FINM 33000: Homework 3 Solutions

October 24, 2024

Problem 1

(a) $\mathbb{P}$ is an equivalent martingale measure iff $(p_u, p_m, p_d) := (\mathbb{P}(\{\omega_u\}), \mathbb{P}(\{\omega_m\}), \mathbb{P}(\{\omega_d\}))$ satisfies

$$\begin{pmatrix} 1 & 50 & 5 \end{pmatrix} = \frac{1}{1.2}\begin{pmatrix} p_u & p_m & p_d \end{pmatrix}\begin{pmatrix} 1.2 & 60 & 0 \\ 1.2 & 30 & 30 \\ 1.2 & 70 & 30 \end{pmatrix}$$

and $p_u, p_m, p_d > 0$. The solution is

$$p_u = 0.8, \qquad p_m = 0.05, \qquad p_d = 0.15.$$

There is only one equivalent martingale measure, so the market is complete.

(b) Solving

$$\begin{pmatrix} 1.2 & 60 & 0 \\ 1.2 & 30 & 30 \\ 1.2 & 70 & 30 \end{pmatrix}\begin{pmatrix} \theta^B \\ \theta^S \\ \theta^C \end{pmatrix} = \begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix},$$

we find $\theta^B = 175$, $\theta^S = -1.5$, $\theta^C = -3.5$ units of respectively B, S, C together replicate X.

(c) Using the replicating portfolio gives the price $175 \times 1 - 1.5 \times 50 - 3.5 \times 5 = 82.5$.

Using the risk-neutral probabilities gives price $(1/1.2)\times(0.8\times120+0.05\times60+0.15\times0) = 82.5$.

Problem 2

(a) $\mathbb{P}$ is an equivalent martingale measure iff $(p_u, p_m, p_d) := (\mathbb{P}(\{\omega_u\}), \mathbb{P}(\{\omega_m\}), \mathbb{P}(\{\omega_d\}))$ satisfies

$$\begin{pmatrix} 1 & 50 & 50 \end{pmatrix} = \frac{1}{1.2}\begin{pmatrix} p_u & p_m & p_d \end{pmatrix}\begin{pmatrix} 1.2 & 60 & 60 \\ 1.2 & 30 & 0 \\ 1.2 & 70 & 80 \end{pmatrix}$$

and $p_u, p_m, p_d > 0$. (The physical probability of each state is nonzero, hence the risk-neutral probability of each state must be nonzero; see the definition of *equivalent* martingale measure in L2.12, or recall its application in L2.23 which says p_u, p_m, p_d are *positive*.)

The solutions (*all* equivalent martingale measures) are

$$p_u = 0.8 - 4k \qquad p_m = 0.05 + k, \qquad p_d = 0.15 + 3k$$

for all k with $-0.05 < k < 0.2$. There is more than one equivalent martingale measure, so the market is incomplete.

(b) No, because there is no solution to

$$\begin{pmatrix} 1.2 & 60 & 60 \\ 1.2 & 30 & 0 \\ 1.2 & 70 & 80 \end{pmatrix} \begin{pmatrix} \theta^B \\ \theta^S \\ \theta^C \end{pmatrix} = \begin{pmatrix} 120 \\ 60 \\ 0 \end{pmatrix}.$$

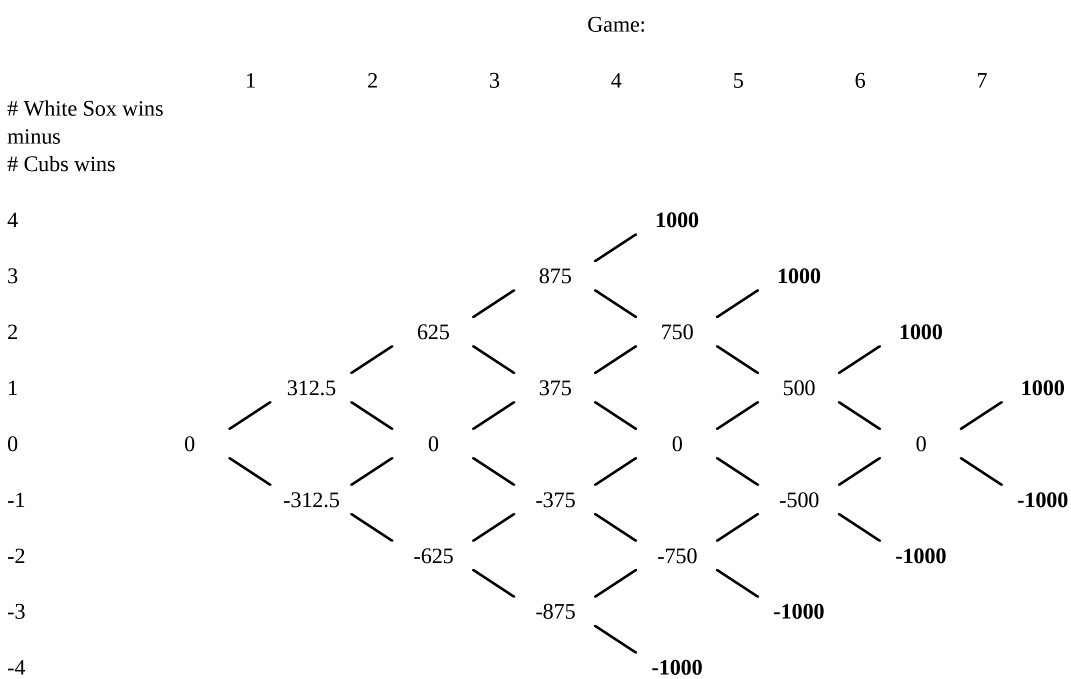
Problem 3

This solution takes mainly the martingale approach, and considers the question of replication only in regard to game 1. Alternatively, you could take entirely the replication approach in regard to all games.

The derivative contract that we wish to replicate is a bet paying 1000 if the White Sox win the Series, and -1000 if the Cubs win the Series. The basic assets are the individual-game bets.

According to the available individual-game bets, the risk-neutral probability p of White Sox victory in each game satisfies $0 = p(1) + (1-p)(-1)$, so $p = 0.5$.

Using risk-neutral probabilities, we calculate the derivative's value at each time, and in each possible state, by inducting backwards from the desired payoffs shown below in boldface. For example, one obtains the 625 by computing $0.5 \times 875 + 0.5 \times 375$.



Since your replicating portfolio needs to have value 312.5 if the White Sox win Game 1, and -312.5 if the Cubs win Game 1, you should bet 312.5 on the White Sox in Game 1 (or in other words, initially you should hold 312.5 units of "White Sox in Game 1" bets).

Another acceptable and equivalent answer is to hold -312.5 units of "Cubs in Game 1" bets.