

FINM 34500/STAT 39000

Problem Set 1 (due January 13)

Reading: Notes through Section 2.8. Chapter 1 as well as Chapter 2.1 and 2.2 should be review, they were covered in FINM 34000. The material in the small font including all of Section 2.5, is optional.

Exercise 1 Let B_t be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.

1. $\mathbb{P}\{B_3 \leq 2\}$
2. $\mathbb{P}\{B_1 \leq 1, B_3 - B_1 \leq 1\}$
3. $\mathbb{P}(E)$ where E is the event that the path stays below the line $y = 2$ up to time $t = 4$.
4. $\mathbb{P}\{B_4 \geq 0 \mid B_8 \geq 0\}$.
5. $\mathbb{P}\{B_1(B_3 - B_1) \geq 0 \mid B_1 \leq 1, (B_3 - B_1)^2 \geq 2\}$.

Exercise 2 Suppose B_t is a standard Brownian motion, $\lambda \in \mathbb{R}$, and let \mathcal{F}_t be its corresponding filtration. Let

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}.$$

1. Show that M_t is a martingale with respect to \mathcal{F}_t . In other words, show that if $s < t$, then

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s.$$

You may use the following fact. If $X \sim N(0, 1)$, then the moment generating function of X is given by

$$m(t) = \mathbb{E}[e^{tX}] = e^{t^2/2}. \quad (1)$$

2. Find $\mathbb{E}[M_7]$ and $\mathbb{E}[M_{43} - 2M_9]$.

Exercise 3 Textbook, Exercise 2.6 (You may wish to use (1) when doing part 4.)

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Exercise 4 Prove Theorem 2.6.3 from the notes. In other words, show that if B_t is a standard Brownian motion, $a > 0$, and

$$Y_t = a^{-1/2} B_{at},$$

then Y_t is a standard Brownian motion. (You need to show that Y_t satisfies the four properties that a standard Brownian motion satisfies.)

Exercise 5 Write a program that will sample from a standard Brownian motion using step size $\Delta t = 1/500$.

- Make a graph of $B_t, 0 \leq t \leq 3$ from your simulation for one trial.
- Repeat the simulation 1000 times to give estimates for the following:

$$\mathbb{P}\{\min_{0 \leq t \leq 3} B_t < -1/2\},$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 < 0\}.$$

Find the exact values of these probabilities and compare your simulations with the exact values. You may use any computer language and/or package that you choose.