Problem Set 7
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Exercise 1 (Book Exercise 5.1)

Continuous distributions: Normal, Uniform, and Exponential distributions
Discrete distributions: Binomial, Poisson distributions

Class Notes page 145: If v is absolutely continuous with respect to μ $(v \ll \mu)$, then, if $\mu(A) = 0, v(A) = 0$, for all $A \in \mathcal{F}$
This means that for absolute continuity $(\mu_j \ll \mu_k)$:
If μ_k assign a zero probability for a set, μ_j must also assigns a zero probability for that set.

X_1, X_4, X_5 follow continuous distributions
X_2, X_3 follow discrete distributions
X_6 follows a discrete distribution because it is a mixture of discrete and continuous random variables

X_1 and X_2:
μ_1 can assign probabilities to all numbers in $\mathbb{R}$ (continuous distribution)
μ_2 can only assign probabilities to $\{0, 1, 2, \ldots, 6,7\}$
There are sets where μ_2 assigns a 0 probability, but μ_1 assign a positive probability.
μ_1 NOT $\ll$ μ_2

A discrete distribution cannot be absolutely continuous to any distribution
μ_2 is not absolutely continuous with respect to μ_1 → **μ_2 NOT $\ll$ μ_1**

X_1 and X_3:
μ_1 can assign probabilities to all numbers in $\mathbb{R}$ (continuous distribution)
μ_3 will assign 0 probabilities to non-integers (discrete distribution)
There are sets where μ_3 assigns a 0 probability, but μ_1 assign a positive probability.
μ_1 NOT $\ll$ μ_3

A discrete distribution cannot be absolutely continuous to any distribution
μ_3 is not absolutely continuous with respect to μ_1 → **μ_3 NOT $\ll$ μ_1**

X_1 and X_4:
X_4 derives its value from X_1 (a continuous random variable).
If μ_1 assign a 0 probability for a set, μ_4 must also assigns a 0 probability for that set.
Therefore, μ_4 is absolutely continuous with respect to μ_1 → **$\mu_4 \ll \mu_1$**

μ_4 is an exponential distribution, which means it assigns 0 probabilities for negative values
μ_1 can assign probabilities to all numbers in $\mathbb{R}$ (continuous distribution)

There are sets where μ_4 assigns a 0 probability, but μ_1 assigns a positive probability.
μ_1 is NOT absolutely continuous with respect to μ_4 → $\boldsymbol{\mu_1}$**NOT** $\ll$ $\boldsymbol{\mu_4}$

X_1 and X_5:
μ_1 can assign probabilities to all numbers in $\mathbb{R}$ (continuous distribution)
μ_5 is continuous on [0,1].
If μ_1 assigns 0 probability for an set, μ_5 must also assign a 0 probability for that set.
Therefore, μ_5 is absolutely continuous with respect to μ_1 → $\boldsymbol{\mu_5 \ll \mu_1}$

There are sets where μ_5 assigns a zero probability (outside [0,1]), but μ_1 assigns a positive probability for those sets → $\boldsymbol{\mu_1}$ **NOT** $\ll$ $\boldsymbol{\mu_5}$

X_1 and X_6:
μ_1 can assign probabilities to all sets in $\mathbb{R}$ (continuous distribution)
μ_6 cannot assign probabilities to all sets in $\mathbb{R}$ (has discrete properties)
There are sets where μ_6 assigns 0 probability, but μ_1 assigns a positive probability
$\boldsymbol{\mu_1}$ **NOT** $\ll$ $\boldsymbol{\mu_6}$

μ_6 has both continuous and discrete properties
μ_6 is not absolutely continuous with respect to μ_1 → $\boldsymbol{\mu_6}$ **NOT** $\ll$ $\boldsymbol{\mu_1}$

X_2 and X_3:
A discrete distribution cannot be absolutely continuous to any distribution
Discrete vs discrete → $\boldsymbol{\mu_2}$ **NOT** $\ll$ $\boldsymbol{\mu_3}$ and $\boldsymbol{\mu_3}$ **NOT** $\ll$ $\boldsymbol{\mu_2}$

X_2 and X_4:
A discrete distribution cannot be absolutely continuous to any distribution
μ_2 is not absolutely continuous with respect to μ_4 → $\boldsymbol{\mu_2}$ **NOT** $\ll$ $\boldsymbol{\mu_4}$

μ_4 is a continuous distribution
μ_2 can only assign probabilities to $\{0, 1, 2, \dots, 6, 7\}$ (discrete distribution)
There are sets where μ_2 assigns 0 probability but μ_4 assign positive probability
μ_4 is not absolutely continuous with respect to μ_2 → $\boldsymbol{\mu_4}$ **NOT** $\ll$ $\boldsymbol{\mu_2}$

X_2 and X_5:
A discrete distribution cannot be absolutely continuous to any distribution: $\boldsymbol{\mu_2}$ **NOT** $\ll$ $\boldsymbol{\mu_5}$

μ_5 can assign probabilities to integers and non-integers in [0,1]
μ_2 can only assign probabilities to integers $\{0, 1, 2, \dots, 6, 7\}$

$$\mu_2(0.1) = \mu_2(0.3) = 0$$

There are sets where μ_2 assigns 0 probability but μ_5 assigns positive probability (Because μ_5 is continuous on all points between [0,1])
$\boldsymbol{\mu_5}$ **NOT** $\ll$ $\boldsymbol{\mu_2}$

$\boldsymbol{X_2}$ **and** $\boldsymbol{X_6}$**:**
A discrete distribution cannot be absolutely continuous to any distribution
$\boldsymbol{\mu_2}$ **NOT** $\ll$ $\boldsymbol{\mu_6}$

μ_6 has both continuous and discrete properties
μ_6 is not absolutely continuous with respect to μ_2 → $\boldsymbol{\mu_6}$ **NOT** $\ll$ $\boldsymbol{\mu_2}$

$\boldsymbol{X_3}$ **and** $\boldsymbol{X_4}$**:**
A discrete distribution cannot be absolutely continuous to any distribution
$\boldsymbol{\mu_3}$ **NOT** $\ll$ $\boldsymbol{\mu_4}$

μ_4 is a continuous distribution
μ_3 cannot assign probabilities to all sets in $\mathbb{R}$ (discrete distribution)
There are sets where μ_3 assigns 0 probability but μ_4 assign positive probability
μ_4 is not absolutely continuous with respect to μ_3 → $\boldsymbol{\mu_4}$ **NOT** $\ll$ $\boldsymbol{\mu_3}$

$\boldsymbol{X_3}$ **and** $\boldsymbol{X_5}$**:**
A discrete distribution cannot be absolutely continuous to any distribution: $\boldsymbol{\mu_3}$ **NOT** $\ll$ $\boldsymbol{\mu_5}$

μ_5 can assign probabilities to integers and non-integers in [0,1]
μ_3 will assign 0 probabilities to non-integers (discrete distribution)
There are sets where μ_3 assigns 0 probability but μ_5 assign positive probability
μ_5 is not absolutely continuous with respect to μ_3 → $\boldsymbol{\mu_5}$ **NOT** $\ll$ $\boldsymbol{\mu_3}$

$\boldsymbol{X_3}$ **and** $\boldsymbol{X_6}$**:**
A discrete distribution cannot be absolutely continuous to any distribution
$\boldsymbol{\mu_3}$ **NOT** $\ll$ $\boldsymbol{\mu_6}$

μ_6 has both continuous and discrete properties
μ_6 is not absolutely continuous with respect to μ_3 → $\boldsymbol{\mu_6}$ **NOT** $\ll$ $\boldsymbol{\mu_3}$

$\boldsymbol{X_4}$ **and** $\boldsymbol{X_5}$**:**
μ_4 is continuous distribution that assigns 0 probabilities for negative values
μ_5 can assign probabilities to numbers in [0,1]
There are no sets where μ_4 assigns 0 probability but μ_5 assign positive probability

$$\boldsymbol{\mu_5 \ll \mu_4}$$

μ_5 assigns 0 probabilities for values not in [0,1]
μ_4 is continuous distribution that assigns 0 probabilities for negative values
There are sets where μ_5 assigns 0 probability but μ_4 assign positive probability

$$\boldsymbol{\mu_4\ NOT \ll \mu_5}$$

X_4 and X_6:
μ_4 is continuous distribution that strictly assigns positive probability for positive values
μ_6 has both continuous and discrete properties
There are sets where μ_6 assigns 0 probability but μ_4 assign positive probability
μ_4 NOT $\ll$ μ_6

μ_4 is assigns 0 probability for negative values
μ_6 can assign positive probability for negative values
There are sets where μ_4 assigns 0 probability but μ_6 assign positive probability
μ_6 is not absolutely continuous with respect to μ_4 → **μ_6 NOT $\ll$ μ_4**

X_5 and X_6:
μ_5 assigns 0 probability for sets not in [0,1]
μ_6 has both continuous and discrete properties
There are sets where μ_5 assigns 0 probability but μ_6 assign positive probability
μ_6 NOT $\ll$ μ_5

μ_6 can assign 0 probabilities for values in [0,1]
μ_5 assigns positive probability for sets in [0,1]
There are sets where μ_6 assigns 0 probability but μ_5 assign positive probability
μ_5 NOT $\ll$ μ_6

Homework 7 - Matheus Raka - Stochastic

Exercise 2

2) 1) Page 13, for Martingale Betting Strategy:

$E[W_n] = 0$, W_n is a martingale : $E[W_{n+1} | F_n] = W_n$

$M_{n+1} = 1 - W_{n+1}$

$E[M_{n+1} | F_n] = 1 - E[W_{n+1} | F_n] = 1 - W_n = M_n \rightarrow M_n$ is a martingale

$$M_n = \begin{cases} 1 - W_n = 1 - 1 = 0 \\ 1 - W_n = 1 - [-2^n + 1] = 2^n \end{cases}$$ M_n is always nonnegative at each n

Therefore, M_n is a nonnegative martingale //

2) 2) From page 150 (5.4)

$Q_n(V) = E[1_V \cdot M_n] = E[E[1_V M_n | F_m]] = E[1_V E[M_n | F_m]]$

because M_n is a martingale, $E[M_n | F_m] = M_m$, for $m < n$

$Q_n(V) = E[1_V \cdot M_m]$:

$Q_n(V) = Q_m(V)$, for $m < n$ and V is F_m-measurable

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2) 3)

$$Q\{M_{n+1} = 2^{n+1} \mid M_n = 2^n\} = \frac{E[M_{n+1} = 2^{n+1} \cdot 1_{M_n = 2^n}]}{E[M_n \cdot 1_{M_n = 2^n}]}$$

$$W_n = \begin{cases} 1 & \text{with prob. } 1-2^{-n} \\ -[2^n - 1] & \text{with prob } \left(\frac{1}{2}\right)^n \end{cases}$$

$$M_n = \begin{cases} 0 & \text{with prob. } 1-2^{-n} \\ 2^n & \text{with prob. } \left(\frac{1}{2}\right)^n \end{cases}$$

$$E[M_n \cdot 1_{M_n = 2^n}] = 2^n \cdot 1 = 2^n$$

If $M_n = 2^n$, and at the $(n+1)$th round I lose again,

$M_{n+1} = 2^{n+1}$ with probability of losing again of $\frac{1}{2}$

$$E[M_{n+1} = 2^{n+1} \cdot 1_{M_n = 2^n}] = 2^{n+1} \cdot (1/2) = 2^n$$

$$Q\{M_{n+1} = 2^{n+1} \mid M_n = 2^n\} = \frac{2^n}{2^n} = 1 \quad //$$

2) 4) $Q(T<\infty) = ?$ The probability that the process reaches 0?

earlier, we saw that the process always grow

from $M_n = 2^n$, to $M_{n+1} = 2^{n+1}$

$Q\{M_{n+1} = 2^{n+1} \mid M_n = 2^n\} = 1.$

Therefore, there is 0 probability that the process will reach 0.

$Q(T<\infty) = 0$ //

2) 5) $E_Q[M_{n+1} \mid \mathcal{F}_n] = E_Q[M_{n+1} \mid M_n = 2^n]$

$= 2^{n+1}$ → the process always becomes 2^{n+1}

$E_Q[M_{n+1} \mid \mathcal{F}_n] = 2 \cdot 2^n = 2 \cdot M_n$, $M_n = 2^n$

$E_Q[M_{n+1} \mid \mathcal{F}_n] \neq M_n$

M_n is not a martingale with respect to the measure Q.

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Exercise 3

Girsanov Theorem: Page 153-154

$M_t = e^{Y_t}$, where $Y_t = \int_0^t A_s dB_s - \frac{1}{2}\int_0^t A_s^2 ds$, $\frac{dQ}{dP} = M_t$

$dW_t = -A_t dt + dB_t$ where W is a Brownian motion under Q

Case 1

$dX_t = 2dt + dB_t \longrightarrow dW_t = dX_t$

$-A_t = 2 \rightarrow A_t = -2$, $A_t^2 = 4$

$Y_t = \int_0^t -2 dB_s - \frac{1}{2}\int_0^t 4 ds = -2\int_0^1 dB_s - 2\int_0^1 ds = -2\left[B_1 - \overset{\nearrow 0}{B_0} + 1 - 0\right]$

$Y_t = -2B_1 - 2$

$\frac{dQ}{dP} = M_t = e^{Y_t} = e^{-2B_1 - 2}$, at $t=1$ //

There is Q such that X_t is a standard Brownian motion under Q

Case 2

$dX_t = 2dt + 6dB_t$

$\frac{1}{6}dX_t = \frac{1}{3}dt + dB_t = dW_t$

$-A_t = 1/3$

$A_t = -1/3$, $A_t^2 = 1/9$

$Y_t = \int_0^t -\frac{1}{3}dB_s - \frac{1}{2}\int_0^t \frac{1}{9}ds = -\frac{1}{3}(B_1 - B_0) - \frac{1}{18}(1-0)$

$Y_t = -\frac{1}{3}B_1 - \frac{1}{18}$

$\frac{dQ}{dP} = M_t = \exp\left\{-\frac{1}{3}B_1 - \frac{1}{18}\right\}$ at $t=1$ //

there is a probability measure Q such that X_t is a standard Brownian motion under Q. //

Exercise 3) Case 3

$dX_t = 2B_t\,dt + dB_t = dW_t$

$-A_t = 2B_t$

$A_t = -2B_t \,; \quad A_t^2 = 4B_t^2$

$Y_t = \int_0^t A_s\,dB_s - \frac{1}{2}\int_0^t A_s^2\,ds = \int_0^t -2B_s\,dB_s - \frac{1}{2}\int_0^t 4B_s^2\,ds$

$Y_t = -2\int_0^t B_s\,dB_s - 2\int_0^t B_s^2\,ds$

$\frac{dQ}{dP} = \exp\left\{-2\int_0^t B_s\,dB_s - 2\int_0^t B_s^2\,ds\right\}$

There is no equivalent probability measure Q such ~~that~~ that X_t is a standard Brownian motion in the new measure.

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Exercise 4.1)

Ito's Lemma: $d\langle X,Y\rangle_t = A_t \cdot C_t \cdot dt = (dX_t)(dY_t)$

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + d\langle X,Y\rangle_t$$

$$M_t = X_t Y_t = X_t \cdot \exp\left\{\int_0^t g(B_s)\,ds\right\}$$

$$X_t = e^{-m \cdot B_t^2} = f(B_t)$$

$$f'(B_t) = -m \cdot 2B_t e^{-m \cdot B_t^2} = -2mB_t X_t$$

$$f''(B_t) = -2me^{-mB_t^2} + (-2mB_t)^2 e^{-m \cdot B_t^2} = -2mX_t + 4m^2B_t^2X_t$$

$$= -2mX_t[1-2mB_t^2] = 2mX_t[2mB_t^2-1]$$

$dX_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$ → Ito's formula I

$$dX_t = -2mB_tX_t dB_t + m \cdot X_t[2mB_t^2-1]\,dt$$

$Y_t = \exp\left\{\int_0^t g(B_s)\,ds\right\}$ → not Ito's integral

$dY_t = g(B_t) \cdot Y_t\,dt$, $(dX_t)(dY_t) = 0$ → no dB_t term on dY_t

$$d(X_tY_t) = X_t \cdot g(B_t)Y_t\,dt + Y_t \cdot \left[-2mB_tX_t\,dB_t + mX_t[2mB_t^2-1]\,dt\right]$$

$$d(X_tY_t) = X_tY_t[g(B_t) + 2m^2B_t^2 - m]\,dt - 2X_tY_t \cdot mB_t\,dB_t$$

For M_t to be a local martingale, drift should be 0 (page 56):

$$g(B_t) + 2m^2B_t^2 - m = 0$$

$$g(B_t) = m - 2m^2B_t^2 \quad //$$

4.2) If M_t is a local martingale, the SDE that M_t satisfies is:

$$d(M_t) = -2X_tY_t \cdot m \cdot B_t\,dB_t$$

$$d(M_t) = -2m \cdot B_t \cdot M_t\,dB_t \quad //$$

4.3) Using Girsanov's theorem (page 154), since M_t is a nonnegative martingale, $M_t = \frac{dQ}{dP}$

$$dM_t = A_t M_t \, dB_t = -2mB_t M_t \, dB_t$$

$$A_t = -2m \cdot B_t$$

$dB_t = A_t dt + dW_t = -2m \cdot B_t dt + dW_t$, where W is a Q-Brownian motion.

↳ this is the SDE satisfied by B_t with respect to a Q-Brownian motion.

4.4) (Theorem 5.3.2) page 156

For $M_t = e^{Y_t}$ to be a martingale, $E\left(e^{\langle Y\rangle_t/2}\right) < \infty$

Page 110: $\langle Y\rangle_t = \int_0^t A_s^2 \, ds$

For $M_t = e^{Y_t}$, $Y_t = \int_0^t A_s \, dB_s - \frac{1}{2}\int_0^t A_s^2 \, ds$

$$Y_t = \int_0^t -2mB_s \, dB_s - \frac{1}{2}\int_0^t 4m^2 B_s^2 \, ds$$

$$\langle Y\rangle_t = \int_0^t A_s^2 \, ds = \int_0^t 4m^2 B_s^2 \, ds = 4m^2 \int_0^t B_s^2 \, ds$$

$$E\left[\exp\left\{\frac{1}{2} \cdot 4m^2 \int_0^t B_s^2 \, ds\right\}\right] = E\left[\exp\left\{2m^2 \int_0^t B_s^2 \, ds\right\}\right] < \infty$$

the Novikov condition holds for finite t ∴

M_t is actually a martingale, not just a local martingale.

Exercise 5.1

$X_t = B_t^r = f(B_t)$

$f'(B_t) = r \cdot B_t^{r-1}$, $f''(B_t) = r(r-1) B_t^{r-2}$

$dX_t = f'(B_t)\, dB_t + \frac{1}{2} f''(B_t)\, dt = r \cdot B_t^{r-1} dB_t + \frac{1}{2} r(r-1) B_t^{r-2} dt$

$dX_t = r \cdot X_t \cdot B_t^{-1} dB_t + \frac{1}{2} r(r-1) X_t \cdot B_t^{-2} dt$

$Y_t = e^{\int_0^t g(B_s)\, ds}$

$dY_t = g(B_t) \cdot Y_t\, dt$, $d\langle X,Y\rangle_t = (dX_t)(dY_t) = 0$ because dY_t has no dB_t term

$d(X_t Y_t) = X_t \cdot dY_t + Y_t\, dX_t + d\langle X,Y\rangle_t$

$= X_t \cdot g(B_t) Y_t\, dt + Y_t \cdot X_t \left[r \cdot B_t^{-1} dB_t + \frac{1}{2} r(r-1) B_t^{-2} dt \right]$

$d(X_t Y_t) = X_t Y_t \left[g(B_t) + \frac{1}{2} r(r-1) B_t^{-2} \right] dt + X_t Y_t \left[r \cdot B_t^{-1} dB_t \right]$

For M_t to be a local martingale,

$g(B_t) + \frac{1}{2} r(r-1) B_t^{-2} = 0$

$g(B_t) = -\frac{1}{2} r(r-1) B_t^{-2}$ //

5.2) Since M_t is a local martingale, the SDE that M_t satisfies is:

$dM_t = M_t \left[r B_t^{-1} \right] dB_t$

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5.3) Using Girsanov's theorem, since M_t is a nonnegative martingale,

$$M_t = \frac{dQ}{dP}$$

$$dM_t = A_t \cdot M_t \, dB_t = \left[r \cdot B_t^{-1}\right] M_t \, dB_t$$

$$A_t = \frac{r}{B_t}$$

$$dB_t = A_t \, dt + dW_t$$

$$dB_t = \frac{r}{B_t} dt + dW_t$$, where W is a Q-brownian motion //

↳ this is the SDE satisfied by B_t with respect to a Q-brownian motion.

5.4) $dB_t = \frac{r}{B_t} dt + dW_t$ is a Bessel process

→ page 158

For the Bessel process, for $r \geq \frac{1}{2}$, $Q\{T = \infty\} = 1$,

meaning that the process will never reach 0, for $T = \min\{t : B_t = 0\}$.

Therefore, for $r \geq \frac{1}{2}$, $Q\{T < \infty\} = 0$ //

meaning the process will never reach 0. //