathbb{E}_t(M_{t+1} - M_t) = 0$
- In discrete-time $(t = 0, 1, 2 \ldots)$ models, $\mathbb{E}_t M_{t+1} = M_t$ is enough

Example: Let S be a simple random walk. Is S a martingale?

No, if $p \neq \frac{1}{2}$. Yes if $p = \frac{1}{2}$ (symmetric random walk)

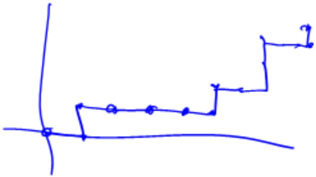
Martingales

as we flip the coin, the number of H can only increase over time

- Let X_t be the number of heads in flips 1 through t. of a fair coin.

  Is X a martingale? No.

$$\mathbb{E}_t(X_{t+1} - X_t) = +0.5$$



- Let $X_0 = 0$. Let $\mathsf{P}(X_1 = 1) = \mathsf{P}(X_1 = -1) = 0.5$.

  Let $X_t = tX_1$ for $t = 2, 3, 4, \ldots$.

  Is $\mathbb{E}X_T = X_0$ for all T? Is X a martingale?

  skip

Optional stopping theorem

- If M is a martingale and $S \leq T$ are bounded stopping times then

$$\mathbb{E}_S M_T = M_S$$

- Still true if S, T are unbounded, provided that M is *uniformly integrable* (UI). Sufficient conditions for UI (either):

  There exists Y such that $\mathbb{E}|Y| < \infty$ and for all t, $|M_t| \leq Y$.

  There exists Z such that $\mathbb{E}|Z| < \infty$ and for all t, $M_t = \mathbb{E}_t Z$.

- Still true if M_n is discrete sum of IID and $\mathbb{E}T < \infty$.

In general $\mathbb{E}M_T = M_0$ can fail: Let $M_t = \exp(W_t - t/2)$.

Let $S = 0$ and let T be the first time that M hits 0.1. (Such time exists with probability 1, because $\log M_t$ is Normal with mean $-t/2$ and standard deviation $\sqrt{t}$.) Then $\mathbb{E}M_T = 0.1 \neq M_0 = 1$.

Optional stopping theorem: Random walk

A drunken guy is trying to cross a 100ft bridge. He has crossed 23ft. The probabilities of going one step forward and one step backward are equal. The steps have equal length. Find the probability p he ends up at the bridge's other side before returning to the bridge's starting point.

Solution:

Let T be his exit time from [either end of] the interval $(0, 100)$.

His time-t position M_t is a bounded (hence UI) martingale.

Then ...

0 23 100 103

$$23 = M_0 = \mathbb{E} M_T = p \times 100 + (1-p) \times 0$$

assuming step $= \pm 1$ foot

$$p = \frac{23}{100}$$

Multi-period model

Stochastic processes

Multi-period models: Arbitrage and Fundamental Theorem

If M_t is martingale

$X_t := M_{t \wedge \tau}$ is also a martingale.

Trading strategy

A *trading strategy* on $t = 0, 1, \ldots, T$ is a sequence $\mathbf{\Theta}_t$ adapted to $\mathcal{F}_t$.

Let us agree to view $\mathbf{\Theta}_t$ as the vector of quantities of the tradeable assets held *after* all time-t trading at prices $\mathbf{X}_t$.

Say that the trading strategy is *self-financing* if for all $t > 0$,

$$\mathbf{\Theta}_{t-1} \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot \mathbf{X}_t$$

with probability 1.

This implies that the change in portfolio value from time t to $t+1$ is

$$V_{t+1} - V_t = \mathbf{\Theta}_{t+1} \cdot \mathbf{X}_{t+1} - \mathbf{\Theta}_t \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot \mathbf{X}_{t+1} - \mathbf{\Theta}_t \cdot \mathbf{X}_t = \mathbf{\Theta}_t \cdot (\mathbf{X}_{t+1} - \mathbf{X}_t)$$

So the change in value is fully attributable to

$$\sum \text{quantity} \times (\text{gain or loss in asset price})$$

Total P&L

Note that we can sum from $t = 0$ to $t = T - 1$

$$V_T - V_0 = \sum_{t=0}^{T-1} \mathbf{\Theta}_t \cdot (\mathbf{X}_{t+1} - \mathbf{X}_t)$$

This looks like a discrete version of the stochastic integral

$$\int_0^T \mathbf{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t$$

Idea: P&L from a self-financing trading strategy is a stochastic integral, namely the integral of quantity with respect to price.

Arbitrage in a multi-period model

Arbitrage is a self-financing trading strategy $\boldsymbol{\Theta}_t$ whose value $V_t := \boldsymbol{\Theta}_t \cdot \mathbf{X}_t$ satisfies

$$V_0 = 0 \quad \text{and both:} \quad \begin{array}{l} \mathsf{P}(V_T \geq 0) = 1 \\ \mathsf{P}(V_T > 0) > 0 \end{array}$$

or

$$V_0 < 0 \quad \text{and} \quad \mathsf{P}(V_T \geq 0) = 1$$

Note that static portfolios are a special case of self-financing trading strategies, so the previous definition is consistent with this one. This definition extends the notion of arbitrage beyond static strategies, to self-financing ones.

Properties of arbitrage-free prices

In the absence of arbitrage, the following properties hold:

- If Θ_t^a and Θ_t^b are self-financing, and if $\mathsf{P}(V_T^a \geq V_T^b) = 1$, then $V_0^a \geq V_0^b$, or else arbitrage would exist, namely $\Theta_t^a - \Theta_t^b$.
- "Law of one price": If Θ_t^a and Θ_t^b are self-financing, and if $\mathsf{P}(V_T^a = V_T^b) = 1$, then $V_0^a = V_0^b$, or else arbitrage would exist.

By defnition, we say that a replicates b

Fundamental Thm of Asset Pricing: multi-period

The fundamental theorem still holds in the multi-period case.

No arb $\Rightarrow \underbrace{\exists}_{\text{there exist}}$ equivalent martingale measure $\mathbb{P}$:

Idea: In each one-period sub-tree,

apply the one-period Fundamental Theorem to get conditional risk-neutral probabilities of each single step. Define the probability of each path to be the product of conditional probabilities of the path's individual steps. Can verify, that this is a martingale measure.

Fundamental Thm of Asset Pricing: multi-period

$\exists$ equivalent martingale measure $\mathbb{P} \Rightarrow$ No arb:

We are given that $\tilde{\mathbf{X}}_t := B_t^{-1}\mathbf{X}_t$ is a vector of martingales under $\mathbb{P}$.

For self-financing $\boldsymbol{\Theta}$, then, $V_t := \boldsymbol{\Theta}_t \cdot \tilde{\mathbf{X}}_t$ is also a martingale because

$\mathbb{E}_t(V_{t+1} - V_t) = \mathbb{E}_t \sum_n \theta_t^n \cdot (\tilde{X}_{t+1}^n - \tilde{X}_t^n) = \sum_n \theta_t^n \cdot \mathbb{E}_t(\tilde{X}_{t+1}^n - \tilde{X}_t^n) = 0$

(Intuition: a martingale is a wealth process generated by playing zero-expectation games. Varying the amounts θ_t^n bet, across games n and times t, still makes collectively a zero-expectation game.)

The martingale property implies $V_0/B_0 = \mathbb{E}(V_T/B_T)$, so the reasoning we gave in the one-period case again shows that no arbitrage exists.

(You can't risklessly make money by playing zero-expectation games!)

Incomplete market

In an incomplete market (example: the one-period "trinomial" model with 3 possible outcomes and two assets B, S):

- From a replication standpoint: Some payoffs can't be replicated. (In the trinomial example, the only replicable payoffs are linear combinations of 1 and S_T, or equivalently, affine functions of S_T). For payoffs having no replicating portfolio, no-arbitrage alone may not be able to determine a unique price for the payoff.
- From the martingale / risk-neutral valuation standpoint: There are many martingale measures consistent with the prices of the basic assets. Different martingale measures can give different valuations for a derivative asset's payoff.

Completing a market

If we change the assumptions of the model, then we may be able to complete the market, and thus price all payoffs using no-arbitrage.

- Could change assumptions by adding more basic assets.
  In the trinomial example, we could complete the market by adding a third asset outside the span of B and S.
- Could change assumptions by adding more trading opportunities.
  In the trinomial example, we could complete the market by allowing trading of B and S at one intermediate time point.

Next we will build models with infinitely many outcomes. Hope to replicate general payoffs by trading B and S *continuously* in time.