FINM 33000: Homework 4

Due Thursday, October 31, 2024 at 11:59pm

Problem 1

Let S_t, where $t = 0, 1, 2, \ldots, T$, denote the time-t price of a tradeable non-dividend-paying asset.

Let $S_0 = 100$ and let each random increment $S_{t+1} - S_t$ take value $+1$ with physical probability 40%, and value -1 with physical probability 60%, independently of all other increments.

There exists a bond, with constant price 1 at all times.

There exists a call on S, with strike $K = 105$ and expiry at time $T = 12$.

(a) The following argument tries to prove that the no-arbitrage time-0 price of the call is *zero*.

> Consider the trading strategy which holds at each time t the portfolio Θ_t^{rep}, defined as follows. [Recall the convention stated in class: at each time t, the price S_t is revealed, then trading (if any) occurs to change the old holdings $\Theta_{t-1}^{\text{rep}}$ to the new holdings Θ_t^{rep}.]
>
> $$\Theta_t^{\text{rep}} := \begin{cases} (0 \text{ share of asset},\ 0 \text{ bonds}) & \text{if } S_t \leq K \\ (1 \text{ share of asset},\ -K \text{ bonds}) & \text{if } S_t > K \end{cases}$$
>
> This trading strategy matches the call payoff with probability 1, because either $S_T > K$, in which case the call payoff matches the time-T portfolio value $S_T - K$, or else $S_T \leq K$, in which case the call payoff matches the time-T portfolio value 0.
>
> Therefore, by the law of one price, the no-arbitrage time-0 call price must equal the time-0 value of the replicating portfolio Θ^{rep}, which is *zero*, because $\Theta_0^{\text{rep}} = (0, 0)$ given that $S_0 = 100 < 105 = K$.

Identify and explain the specific flaw in this "proof".

(b) Find the true time-0 value of the call.

Do not do a 12-step backwards induction, and do not use a computer (unless you want to check your answer).

Although your answer should be explicit, you may leave it unsimplified. You may leave binomial coefficients (numbers of the form: $\binom{n}{k}$, pronounced "n choose k") unsimplified. For example, you may write $\binom{12}{2}$ without actually calculating it.

Problem 2

An asset price S follows a random walk $S_0, S_1, S_2, \ldots$, starting at $S_0 = 2.16$, with step sizes ± 0.01.

Assume that you buy S at time 0 for 2.16 dollars, with a stop-loss level of 2 dollars, and a take-profit level of 3 dollars. Thus you will exit the trade (sell the asset) at the first time that the stock price hits either 2 dollars or 3 dollars (and you will sell for exactly 2 dollar or 3 dollars; ignore spreads, slippage, market impact, fees).

(a) Assume that the steps ± 0.01 are with probability 50% each.

Find the probability that you will exit the trade with a positive profit.

(It can be shown that, with probability 1, you will eventually exit the trade; S cannot stay forever inside the interval between 2 and 3. So the only two possibilities are that you exit at 2 dollars for a loss, or you exit at 3 dollars for a positive profit).

Hint: S is a martingale.

(b) Now assume the asymmetric case that the step sizes $+0.01$ and -0.01 are with probability u and $1-u$ respectively, where $u > 1/2$. Under this assumption, S is not a martingale.

However there is a constant A, where $0 < A < 1$, such that

$$A^{S_t}$$

is a martingale. Solve for A in terms of u.

Hint: Let $M_t = A^{S_t}$. The general condition for M to be a martingale is $\mathbb{E}_t(M_T - M_t) = 0$ for all $T > t$, but because this problem is in discrete time ($t = 0, 1, 2, 3, \ldots$) it will be enough if you simply consider the case $T = t+1$, and solve the equation

$$\mathbb{E}_t(M_{t+1} - M_t) = 0$$

for A in terms of u.

(c) Under the assumptions of (b), find the probability that you will exit the trade with a profit. You may leave your answer in terms of either A or u.

You may assume that the optional stopping theorem applies to the martingale S in part (a), and to the martingale M in part (b), at time 0 and time τ, the exit time.

Hint for all parts: It may be more convenient to express everything in cents instead of dollars, so the step sizes become ± 1.