Stochastic Calculus - Homework 1

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1 Exercise 1

1.1

I will be using the standard normal distribution table from:
https://math.arizona.edu/~rsims/ma464/standardnormaltable.pdf

$$P\{B_3 \leq 2\}$$
$$m = 0, \sigma^2 = 1$$
$$B_3 \sim N(0 * t, 1 * t) = N(0 * 3, 1 * 3) = N(0, 3)$$
$$Z = \frac{B_3 - \mu}{\sigma} = \frac{2 - 0}{\sqrt{3}} = 1.15$$
$$P\{B_3 \leq 2\} = P\{Z \leq 1.15\} = 0.875$$

1.2

According to the independent increments rule, if $s < t$, the random variable $B_t - B_s$ is independent of the values for $\{B_r : r \leq s\}$.

Therefore, since $1 < 3$, the random variable $B_3 - B_1$ is independent of the values for $\{B_1 : 1 \leq 1\}$.

$$P\{B_1 \leq 1\}$$
$$B_1 \sim N(0, 1)$$
$$Z = \frac{B_1 - \mu}{\sigma} = \frac{1 - 0}{\sqrt{1}} = 1$$
$$P\{B_1 \leq 1\} = P\{Z \leq 1\} = 0.84134$$

$$m * (t - s) = 0$$
$$\sigma^2 * (t - s) = 1 * (3 - 1) = 2$$
$$B_3 - B_1 \sim N(0, 2)$$
$$Z = \frac{B_3 - B_1 - \mu}{\sigma} = \frac{1 - 0}{\sqrt{2}} = 0.71$$
$$P\{B_3 - B_1 \leq 1\} = P\{Z \leq 0.71\} = 0.76115$$

$$P\{B_1 \leq 1, B_3 - B_1 \leq 1\} = P\{B_1 \leq 1\} * P\{B_3 - B_1 \leq 1\}$$
$$P\{B_1 \leq 1, B_3 - B_1 \leq 1\} = 0.84134 * 0.76115 = 0.6404$$

1.3

$$P(E) = P\left(\max_{0 \leq s \leq 4} B_s < 2\right) = 1 - P\left(\max_{0 \leq s \leq 4} B_s \geq 2\right)$$

Using the reflection principle in page 61 in the class notes,

$$P\left(\max_{0 \leq s \leq 4} B_s \geq 2\right) = 2 * P\{B_4 > 2\} = 2\left[1 - \Phi\left(\frac{2}{\sqrt{4}}\right)\right] = 2\{1 - \Phi(1)\} = 2(1 - 0.84134) = 0.31732$$

$$P(E) = 1 - 0.31732 = 0.683$$

Exercise 1.4

$$P\{B_4 \geq 0 \mid B_8 \geq 0\} = \frac{P(B_4 \geq 0, B_8 \geq 0)}{P(B_8 \geq 0)}$$

$$P(B_4 \geq 0, B_8 \geq 0) = P(B_8 \geq 0 \mid B_4 = x) \cdot P(B_4 = x)$$

$$= \int_0^\infty P(B_8 \geq 0 \mid B_4 = x) \cdot dP(B_4 = x)$$

Page 70 of class notes: '$B_t \sim N(0,t)$, $dP(B_t = x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$

$$B_4 \sim N(0,4) \rightarrow dP(B_4 = x) = \frac{1}{\sqrt{2\pi 4}} e^{-\frac{x^2}{2\cdot 4}} dx = \frac{1}{\sqrt{8\pi}} e^{-x^2/8} dx$$

$B_8 \geq 0$

$B_8 - B_4 \geq -B_4$

$B_8 - B_4 \geq -x$

$$P(B_8 \geq 0 \mid B_4 = x) = P(B_8 - B_4 \geq -x) = \int_{-x}^\infty \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{y^2}{2\cdot 4}} dy$$

$$= \int_{-x}^\infty \frac{1}{\sqrt{8\pi}} e^{-y^2/8} dy$$

$$P(B_4 \geq 0, B_8 \geq 0) = \int_0^\infty \int_{-x}^\infty \frac{1}{8\pi} e^{-\frac{(x^2+y^2)}{8}} dy\, dx$$

$x = r\cos\theta$

$y = r\sin\theta$

① x goes from $0 \rightarrow \infty$

$x > 0$

$r\cos\theta > 0$

$\cos\theta > 0$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

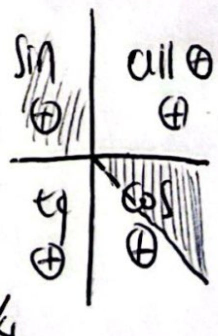
② y goes from $-x$ to ∞

$y > -x$

$\frac{r\sin\theta}{r\cos\theta} > -1$

$tg\theta > -1$

$\theta > -\pi/4$



$-\pi/4 < \theta < \pi/2$

$$P(B_4 \geq 0, B_8 \geq 0) = \int_0^\infty \int_{-\pi/4}^{\pi/2} \frac{1}{8\pi} e^{-r^2/8}\, r\,dr\,d\theta$$

$$= \frac{1}{8\pi} \int_0^\infty e^{-r^2/8}\, r\,dr \int_{-\pi/4}^{\pi/2} d\theta$$

$$= \frac{1}{8\pi} \int_0^\infty e^{-r^2/8}\, r\,dr \left\{ \pi/2 + \pi/4 \right\}$$

$$= \frac{3}{32\pi} \int_0^\infty e^{-r^2/8}\, r\,dr$$

$$= \frac{3}{32\pi} \cdot \int_0^\infty e^{u} \cdot (-4)\,du = -\frac{3}{8} \int_0^\infty e^{u}\,du$$

$u = -r^2/8$

$du = -\frac{2r}{8}\,dr$

$du = -\frac{1}{4} r\,dr$

$r\,dr = -4\,du$

$$P(B_4 \geq 0, B_8 \geq 0) = -\frac{3}{8} \left(\frac{1}{e^\infty} - \frac{1}{e^0} \right) = -\frac{3}{8}(0-1) = \frac{3}{8}$$

Page 60 class notes: $P(B_t \geq r) = 1 - \Phi(r/\sqrt{t})$

$$P(B_8 \geq 0) = 1 - \Phi(0) = 1 - 0.5 = 0.5$$

$$P\{B_4 \geq 0 \mid B_8 \geq 0\} = \frac{3/8}{1/2} = \frac{3}{4}$$

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Exercise 1

5) $P\{B_1(B_3-B_1)\geqslant 0 \mid B_1\leqslant 1, (B_3-B_1)^2\geqslant 2\}$

$$= \frac{P\{B_1(B_3-B_1)\geqslant 0,\ B_1\leqslant 1,\ (B_3-B_1)^2\geqslant 2\}}{P(B_1\leqslant 1, (B_3-B_1)^2\geqslant 2)}$$

$B_t \sim N(0,t)$

$B_1 \sim N(0,1)$

$P(B_1\leqslant 1) = \Phi(1/\sqrt{1}) = \Phi(1) = 0.84134$

$P\{(B_3-B_1)^2\geqslant 2\} = P(B_3-B_1\geqslant\sqrt{2}) + P(B_3-B_1\leqslant -\sqrt{2})$

$B_3-B_1 \sim N(0,2)$

$P(B_3-B_1\geqslant\sqrt{2}) = 1-\Phi(\sqrt{2}/\sqrt{2}) = 1-\Phi(1)$

$P(B_3-B_1\leqslant -\sqrt{2}) = \Phi(-\sqrt{2}/\sqrt{2}) = \Phi(-1) =$

$P\{(B_3-B_1)^2\geqslant 2\} = 1-\Phi(1)+\Phi(-1)$

$= 1-0.84134+0.15866$

$= 0.31732$

denominator

$P(B_1\leqslant 1,(B_3-B_1)^2\geqslant 2) = \underbrace{P(B_1\leqslant 1)\cdot P((B_3-B_1)^2\geqslant 2)}_{\text{independent}}$

$= (0.84134)(0.31732)$

$= 0.267$

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Numerator

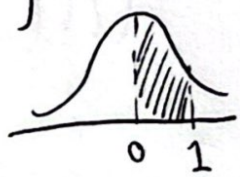
For $B_1(B_3-B_1)\geq 0$,

$$(B_3-B_1)^2\geq 2 \begin{cases} (B_3-B_1)\geq\sqrt{2}\ , & B_1\geq 0 \ \text{①} \rightarrow B_1 \text{ can't be negative} \\ (B_3-B_1)\leq-\sqrt{2}\ , & B_1\leq 0 \ \text{②} \rightarrow B_1 \text{ can't be positive} \end{cases}$$

① $P\{B_1\geq 0,\ B_1\leq 1,\ (B_3-B_1)\geq\sqrt{2}\}$

$$= P\{B_1\geq 0,\ B_1\leq 1\}\cdot P\{(B_3-B_1)\geq\sqrt{2}\}$$

$$= \{\Phi(1)-\Phi(0)\}\{1-\Phi(1)\}$$



$$= (0.84134-0.5)(1-0.84134) = 0.0542$$

② $P\{B_1\leq 0,\ B_1\leq 1,\ (B_3-B_1)\leq-\sqrt{2}\}$

$$= P\{B_1\leq 0\}\cdot P\{(B_3-B_1)\leq-\sqrt{2}\}$$

$$= \Phi(0)\cdot\Phi(-1) = (0.5)(0.15866) = 0.0793$$

$P(B_1(B_3-B_1)\geq 0 \mid B_1\leq 1,\ (B_3-B_1)^2\geq 2\}$

$$= \frac{0.0542+0.0793}{0.267} = 0.5$$

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Exercise 2.1

From le.ac.uk/users/dsgp1/COURSES/MATHSTAT/6normgf.pdf,

m.g.f of x with mean M, variance σ^2,

$$M_x(a) = E(e^{x\cdot a}) = e^{M.a + \frac{1}{2}\sigma^2 a^2}$$

$$E(M_t \mid F_s) = E(e^{\lambda\cdot B_t - (\lambda^2/2)t} \mid F_s)$$

$$= E(e^{\lambda(B_s + B_t - B_s) - (\lambda^2/2)t} \mid F_s)$$

$$= E(e^{\lambda\cdot B_s} \cdot e^{\lambda(B_t - B_s)} \cdot e^{(-\lambda^2/2)t} \mid F_s)$$

$$B_t - B_s \sim N(\overset{M}{0}, \underset{\sigma^2}{t-s})$$

$$m_{B_t - B_s}(\lambda) = E(e^{\lambda\cdot(B_t - B_s)}) = e^{0 + \frac{1}{2}(t-s)\cdot\lambda^2}$$

$$= e^{\frac{\lambda^2}{2}\cdot t} \cdot e^{-\frac{\lambda^2}{2}s}$$

$$E(M_t \mid F_s) = e^{\lambda\cdot B_s} E(\cancel{e^{\frac{\lambda^2}{2}t}} \cdot e^{-\frac{\lambda^2}{2}s} \cdot \cancel{e^{-(\lambda^2/2)t}} \mid F_s)$$

$$= e^{\lambda\cdot B_s} E(e^{-(\lambda^2/2)s} \mid F_s)$$

$$= e^{\lambda\cdot B_s} \cdot e^{-(\lambda^2/2)s} = e^{\lambda\cdot B_s - (\lambda^2/2)s}$$

$$E(M_t \mid F_s) = M_s \rightarrow M_t \text{ is a martingale}$$

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Exercise 2.2

Since M_t is a martingale, $E(M_t) = E(M_0)$

$M_0 = e^{\lambda \cdot B_0 - (\lambda^2/2) \cdot 0} = e^0 = 1 \rightsquigarrow E(M_0) = 1$

For a brownian motion, $B_0 = 0$ (page 44 class notes)

$E(M_7) = M_0 = 1$ //

$E(M_{43}) = E(M_9) = 1$

$E(M_{43} - 2M_9) = E(M_{43}) - 2E(M_9) = 1 - 2(1)$

$= -1$ //

Exercise 3.1

$$B_t = B_s + B_t - B_s$$

$$B_t^2 = (B_s + B_t - B_s)^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2$$

$$E(B_t^2 | F_s) = E(B_s^2 | F_s) + E(2B_s(B_t - B_s) | F_s) + E\{(B_t - B_s)^2 | F_s\}$$

$$E(B_s^2 | F_s) = B_s^2$$

$$E(2B_s(B_t - B_s) | F_s) = 2B_s \cdot E\{(B_t - B_s) | F_s\}$$

$$= 2B_s \cdot E(B_t - B_s)$$

$$= 2B_s \cdot 0 = 0$$

$$(B_t - B_s) \sim N(0, t-s)$$

$$Var(B_t - B_s) = E\{(B_t - B_s)^2\} - \{E(B_t - B_s)\}^2$$

$$t - s = E\{(B_t - B_s)^2\} - 0$$

$$E((B_t - B_s)^2 | F_s) = E\{(B_t - B_s)^2\} = t - s$$

$$E(B_t^2 | F_s) = B_s^2 + 0 + t - s$$

$$= B_s^2 + t - s$$

//

$\cdot S$

Exercise 3. (2.6.2)

$$B_t^3 = (B_s + B_t - B_s)^3 = B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3$$

$$E(B_t^3 \mid F_s) = E\{B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3 \mid F_s\}$$

$$E(B_s^3 \mid F_s) = B_s^3$$

$$E\{3B_s^2(B_t - B_s) \mid F_s\} = 3B_s^2 \, \overset{\to 0}{E(B_t - B_s \mid F_s)} = 0$$

$$E\{(B_t - B_s)^2 \mid F_s\} = t - s \quad \text{(from previous 3.1)}$$

$$E(3B_s(B_t - B_s)^2 \mid F_s) = 3B_s \cdot (t - s)$$

$$(B_t - B_s) \sim N(\mu = 0, \sigma^2 = t - s)$$

$$E[(B_t - B_s - \underbrace{\mu}_{\to 0})^3] = E[(B_t - B_s)^3] = 0$$

$$E\{(B_t - B_s)^3 \mid F_s\} = E\{(B_t - B_s)^3\} = 0$$

$$E(B_t^3 \mid F_s) = B_s^3 + 3B_s(t - s)$$

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From en.wikipedia.org/wiki/Normal_distribution,

$$E[(X-\mu)^p] = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \sigma^p (p-1)!! & \text{if } p \text{ is even} \end{cases}$$

Exercise 3 (2.6.3)

$$B_t^4 = (B_s + B_t - B_s)^4$$
$$= B_s^4 + 4B_s^3(B_t - B_s) + 6B_s^2(B_t - B_s)^2 + 4B_s(B_t - B_s)^3 + (B_t - B_s)^4$$

$$E(B_s^4 \mid F_s) = B_s^4$$

$$E(4B_s^3(B_t - B_s) \mid F_s) = 4B_s^3 E(B_t - B_s \mid F_s) = 0$$

$$E(6B_s^2(B_t - B_s)^2 \mid F_s) = 6B_s^2 \cdot E((B_t - B_s)^2 \mid F_s) = 6B_s^2(t-s)$$

$$E(4B_s(B_t - B_s)^3 \mid F_s) = 4B_s \cdot E\{(B_t - B_s)^3\} = 0$$

$$E\{(B_t - B_s)^4 \mid F_s\} = E\{(B_t - B_s)^4\}$$

based on the previous wikipedia source,

$$E[(B_t - B_s - \mu)^4] = \sigma^4(4-1)!!$$
$$(B_t - B_s) \sim N(\mu = 0, \sigma^2 = t-s)$$

$$E[(B_t - B_s)^4] = (t-s)^2 \cdot 3!! = (3)(1)(t-s)^2 = 3(t-s)^2$$

$$E[B_t^4 \mid F_s] = B_s^4 + 3(t-s)^2 + 6B_s^2(t-s)$$

// //

Exercise 3 (2.6.4)

$$E[e^{4B_t-2} \mid F_s] = e^{-2} E[e^{4(B_s+B_t-B_s)} \mid F_s]$$
$$= e^{-2} \cdot e^{4B_s} \cdot E[e^{4(B_t-B_s)} \mid F_s]$$

$$B_t - B_s \sim N(0, t-s)$$
$$m_{B_t-B_s}(4) = E[e^{4(B_t-B_s)}] = e^{0+\frac{1}{2}(t-s)\cdot 4^2}$$
$$= e^{8(t-s)}$$

$$E[e^{4B_t-2} \mid F_s] = e^{4B_s-2} \cdot E[e^{4(B_t-B_s)}]$$
$$= e^{4B_s-2} \cdot e^{8(t-s)}$$
$$= \frac{e^{4B_s+8(t-s)}}{e^2}$$

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Exercise 4

$$Y_t = \frac{B_{at}}{\sqrt{a}}$$

There are 4 properties to prove:

1. $Y_0 = 0$
2. Independent increment: If $s < t$, the random variable $Y_t - Y_s$ is independent of the values Y_r for $r \leq s$
3. For $s < t$, the distribution of $Y_t - Y_s$ is normal with mean $m(t-s)$ and variance $\sigma^2(t-s)$
4. With probability one, the function $t \to B_t$ is a continuous function of t

Property 1:

Since B_t is a standard Brownian motion, $B_0 = 0$

$$Y_0 = \frac{B_{a*0}}{\sqrt{a}} = \frac{B_0}{\sqrt{a}} = 0$$

We have proven that

$$Y_0 = 0$$

And that the first property is satisfied.

Property 2:

We know that $B_t - B_s$ is independent of the values B_r for $r \leq s$

$$Y_t - Y_s = \frac{B_{at}}{\sqrt{a}} - \frac{B_{as}}{\sqrt{a}} = \frac{1}{\sqrt{a}}(B_{at} - B_{as})$$

Since the random variable $B_{at} - B_{as}$ has the independent increment property, the random variable $Y_t - Y_s$ also has the independent increment property. The second property is satisfied.

Property 3:

The distribution of $B_t - B_s$ is normal with mean $m(t-s)$ and variance $\sigma^2(t-s)$

We know that $B_t \sim N(0,t)$

Therefore,

$$B_{at} \sim N(0, at)$$

$$\frac{B_{at}}{\sqrt{a}} \sim N\left(\frac{0}{\sqrt{a}} = 0, \frac{at}{\left(\sqrt{a}\right)^2} = t\right)$$

$$Y_t \sim N(0,t)$$

Y_t is identically distributed as B_t

$$Y_s \sim N(0, s)$$

Therefore, the distribution of $Y_t - Y_s$ is normal with mean $m(t-s)$ and variance $\sigma^2(t-s)$. The third property is satisfied.

Property 4:
We know that the function $t \to B_t$ is a continuous function of t. Since Y_t is only a scaled and time-changed version of B_t, the function $t \to Y_t$ is also a continuous function of t. The fourth property is satisfied.

Conclusion: Since Y_t satisfies all 4 properties that a standard Brownian motion satisfies, Y_t is a standard Brownian motion.

Exercise 5 , exact values

$$P\{\min_{0\leq s\leq t} B_s \leq -a\} = P\{\max_{0\leq s\leq t} B_s \geq a\} = 2P\{B_t > a\}$$

$$= 2[1-\Phi\, a/\sqrt{t}]$$ (page 61 class notes)

$$P\{\min_{0\leq t\leq 3} B_t \leq -1/2\} = 2P\{B_3 > 1/2\}$$

$$= 2[1-\Phi\,(1/2)/\sqrt{3}]$$

$$= 2[1-\Phi(0.29)] =$$

$$P\{\min_{0\leq t\leq 3} B_t \leq -\tfrac{1}{2}\} = 2\{1-0.61409\} = 0.7718$$

$$P\{B_{1.5} > 0, B_3 < 0\} = P\{B_3 < 0 \mid B_{1.5} = x\} \cdot P\{B_{1.5} = x\}$$

$$= \int_0^\infty P\{B_3 < 0 \mid B_{1.5} = x\} \cdot dP\{B_{1.5} = x\}$$

$$dP(B_{1.5} = x) = \frac{1}{\sqrt{2\pi \cdot 3/2}} \cdot e^{-x^2/2\cdot 3/2}\, dx = \frac{1}{\sqrt{3\pi}} e^{-x^2/3}\, dx$$

$$P\{B_3 < 0 \mid B_{1.5} = x\} = P\{B_3 - B_{1.5} < -x\}$$

$$= \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi \cdot 3/2}} e^{-y^2/2\cdot 3/2}\, dy$$

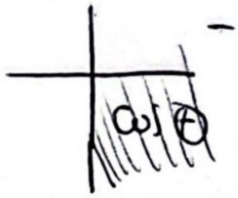
$$= \int_{-\infty}^{-x} \frac{1}{\sqrt{3\pi}} e^{-y^2/3}\, dy$$

Exercise 5 (continued)

① $x > 0$

$r\cos\theta > 0$

$\cos\theta > 0$



$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

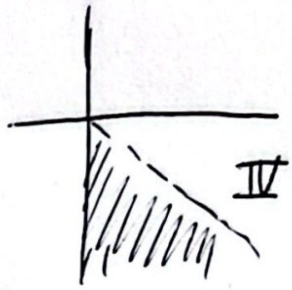
② $y < -x$

$\frac{r\sin\theta}{r\cos\theta} < -1$

$tg\,\theta < -1$

$\theta < -\pi/4$

so, $-\frac{\pi}{2} < \theta < -\pi/4$



$$P\{B_{1.5} > 0, B_3 < 0\} = \int_0^\infty \int_{-\infty}^{-x} \frac{1}{3\pi} e^{-(x^2+y^2)/3}\, dy\, dx$$

$$= \int_0^\infty \int_{-\pi/2}^{-\pi/4} \frac{1}{3\pi} e^{-r^2/3} r\, dr\, d\theta$$

$$= \int_0^\infty \frac{1}{3\pi} e^{-r^2/3} \cdot \left(-\frac{\pi}{4} + \frac{\pi}{2}\right) r\, dr$$

$$= \frac{1}{3\pi} \int_0^\infty e^{-r/3} \cdot \frac{\pi}{4}\, r\, dr$$

$$= \frac{1}{12} \int_0^\infty e^{-r^2/3}\, r\, dr$$

$$= \frac{1}{\cancel{12}\,4} \cdot \left(-\frac{\cancel{3}^{\,1}}{2}\right) \int_0^\infty e^{u}\, du$$

$$= -\frac{1}{8}\left(\frac{1}{e^\infty} - \frac{1}{e^0}\right) = \frac{1}{8}$$

$u = -\frac{1}{3} r^2$

$du = -\frac{2}{3} r\, dr$

$r\, dr = -\frac{3}{2}\, du$

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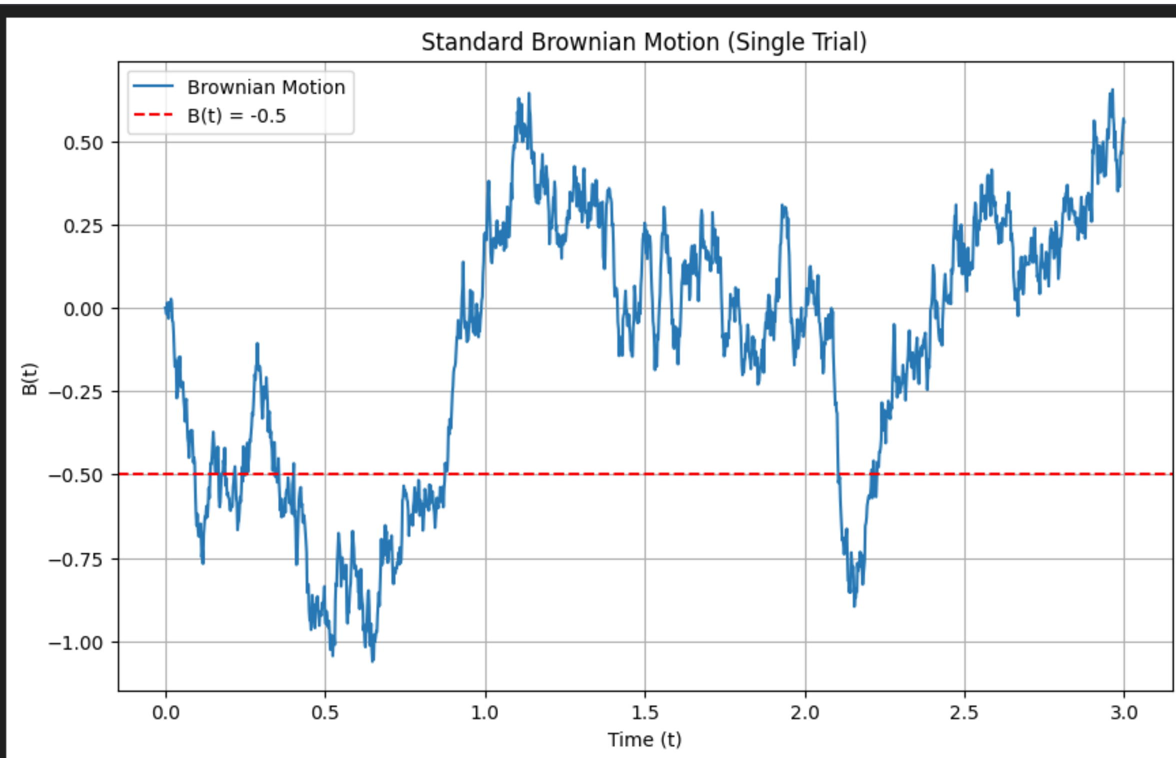
Exercise 5:

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Estimated P(min(0 <= t <= 3) B_t < -1/2): 0.756
Estimated P(B_1.5 > 0, B_3 < 0): 0.146
```

For $P\left(\min_{0\le t\le 3} B_t < -\frac{1}{2}\right)$, the simulated probability is 0.756 which is similar to the exact probability of 0.7718.

For $P(B_{1.5} > 0, B_3 < 0)$, the simulated probability is 0.146 which is similar to the exact probability of 1/8 which is 0.125.

The calculation for the exact probabilities can be found in the previous pages. The code was written with the help of OpenAI's chatGPT.com.

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
T = 3  # Total time
dt = 1/500  # Time step
n_steps = int(T / dt)  # Number of steps
n_simulations = 1000  # Number of simulations

# Function to generate Brownian motion
def generate_brownian_motion(n_steps, dt):
    dB = np.sqrt(dt) * np.random.randn(n_steps)  # Increments
    B = np.cumsum(dB)  # Cumulative sum for Brownian motion
    B = np.insert(B, 0, 0)  # Insert B(0) = 0
    return B

# Single simulation and plot
t = np.linspace(0, T, n_steps + 1)  # Time grid
B_single = generate_brownian_motion(n_steps, dt)

plt.figure(figsize=(10, 6))
plt.plot(t, B_single, label="Brownian Motion")
plt.title("Standard Brownian Motion (Single Trial)")
plt.xlabel("Time (t)")
plt.ylabel("B(t)")
plt.axhline(-0.5, color='red', linestyle='--', label="B(t) = -0.5")
plt.legend()
plt.grid()
plt.show()

# 1000 simulations for probabilities
min_less_than_neg_half = 0
b1_5_greater_zero_b3_less_zero = 0

for i in range(n_simulations):
    B = generate_brownian_motion(n_steps, dt)
    if np.min(B) < -0.5:
        min_less_than_neg_half += 1
    if B[int(1.5 / dt)] > 0 and B[-1] < 0:
        b1_5_greater_zero_b3_less_zero += 1

# Calculate probabilities
p_min_less_than_neg_half = min_less_than_neg_half / n_simulations
p_b1_5_and_b3 = b1_5_greater_zero_b3_less_zero / n_simulations

print(f"Estimated P(min(0 <= t <= 3) B_t < -1/2): {p_min_less_than_neg_half}")
print(f"Estimated P(B_1.5 > 0, B_3 < 0): {p_b1_5_and_b3}")
```