HW4 Stochastic - Matheus Raka Pradnyatama

Theorem 3.4.1: $df(t,B_t) = \partial_x f(t,B_t)\,dt + \left[\dot{f}(t,B_t) + \frac{1}{2}f''(t,B_t)\right]dt$
(page 107)

1.1) $f(t,x) = xe^{-x}$, $\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$f'(t,x) = e^{-x} + (x)(-1)e^{-x} = (1-x)e^{-x}$

$f''(t,x) = (-1)(1-x)e^{-x} + e^{-x}(-1) = (x-1)e^{-x} - e^{-x}$
$= (x-2)e^{-x}$

$\dot{f}(t,x) = 0$

$df(t,B_t) = (1-B_t)e^{-B_t}dB_t + 0 + \frac{1}{2}(B_t-2)e^{-B_t}dt$

$= e^{-B_t}\left[(1-B_t)dB_t + \frac{1}{2}(B_t-2)\,dt\right]$ //

1.2) $f(t,x) = xte^{-tx}$

$\dot{f}(t,x) = xe^{-tx} + (-x)xe^{-tx}t = xe^{-tx} - x^2e^{-tx}t$
$= xe^{-tx}(1-xt)$

$f'(t,x) = te^{-tx} + (-t)(xt)e^{-xt} = te^{-tx} - xt^2e^{-tx}$
$= te^{-tx}(1-tx)$

$f''(t,x) = (-t)\{te^{-tx}(1-tx)\} + (-t)te^{-tx}$
$= -t^2e^{-tx}((1-tx)+1) = -t^2e^{-tx}(2-tx)$
$= t^2e^{-tx}(tx-2)$

1.2) continued , $X = B_t$

$$d f(t, B_t) = t e^{-tX}(1-tX)\,dB_t + \left[X e^{-tX}(1-Xt) + \frac{1}{2} t^2 e^{-tX}(tX-2)\right] dt$$

$$= e^{-tX}\left\{t(1-tX)\,dB_t + \left[X(1-Xt) + \frac{1}{2}t^2(tX-2)\right] dt\right\}$$

$$= e^{-tB_t}\left\{t(1-B_t)\,dB_t + \left[B_t - t\cdot B_t^2 + \frac{1}{2}t^2(tB_t-2)\right] dt\right\}$$

//

1.3) $f(t, X) = t + (\sin X)^3$

$\dot{f}(t, X) = 1$

$f'(t, X) = 3\sin^2(X)\cdot\cos(X)$

$f''(t, X) = 6\sin(X)\cdot\cos X(\cos(X)) + 3\sin^2(X)\{-\sin(X)\}$

$= 6\sin(X)\cos^2(X) - 3\sin^3(X)$

$$d f(t, B_t) = 3\sin^2(B_t)\cos(B_t)\,dB_t + \left[1 + \frac{1}{2}\left\{6\sin(B_t)\cos^2(B_t) - 3\sin^3(B_t)\right\}\right] dt$$

//

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Exercise 1.4) (Repeating)

Page 111: theorem 3.4.2

$$dX_t = R_t\,dt + A_t\,dB_t$$

$$df(t,X_t) = \left\{\dot{f}(t,X_t) + R_t f'(t,X_t) + \frac{A_t^2}{2} f''(t,X_t)\right\}dt + A_t * f'(t,X_t)\,dB_t$$

$$R_t = 2X_t, \quad A_t = 2X_t \rightsquigarrow A_t^2 = 4X_t^2$$

1.1) $f(t,x) = xe^{-x}$

$$df(t,X_t) = \left\{0 + 2X_t(1-X_t)e^{-X_t} + 2X_t^2(X_t-2)e^{-X_t}\right\}dt + 2X_t(1-X_t)e^{-X_t}\,dB_t$$

$$= 2X_t e^{-X_t}\left[(1-X_t) + X_t(X_t-2)\right]dt + 2X_t(1-X_t)e^{-X_t}\,dB_t$$

$$= 2X_t e^{-X_t}\left[(X_t^2-3X_t+1)\,dt + (1-X_t)\,dB_t\right]$$

//

1.2) $f(t,x) = x \cdot t e^{-t \cdot x}$

$$df(t,X_t) = \Big[X_t e^{-tX_t}(1-X_t) + 2X_t \cdot t e^{-tX_t}(1-t\cdot X_t) + 2X_t^2 \cdot t^2 e^{-t\cdot X_t}(t\cdot X_t - 2)\Big]dt + 2X_t\left[te^{-tX_t}(1-t\cdot X_t)\right]dB_t$$

$$= X_t \cdot e^{-t\cdot X_t}\left[(1-X_t) + 2t(1-t\cdot X_t) + 2X_t \cdot t^2(t\cdot X_t-2)\right]dt + 2X_t \cdot t\, e^{-t\cdot X_t}(1-t\cdot X_t)\,dB_t$$

//

1.3) (Repeated) $f(t,x) = t + (\sin x)^3$

$$d f(t, X_t) = \{1 + 2X_t \cdot 3\sin^2(X_t)\cos(X_t) + 2X_t^2(6\sin(X_t)\cos^2(X_t) - 3\sin^3(X_t))\} dt + 2X_t(3\sin^2(X_t)\cos(X_t))\, dB_t$$

$$= [1 + 6X_t\sin^2(X_t)\cos(X_t) + X_t^2(12\sin(X_t)\cos^2(X_t) - 6\sin^3(X_t)] dt + [6X_t\sin^2(X_t)\cos(X_t)]\, dB_t$$ //

Exercise 2

$$d(M_t) = d(Z_t^2) - d(\langle Z\rangle_t)$$

Page 110: $\langle Z\rangle_t = \int_0^t A_s^2 \, ds$

$$d(\langle Z\rangle_t) = A_t^2 \, dt$$

Ito's formula III: (page 111)

$$df(t, Z_t) = \left[\dot{f}(t, Z_t) + R_t f'(t, Z_t) + \frac{A_t^2}{2} f''(t, Z_t)\right] dt + A_t f'(t, Z_t) \, dB_t$$

$f(t, Z_t) = Z_t^2$ | $f''(t, Z_t) = 2$ | $dZ_t = A_t \, dB_t$

$f'(t, Z_t) = 2Z_t$ | $\dot{f}(t, Z_t) = 0$ | $\rightarrow R_t = 0$ (drift 0)

$$df(t, Z_t) = \left[0 + 0 + \frac{A_t^2}{2} \cdot 2\right] dt + A_t \cdot 2 Z_t \, dB_t$$

$$d(Z_t^2) = 2 A_t \cdot Z_t \cdot dB_t + A_t^2 \, dt$$

$$d(M_t) = 2 A_t \cdot Z_t \, dB_t + A_t^2 \, dt - A_t^2 \, dt$$

$$d(M_t) = 2 A_t Z_t \, dB_t$$

$$M_{t+1} - M_t = \int_t^{t+1} 2 A_t Z_t \, dB_t$$

$$M_{t+1} = M_t + \int_t^{t+1} 2 A_t Z_t \, dB_t$$

$$E[M_{t+1} | F_t] = E[M_t | F_t] + \underbrace{E\left[\int_t^{t+1} 2 A_t Z_t \, dB_t \,\Big|\, F_t\right]}_{\searrow 0}$$

expectation of Ito's integral with respect to Brownian Motion is 0

$$E[M_{t+1} | F_t] = M_t$$

$\hookrightarrow M_t$ is a martingale

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Exercise 2

We have proven earlier that M_t is a martingale.

According to page 56, a martingale is a continuous martingale if with probability 1, the function $t \to M_t$ is a continuous function.

$$M_t = Z_t^2 - \langle Z \rangle_t$$

$$Z_t = \int_0^t A_s dB_s$$

Z_t is an integral of A_s, where A_s is a bounded, adapted process with continuous paths. The function $t \to A_s$ is a continuous function. Therefore, the function $t \to Z_t$ also has continuous paths (a continuous function).

Since Z_t^2 is only the squared version of Z_t, the function $t \to Z_t^2$ is also a continuous function.

$$\langle Z \rangle_t = \int_0^t A_s^2 ds$$

Since the function $t \to A_s$ is a continuous function, and A_s^2 is only the squared version of A_s, then the function $t \to A_s^2$ is also continuous function.
Since the integral of a continuous function is also a continuous function, the function $t \to \langle Z \rangle_t$ is a continuous function.

$$M_t = Z_t^2 - \langle Z \rangle_t$$

Since M_t is a linear combination of Z_t^2 and $\langle Z \rangle_t$, then both components are continuous processes, then with probability 1, the function $t \to M_t$ is a continuous function.

Therefore, M_t is a continuous martingale (with respect to the filtration where $\mathcal{F}_t$ is the information in $\{B_s: s \leq t\}$.

Exercise 3

Page 108: Geometric BM:

SDE : $dX_t = X_t [m\,dt + \sigma\, dB_t]$

$X_t = X_0 \exp\{(m - \frac{\sigma^2}{2})t + \sigma B_t\}$

here, $m = 2$, $\sigma = 1$

$m - \frac{\sigma^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$

3.1) $X_t = X_0 \cdot \exp\{\frac{3}{2}t + B_t\}$ //

3.2) $X_0 = 1$

$X_t = \exp\{\frac{3}{2}t + B_t\}$

$X_1 = \exp\{\frac{3}{2} + B_1\}$

$\ln(X_1) = \frac{3}{2} + B_1$

$P(B_t \geq r) = 1 - \Phi(r/\sqrt{t})$

$P(X_1 > 3) = P(\ln(X_1) > \ln(3)) = P(1.5 + B_1 > \ln 3)$

$= P(B_1 > \ln(3) - 1.5) = 1 - \Phi\left(\frac{\ln(3) - 1.5}{1}\right)$

$= 1 - \Phi(-0.40) = 1 - 0.34458$

$P(X_1 > 3) = 0.6554$ //

Exercise 3

3.3) $X_0 = 1/2$

$$2X_t = \exp\left\{\frac{3}{2}t + B_t\right\} \quad \Big| \quad \ln(X_2) = B_2 + 3 - \ln(2)$$

$$2X_2 = \exp\{3 + B_2\}$$

$$\ln(2) + \ln(X_2) = 3 + B_2$$

$$P(X_2 < 3) = P(\ln(X_2) < \ln(3)) = P\{B_2 + 3 - \ln(2) < \ln(3)\}$$

$$= P\{B_2 < \ln(3) + \ln(2) - 3\} = \Phi\left(\frac{\ln(3) + \ln(2) - 3}{\sqrt{2}}\right)$$

$$P(X_2 < 3) = \Phi(-0.85) = 0.1977 \quad //$$

3.4) $dX_t = 2X_t\,dt + X_t\,dB_t$

$R_t = 2X_t, \; A_t = X_t, \; A_t^2 = X_t^2$

$f(t, X_t) = Y_t = \log(X_t)$

$f'(t, X_t) = \frac{1}{X_t}, \quad f''(t, X_t) = -\frac{1}{X_t^2}, \quad \dot{f}(t, X_t) = 0$

Using Ito's formula III (page 111)

$$d\,f(t, X_t) = \left[0 + 2X_t \cdot \frac{1}{X_t} + \frac{X_t^2}{2}\left(-\frac{1}{X_t^2}\right)\right]dt + X_t \cdot \frac{1}{X_t}\,dB_t$$

$$d(Y_t) = \left(2 - \frac{1}{2}\right)dt + 1 \cdot dB_t$$

$$d(Y_t) = \frac{3}{2}\,dt + 1 \cdot dB_t \quad //$$

Exercise 4 reference page 118 – 119 class notes

4.1) $dX_t = 2X_t\,dt + 2X_t\,dB_t$

$H_t = 2X_t = A_t$

$dY_t = 3Y_t\,dt - Y_t\,dB_t$

$h_t = 3Y_t \quad , \quad C_t = -Y_t$

$d(\langle X,Y\rangle_t) = A_t C_t\,dt = -2X_tY_t\,dt$

$d(X_tY_t) = X_t dY_t + Y_t dX_t + d\langle X,Y\rangle_t$

$$d(Z_t) = X_t(3Y_t\,dt - Y_t\,dB_t) + Y_t(2X_t)(dt + dB_t) - 2X_tY_t\,dt$$

$$= X_tY_t(3\,dt - dB_t) + 2X_tY_t(dt + dB_t) - 2X_tY_t\,dt$$

$$d(Z_t) = 3X_tY_t\,dt + X_tY_t\,dB_t$$

$$d(Z_t) = Z_t(3\,dt + dB_t) \quad //$$

Exercise 4.2

$$d(Y_t^{-1}) = -1 \cdot Y_t^{-2}\, dY_t = -\frac{1}{Y_t^2}(3Y_t\, dt - Y_t\, dB_t)$$

$$d(Y_t^{-1}) = -\frac{3}{Y_t} dt + \frac{1}{Y_t} dB_t$$

$$d(X_t)\, d(Y_t^{-1}) = 2X_t \cdot \left(\frac{1}{Y_t}\right)(dB_t^2) = 2\frac{X_t}{Y_t} dt$$

$$d(X_t/Y_t) = X_t \cdot (dY_t^{-1}) + Y_t^{-1} \cdot dX_t + (dX_t)(dY_t^{-1})$$

$$= X_t\left(\frac{1}{Y_t}\right)(-3dt + dB_t) + \frac{1}{Y_t}(X_t)(2dt + 2B_t)$$

$$+ 2\frac{X_t}{Y_t} dt$$

$$= (-3+2+2) Z_t\, dt + (1+2)\, Z_t\, dB_t$$

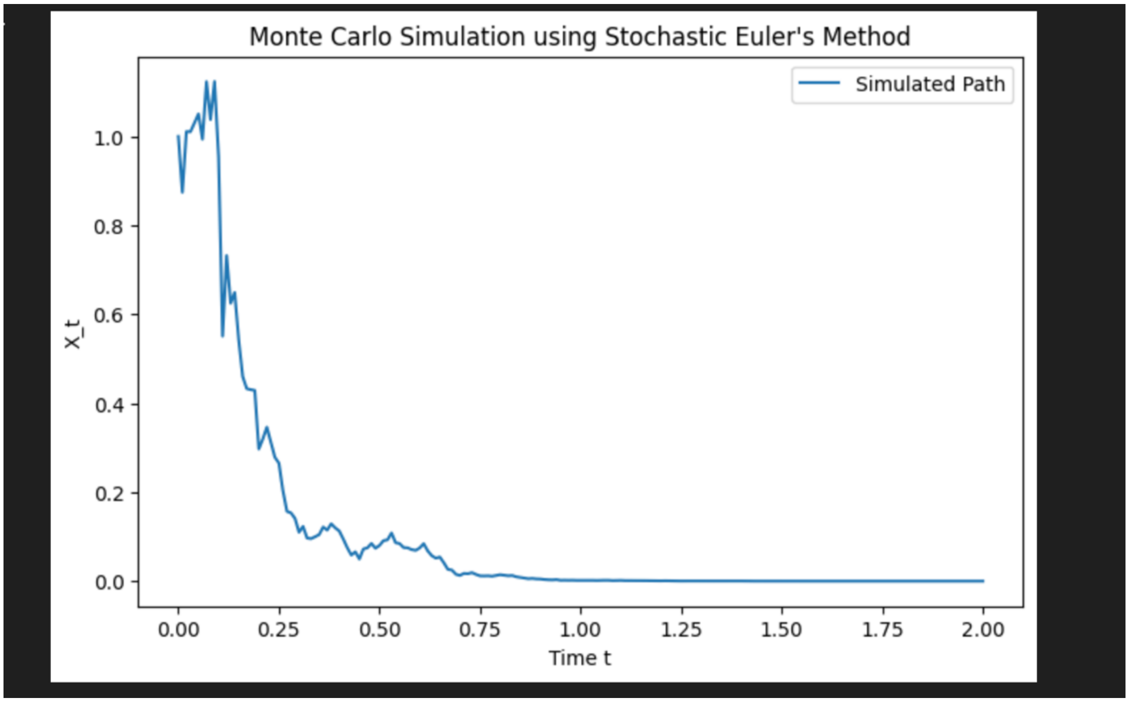
$$= Z_t\, dt + 3Z_t\, dB_t$$

$$d(Z_t) = Z_t\,(dt + 3\, dB_t)$$

//

Exercise 5
The code is written with the help of OpenAI ChatGPT.

Exercise 5.1)



Exercise 5.2) The estimated probability for $P(X_2 \geq 3)$ is 0.0006.

```python
import numpy as np

# Given parameters
X0 = 1          # Initial condition
mu = -2         # Drift coefficient
sigma = 2       # Diffusion coefficient
T = 2           # Time horizon
dt = 0.01       # Time step size
N = int(T / dt)  # Number of time steps
M = 10000       # Number of simulations (use more than 1000 for better accuracy)

# Initialize array for X_t simulations
X = np.zeros((M, N+1))
X[:, 0] = X0  # Set initial value

# Perform Euler-Maruyama simulation
for i in range(1, N+1):
    dB = np.sqrt(dt) * np.random.randn(M)  # Brownian increments
    X[:, i] = X[:, i-1] * (1 + mu * dt + sigma * dB)

# Estimate probability P(X_2 >= 3)
probability = np.mean(X[:, -1] >= 3)
print(f"Estimated Probability P(X_2 ≥ 3): {probability:.5f}")
```

[3]

```
Estimated Probability P(X_2 ≥ 3): 0.00060
```

Exercise 5 3.7.3 Exact Probability

$dX_t = X_t[-2\,dt + 2\,dB_t]$, $X_0 = 1$

$X_t = X_0 \exp\{(m - \frac{\sigma^2}{2})t + \sigma \cdot B_t\}$

$m = -2, \quad \sigma = 2$

$m - \frac{\sigma^2}{2} = -2 - \frac{4}{2} = -4$

$X_t = 1 \exp\{-4t + 2 \cdot B_t\}$

$\ln(X_t) = -4t + 2B_t$

$\ln(X_2) = -4(2) + 2B_2 = 2B_2 - 8$

$P(X_2 \geq 3) = P(\ln(X_2) \geq \ln(3)) = P(2B_2 - 8 \geq \ln(3))$

$= P(2B_2 \geq 8 + \ln(3)) = P\left(B_2 \geq 4 + \frac{\ln(3)}{2}\right)$

$= 1 - \Phi\left(\frac{\frac{\ln(3)}{2} + 4}{\sqrt{2}}\right) = 1 - 0.99936$

$P(X_2 \geq 3) = 0.00064$ // $\rightarrow 3.22$

Appendix 1: Code for Monte Carlo Simulation

```
import numpy as np
import matplotlib.pyplot as plt

# Given parameters
X0 = 1          # Initial condition
mu = -2         # Drift coefficient
sigma = 2       # Diffusion coefficient
T = 2           # Time horizon
dt = 0.01       # Time step size
N = int(T / dt)  # Number of time steps
M = 1000        # Number of simulations

# Time grid
t = np.linspace(0, T, N+1)

# Initialize array for X_t simulations
X = np.zeros((M, N+1))
X[:, 0] = X0  # Set initial value

# Perform Euler-Maruyama simulation
for i in range(1, N+1):
    dB = np.sqrt(dt) * np.random.randn(M)  # Brownian increments
    X[:, i] = X[:, i-1] * (1 + mu * dt + sigma * dB)

# Plot one sample path
plt.figure(figsize=(8, 5))
plt.plot(t, X[0], label="Simulated Path")
plt.xlabel("Time t")
plt.ylabel("X_t")
plt.title("Monte Carlo Simulation using Stochastic Euler's Method")
plt.legend()
plt.show()
```