Homework 5 Stochastic - Matheus Raka

Exercise 1.1)

$$f(t, B_t) = Y_t = B_t^3 + 2t^2$$

$$df(t, B_t) = f'(t, B_t)\,dB_t + \left[\dot{f}(t, B_t) + \frac{1}{2} f''(t, B_t)\right] dt$$

$$\dot{f}(t, B_t) = 4t$$

$$f'(t, B_t) = 3B_t^2 \quad , \quad f''(t, B_t) = 6B_t$$

$$df(t, B_t) = 3B_t^2\,dB_t + \left[4t + \frac{6B_t}{2}\right] dt$$

$$d(Y_t) = \left[4t + 3B_t\right] dt + 3B_t^2\,dB_t$$

$$d\langle Y\rangle_t = \left[d(Y_t)\right]^2 = 9B_t^4\,dt \rightarrow A_t = 9B_t^4 \quad //$$

$$d\langle Y, X\rangle_t = (dY_t)(dX_t) = \left((4+3B_t)\,dt + 3B_t^2\,dB_t\right)\left(3X_t\,dt - 2\sqrt{X_t}\,dB_t\right)$$

$$= -6B_t^2\sqrt{X_t}\,dt$$

$$C_t = -6B_t^2\sqrt{X_t} \quad //$$

Exercise 1.2

$$d(Y_t) = dX_t + X_t^3\,dt = 3X_t\,dt - 2\sqrt{X_t}\,dB_t + X_t^3\,dt$$

$$d(Y_t) = (X_t^3 - 3X_t)\,dt - 2\sqrt{X_t}\,dB_t$$

$$d\langle Y\rangle_t = \left(-2\sqrt{X_t}\right)^2 dt = 4X_t\,dt \rightarrow A_t = 4X_t \quad //$$

$$d\langle Y, X\rangle_t = (dX_t)(dY_t) = \left(-2\sqrt{X_t}\,dB_t\right)\left(-2\sqrt{X_t}\,dB_t\right)$$

$$d\langle Y, X\rangle_t = 4X_t\,dt \rightarrow C_t = 4X_t \quad //$$

Exercise 1.3

$$Y_t = X_t^3 + \exp\{2\int_0^t X_s^2 ds\}$$

$$d(Y_t) = d(X_t^3) + d[\exp\{2\int_0^t X_s^2 ds\}$$

$$f(X_t) = X_t^3$$

$$f'(X_t) = 3X_t^2, \quad f''(X_t) = 6X_t, \quad d\langle X\rangle_t = 4X_t dt \rightarrow (dX_t)^2$$

$$d f(X_t) = 3X_t^2 dX_t + \frac{1}{2}\cdot 6X_t d\langle X\rangle_t = 3X_t^2(3X_t dt - 2\sqrt{X_t}\, dB_t) + 3X_t d\langle X\rangle_t$$

$$d(X_t^3) = (9X_t^3 + 12X_t^2)dt - 6X_t^2\sqrt{X_t}\, dB_t$$

$$d(Y_t) = (9X_t^3 + 12X_t^2)dt - 6X_t^2\sqrt{X_t}\, dB_t + 2X_t^2\exp\{2\int_0^t X_s^2 ds\} dt$$

$$d(Y_t) = [9X_t^3 + 12X_t^2 + 2X_t^2\exp\{2\int_0^t X_s^2 ds\}]dt - 6X_t^2\sqrt{X_t}\, dB_t$$

$$d\langle Y\rangle_t = (-6X_t^2\sqrt{X_t})dt = 36X_t^4\cdot X_t dt = 36X_t^5 dt$$

$$A_t = 36X_t^5 \;//$$

$$d\langle Y,X\rangle_t = (-6X_t^2\sqrt{X_t}\, dB_t)(-2\sqrt{X_t}\, dB_t)$$

$$= 12X_t^2 X_t dt = 12X_t^3 dt$$

$$C_t = 12X_t^3 \;//$$

Exercise 2.1

$f(X_t) = X_t^3$

$f'(X_t) = 3X_t^2$, $f''(X_t) = 6X_t$

$d\langle X\rangle_t = (dX_t)^2 = X_t^2(\sigma_1^2 dt + \rho_1^2 dt) = X_t^2(\sigma_1^2 + \rho_1^2)\, dt$

$\frac{1}{2} f''(X_t) = \frac{1}{2} \cdot 6X_t = 3X_t$

$$d(Z_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t) \cdot d\langle X\rangle_t$$

$$= 3X_t^2 (X_t)(m_1 dt + \sigma_1 dB_t^1 + \rho_1 dB_t^2) +$$

$$3X_t (X_t^2)(\sigma_1^2 + \rho_1^2)\, dt$$

$$d(Z_t) = 3X_t^3 [m_1(B_t) + \sigma_1^2(B_t) + \rho_1^2(B_t)]\, dt +$$

$$3X_t^3 \sigma_1(B_t)\, dB_t^1 + 3X_t^3 \cdot \rho_1(B_t)\, dB_t^2$$

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Exercise 2.2

$$d(X_tY_t) = X_t\,dY_t + Y_t\,dX_t + d\langle X,Y\rangle_t$$

Page 120: $d\langle B^i, B^j\rangle = 0,\ i \neq j$

$$d\langle B_t^1, B_t^2\rangle = (dB_t^1)(dB_t^2) = 0$$

$$d\langle B_t^i, B_t^j\rangle = dt,\ i = j$$

$$d\langle X,Y\rangle_t = (d(X_t))(dY_t)$$
$$= X_tY_t(\sigma_1\sigma_2 + \rho_1\rho_2)\,dt$$

$$X_t\,dY_t = X_tY_t(m_2\,dt + \sigma_2\,dB_t^1 + \rho_2\,dB_t^2)$$

$$Y_t\,dX_t = X_tY_t(m_1\,dt + \sigma_1\,dB_t^1 + \rho_1\,dB_t^2)$$

$$d(Z_t) = X_tY_t\big[m_1(B_t) + m_2(B_t) + \sigma_1(B_t)\sigma_2(B_t) + \rho_1(B_t)\rho_2(B_t)\big]dt + X_tY_t\big[\sigma_1(B_t) + \sigma_2(B_t)\big]dB_t^1 + X_tY_t\big[\rho_1(B_t) + \rho_2(B_t)\big]dB_t^2$$

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Exercise 3.1

Page 121:

$$df(t,\bar{X}_t) = \dot{f}(t,\bar{X}_t) + \nabla f(t,\bar{X}_t)\cdot d\bar{X}_t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i x_j} f(t,\bar{X}_t)\, d\langle X^i, X^j\rangle_t$$

$$df(t,\bar{Z}_t) = \dot{f}(t,Z_t) + \nabla f(t,Z_t)\cdot dZ_t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} f(t,Z_t)\, d\langle X, Y\rangle_t$$

$$f(t,Z_t) = e^{X_t+Y_t}$$

$$\dot{f}(t,Z_t) = 0$$

$$dX_t = dB_t^1 + dB_t^2 \quad , \quad dY_t = 2dB_t^1 - dB_t^2$$

$$d\langle X,X\rangle_t = (dX_t)^2 = (dB_t^1)^2 + (dB_t^2)^2 = 2\,dt$$

$$d\langle Y\rangle_t = (dY_t)^2 = (2dB_t^1)^2 + (-dB_t^2)^2 = 4dt + dt = 5dt$$

$$d\langle X,Y\rangle_t = (dX_t)(dY_t) = (dB_t^1)(2B_t^1) + (dB_t^2)(-dB_t^2) = 2dt - dt = dt$$

$$\nabla f(t,Z_t)\cdot dZ_t = \frac{df}{dX_t}(dX_t) + \frac{df}{dY_t}(dY_t)$$

$$= e^{X_t+Y_t}\cdot dX_t + e^{X_t+Y_t}\, dY_t$$

$$= e^{X_t+Y_t}\left[dB_t^1 + dB_t^2 + 2B_t^1 - dB_t^2\right] = e^{X_t+Y_t}\left[3dB_t^1\right]$$

$$\frac{d^2f}{dX} = \frac{d^2f}{\partial^2 Y} = e^{X_t+Y_t} = \frac{d^2f}{dXdY}$$

$$\frac{d^2f}{dX}\cdot(dX_t)^2 = e^{X_t+Y_t}[2dt] \quad , \quad \frac{d^2f}{dY}(dY_t)^2 = e^{X_t+Y_t}[5dt]$$

$$\frac{d^2f}{dXdY}(d\langle X,Y\rangle_t) = e^{X_t+Y_t}(dt)$$

$$d(Z_t) = \nabla f\, dZ_t + \frac{1}{2}\left[\frac{d^2f}{dX^2}(dX_t)^2 + \frac{d^2f}{dY^2}(dY_t)^2 + 2\frac{d^2f}{dXdY}\, d\langle X,Y\rangle_t\right]$$

$$d(Z_t) = e^{X_t+Y_t}\cdot 3dB_t^1 + \frac{1}{2}(e^{X_t+Y_t})(2+5+2)\,dt$$

$$d(Z_t) = \frac{9}{2}e^{X_t+Y_t}\,dt + 3e^{X_t+Y_t}\,dB_t^1 + 0\, dB_t^2 \quad //$$

Exercise 3.2

$$Z_t = \int_0^t e^{X_s + Y_s}\, ds$$

$$dZ_t = e^{X_t + Y_t}\, dt$$

$$dZ_t = e^{X_t + Y_t}\, dt + 0 \cdot dB_t^1 + 0 \cdot dB_t^2$$

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Exercise 3.3

$$f(t, X_t) = Z_t = \exp\{t \cdot X_t\} = e^{t \cdot X_t}$$

$$\dot{f}(t, X_t) = X_t e^{t \cdot X_t}$$

$$f'(t, X_t) = t e^{t \cdot X_t}, \quad f''(t, X_t) = t^2 e^{t \cdot X_t}$$

$$d(Z_t) = \dot{f}(t, X_t)\, dt + \nabla f(t, X_t) \cdot dX_t + \frac{1}{2}\left(\frac{d^2 f}{dX} \cdot (dX_t)^2\right)$$

$$\nabla f\, dX_t = t \cdot e^{t \cdot X_t}(dB_t^1 + dB_t^2)$$

$$\frac{1}{2}\frac{d^2 f}{dX}(dX_t)^2 = \frac{1}{2} \cdot t^2 e^{t \cdot X_t}(2dt) = t^2 e^{t \cdot X_t}\, dt$$

$$(dX_t)^2 = dt + dt = 2dt$$

$$d(Z_t) = X_t \cdot e^{t \cdot X_t}\, dt + t e^{t \cdot X_t}(dB_t^1 + dB_t^2) + t^2 e^{t \cdot X_t}\, dt$$

$$d(Z_t) = e^{t \cdot X_t}(X_t + t^2)\, dt + t e^{t \cdot X_t}\, dB_t^1 + t e^{t \cdot X_t}\, dB_t^2$$

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Exercise 4.1

$$dX_t = dt + dB_t$$

$$d\langle X\rangle_t = (dX_t)^2 = (dB_t)^2 = dt$$

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle X\rangle_t$$

$$= f'(X_t)[dt + dB_t] + \frac{1}{2} \cdot f''(X_t)\,dt$$

$$df(X_t) = \left[f'(X_t) + \frac{1}{2} f''(X_t)\right] dt + f'(X_t)\,dB_t$$

Page 56 → for brownian motion to be martingale, $\overset{\text{drift}}{m} = 0$

$$f'(X_t) + \frac{1}{2} f''(X_t) = 0$$

$$g(X_t) + \frac{1}{2} g'(X_t) = 0$$

$$\frac{1}{2} g'(X_t) = -g(X_t)$$

$$\frac{g'(X_t)}{g(X_t)} = -2$$

$$\int \frac{g'(X_t)}{g(X_t)} = \int -2$$

$$\ln[g(X_t)] = -2X_t + C$$

$$g(X_t) = e^{C-2X_t} = C \cdot e^{-2X_t}$$

$$f'(X_t) = C \cdot e^{-2X_t}$$

$$\int f'(X_t) = \int C e^{-2X_t} = \frac{C}{-2} e^{-2X_t} + D$$

$$f(X_t) = -\frac{C}{2} \cdot e^{-2X_t} + D$$

$$f(0) = 0$$

$$-\frac{C}{2} e^{-2(0)} + D = 0$$

$$-\frac{C}{2} + D = 0$$

$$D = C/2$$

$$f(X_t) = -\frac{C}{2} e^{-2X_t} + \frac{C}{2}$$

$$f(X_t) = \frac{C}{2}\left(1 - e^{-2X_t}\right) = M_t$$

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where C is some constant

Exercise 4.2

$$d f(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, d\langle X\rangle_t$$

$$= f'(X_t)\,(X_t\, dt + dB_t) + \frac{1}{2} f''(X_t)\, dt$$

$$= \left[X_t f'(X_t) + \frac{1}{2} f''(X_t)\right] dt + f'(X_t)\, dB_t$$

$$X_t f'(X_t) + \frac{1}{2} f''(X_t) = 0$$

$$X_t \cdot g(X_t) + \frac{1}{2} g'(X_t) = 0$$

$$\frac{1}{2} g'(X_t) = - X_t \cdot g(X_t)$$

$$\frac{g'(X_t)}{g(X_t)} = -2X_t$$

$$\int \frac{g'(X_t)}{g(X_t)} = \int -2X_t$$

$$\ln[g(X_t)] = -X_t^2 + C$$

$$g(X_t) = e^{C - X_t^2} = C \cdot e^{-X_t^2}$$

$$f'(X_t) = C \cdot e^{-X_t^2}$$

$$\int f'(X_t) = \int C \cdot e^{-X_t^2}\, d(X_t)$$

$$f(X_t) = C \cdot \int_0^{X_t} e^{-y^2}\, dy = M_t$$, where c is a constant

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Exercise 4.3

$$d f(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t) \cdot d\langle X\rangle_t$$

$$= f'(X_t)\left[\frac{1}{X_t} dt + dB_t\right] + \frac{1}{2} f''(X_t)\, dt$$

$$= \left[\frac{1}{X_t} f'(X_t) + \frac{1}{2} f''(X_t)\right] dt + f'(X_t)\, dB_t$$

$$\frac{1}{X_t} g(X_t) + \frac{1}{2} g'(X_t) = 0$$

$$\frac{1}{2} g'(X_t) = -\frac{1}{X_t} g(X_t)$$

$$\frac{g'(X_t)}{g(X_t)} = -\frac{2}{X_t}$$

$$\int \frac{g'(X_t)}{g(X_t)} = \int -\frac{2}{X_t} = \int -2 X_t^{-1}$$

$$\ln[g(X_t)] = -2\ln(X_t) + c$$

$$g(X_t) = e^{c - 2\ln(X_t)} = c e^{-2\ln(X_t)}$$

$$g(X_t) = c \cdot (X_t)^{-2} = \frac{c}{X_t^2}$$

$$f'(X_t) = c / X_t^2$$

$$\int f'(X_t) = \int \frac{c}{X_t^2} = -c \cdot X_t^{-1} + D$$

$$f(X_t) = -\frac{c}{X_t} + D$$

$$f(1) = 0$$

$$f(1) = -\frac{c}{1} + D = 0$$

$$c = D$$

$$f(X_t) = -\frac{c}{X_t} + c$$

$$f(X_t) = c\left(1 - \frac{1}{X_t}\right) = |Y|_t$$

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Where c is some constant.

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