$P(B_t \geq r) = P(B_1 \geq r/\sqrt{t}) = 1 - \Phi(r/\sqrt{t}) = \int_{r/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$P(B_t \leq r) = P(B_1 \leq r/\sqrt{t}) = \Phi(r/\sqrt{t}) = P(Z \leq r/\sqrt{t})$

$B_t \sim N(0,t) : dP(B_t = x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$

$B_0 = 0,$

$B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$

$Z_t = (X_t, Y_t)$

cov matrix for Z_t

$$T_t = \begin{pmatrix} Var(X_t) & Cov(X_t, Y_t) \\ Cov(X_t, Y_t) & Var(Y_t) \end{pmatrix}$$

$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$

$P(B_8 \geq 0, B_4 \geq 0) = P(B_8 \geq 0 \mid B_4 \geq 0) \cdot P(B_4 \geq 0)$

$P\{\min_{0 \leq s \leq t} B_s \leq -a\} = P\{\max_{0 \leq s \leq t} B_s \geq a\} = 2P(B_t > a)$

$= \int_0^\infty P(B_8 \geq 0 \mid B_4 = x)\, dP(B_4 = x)$

$m_X(a) = E(e^{X \cdot a}) = \exp\{M \cdot a + \frac{1}{2}\sigma^2 a^2\}$

$Cov(aX+bY, cW+dV)$

$= ac\, Cov(X,W) + ad\, Cov(X,V) + bc\, Cov(Y,W) + bd\, Cov(Y,V)$

$E[|X|^p] = E[|\sigma \cdot Z|^p] = \sigma^p E[Z^p]$

$Z \sim N(0,1)$

$$E[(X-M)^p] = \begin{cases} 0 & , \text{odd } p \\ \sigma^p (p-1)!! & , \text{even } p \end{cases}$$

$n!! = (n)(n-2)\cdots(3)(1)$

$P(B_t \geq a \mid \max_{a \leq s \leq t} B_s \geq a) = \frac{1}{2}$

$\langle W \rangle_t = \sigma^2 t$, for $W_t \sim (mt, \sigma^2 t)$ BM

$\langle B^i, B^k \rangle_t = T_{ik} t$, for $B_t \sim (mt, Tt)$ BM

Ref $T_a = \min\{s : B_s \geq a\} = \min\{s : B_s = a\}$

$P(\max_{0 \leq s \leq t} B_s \geq a) = P(T_a \leq t) = P(T_a < t)$

$B_t \sim$ SBM

$\partial_t\, p_t(x) = \frac{1}{2} \partial_{xx}\, p_t(x)$

$\nabla^2 f = 0$

f is harmonic

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$S_n = X_1 + \cdots + X_n$

$S_n \sim N(0,n) \rightarrow E(S_n^2) = n$

$\frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$dX_t = X_t[m_1 dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2]$, B_t is 2-d SBM

$dY_t = Y_t[m_2 dt + \rho_1 dB_t^2 + \rho_2 dB_t^2]$, $B_t = (B_t^1, B_t^2)$

$B_t \sim BM(mt, \sigma^2 t)$

$\dot{p}_t(x) = -m p_t'(x) + \frac{\sigma^2}{2} p_t''(x)$

$\dot{p}_t(x) = L^* p_t(x)$, Forward

$d(Z_t) = \frac{df}{dX}(dX_t) + \frac{df}{dY}(dY_t) + \frac{1}{2}\left[\frac{d^2 f}{dX^2}(dX_t)^2 + \frac{d^2 f}{dY^2}(dY_t)^2 + 2\frac{d^2 f}{dX dY} d\langle X,Y \rangle_t\right]$

Backward

$\dot{p}_t(x) = L\, p_t(x)$

$= m \cdot p_t'(x) + \frac{\sigma^2}{2} p_t''(x)$

$\lim_{n\to\infty} \sum_{j=1}^{n} [Y(\frac{j}{n}) - Y(\frac{j-1}{n})]^2 = \langle Y \rangle$, $\lim_{n\to\infty} \sum_{j=1}^{n} [Y(\frac{j}{n}) - Y(\frac{j-1}{n})][X(\frac{j}{n}) - X(\frac{j-1}{n})] = \langle X,Y \rangle = \int dX_t dY_t$

B_t = d-dimensional BM, drift $\vec{m} = (m_1, \ldots, m_d)$ and Cov. mat T,

$Lf(x) = m \cdot \nabla f(x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} T_{jk} \cdot \partial_{jk} f(x)$ → matrix element

$L^* f(x) = -m \cdot \nabla f(x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} T_{jk} \cdot \partial_{jk} f(x)$ → 2nd deriv of f w.r.t. each vector component

Expected Value Back

$\phi(t,x) = E(f(B_t) \mid B_0 = x)$

$\dot{\phi}(t,x) = L\,\phi(t,x)$

$\dot{\phi}(t,x) = L^* \phi(t,x)$, for $t > 0$

$F(t,x) = E(F(B_T) \mid B_t = x) = \phi(T-t, x)$

$\dot{F}(t,x) = -L\, F(t,x)$ Back

Suppose $B_t = (B_t^1, \ldots, B_t^d)$ is a SBM in $\mathbb{R}^d$, then

$df(t, B_t) = \nabla f(t, B_t)\, dB_t + [\dot{f}(t, B_t) + \frac{1}{2}\nabla^2 f(t, B_t)]\, dt$

$\nabla^2 f(\bar{x}) = \sum_{j=1}^{d} \partial_{x_j x_j} f(\bar{x})$

Ito I: $f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$

$d\langle B^i, B^j\rangle = 0,\ i \neq j$

$d\langle B^i, B^j\rangle = dt,\ i = j$

$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$

$\int_0^t B_s dB_s = \frac{1}{2}[B_t^2 - \underbrace{B_0^2}_{\to 0} - t]$

X_t = GBM, m, σ vol

$X_t = X_0 \exp\{(m - \frac{\sigma^2}{2})t + \sigma B_t\}$

$\int \frac{f'(x)}{f(x)} = \ln[f(x)]$

$e^{2\ln(x)} = e^{[\ln(x)]^2} = x^2$

Ito II: $df(t,B_t) = f'(t,B_t)dB_t + [\dot{f}(t,B_t) + \frac{1}{2}f''(t,B_t)]dt$

$f(t,B_t) = f(0,B_0) + \int_0^t f'(t,B_s)dB_s + \int_0^t [\partial_s f(s,B_s) + \frac{1}{2}f''(s,B_s)]ds$

For $f(X_t) = M_t$

$df(X_t) = m \cdot dt + \sigma dB_t$

$m = 0$

X_t = GBM, drift m, vol σ

$dX_t = mX_t dt + \sigma X_t dB_t$

$(dB_t)^2 = dt$, $(dB_t)(dt) = 0$

$(dt)^2 = 0$

$Q_n = \sum_{j=1}^n [B(\frac{j}{n}) - B(\frac{j-1}{n})]^2 = \frac{1}{n}\sum_{j=1}^n Y_j$

$Y_j = [\frac{B(j/n) - B(j-1/n)}{1/\sqrt{n}}]^2$

$dX_t = R_t dt + A_t dB_t$

Ito III

$df(t,X_t) = [\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{A_t^2}{2}f''(t,X_t)]dt + A_t f'(t,X_t)dB_t$

$E(Q_n) = \frac{1}{n}\sum_{j=1}^n E(Y_j) = 1$

$\langle X\rangle_t = \int_0^t A_s^2 ds$

$d\langle X\rangle_t = A_t^2 dt$

expectation of Ito's integral = 0

$E[\int_0^t X_t dB_t] = 0$

one d, prop of SBM, SBM: $m=0, \sigma^2=1$

1) $B_0 = 0 \to$ start at origin

2) For $s<t$, $B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$

3) the r.v. $Y_t - Y_s$ is indep. of Y_r for $r<s$

↳ indep increment

4) W/ prob. 1, $t \to B_t$ is a cont. function of t

d-dimension, SBM d-dim: $m=0$, $T^{-1} = I$

2) for $s<t$, $B_t - B_s \sim$ JN mean $m(t-s)$, cov mat $T(t-s)$

$dX_t = H_t dt + A_t dB_t$

$dY_t = K_t dt + C_t dB_t$

$d\langle X,Y\rangle_t = A_t C_t dt = (dX_t)(dY_t)$

$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X,Y\rangle_t$

For $Z_t = \int_0^t A_s dB_s$

$dZ_t = A_t dB_t$

For $dX_t = R_t dt + A_t dB_t$

$d\langle X\rangle_t = A_t^2 dt = (dX_t)^2$

Ito III

$df(t,X_t) = \dot{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)d\langle X\rangle_t$

B_t is SBM, A_t, C_t simple process

1) Linearity: $\int_0^t (aA_s + bC_s)dB_s = a\int_0^t A_s dB_s + b\int_0^t C_s dB_s$, $\int_0^t A_s dB_s = \int_0^r A_s dB_s + \int_r^t A_s dB_s$

2) $Z_t = \int_0^t A_s dB_s$ is a martingale w.r.t. F_t

3) Var rule: Z_t is square integrable &

$Var(Z_t) = E[Z_t^2] = \int_0^t E[A_s^2]ds$

4) W/ prob 1, $t \to Z_t$ is a cont. func.

$f(X_t)$

$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t$

If $\bar{X}_t = (X_t^1, \dots, X_t^n)$, $\nabla f(t,\bar{X}_t) \cdot d\bar{X}_t = \sum_{k=1}^n \partial_{x_k} f(t,\bar{X}_t)dX_t^k$

Suppose $B_t^1, \dots, B_t^d$ are indep. SBM, and $X_t^1, \dots X_t^n$ are processes satisfying

$dX_t^k = H_t^k dt + \sum_{i=1}^d A_t^{i,k} dB_t^i$, then

$df(t,\bar{X}_t) = \dot{f}(t,\bar{X}_t)dt + \nabla f(t,\bar{X}_t)d\bar{X}_t + \frac{1}{2}\sum_{j=1}^n\sum_{k=1}^n \partial_{x_j x_k} f(t,\bar{X}_t)d\langle X^j, X^k\rangle_t$

$E[Y_j] = E[Z^2] = 1$

$E[Y_j^2] = E[Z^4] = 3$

$Var(Q_n) = \frac{2}{n}$

$P(B_t \le r) = P(Z < r/\sqrt{t}) = \Phi(r/\sqrt{t})$

$B_t \sim N(0,t)$, $dP(B_t = x) = \frac{1}{\sqrt{2\pi t}} e^{(-x^2/2t)} dx$

$B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$

$P(\min_{0\le s\le t} B_s \le -a) = P(\max_{0\le s\le t} B_s \ge a) = 2P(B_t \ge a)$

$P(B_s \le a \ \forall s \le t) = 1 - 2P(B_s \ge a \text{ for some } s \le t) = 1 - 2P(B_t \ge a)$

$P(B_8 \ge 0 \mid B_4 \ge 0) \cdot P(B_4 \ge 0) = \int_0^\infty P(B_8 \ge 0 \mid B_4 = x)\, dP(B_4 = x)$

B_t is SBM; A_t, C_t are simple processes

- Linearity: $\int_0^t (aA_s + bC_s)dB_s = a\int_0^t A_s dB_s + b\int_0^t C_s dB_s$
  $\int_0^t A_s dB_s = \int_0^r A_s dB_s + \int_r^t A_s dB_s$
- Martingale prop: $Z_t = \int_0^t A_s dB_s$ is martingale w.r.t. $\{F_t\}$
- Var rule: Z_t is square integrable, and $Var(Z_t) = E[Z_t^2] = \int_0^t E[A_s^2] ds$
- Continuity: w/ $p=1$, the function $t \to Z_t$ is a continuous function

$E(e^{aX}) = \exp\{\mu a + \frac{1}{2}\sigma^2 a^2\}$

$E(X-\mu)^p = \begin{cases} 0, & \text{odd } p \\ \sigma^p (p-1)!!, & \text{even } p \end{cases}$

$Z_t = (X_t, Y_t)$, $\Gamma_t = \begin{pmatrix} Var(X_t) & Cov(X_t,Y_t) \\ Cov(X_t,Y_t) & Var(Y_t) \end{pmatrix}$

$W_t \sim BM(mt, \sigma^2 t)$, $\langle W \rangle_t = \sigma^2 t$

Binom: $P(k \text{ success}) = {}_nC_k \cdot p^k \cdot (1-p)^{n-k}$

$X_{k\Delta t} = X_{(k-1)\Delta t} + m\Delta t + \sigma\sqrt{\Delta t}\, \hat{N}$ $(0,1)$

$P(X_{k\Delta t} - X_{k-1\,\Delta t} = a) = p$, $a = \sigma\sqrt{\Delta t}$, $E(X_{k\Delta t} - X_{k-1}) = m\Delta t$ $\hookrightarrow m\Delta t$

$B_t \sim (mt, \Gamma t)$ BM, $\langle B^i, B^k \rangle_t = \Gamma_{ik} t$

$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$

$Cov(aX+bY, cW+dV) = ac\, Cov(X,W) + ad\, Cov(X,W) + \ldots + bd\, Cov(Y,V)$

$Y_t = \exp\{\int_0^t g(B_s)ds\}$, $dY_t = g(B_t) Y_t \cdot dt$

Backward

Expected Value: $\phi = E(f(B_T) \mid B_0 = x)$

$\dot{p} = Lp(x) = mp' + \frac{\sigma^2}{2} p''$, $\dot{\phi} = L\phi$

$d(\int_0^t X_s^3 ds) = X_t^3 dt$

Not Ito: $d(Y_t) = d(\exp\{a\int_0^t X_s^2 ds\}) = a \cdot X_t^2 Y_t \cdot dt$

Forward

$\dot{p} = L^* p(x) = -mp' + \frac{\sigma^2}{2} p''$

$F = E[F(B_T) \mid B_t = x]$, $\dot{F} = -L F(t,x)$

$(dB_t)^2 = dt$, $(dB_t)(dt) = (dt)^2 = 0$

1-D properties of BM: 1) $B_0 = 0 \to$ start at origin
2) For $s<t$, $B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$
3) Indep. increment: the r.v. $(Y_t - Y_s)$ is indep. of Y_r for $r \le s$
4) Continuous paths: w/ $p=1$, $t \to B_t$ is a cont. function of t

$e^{2\ln(x)} = e^{(\ln(x))^2} = x^2$

$\int \frac{f'}{f} = \ln(f)$

d-dim BM, $\vec{m} = (m_1, \ldots, m_d)$, cov mat Γ

$Lf = m\nabla f + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d} \Gamma_{jk} f''$ $\to$ 2nd deriv wrt each vector comp. $\hookrightarrow$ matrix element

$L^* f = -m \cdot \nabla f + \text{—〃—}$, $\dot{\phi} = L^*\phi$

$\int dM_t = M_{t+1} - M_t$

d-dim 2) For $s<t$, $(B_t - B_s) \sim JN$ w/ mean $m(t-s)$, cov. mat. $\Gamma(t-s)$

SBM: $m=0$, $\Gamma = I$

$dX_t = m(X_t)dt + \sigma dB_t$, $\dot{\phi} = -m'\phi - m \cdot \phi' + \frac{\sigma^2}{2}\phi'' = L^*\phi$

$df(B_t) = f' dB_t + \frac{1}{2} f'' dt$

$df(t, B_t) = f' dB_t + [\dot{f} + \frac{1}{2} f''] dt$

$\langle X \rangle_t = \int_0^t A_s^2 ds$, $d\langle X \rangle_t = A_t^2 dt = (dX_t)^2$

$dX_t = R_t dt + A_t dB_t$

$df(t, X_t) = \dot{f} dt + f' dX_t + \frac{1}{2} f'' d\langle X \rangle_t$

$df(X_t) = f' dX_t + \frac{1}{2} f'' d\langle X \rangle_t$

GBM: $dX_t = X_t[m\,dt + \sigma\, dB_t]$

$X_t = X_0 \exp\{(m - \frac{\sigma^2}{2})t + \sigma B_t\}$

SBM: $m=0, \sigma^2 = 1$

GBM: $dX_t = X_t dW_t$, $X_t = X_0 \exp\{W_t - \frac{1}{2}t\}$

$(dX_t)(dY_t)$

$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t$

$dX_t = X_t[m_1 dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2]$, B_t is 2-d SBM

$dY_t = Y_t[m_2 dt + \rho_1 dB_t^1 + \rho_2 dB_t^2]$, $B_t = (B_t^1, B_t^2)$

$d\langle B^i, B^j \rangle = 0, i \ne j$

$d\langle B^i, B^j \rangle = dt, i = j$

$dZ_t = \frac{df}{dX} dX_t + \frac{df}{dY} dY_t + \frac{1}{2}\left[\frac{d^2 f}{dX^2}(dX_t)^2 + \frac{d^2 f}{dY^2}(dY_t)^2 + 2\frac{d^2 f}{dX\,dY} d\langle X, Y \rangle_t\right]$

$E[F(X_0)] = E[F(X_T)]$

$F(1) \cdot P(X_0 = 1) = F(0) \cdot P(X_T = 0) + F(3) \cdot P(X_T = 3)$

$X_0 = x$

$T = \min(t: X_t = -2 \text{ or } 3)$

$E(X_0) = E(X_T)$

$x = (-2)P(X_T = -2) + 3P(X_T = 3)$

$Q_n = \sum_{j=1}^{n}[B(\frac{j}{n}) - B(\frac{j-1}{n})]^2 = \frac{1}{n}\sum_{j=1}^{n} Y_j$, $E(Q_n) = 1$, $Var(Q_n) = \frac{2}{n}$, $E(Y_j) = 1$, $E(Y_j^2) = 3$

Bessel Process (BP)

$dX_t = \frac{a}{X_t}dt + dB_t$, $X_0 = x > 1$

$T = \min\{t : X_t = r \text{ or } R\}$

$\phi(x) = P(X_t = R \mid X_0 = x)$

If $a \neq 1/2$, $\phi(x) = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}$

If $a = 1/2$,

$\phi(x) = \frac{\ln(x) - \ln(r)}{\ln(R) - \ln(r)}$, $X_t > 0\ \forall t$ w/ $p = 1$

$a < 1/2$, BP reach 0 w/ $p = 1$, $P(T < \infty) = 1$

$a \geq 1/2$, BP never reach 0, $P(T = \infty) = 1$, $P(T < \infty) = 0$

SBM reach 0 w/ $p = 1$, $P(T < \infty) = 1$

MBS: $E[W_n] = 0$, $E[W_{n+1} \mid F_n] = W_n$

$W_n = \begin{cases} 1 & \text{w/ } p = 1 - (1/2)^n \\ (1 - 2^n) & \text{w/ } p = (1/2)^n \end{cases}$

$Q[M_{n+1} = 2^{n+1}] = E[\{1_{M_{n+1} = 2^{n+1}}\} M_{n+1}]$

$= E[\{1_{M_{n+1} = 2^{n+1}}\} 2^{n+1}] = 2^{n+1} P\{M_{n+1} = 2^{n+1}\}$

$= 2^{n+1} \cdot 2^{-(n+1)} = 1$

If any conditions hold, M_t is martingale:

- $P^*(T > t) = 1$
- $E[M_t] = 1$
- $E\left[\exp\left\{\frac{\langle Y \rangle_t}{2}\right\}\right] < \infty$

Girsanov (continued)

To find equivalent prob. measure Q s.t. X_t is SBM in Q, $\langle X \rangle_t = \langle B \rangle_t$, where $dX_t = \ldots dt + dB_t$

$\tilde{S}_t = \frac{S_t}{R_t} = e^{-rt} \cdot S_t$

$d\tilde{S}_t = \sigma_t \cdot \tilde{S}_t\, dW_t$

$a_t = \frac{A_t}{\sigma_t \cdot \tilde{S}_t}$

$b_t = \tilde{V}_t - \frac{A_t}{\sigma_t}$

$\tilde{V}_t = E_Q[\tilde{V} \mid F_t]$

$\int_{\tilde{S}_t}^{\tilde{S}_T} \frac{d\tilde{S}}{\tilde{S}^2} = \int_t^T dt$

$V_t = a_t S_t + b_t R_t = e^{rt} \cdot \tilde{V}_t$

$d\tilde{V}_t = A_t\, dW_t$

$P(T > t) = 1$

V is contingent claim if $V \geq 0$ & $E_Q[\tilde{V}^2] < \infty$

ex: $V = S_2^2$, $\tilde{V}_2 = \tilde{S}_2^2$, $\tilde{S}_2 = S_0 \exp\{W_2 - 2/2\}$

$R_t = e^{rt} R_0$

$E[\tilde{S}_2^2 \mid F_t] = S_0^2\, E((\exp\{2(W_2 - 1)\}) \mid F_t)$, $W_2 - W_t \sim N(0, 2 - t)$

$\tilde{V}_T = e^{-rT} \cdot V_T = \ldots \tilde{S}_t$ / $\tilde{S}_t = S_0 \exp\{W_t - t/2\}$

Feynman Kac: $F(x) = (x - S)^+$

$V_t = \phi = E[e^{-r(T-t)} F(X_T) \mid X_t = x]$

$dX_t = m\,dt + \sigma\, dB_t$

$\dot{\phi} = -m\phi' - \frac{1}{2}\sigma^2 \phi'' + r\phi$

terminal cond: $\phi(T, x) = F(x)$

$G(z) = P(e^{aN + y} < z) = \Phi\left(\frac{\ln(z) - y}{a}\right)$

$g(z) = G'(z) = \frac{1}{az}\phi\left(\frac{\ln z - y}{a}\right)$

FK: $dX_t = m \cdot dt + \sigma\, dB_t$

$\dot{f} = -L \cdot f$

$Lf = m \cdot f' + \frac{1}{2}\sigma^2 f''$

SBM is always a martingale

$\int_0^t B_s\, dB_s = \frac{1}{2}[B_t^2 - t]$

$\int_b^\infty \phi(w)\, dw = \Phi(-b)$

$\int_b^\infty e^{aw} \cdot \phi(w)\, dw = e^{a^2/2}\, \Phi(-b + a)$

Small support ≪ big support

discrete ≪ continuous

M_n is square integrable if $\forall n$, $E[M_n^2] < \infty$

$Z_n = \int_0^t A_s\, dB_s$

$Var(Z_t) = E[Z_t^2] = \int_0^t E(A_s^2)\, ds$

If $\mu_k(A) = 0$, $\mu_j(A) = 0$, $\mu_j \ll \mu_k$

μ_j & μ_k r singular measures: $\mu \perp V$, $\mu_j(E) = 0$, $\mu_k(\Omega \setminus E) = 0$

If $\langle M \rangle_t = t$, M_t cont mart., $M_0 = 0$ ⟹ SBM

Lebesgue measure, μ_L, μ_L = length of V, $\mu_L(\mathbb{R}) = \infty$

Cont.: normal: $(-\infty, \infty)$

$X_2 = e^{X_1} \to$ Lognormal: $[0, \infty)$

exponen: ↗

unif: $[a, b]$

0 for individual point

Discrete

Binom: $[0, 1, \ldots, n]$

Poisson: $[0, \infty)$

0 for non-integer

Girsanov: For M_t nonnegative mart.

$dM_t = A_t M_t\, dB_t$, $M_0 = 1$

$\frac{dQ}{dP} = M_t = e^{Y_t}$, $Y_t = \int_0^t A_s\, dB_s - \frac{1}{2}\int_0^t A_s^2\, ds$

$dW_t = -A_t\, dt + dB_t$, W is SBM in Q

$dB_t = A_t\, dt + dW_t$,

B_t is not SBM in Q, it has drift A_t

$Q_n(V) = E[1_V M_n] = Q_m(V) = Q(V)$

$Z_t = \int_0^t A_s\, dB_s$

Z_t is martingale if $\int_0^t A_s^2\, ds < \infty$

Novikov ↗

$\langle Z \rangle_t = \int_0^t A_s^2\, ds$

$E[Y_0] = E[Y_n]$,

obtain $P(Y_n = a)$

$Q(Y_n = a) = \frac{E[1_{Y_n = a} \cdot Y_n]}{E[Y_n]}$

$a \cdot P(Y_n = a)$

Wiener measure $\begin{cases} \langle B^1 \rangle_t = \langle B^2 \rangle_t, & \text{mutually abs cont.} \\ \langle B^1 \rangle_t \neq \langle B^2 \rangle_t, & \text{singular measures} \end{cases}$