Financial Mathematics 33000

Lecture 6

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Fundamental theorem in continuous time

Black-Scholes

Replication and Expectation

Fundamental theorem in continuous time

$\boxed{\text{Existence of equivalent martingale measure } \mathbb{P} \Rightarrow \text{No arb}}$

Proof:

- Given: under $\mathbb{P}$, each member of the discounted asset prices

$$\frac{1}{B_t}\mathbf{X}_t = \left(\frac{X_t^1}{B_t}, \frac{X_t^2}{B_t}, \cdots, \frac{X_t^N}{B_t}\right)$$

  is a martingale Itô process. Need to prove: No arbitrage
- For any self-financing trading strategy $\mathbf{\Theta}_t$ with value V_t,

  we'll show that V/B is a martingale, where $V_t := \mathbf{\Theta}_t \cdot \mathbf{X}_t$.

  After we show this, we're done, by the familiar argument:

  If $V_0 = 0$, then $V_0/B_0 = 0$, hence $\mathbb{E}(V_T/B_T) = 0$.

  If also $V_T \geq 0$, then $V_T/B_T \geq 0$, so $V_T/B_T = 0$, hence $V_T = 0$

  Conclusion: $\mathbf{\Theta}$ is not an arbitrage.

Fundamental theorem

- To see that V/B is a martingale, let $A_t := 1/B_t$. Then

$$\begin{aligned} \mathrm{d}(V_t/B_t) = \mathrm{d}(A_t V_t) &= V_t \mathrm{d}A_t + A_t \mathrm{d}V_t + \mathrm{d}A_t \mathrm{d}V_t \\ &= \boldsymbol{\Theta}_t \cdot \mathbf{X}_t \mathrm{d}A_t + A_t(\boldsymbol{\Theta} \cdot \mathrm{d}\mathbf{X}_t) + (\mathrm{d}A_t)(\boldsymbol{\Theta}_t \cdot \mathrm{d}\mathbf{X}_t) \\ &= \boldsymbol{\Theta}_t \cdot (\mathbf{X}_t \mathrm{d}A_t + A_t \mathrm{d}\mathbf{X}_t + \mathrm{d}A_t \mathrm{d}\mathbf{X}_t) \\ &= \boldsymbol{\Theta}_t \cdot \mathrm{d}(A_t \mathbf{X}_t) = \sum_{n=1}^{N} \theta_t^n \mathrm{d}(A_t X_t^n) \end{aligned}$$

  Since each $A_t X_t^n$ is a martingale, V/B is a martingale also.

- Idea: A martingale is the cumulative PnL from betting on zero-$\mathbb{E}$ games. Varying your bet size across games and across time still produces, collectively, a zero-expectation game. Can't risklessly make something from nothing by playing zero-expectation games.

Fundamental theorem

$\boxed{\text{No arb} \Rightarrow \text{Existence of equivalent martingale measure } \mathbb{P}}$

Intuition of proof: Same as in L2, L3.

- Define $\mathbb{P}$ by defining the $\mathbb{P}_t$-probability of an event to be the time-t **price**, in units of B, of a binary ("Arrow-Debreu") asset that pays 1 unit of B at time T if the event occurs, else 0. (But what if the binary asset does not exist and can't be replcated?)
- Martingale property holds because any asset X can be replicated by portfolio of $X_T(\omega_j)/B_T(\omega_j)$ units of the binary asset for each ω_j. Value portfolio by summing quantity $\times$ price.

$$\frac{X_0}{B_0} = \sum_{j=1}^{J} \frac{X_T(\omega_j)}{B_T(\omega_j)} \cdot \mathbb{P}(\omega_j) = \mathbb{E}\left(\frac{X_T}{B_T}\right)$$

Fundamental theorem: Comments

- Idea: The $\mathbb{P}$ probability of an event is simply the *price* (in units of B) of a asset that pays 1 unit of B if that event occurs, else 0.
- Note: In this entire proof, we never assumed that B is the bank account, and never assumed that it is riskless. It is enough to assume that B is some asset with positive price process.

  In some applications, it may be easier to normalize using some such asset (some *numeraire*) that is *not* the bank account.

  By default, if we say risk-neutral or martingale measure without specifying the numeraire, it is understood to be the bank account.

Option pricing

In L5, we did this by *replication*.

In L6, let's do it by martingale methods: Option price equals the *expected discounted payoff*, under a martingale measure $\mathbb{P}$. Why?

- By the Fundamental theorem.

How do we calculate $\mathbb{P}$-expectations (denoted by $\mathbb{E}$)?

- In many cases, a model is already specified under risk-neutral measure. Then simply work directly under the given measure.

But what if the model is specified under physical measure?

- We know how S behaves with respect to physical measure P. How does S behave wrt $\mathbb{P}$? All risk driven by W. So let's see what changing measure does to W, then find what it does to S.

Girsanov's theorem

Theorem: If W is a Brownian motion under P,

and if $\mathbb{P}$ is a probability measure on $\mathcal{F}_T^W$ that is equivalent to P,

then there exists an adapted process λ such that for all $t \in [0, T]$,

$$\tilde{W}_t := W_t + \int_0^t \lambda_s \mathrm{d}s$$

is Brownian motion under $\mathbb{P}$. Therefore:

- $\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \lambda_t \mathrm{d}t$, and $\tilde{W}$ is BM under $\mathbb{P}$ but not under P
- $\mathrm{d}W_t = \mathrm{d}\tilde{W}_t - \lambda_t \mathrm{d}t$, and W is BM under P but not under $\mathbb{P}$

Girsanov: an analogy

No proof, but here is an *analogy* on a sample space $\Omega = \{\omega_1, \ldots, \omega_6\}$.

Let $X(\omega_1) = X(\omega_2) = X(\omega_3) = 25$, $X(\omega_4) = X(\omega_5) = X(\omega_6) = 10$.

- Let $\mathsf{P}(\omega) = 1/6$ for each ω. Then $X \sim \text{Uniform}\{10, 25\}$ under P.
- But if $\mathbb{P}$ assigns probability

  $1/12$ to each of $\omega_1, \omega_2, \omega_3$,

  and $1/4$ to each of $\omega_4, \omega_5, \omega_6$,

  then X is *not* $\text{Uniform}\{10, 25\}$ under $\mathbb{P}$.
- However, $\tilde{X} := X + \lambda$ *is* $\text{Uniform}\{10, 25\}$ under $\mathbb{P}$,

  where $\lambda(\omega_4) := 15$ and $\lambda(\omega) := 0$ for $\omega \neq \omega_4$.

X under $\mathbb{P}$ does not have the same distribution as X under P.

But X *plus drift* under $\mathbb{P}$ has the same distribution as X under P.

Girsanov: some intuition

No proof, but here is some *intuition*:

- W is BM under P. After changing measure to $\mathbb{P}$, the W may not still be BM, but it is plausible that it is a martingale plus drift:

$$\mathrm{d}W_t = \lambda_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$$

where B is a BM under $\mathbb{P}$, and σ_t is some adapted process. So

$$(\mathrm{d}W_t)^2 = (\lambda_t \mathrm{d}t + \sigma_t \mathrm{d}B_t)^2$$

hence $\mathrm{d}t = \sigma_t^2 \mathrm{d}t$, so $\sigma_t = \pm 1$. Define $\tilde{W}$ by $\mathrm{d}\tilde{W}_t = \sigma_t \mathrm{d}B_t$.

- Then W can be shown to be $\mathbb{P}$-BM. And, as claimed,

$$\mathrm{d}W_t = \lambda_t \mathrm{d}t + \mathrm{d}\tilde{W}_t.$$

Fundamental theorem in continuous time

Black-Scholes

Replication and Expectation

Black-Scholes via martingale approach

Black-Scholes dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t \qquad\qquad B_0 = 1$$

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \qquad\qquad S_0 > 0$$

where W is BM under physical measure P, and $\sigma > 0$.

No arb implies that $\exists\ \mathbb{P}$, equivalent to P, such that S/B is a $\mathbb{P}$-MG.

Hence by Girsanov, $\exists\ \lambda$ such that $\tilde{W}_t := W_t + \int_0^t \lambda_s \mathrm{d}s$ is $\mathbb{P}$-BM.

Substitute $\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \lambda_t\mathrm{d}t$ into the SDE of S:

$$\begin{aligned}\mathrm{d}S_t &= \mu S_t\mathrm{d}t + \sigma S_t(\mathrm{d}\tilde{W}_t - \lambda_t\mathrm{d}t)\\ &= (\mu - \lambda_t\sigma)S_t\mathrm{d}t + \sigma S_t\mathrm{d}\tilde{W}_t\end{aligned}$$

But can we say anything about $\mu - \lambda_t\sigma$?

Under $\mathbb{P}$, every tradeable asset X has drift rX

This page does not assume that X is a GBM.

Assume only that X is an Itô process.

- Under $\mathbb{P}$, the discounted price X/B is a MG, hence has zero drift.
- By Itô's rule, X/B has dynamics

$$\begin{aligned} \mathrm{d}(X_t/B_t) = \mathrm{d}(e^{-rt}X_t) &= e^{-rt}\mathrm{d}X_t - re^{-rt}X_t\mathrm{d}t + \mathrm{d}(e^{-rt})\mathrm{d}X_t \\ &= e^{-rt}(\mathrm{d}X_t - rX_t\mathrm{d}t), \end{aligned}$$

  so $\mathrm{d}X_t - rX_t\mathrm{d}t$ has no drift term.
- Therefore the drift term of $\mathrm{d}X_t$ must be $rX_t\mathrm{d}t$.

Under $\mathbb{P}$, the GBM S is still GBM, but with drift r

- Applying this to S, we have $(\mu - \lambda_t \sigma) S_t = r S_t$. So, for $\mathbb{P}$-BM $\tilde{W}$,

$$\boxed{\mathrm{d}S_t = r S_t \mathrm{d}t + \sigma S_t \mathrm{d}\tilde{W}_t}$$

Rate of growth changes from μ to r. Volatility stays the same. Often, option model specifications start here, bypassing physical measure, to work directly under risk-neutral measure with drift r.

- By L4, therefore, under $\mathbb{P}$, conditional on $\mathcal{F}_t^W$

$$\boxed{\log S_T \sim \ \mathrm{Normal}(\log S_t + (r - \sigma^2/2)(T-t),\ \sigma^2(T-t))}$$

Compare: under P, conditional on $\mathcal{F}_t^W$,

$$\log S_T \sim \ \mathrm{Normal}(\log S_t + (\mu - \sigma^2/2)(T-t),\ \sigma^2(T-t))$$

Lognormal distribution

Here's a more general calculation, allowing different rates for growth and discounting, on an underlying X, not necessarily a stock price.

- Let $t < T$. Let R_{grow} and r be constants.
- Assume that (conditional on the time-t information $\mathcal{F}_t$) the random variable X_T has lognormal $\mathbb{P}$-distribution

  $$\log X_T \sim \text{Normal}(\log X_t + (R_{grow} - \sigma^2/2)(T-t),\ \sigma^2(T-t))$$

  where $X_t > 0$, and $\sigma > 0$ is a constant.
- One way that this distribution could arise is from the dynamics

  $$\mathrm{d}X_t = R_{grow} X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \qquad X_0 > 0$$

  where W is $\mathbb{P}$-BM.

Conclusion: the Black-Scholes call price formula

Then, letting $\mathbb{E}$ denote expectation wrt $\mathbb{P}$,

$$\boxed{e^{-r(T-t)}\mathbb{E}_t(X_T - K)^+ = C^{BS}(X_t, t, K, T, R_{grow}, r, \sigma)}$$

where the function C^{BS} is defined for $X > 0, K > 0, \sigma > 0, t < T$ by

$$C^{BS}(X, t, K, T, R_{grow}, r, \sigma) := e^{-r(T-t)}\big[FN(d_1) - KN(d_2)\big],$$

and

$$F := Xe^{R_{grow}(T-t)} = \mathbb{E}_t X_T$$

and

$$d_{1,2} := d_{+,-} := \frac{\log(F/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}.$$

Proof of formula: preliminaries

For any normal random variable Y with mean m and variance v,

$$\mathbb{E}\mathbf{1}_{Y>k} = \mathbb{P}(Y > k) = \mathbb{P}\left(\frac{Y-m}{\sqrt{v}} > \frac{k-m}{\sqrt{v}}\right) = N\left(\frac{m-k}{\sqrt{v}}\right)$$

and

$$\mathbb{E}(e^Y \mathbf{1}_{Y>k}) = e^m \mathbb{E}(e^{Y-m}\mathbf{1}_{Y-m>k-m}) = e^m \int_{k-m}^{\infty} e^y \frac{1}{\sqrt{2\pi v}} e^{-y^2/(2v)} \mathrm{d}y$$

$$= \frac{e^{m+v/2}}{\sqrt{2\pi v}} \int_{k-m}^{\infty} e^{-(y^2-2vy+v^2)/(2v)} \mathrm{d}y$$

we want to calculate the integral
only if it's bigger than k-m

$$= \frac{e^{m+v/2}}{\sqrt{2\pi}} \int_{\frac{k-m-v}{\sqrt{v}}}^{\infty} e^{-z^2/2} \mathrm{d}z \quad \text{where } z := \frac{y-v}{\sqrt{v}}$$

y = k-m, plug this to z to make it k-m-v/v^0.5

$$= e^{m+v/2} N\left(\frac{m-k}{\sqrt{v}} + \sqrt{v}\right) \text{ and, in } k = -\infty \text{ case, } \mathbb{E}e^Y = e^{m+v/2}$$

take the negative of the lower bound

So $\mathbb{E}(e^Y \mathbf{1}_{Y>k}) = N\left(\frac{m-k}{\sqrt{v}} + \sqrt{v}\right)\mathbb{E}e^Y$.

Proof of formula: decompose $(X_T - K)^+$

Proof: Using the fact that $x^+ = x\mathbf{1}_{x>0}$ for all x,

$$
\begin{aligned}
\mathbb{E}_t(X_T - K)^+ &= \mathbb{E}_t(X_T - K)\mathbf{1}_{X_T>K} = \mathbb{E}_t(X_T\mathbf{1}_{X_T>K}) - K\mathbb{E}_t\mathbf{1}_{X_T>K} \\
&= \mathbb{E}_t\left(e^{Y_T}\mathbf{1}_{Y_T>\log K}\right) - K\mathbb{E}_t\mathbf{1}_{Y_T>\log K} \\
\text{(By L6.17)} &= \left(\mathbb{E}_t e^{Y_T}\right)N\left(\frac{m-k}{\sqrt{v}} + \sqrt{v}\right) - KN\left(\frac{m-k}{\sqrt{v}}\right) \\
&= F_t N(d_1) - KN(d_2)
\end{aligned}
$$

where $k := \log K$ and $Y_T := \log X_T \sim \text{Normal}(m, v)$ conditional on $\mathcal{F}_t$, where $m = \log X_t - (\sigma^2/2)(T-t)$ and $v = \sigma^2(T-t)$, and

$$F_t := X_t e^{R_{grow}(T-t)} = \mathbb{E}_t X_T.$$

Conclude by multiplying by discount factor.

Probabilistic interpretation of $N(d_1)$ and $N(d_2)$

Call option payout = Asset or nothing call payout - Binary call payout

In summary,

$$
\begin{aligned}
C_t &= e^{-r(T-t)}\Big(\mathbb{E}_t[S_T \mathbf{1}_{S_T>K}] - \mathbb{E}_t[K\mathbf{1}_{S_T>K}]\Big) \\
&= e^{-r(T-t)}\left(F_t \mathbb{E}_t\left[\frac{S_T}{F_t}\mathbf{1}_{S_T>K}\right] - K\mathbb{P}_t(S_T > K)\right) \\
&= e^{-r(T-t)}\Big(F_t N(d_1) - K N(d_2)\Big)
\end{aligned}
$$

The $N(d_2)$ is the $\mathbb{P}_t$ that the call option expires in the money.

(So $e^{-r(T-t)}N(d_2)$ = time-t price of K-strike T-expiry binary call.)

The $N(d_1)$ is the "share measure" $\mathbb{P}_t^S$ that the call expires ITM.

And time-t price of asset-or-nothing call paying $X_T\mathbf{1}_{X_T>K}$ is $e^{-r(T-t)}F_t N(d_1)$. If $X = S$ is a no-dividend stock, this is $S_t N(d_1)$

Probabilistic analysis of effect of r

Increasing r, while keeping everything else fixed, has what effect on time-0 call prices?

- Martingale methods make the answer clear:

$$
\begin{aligned}
C_0 &= e^{-rT}\mathbb{E}C_T \\
&= e^{-rT}\mathbb{E}(S_T - K)^+ \\
&= e^{-rT}\mathbb{E}(S_0 e^{(r-\sigma^2/2)T+\sigma\tilde{W}_T} - K)^+ \\
&= \mathbb{E}(S_0 e^{(-\sigma^2/2)T+\sigma\tilde{W}_T} - Ke^{-rT})^+
\end{aligned}
$$

so the call option will increase
(C_0 will increase)
because we have -K*e^(-r), the amount we substract will be smaller
denominator gets bigger
but this is assuming that S_0 stays fixed

In reality, if r increase, S will also decrease (stock markets go down),
so both terms go down, and the call option price goes down in aggregate

Probabilistic intuition about impact of σ

The *vega*, at time-t, of an asset or portfolio with value $C_t = C(S_t, t; \sigma)$ is $\frac{\partial C}{\partial \sigma}(S_t, t; \sigma)$. For a call or put in the B-S model,

$$\text{vega} := \frac{\partial C^{BS}}{\partial \sigma} = S\sqrt{T-t}N'(d_1) > 0$$

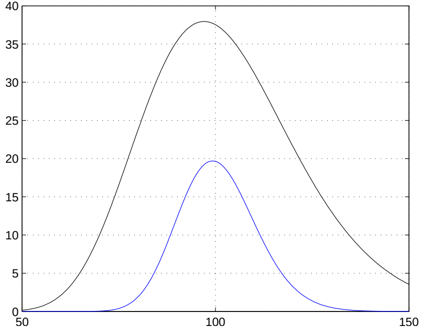
Why positive? Let's take $r = 0$.

- For a linear payout $a + bS_T$, the time-0 value is $a + bS_0$ regardless of σ. So vega for a linear payout is zero.
- For a convex payout $f(S_T)$, such as a call, the payout dominates the linear tangent to f at S_0. So the contract's time-0 value is bigger than $f(S_0)$. By how much? Depends on σ. The larger the σ, the larger the convex contract's time-0 value, because the larger the chance that S goes to where $f >$ linear. Large $\sigma \Rightarrow$ more convexity is reachable.

higher volatility, higher value of the option

Vega of a call

Vega of a call, plotted against S_t, for $T - t = 1$, and $T - t = 0.25$.



Under B-S dynamics, vega of a call is positive.

A call and a put option with the same strike price and expiry date, has the same vega due to put-call parity that doesn't depend on vol

Probabilistic analysis of joint effect of σ and T

Halving σ and quadrupling T, while keeping everything else fixed, has what effect on time-0 call prices?

Appendix

On an interval I a function $f : I \to \mathbb{R}$ is said to be *convex* if its graph lies on or below all of its chords: for all $x, y \in I$, all $\alpha \in [0, 1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

Properties:

- A twice differentiable function f is convex iff $f'' \geq 0$ everywhere.
- A convex function's graph lies on or above all of its tangents.
- 

  Jensen's inequality:

  If f is convex on I and X is an integrable random variable taking values in I then $\mathbb{E}f(X) \geq f(\mathbb{E}X)$.

  Johan Jensen (1859-1925)

Replication and Expectation paths to solutions

Underlying dynamics (example: GBM in the B-S case)

Contract (example: call or put)

risk-neutral pricing/fundamental theorem/martingale method

Replication

Or risk-neutral pricing

PDE with terminal condition

$\mathbb{E}$(contract's discounted payout)

Solve PDE

Compute E

Pricing formula

Differentiate: dC/dS

How many shares to hold in replicating portfolio at time 0

PDE can come from probabilistic approach too

Recall: under $\mathbb{P}$, "every tradeable asset's proportional drift rate is r".

- Apply this to S (assuming GBM $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$) to get

  $$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}\tilde{W}_t$$

  where $\mathrm{d}B_t = rB_t\mathrm{d}t$, and $\tilde{W}$ is a $\mathbb{P}$-BM with $\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \lambda_t \mathrm{d}t$.

  The drift changes (to rS_t), but the *volatility does not.*

- Apply this to the option price C, assuming $C_t = C(S_t, t)$. By Itô

  $$\mathrm{d}C_t = \frac{\partial C}{\partial t}\mathrm{d}t + \frac{\partial C}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\mathrm{d}S)^2.$$

  Equate the drift of C to rC:

  $$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

  This is the B-S PDE. Terminal condition: $C(S,T) = (S-K)^+$

Same formula, multiple interpretations

Black-Scholes $\boxed{N(d_2)}$ at time t, where $d_2 = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}$:

- $N(d_2)$ is the risk-neutral probability of $S_T > K$.
- $e^{-r(T-t)}N(d_2)$ is the value of a binary call that pays $\mathbf{1}_{S_T>K}$
- $e^{-r(T-t)}N(d_2)$ is $-\partial C/\partial K$, where C is vanilla call value.
- $-Ke^{-r(T-t)}N(d_2)$ is value of vanilla-call replicator's B holdings.

Black-Scholes $\boxed{N(d_1)}$ at time t, where $d_1 = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$:

- $N(d_1)$ is the *share-measure* probability of $S_T > K$. It's the time-t price, in shares, of an asset that pays 1 share if $S_T > K$.
- $S_t N(d_1)$ is value of an asset-or-nothing call that pays $S_T\mathbf{1}_{S_T>K}$
- $N(d_1)$ is $\partial C/\partial S$, the delta of a vanilla call.
- $S_t N(d_1)$ is value of vanilla-call replicator's share holdings.

General payoffs $f(S_T)$: Price by PDE or Expectation

To find time-t price:

- PDE approach: To price instead an option paying $f(S_T)$, use the PDE that comes from the dynamics, changing only the terminal condition to $C(S,T) = f(S)$.
- Expectations approach: Calculate

$$e^{-r(T-t)} \int_0^\infty f(s)p(s)\mathrm{d}s$$

where p is the time-t conditional probability density of S_T. Or,

$$e^{-r(T-t)} \int_{-\infty}^\infty f(e^x)p_L(x)\mathrm{d}x$$

where p_L is the time-t conditional probability density of $\log S_T$.

General payoffs $f(S_T)$: Replication

- The replication argument showed: if $C(S,t)$ is a function that satisfies the B-S PDE with terminal condition $C(S,T) = f(S)$, then a portfolio of $\frac{\partial C}{\partial S}$ shares and $(C - \frac{\partial C}{\partial S} S_t)/B_t$ units of B replicates a $f(S_T)$ payoff, and self-finances.
- So to hedge a contract on $f(S_T)$, can use PDE *or* risk-neutral $\mathbb{E}$, to find the option pricing function C which satisfies B-S PDE. We can then calculate $\partial C/\partial S$ to find the delta hedge.

Example: binary put

Example: time-t price of K-strike T-expiry binary put in L6.27 model.

Solution 1a: Let $k = \log K$.

$$e^{-r(T-t)} \int_{-\infty}^{k} 1 \times \frac{1}{\sqrt{2\pi}\sigma\sqrt{T-t}} e^{-\frac{(x-\log S_t-(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}} \mathrm{d}x = e^{-r(T-t)} N(-d_2)$$

because $y := \frac{x-\log S_t-(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ simplifies $\int$ to $\int_{-\infty}^{-d_2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y$.

Solution 1b: Let $m = \log X_t - (\sigma^2/2)(T-t)$ and $v = \sigma^2(T-t)$

$$e^{-r(T-t)} \mathbb{P}_t(S_T < K) = e^{-r(T-t)} \mathbb{P}_t\left(\frac{X_T - m}{\sqrt{v}} < \frac{k-m}{\sqrt{v}}\right) = e^{-r(T-t)} N(-d_2)$$

Solution 2: Can verify that $C(S,t) = e^{-r(T-t)} N(-d_2)$ solves PDE in L6.27 with terminal condition $C(S,T) = \mathbf{1}_{S<K}$.

In any case, replicating portfolio has $\frac{\partial C}{\partial S} = -e^{-r(T-t)} \frac{N'(d_2)}{S_t \sigma \sqrt{T-t}}$ shares.