Exercise 1.1, HW2 Stochastic, Matheus Raka Pradnyatama

1.1) $f\left(\frac{j}{n}\right) = \frac{j^2}{n^2}$, $f\left(\frac{j-1}{n}\right) = \frac{(j-1)^2}{n^2} = \frac{j^2-2j+1}{n^2}$

$$f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) = \frac{j^2-(j^2-2j+1)}{n^2} = \frac{2j-1}{n^2}$$

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left|\frac{2j-1}{n^2}\right|^{5/4} = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{(2j-1)^{5/4}}{n^{5/2}}$$

as $n \to \infty$,

$$Q \longrightarrow \frac{\sum_{j=1}^{\infty} (2j-1)^{5/4}}{\infty} = 0$$

//

as $n \to \infty$, Q goes to 0

//

Exercise 1.2

For a standard Brownian motion $B_t \sim N(0,t)$,

$B_t - B_s \sim N(0, (t-s))$

$B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \sim N\left(0, \frac{j}{n} - \frac{j-1}{n}\right) = N\left(0, \frac{1}{n}\right) \to \sigma^2 = \frac{1}{n}$

To scale $X \sim N(0, \sigma^2)$ to $Z \sim N(0,1)$

$Z = \frac{X-\mu}{\sigma} = \frac{X-0}{\sigma} = \frac{X}{\sigma} \to X = \sigma \cdot Z$

$\sigma = \left(\frac{1}{n}\right)^{1/2}$

$E[|X|^p] = E[|\sigma Z|^p] = \sigma^p \cdot E[|Z|^p]$

$$E\left[\left|f\left(\tfrac{j}{n}\right) - f\left(\tfrac{j-1}{n}\right)\right|^{5/4}\right] = E\left[\left|B\left(\tfrac{j}{n}\right) - B\left(\tfrac{j-1}{n}\right)\right|^{5/4}\right] = \sigma^{5/4} E\left[|Z|^{5/4}\right]$$

$$= \left(\left(\frac{1}{n}\right)^{1/2}\right)^{5/4} \cdot E\left[|Z|^{5/4}\right] = \left(\frac{1}{n}\right)^{5/8} E\left[|Z|^{5/4}\right]$$

where $Z \sim N(0,1)$

$$E(Q) = \sum_{j=1}^{n} E\left[\left|f\left(\tfrac{j}{n}\right) - f\left(\tfrac{j-1}{n}\right)\right|^{5/4}\right] = (n)\left(\frac{1}{n}\right)^{5/8} E\left[|Z|^{5/4}\right]$$

$E(Q) = n^{3/8} E\left[|Z|^{5/4}\right] \to \infty$ as $n \to \infty$

$Q = \lim_{n \to \infty} E(Q) = \infty$ //

Q diverges as $n \to \infty$ //

Stochastic Calculus – Homework 2
Matheus Raka Pradnyatama

Exercise 2.1

$$Y_t = 2B_t - W_t - t$$

(Page 68 of class notes) There are 4 properties to prove:

1. $Y_0 = 0$
2. If $s < t$, the distribution of $Y_t - Y_s$ is joint normal with mean $m(t-s)$ and covariance matrix $(t-s)\Gamma$
3. If $s < t$, the random vector $Y_t - Y_s$ is independent of $\mathcal{F}_s$
4. With probability one, the function $t \to Y_t$ is continuous

Property 1

Since B_t is a standard Brownian motion, $B_0 = 0$
Since W_t is a standard Brownian motion, $W_0 = 0$

$$Y_0 = 2B_0 - W_0 - 0 = 2 * 0 - 0 = 0$$

We have proven that $Y_0 = 0$, meaning that Y_t starts at the originhe first property is satisfied.

Property 2

$$Y_t - Y_s = 2B_t - W_t - t - 2B_s + W_s + s$$
$$Y_t - Y_s = 2(B_t - B_s) - (W_t - W_s) - (t - s)$$

The distribution of $(Y_t - Y_s)$ depends on the distribution of $(B_t - B_s)$ and $(W_t - W_s)$.

Since B_t is a standard Brownian motion, $B_t - B_s$ has variance $(t - s)$
Since W_t is a standard Brownian motion, $W_t - W_s$ has variance $(t - s)$
$(t - s)$ is a constant so it has a variance of 0

$Var(Y_t - Y_s) = Var\big(2(B_t - B_s)\big) + Var\big(-(W_t - W_s)\big) + Var\big(-(t - s)\big)$
$Var(Y_t - Y_s) = 4Var(B_t - B_s) + Var(W_t - W_s) + 0$
$Var(Y_t - Y_s) = 4(t - s) + (t - s) = 5(t - s)$

$$E(Y_t - Y_s) = m(t - s) = -1(t - s)$$

Since Y_t is a one-dimensional Brownian motion, the covariance matrix is equal to the variance of Y_t.

Since the distributions of $(B_t - B_s)$ and $(W_t - W_s)$ are normal with mean 0 and variance $(t - s)$, the distribution of $(Y_t - Y_s)$ is joint normal with mean $-1(t - s)$ and covariance matrix $(t - s) * 5$. The second property is satisfied.

Property 3

$$Y_t - Y_s = 2(B_t - B_s) - (W_t - W_s) - (t - s)$$

The random vector $Y_t - Y_s$ depends on the values of $(B_t - B_s)$, $(W_t - W_s)$, and $(t - s)$.

Since B_t and W_t are Brownian motions, $(B_t - B_s)$ and $(W_t - W_s)$ don't depend on the information up to time s, i.e. $\mathcal{F}_s$.
$(B_t - B_s)$ and $(W_t - W_s)$ are independent of $\mathcal{F}_s$.
The value of $(t - s)$ is independent of $\mathcal{F}_s$.

Since all of the components are independent of $\mathcal{F}_s$, $Y_t - Y_s$ is also independent of $\mathcal{F}_s$. The third property is satisfied.

Property 4
Since B_t and W_t are standard Brownian motions, both are continuous functions of t. Since Y_t is merely a linear combination of B_t and W_t, a linear combination of continuous functions of t, is also a continuous function of t. Therefore, with probability one, the function $t \to Y_t$ is continuous. The fourth property is satisfied.

Conclusion: Since Y_t satisfies all 4 properties, Y_t is a one-dimensional Brownian motion starting at the origin ($Y_0 = 0$).

$$Y_t = 2B_t - W_t - t$$

Since B_t and W_t are standard Brownian motions, B_t and W_t has mean 0 and variance t
t is a constant so it has a variance of 0.

$$E(Y_t) = E(2B_t) - E(W_t) - E(t) = 0 + 0 - t$$
$$E(Y_t) = -t$$
$$m = -1$$

$$Var(Y_t) = Var(2B_t) + Var(-W_t) + Var(-t) = 4Var(B_t) + Var(W_t) + 0 = 4t + t$$
$$Var(Y_t) = 5t$$
$$\sigma^2 = 5$$

For Y_t, drift (m) is -1, variance parameter (σ^2) is 5.

Exercise 2.2

$$Z_t = (Z_t^1, Z_t^2)$$
$$Z_t^1 = Y_t + t = 2B_t - W_t - t + t$$
$$Z_t^1 = 2B_t - W_t$$
$$Z_t^2 = -B_t + 3W_t$$

Property 1

$$Z_0^1 = 2B_0 - W_0 = 2 * 0 + 0 = 0$$
$$Z_0^2 = -B_0 + 3W_0 = 0 + 3 * 0 = 0$$
$$Z_0 = (Z_0^1, Z_0^2) = (0, 0)$$

We have proven that $Z_0 = (0, 0)$, meaning that Z_t starts at the origin, and that the first property is satisfied.

Property 2

$$Z_t - Z_s = (Z_t^1 - Z_s^1, Z_t^2 - Z_s^2)$$

$$Z_t^1 - Z_s^1 = 2B_t - W_t - (2B_s - W_s)$$
$$Z_t^1 - Z_s^1 = 2(B_t - B_s) - 1(W_t - W_s)$$

The distribution of $(Z_t^1 - Z_s^1)$ depends on the distributions of $(B_t - B_s)$ and $(W_t - W_s)$.
The distributions of $(B_t - B_s)$ and $(W_t - W_s)$ are normal with mean 0 and variance $(t - s)$.

$Var(Z_t^1 - Z_s^1) = Var\{2(B_t - B_s)\} + Var\{-1(W_t - W_s)\}$
$Var(Z_t^1 - Z_s^1) = 4Var(B_t - B_s) + 1Var(W_t - W_s)$
$Var(Z_t^1 - Z_s^1) = 4(t - s) + 1(t - s)$
$Var(Z_t^1 - Z_s^1) = 5(t - s)$

$E(Z_t^1 - Z_s^1) = E(Y_t - Y_s) + E(t - s) = -1(t - s) + (t - s) = 0$

The distribution of $(Z_t^1 - Z_s^1)$ is normal with mean 0 and variance $5(t - s)$.

$Z_t^2 - Z_s^2 = -B_t + 3W_t + B_s - 3W_s$
$Z_t^2 - Z_s^2 = -(B_t - B_s) + 3(W_t - W_s)$

The distribution of $(Z_t^2 - Z_s^2)$ depends on the distributions of $(B_t - B_s)$ and $(W_t - W_s)$.
The distributions of $(B_t - B_s)$ and $(W_t - W_s)$ are normal with mean 0 and variance $(t - s)$.

$Var(Z_t^2 - Z_s^2) = Var\{-(B_t - B_s)\} + Var\{3(W_t - W_s)\}$
$Var(Z_t^2 - Z_s^2) = Var(B_t - B_s) + 9Var(W_t - W_s)$
$Var(Z_t^2 - Z_s^2) = (t - s) + 9(t - s)$
$Var(Z_t^2 - Z_s^2) = 10(t - s)$

$E(Z_t^2 - Z_s^2) = -E(B_t - B_s) + 3E(W_t - W_s) = 0 + 3 * 0 = 0$

The distribution of $(Z_t^2 - Z_s^2)$ is normal with mean 0 and variance $10(t-s)$.

Because the distributions of both $(Z_t^1 - Z_s^1)$ and $(Z_t^2 - Z_s^2)$ are normal with mean 0 and some variance, distribution of $Z_t - Z_s$ is joint normal with mean 0 and covariance matrix $\Gamma(\mathrm{t}-\mathrm{s})$. The second property is satisfied.

Property 3

$$Z_t - Z_s = (Z_t^1 - Z_s^1, Z_t^2 - Z_s^2)$$

$$Z_t^1 - Z_s^1 = 2(B_t - B_s) - 1(W_t - W_s)$$
$$Z_t^2 - Z_s^2 = -(B_t - B_s) + 3(W_t - W_s)$$

$(Z_t^1 - Z_s^1)$ and $(Z_t^2 - Z_s^2)$ depends only on the values of $(B_t - B_s)\ and\ (W_t - W_s)$. Since B_t and W_t are Brownian motions, $(B_t - B_s)$ and $(W_t - W_s)$ are independent of $\mathcal{F}_s$.
Since the components are independent of $\mathcal{F}_s$, $(Z_t^1 - Z_s^1)$ and $(Z_t^2 - Z_s^2)$ are also independent of $\mathcal{F}_s$.

Since both $(Z_t^1 - Z_s^1)$ and $(Z_t^2 - Z_s^2)$ are independent of $\mathcal{F}_s$, the random vector $Z_t - Z_s$ is independent of $\mathcal{F}_s$. The third property is satisfied.

Property 4
B_t and W_t are Brownian motions. They are both continuous functions of t.

Since Z_t^1 and Z_t^2 are merely linear combinations of B_t and W_t, a linear combination of continuous functions of t, is also a continuous function of t. Therefore, with probability one, the functions $t \to Z_t^1$ and $t \to Z_t^2$ are continuous.

Since with probability one, the functions $t \to Z_t^1$ and $t \to Z_t^2$ are continuous, then with probability one, the function $t \to Z_t$ is continuous.

Conclusion: Since Z_t satisfies all 4 properties, Z_t is a two-dimensional Brownian motion starting at the origin $\{Z_0 = (0,0)\}$.

<u>Covariance Matrix</u>

$$\Gamma\mathrm{t} = \begin{pmatrix} Var(Z_t^1) & Cov(Z_t^1, Z_t^2) \\ Cov(Z_t^1, Z_t^2) & Var(Z_t^2) \end{pmatrix}$$

$$Z_t^1 = Y_t + t$$
$$Var(Z_t^1) = Var(Y_t) + Var(t) = 5t$$

$$Z_t^2 = -B_t + 3W_t$$
$$Var(Z_t^2) = Var\{-(B_t)\} + Var\{3(W_t) = Var(B_t) + 9Var(W_t) = t + 9t$$
$$Var(Z_t^2) = 10t$$

$$Z_t^1 = Y_t + t = 2B_t - W_t - t + t$$
$$Z_t^1 = 2B_t - W_t$$
$$Z_t^2 = -B_t + 3W_t$$

$$Cov(Z_t^1, Z_t^2) = Cov(2B_t, -B_t) + Cov(2B_t, 3W_t) + Cov(-W_t, -B_t) + Cov(-W_t, 3W_t)$$

Since B_t and W_t are independent Brownian motions,

$$Cov(W_t, B_t) = Cov(2B_t, 3W_t) = Cov(-W_t, -B_t) = 0$$

$$Cov(B_t, B_t) = Var(B_t) = t$$
$$Cov(2B_t, -B_t) = (2) * (-1) * Cov(B_t, B_t) = -2t$$

$$Cov(W_t, W_t) = Var(W_t) = t$$
$$Cov(-W_t, 3W_t) = (-1) * (3) * Cov(W_t, W_t) = -3t$$

$$Cov(Z_t^1, Z_t^2) = -2t + 0 + 0 - 3t = -5t$$

$$\Gamma\mathrm{t} = \begin{pmatrix} 5t & -5t \\ -5t & 10t \end{pmatrix}$$

<u>Covariance Matrix</u>

$$\Gamma = \begin{pmatrix} 5 & -5 \\ -5 & 10 \end{pmatrix}$$

$$E(Z_t^1) = 2E(B_t) - E(W_t) = 0 - 0 = 0$$
$$E(Z_t^2) = -E(B_t) + 3E(W_t) = 0 + 3 * 0 = 0$$

Since $E(Z_t^1) = E(Z_t^2) = 0,\ Z_t$ has 0 drift.

Exercise 2.3

According to theorem 2.8.1 in page 66, since Z_t^1 is a Brownian motion with drift m and variance σ^2,

$$\langle Z^1 \rangle_t = \sigma^2 t = 5t$$

According to theorem 2.9.1 in page 69, since Z_t is a 2-dimensional Brownian motion with drift m and covariance matrix Γ,

$$\langle Z^1, Z^2 \rangle_t = \Gamma_{12} t = -5t$$

Exercise 3

$P(B_t \geq 6W_t - 4) = P(4 \geq 6W_t - B_t) = P(6W_t - B_t \leq 4) = P(X_t \leq 4)$

$E(6W_t - B_t) = 6E(W_t) - E(B_t) = 0 - 0 = 0$

$Var(6W_t - B_t) = Var(6W_t) + Var(-B_t) =$

$= 36Var(W_t) + Var(B_t) = 36t + t = 37t$

$X_t \sim N(0, 37t), \sigma = \sqrt{37}$

$X_t = 6W_t - B_t$

$= P\left(Z_t \leq \frac{4-0}{\sqrt{37}} = \frac{4}{\sqrt{37}}\right)$

Using reflection principle, page 61 class notes

Set Z_t as a standard brownian motion, $Z_t = \frac{X_t}{\sqrt{37}}$

$P(\max_{0 \leq t \leq 3} X_t \geq 4) = P(\max_{0 \leq t \leq 3} Z_t \geq \frac{4}{\sqrt{37}}) = 2\left[1 - \Phi\left(\left(\frac{4}{\sqrt{37}}\right)/\sqrt{3}\right)\right]$

$= 2[1 - \Phi(0.38)] = 2 - 2\Phi(0.38)$

$P(B_t \geq 6W_t - 4 \text{ for all } 0 \leq t \leq 3) = P(\max_{0 \leq t \leq 3} 6W_t - B_t \leq 4)$

$= P(\max_{0 \leq t \leq 3} X_t \leq 4) = 1 - P(\max_{0 \leq t \leq 3} X_t \geq 4)$

$= 2\Phi(0.38) - 1 = 2(0.64803) - 1$

$P(B_t \geq 6W_t - 4 \text{ for all } 0 \leq t \leq 3) = 0.2961$ //

Exercise 4

For $B_t \sim N(0,t) \rightarrow B_t$ is standard Brownian motion, the functions $\phi(t,x)$ satisfies the heat equation:

$$\partial t\, \phi(t,x) = \frac{1}{2} \partial xx\, \phi(t,x), \quad \rightarrow PDE \quad //$$

for each $\phi(t,x)$ in exercise 4.1-4.4 (class notes page 71)

If B has drift 1 and variance parameter 4, the functions $\phi(t,x)$ satisfies the heat equation:

$$dt\, \phi(t,x) = -\bar{m}\, dx\, \phi(t,x) + \frac{\sigma^2}{2} dxx\, \phi(t,x)$$

$$dt\, \phi(t,x) = -dx\, \phi(t,x) + 2\, dxx\, \phi(t,x) \quad \rightarrow PDE \quad //$$

for each function $\phi(t,x)$ in exercise 4.1-4.4

(class notes page 73)

4.1

$\phi(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ (class notes page 70)

initial condition : $B_0 = 0$

Exercise 4 (continued)

For 4.2, initial condition:

at $t=0$, if $B_0 = x \geq 4$, $P(B_t < 4 \mid B_0 = x) = 0$

if $B_0 = x < 4$, $P(B_t < 4 \mid B_0 = x) = 1$

$$\phi(0,x) = \begin{cases} 1 & , \text{ for } x < 4 \\ 0 & , \text{ for } x \geq 4 \end{cases}$$

initial condition

For 4.3, initial condition:

at $t=0$, $B_t = B_0 = x$

$$\phi(0,x) = E[B_0^3 \mid B_0 = x] = E[B_0^3] = E(x^3) = x^3$$

initial condition: $\phi(0,x) = x^3$

For 4.4, final condition:

at $t=4$, $B_t = B_4 = x$

$$\phi(4,x) = E[B_4 - B_4^2 \mid B_4 = x] = E[x - x^2] = x - x^2$$

Final condition: $\phi(4,x) = x - x^2$

These conditions apply when B_t is a standard brownian motion, and also for when B has drift 1 and variance parameter 4.

Exercise 5.1

5.1) $E(Z_t^1) = E(B_t) + E(W_t) - E(t) = 0 + 0 - t = -t = m_1 \cdot t$

$E(Z_t^2) = E(2 \cdot B_t) + (-4)E(W_t) = 0 + 0 = 0 = m_2$

drift for $Z = m = (-1, 0)$ //

$Cov(B_t, B_t) = Var(B_t) = t = Var(W_t) = Cov(W_t, W_t)$

$Cov(B_t, 2B_t) = (1)(2)\,Cov(B_t, B_t) = 2t$

$Cov(B_t, -4W_t) = Cov(W_t, 2B_t) = Cov(W_t, B_t) = 0$,

because B_t and W_t are independent Brownian motions

$Cov(W_t, -4W_t) = (1)(-4)\,Cov(W_t, W_t) = -4t$

$$Cov(Z_t^1, Z_t^2) = Cov(B_t, 2B_t) + Cov(B_t, -4W_t) + Cov(W_t, 2B_t) + Cov(W_t, -4W_t)$$

$$= 2t + 0 - 4t = -2t$$

$Var(Z_t^1) = Var(B_t) + Var(W_t) = t + t = 2t$

$Var(Z_t^2) = Var(2B_t) + Var(-4W_t) = 4Var(B_t) + 16Var(W_t)$

$= 4t + 16t = 20t$

<u>Covariance matrix for Z</u>

$$T_t = \begin{pmatrix} Var(Z_t^1) & Cov(Z_t^1, Z_t^2) \\ Cov(Z_t^1, Z_t^2) & Var(Z_t^2) \end{pmatrix} = \begin{pmatrix} 2t & -2t \\ -2t & 20t \end{pmatrix}$$

$$T = \begin{pmatrix} 2 & -2 \\ -2 & 20 \end{pmatrix}$$

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Exercise 5.2

According to theorem 2.10.3 (page 76 class notes), since Z is a 2-dimensional Brownian motion with drift $m=(-1,0)$ and covariance matrix T', then the infinitesimal generator is

$$L f(x) = m \cdot \nabla f(x) + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} T'_{jk} \, d_{jk} \cdot f(x)$$

$$m \cdot \nabla f(x) = (-1,0) \cdot \left(\frac{df}{dz^1}, \frac{df}{dz^2} \right) = -\frac{df}{dz^1}$$

$$\frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} T'_{jk} \, d_{jk} f(x) = \frac{1}{2} \left\{ 2 \cdot \frac{d^2 f}{d(z^1)^2} + (-2) \cdot 2 \cdot \frac{d^2 f}{dz^1 dz^2} + 20 \cdot \frac{d^2 f}{d(z^2)^2} \right\}$$

$$= \frac{d^2 f}{d(z^1)^2} - 2 \cdot \frac{d^2 f}{dz^1 dz^2} + 10 \cdot \frac{d^2 f}{d(z^2)^2}$$

$$L f(x) = -\frac{df}{dz^1} + \frac{d^2 f}{d(z^1)^2} - 2 \cdot \frac{d^2 f}{dz^1 dz^2} + 10 \cdot \frac{d^2 f}{d(z^2)^2} \quad //$$

↳ L operator associated to Z

$$L^* f(x) = \frac{df}{dz^1} + \frac{d^2 f}{d(z^1)^2} - 2 \cdot \frac{d^2 f}{dz^1 dz^2} + 10 \cdot \frac{d^2 f}{d(z^2)^2} \quad //$$

↳ L^* operator associated to Z

(page 79 class notes)

Exercise 5.3

According to page 79 (class notes), $\phi(t,x)$ satisfies the equation $\quad d_t\,\phi(t,x) = L_x^*\,\phi(t,x)$

$$d_t\,\phi(t,x) = \frac{d\phi(x)}{dz^1} + \frac{d^2\phi(x)}{d(z^1)^2} - 2\cdot\frac{d^2\phi(x)}{dz^1\,dz^2} + 10\cdot\frac{d^2\phi(x)}{d(z^2)^2}$$

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