FINM 33000 – Homework 3: Matheus Raka Pradnyatama

Problem 1

Part a)

For S/B to be a martingale,

$$E\left(\frac{S_T}{B_T}\right) = \frac{S_0}{B_0} = \frac{50}{1} = 50$$

$$\frac{S_T(\omega_u)}{B_T(\omega_u)} * p_u + \frac{S_T(\omega_m)}{B_T(\omega_m)} * p_m + \frac{S_T(\omega_d)}{B_T(\omega_d)} * p_d = 50$$

$$\frac{60}{1.2} * p_u + \frac{30}{1.2} * p_m + \frac{70}{1.2} * p_d = 50$$

$$50p_u + 25p_m + \frac{175}{3}p_d = 50$$

$$2p_u + p_m + \frac{7}{3}p_d = 2$$

$$6p_u + 3p_m + 7p_d = 6$$

$$p_u + p_m + p_d = 1$$

$$7p_u + 7p_m + 7p_d = 7$$

$7p_u + 7p_m + 7p_d = 7$

$\underline{6p_u + 3p_m + 7p_d = 6}$ $-$

$p_u + 4p_m = 1$

$p_u = 1 - 4p_m$

For C/B to be a martingale,

$$E\left(\frac{C_T}{B_T}\right) = \frac{C_0}{B_0} = \frac{5}{1} = 5$$

$$\frac{C_T(\omega_u)}{B_T(\omega_u)} * p_u + \frac{C_T(\omega_m)}{B_T(\omega_m)} * p_m + \frac{C_T(\omega_d)}{B_T(\omega_d)} * p_d = 5$$

$$\frac{0}{1.2} * p_u + \frac{30}{1.2} * p_m + \frac{30}{1.2} * p_d = 5$$

$$25(p_m + p_d) = 5$$

$$p_m + p_d = \frac{1}{5}$$

$$p_d = \frac{1}{5} - p_m$$

$$p_u + p_m + p_d = 1$$

$$1 - 4p_m + p_m + \frac{1}{5} - p_m = 1$$

$$-4p_m = -\frac{1}{5}$$

$p_m = \frac{1}{20} = 0.05$

$p_u = 1 - 4 * \frac{1}{20} = \frac{4}{5} = 0.8$

$p_d = \frac{1}{5} - \frac{1}{20} = \frac{3}{20} = 0.15$

$$(p_u, p_m, p_d) = (0.8, 0.05, 0.15)$$

Since there is only one unique solution our equations, there is only one martingale measure. Based on the Second Fundamental Theorem, since there is only one martingale measure, the market $\{B, S, C\}$ is complete.

Part b)

Let's say:

- n_B is the number of units of bank account B
- n_S is the number of units of stock S
- n_C is the number of units of option C

For ω_u:

$$n_B * B_T(\omega_u) + n_S * S_T(\omega_u) + n_C * C_T(\omega_u) = X_T(\omega_u)$$
$$n_B * 1.2 + n_S * 60 + n_C * 0 = 120$$
$$1.2n_B + 60n_s = 120$$

For ω_m:

$$n_B * B_T(\omega_m) + n_S * S_T(\omega_m) + n_C * C_T(\omega_m) = X_T(\omega_m)$$
$$n_B * 1.2 + n_S * 30 + n_C * 30 = 60$$
$$1.2n_B + 30n_S + 30n_C = 60$$

For ω_d:

$$n_B * B_T(\omega_d) + n_S * S_T(\omega_d) + n_C * C_T(\omega_d) = X_T(\omega_d)$$
$$n_B * 1.2 + n_S * 70 + n_C * 30 = 0$$
$$1.2n_B + 70n_S + 30n_C = 0$$

$1.2n_B + 70n_S + 30n_C = 0$

$\underline{1.2n_B + 30n_S + 30n_C = 60}$ $-$

$40n_S = -60$

$n_S = -1.5$

$1.2n_B + 60n_s = 120$

$1.2n_B = 120 - 60(-1.5) = 210$

$n_B = 175$

$1.2n_B + 70n_S + 30n_C = 0$
$30n_C = -1.2n_B - 70n_S = -1.2(175) - 70(-1.5) = -105$
$n_C = -3.5$

We can replicate the payoff using a portfolio of B, S, C
The replicating portfolio should be: (175 units of B, -1.5 units of S, -3.5 units of C)

Part c)
Suppose the no-arbitrage time-0 price of payoff X_T is denoted as V_0

Using the replicating portfolio
(175 units of B, -1.5 units of S, -3.5 units of C)
$V_0 = 175 * B_0 - 1.5 * S_0 - 3.5C_0 = 175 * 1 - 1.5 * 50 - 3.5 * 5 = 82.5$
$\boldsymbol{V_0 = 82.5}$

Using the pricing probabilities from part (a)
$(p_u, p_m, p_d) = (0.8, 0.05, 0.15)$

$$V_0 = E\left(\frac{X_T}{B_T}\right) = \frac{X_T(\omega_u)}{B_T(\omega_u)} * p_u + \frac{X_T(\omega_m)}{B_T(\omega_m)} * p_m + \frac{X_T(\omega_d)}{B_T(\omega_d)} * p_d = \frac{120}{1.2} * 0.8 + \frac{60}{1.2} * 0.05 + 0$$
$\boldsymbol{V_0 = 82.5}$

The no-arbitrage time-0 price of payoff $\boldsymbol{X_T}$ is $\mathbf{82.5}$

Problem 2 Part a) From Problem 1, we have

$$p_u + p_m + p_d = 1$$
$$p_u + 4p_m = 1$$
$$p_u = 1 - 4p_m$$

For C/B to be a martingale,

$$E\left(\frac{C_T}{B_T}\right) = \frac{C_0}{B_0} = \frac{50}{1} = 50$$
$$\frac{C_T(\omega_u)}{B_T(\omega_u)} * p_u + \frac{C_T(\omega_m)}{B_T(\omega_m)} * p_m + \frac{C_T(\omega_d)}{B_T(\omega_d)} * p_d = 50$$
$$\frac{60}{1.2} * p_u + \frac{0}{1.2} * p_m + \frac{80}{1.2} * p_d = 50$$
$$50p_u + \frac{200}{3} * p_d = 50$$
$$150p_u + 200p_d = 150$$
$$3p_u + 4p_d = 3$$
$$3p_u = 3 - 4p_d$$
$$p_u = 1 - \frac{4}{3}p_d$$

From Problem 1:

$$6p_u + 3p_m + 7p_d = 6$$
$$6p_u = 6 - 3p_m - 7p_d$$
$$p_u = 1 - \frac{1}{2}p_m - \frac{7}{6}p_d$$

$$1 - \frac{4}{3}p_d = 1 - \frac{1}{2}p_m - \frac{7}{6}p_d$$
$$\frac{7}{6}p_d - \frac{4}{3}p_d = -\frac{1}{2}p_m$$
$$-\frac{1}{6}p_d = -\frac{1}{2}p_m$$
$$p_d = 3p_m$$

$$p_u = 1 - \frac{1}{2}p_m - \frac{7}{6}p_d = 1 - \frac{1}{2}p_m - \frac{7}{6} * 3p_m = 1 - 4p_m$$

$$p_u + p_m + p_d = 1$$
$$1 - 4p_m + p_m + 3p_m = 1$$

It seems we cannot find a single solution for our equations. This means that there more than one martingale measure. Based on the Second Fundamental Theorem, since there is more than one martingale measure, the market $\{B, S, C\}$ is incomplete.

Problem 2 Part b)

Let's say:

- n_B is the number of units of bank account B
- n_S is the number of units of stock S
- n_C is the number of units of option C

For ω_u:

$$n_B * B_T(\omega_u) + n_S * S_T(\omega_u) + n_C * C_T(\omega_u) = X_T(\omega_u)$$
$$n_B * 1.2 + n_S * 60 + n_C * 60 = 120$$
$$1.2n_B + 60n_s + 60n_c = 120$$

For ω_m:

$$n_B * B_T(\omega_m) + n_S * S_T(\omega_m) + n_C * C_T(\omega_m) = X_T(\omega_m)$$
$$n_B * 1.2 + n_S * 30 + n_C * 0 = 60$$
$$1.2n_B + 30n_S = 60$$

For ω_d:

$$n_B * B_T(\omega_d) + n_S * S_T(\omega_d) + n_C * C_T(\omega_d) = X_T(\omega_d)$$
$$n_B * 1.2 + n_S * 70 + n_C * 80 = 0$$
$$1.2n_B + 70n_S + 80n_C = 0$$

$1.2n_B + 70n_S + 80n_C = 0$
$\underline{1.2n_B + 60n_s + 60n_c = 120}$ _
$10n_s + 20n_C = -120$
$20n_C = -120 - 10n_s$
$n_C = -6 - 0.5n_s$

$1.2n_B + 30n_S = 60$
$1.2n_B = -30n_S + 60$

$1.2n_B + 60n_s + 60n_c = 120$
$1.2n_B + 60n_s + 60 * (-6 - 0.5n_s) = 120$
$1.2n_B + 60n_s - 360 - 30n_s = 120$
$1.2n_B + 30n_s = 480$

But we also have: $1.2n_B + 30n_S = 60$

$1.2n_B + 30n_s$ cannot be both 480 and 60. This means there is no solution for our equations. We cannot replicate the payoff using a portfolio of B, S, C because there is no solution for our equations.

Problem 3

From Stochastic Calculus FINM 3400 Class Notes, page 30, in a Martingale Betting Strategy, the possible payoffs are +1 and -1, and the probability of getting +1 and -1 is ½ each.

Example 5.4. Martingale betting strategy. Let $X_1, X_2, \ldots$ be independent random variables with

$$\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = \frac{1}{2}. \tag{8}$$

This means that in setting our bet, in which the bookie will pay me +1 dollar if the White Sox win game n and −1 dollar if the Cubs win game n, the risk-neutral probability is 0.5. Each outcome is equally likely to appear. We can use a binary tree to determine the bets in each game.

Game 7: We know the payoff must be +1,000 or -1,000

Game 6

- The series can end at Game 6, so we must have a payoff of +1,000 and -1,000
- To have a payoff of +1,000 or -1,000 (at Game 7), we can have $0 and decide to bet $1,000 on White Sox winning the game
    - If White Sox wins, payoff = +1,000
    - If White Sox lose, payoff = -1,000

Game 5

- The series can end at Game 5, so we must have a payoff of +1,000 and -1,000
- To have a payoff of +1,000 or 0 (at Game 6), we can have 500 and decide to bet 500 on White Sox winning the game
    - If White Sox wins, payoff = 500 + 500 = +1,000
    - If White Sox lose, payoff = 500 – 500 = 0
- To have a payoff of 0 or -1,000 (at Game 6), we can have -500 and decide to bet 500 on White Sox winning the game
    - If White Sox wins, payoff = -500 + 500 = 0
    - If White Sox lose, payoff = -500 – 500 = -1,000

Game 4

- The series can end at Game 4, so we must have a payoff of +1,000 and -1,000
- To have a payoff of +1,000 or +500 (at Game 5), we can have 750 and decide to bet 250 on White Sox winning the game
    - If White Sox wins, payoff = 750 + 250 = +1,000

- If White Sox lose, payoff = 750 – 250 = +500
- To have a payoff of +500 or -500 (at Game 5), we can have 0 and decide to bet 500 on White Sox winning the game
  - If White Sox wins, payoff = 0 + 500 = +500
  - If White Sox lose, payoff = 0 – 500 = -500
- To have a payoff of -500 or -1,000 (at Game 5), we can have -750 and decide to bet 250 on White Sox winning the game
  - If White Sox wins, payoff = -750 + 250 = -500
  - If White Sox lose, payoff = -750 – 250 = 1,000

Game 3

- To have a payoff of +1,000 or +750 (at Game 4), we can have $\frac{1000+750}{2} = 875$ and decide to bet 125 on White Sox winning the game
  - If White Sox wins, payoff = 875 + 125 = +1,000
  - If White Sox lose, payoff = 875 – 125 = +750
- To have a payoff of +750 or 0 (at Game 4), we can have $\frac{750+0}{2} = 375$ and decide to bet 375 on White Sox winning the game
  - If White Sox wins, payoff = 375 + 375 = +750
  - If White Sox lose, payoff = 375 – 375 = 0
- To have a payoff of 0 or -750 (at Game 4), we can have $\frac{-750+0}{2} = -375$ and decide to bet 375 on White Sox winning the game
  - If White Sox wins, payoff = -375 + 375 = 0
  - If White Sox lose, payoff = -375 – 375 = -750
- To have a payoff of -750 or -1,000 (at Game 4), we can have $\frac{-750-1000}{2} = -875$ and decide to bet 125 on White Sox winning the game
  - If White Sox wins, payoff = -875 + 125 = -750
  - If White Sox lose, payoff = -875 – 125 = -1,000

Game 2

- To have a payoff of +875 or +375 (at Game 3), we can have $\frac{875+375}{2} = 625$ and decide to bet 250 on White Sox winning the game
  - If White Sox wins, payoff = 625 + 250 = +875
  - If White Sox lose, payoff = 625 – 250 = +375
- To have a payoff of +375 or -375 (at Game 3), we can have 0 and decide to bet 375 on White Sox winning the game
  - If White Sox wins, payoff = 0 + 375 = +375
  - If White Sox lose, payoff = 0 – 375 = -375

- To have a payoff of -375 or -875 (at Game 3), we can have $\frac{-375-875}{2} = -625$ and decide to bet 250 on White Sox winning the game
    - If White Sox wins, payoff = -625 + 250 = -375
    - If White Sox lose, payoff = -625 – 250 = -875

Game 1

- To have a payoff of +625 or 0 (at Game 2), we can have $\frac{625}{2} = 312.5$ and decide to bet 312.5 on White Sox winning the game
    - If White Sox wins, payoff = 312.5 + 312.5 = +625
    - If White Sox lose, payoff = 312.5 – 312.5 = 0
- To have a payoff of 0 or -625 (at Game 2), we can have $\frac{-625}{2} = -312.5$ and decide to bet 312.5 on White Sox winning the game
    - If White Sox wins, payoff = -312.5 + 312.5 = 0
    - If White Sox lose, payoff = -312.5 – 312.5 = -625

This means that to achieve an end result payoff of either +1,000 (if the White Sox wins the series) or -1,000 (if the White Sox loses the series), we need to bet **$312.5 on the White Sox winning Game 1.**