Matheus Raka Pradnyatama HW3 StoChastic

Exercise 1

$$z_1 = \int_0^1 A_s\, dB_s = \int_0^{1/2} A_s\, dB_s + \int_{\frac{1}{2}}^1 A_s\, dB_s .$$

$$= \int_0^{1/2} 1 \cdot dB_s + \int_{1/2}^1 Y\, dB_s = B_s \Big|_{s=0}^{s=1/2} + Y \cdot B_s \Big|_{s=1/2}^{s=1}$$

$$z_1 = B_{1/2} - B_0 + Y(B_1 - B_{1/2})$$

Because B_t is a brownian motion, $B_0 = 0$

$$B_{1/2} \sim N(0, 1/2)$$

$$P(B_t \geq r) = 1 - \Phi(r/\sqrt{t})$$

$$P(B_{0.5} \geq 0) = 1 - \Phi\left(\frac{0}{\sqrt{0.5}}\right) = 1 - \Phi(0) = 1 - 0.5 = 0.5$$

$$P(B_{1/2} < 0) = 1 - P(B_{1/2} \geq 0) = 1 - 0.5 = 0.5$$

Exercise 1.1

$$P(z_1 \geq 0) = P(z_1 \geq 0, B_{1/2} \geq 0) + P(z_1 \geq 0, B_{1/2} < 0)$$

For $B_{1/2} \geq 0$, $Y = 0$,

$$z_1 = B_{1/2} + 0(B_1 - B_{1/2}) = B_{1/2} \geq 0$$ always, since the initial condition is $B_{1/2} \geq 0$

$$P(z_1 \geq 0 \mid B_{1/2} \geq 0) = 1$$

For $B_{1/2} < 0$, $Y = 1$,

$$Z_1 = B_{1/2} + 1(B_1 - B_{1/2}) = B_1$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = P(B_1 \geq 0 ; B_{1/2} < 0) :$$

$$= P(B_1 \geq 0 \mid B_{1/2} = x) \cdot P(B_{1/2} = x)$$

$$= \int_{-\infty}^{0} P(B_1 \geq 0 \mid B_{1/2} = x) \cdot dP(B_{1/2} = x)$$

Page 70 of class notes: $(B_t \sim N(0,t))$, $dP(B_t = x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$

$B_{1/2} \sim N(0, 1/2)$

$$dP(B_{1/2} = x) = \frac{1}{\sqrt{2\pi \cdot 1/2}} e^{-x^2/(2)(1/2)} dx = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

$B_1 \geq 0$

$B_1 - B_{1/2} \geq -B_{1/2}$

$B_1 - B_{1/2} \geq -x$

$$P(B_1 \geq 0 \mid B_{1/2} = x) = P(B_1 - B_{1/2} \geq -x) = \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi \cdot 1/2}} e^{\frac{-y^2}{(2)(1/2)}} dy$$

$$= \int_{-x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_{-\infty}^{0} \int_{-x}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)} dy\, dx$$

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Exercise 1.1 (continued)

$x = r\cos\theta$

$y = r\sin\theta$ ①

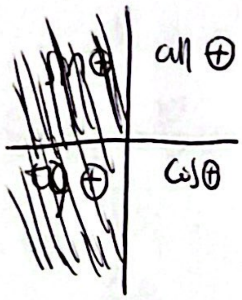
x goes from $-\infty$ to 0

$x < 0$

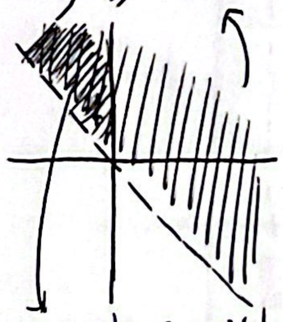
$r\cos\theta < 0$

$\cos\theta < 0$

$\frac{\pi}{2} < \theta < 3\pi/2$



② $y \geq -x$



overlap with ①:

$$\frac{\pi}{2} < \theta < \frac{3\pi}{4}$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_0^\infty \int_{\pi/2}^{3\pi/4} \frac{1}{\pi} e^{-r^2} r\,dr\,d\theta = \int_0^\infty \frac{1}{\pi} e^{-r^2} r\,dr \left(\frac{3\pi}{4} - \frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}\int e^{-r^2} r\,dr \left(\frac{\pi}{4}\right) = \frac{1}{4}\int_0^\infty e^{u}\left(\frac{du}{-2}\right)$$

$u = -r^2$

$du = -2r\,dr$

$r\,dr = \frac{du}{-2}$

$$= -\frac{1}{8} e^{u}\Big|_{r=0}^{r=\infty} = -\frac{1}{8}\left(\frac{1}{e^\infty} - \frac{1}{e^0}\right)$$

$$= -\frac{1}{8}(-1) = \frac{1}{8}$$

$$P(Z_1 \geq 0) = P(Z_1 \geq 0 \mid B_{1/2} \geq 0) \cdot P(B_{1/2} \geq 0) + P(Z_1 \geq 0, B_{1/2} < 0)$$

$$= (1)(0.5) + 1/8$$

$$P(Z_1 \geq 0) = 0.625$$

Exercise 1.2 (Case #2)

For $B_{1/2} \geq 0$, $Y = 0$

$Z_1 = B_{1/2}$ (Same as case #1) $\to P(Z_1 \geq 0 \mid B_{1/2} \geq 0) = 1$

For $B_{1/2} < 0$, $Y = -5$

$Z_1 = B_{1/2} - 5(B_1 - B_{1/2}) = 6B_{1/2} - 5B_1 = B_{1/2} - 5(B_1 - B_{1/2})$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_{-\infty}^{0} P(Z_1 \geq 0 \mid B_{1/2} = x) \cdot dP(B_{1/2} = x)$$

$dP(B_{1/2} = x) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$ (same as before)

$Z_1 \geq 0$, $B_{1/2} = x$

$B_{1/2} - 5(B_1 - B_{1/2}) \geq 0$

$-5(B_1 - B_{1/2}) \geq -B_{1/2}$

$-5(B_1 - B_{1/2}) \geq -x$

$5(B_1 - B_{1/2}) \leq x \to (B_1 - B_{1/2}) \leq x/5$

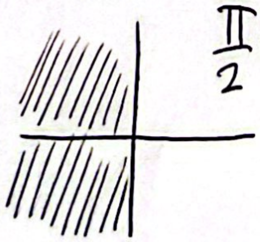
$$P(Z_1 \geq 0 \mid B_{1/2} = x) = P\left(B_1 - B_{1/2} \leq \frac{x}{5}\right) = \int_{-\infty}^{x/5} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_{-\infty}^{0} \int_{-\infty}^{x/5} \frac{1}{\pi} e^{-(x^2+y^2)} dy\,dx$$

① x goes from $-\infty$ to 0

$\cos\theta < 0$

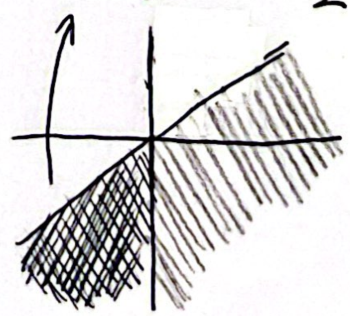
$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$



$y < x/5$

$\hookrightarrow$ quadrant III

$\pi + \operatorname{arctg}\left(\frac{1}{5}\right) < \theta < \frac{3\pi}{2}$



Exercise 1.2 (continued)

$$P(z_1 \geqslant 0, B_{1/2} < 0) = \int_0^{\infty} \int_{\pi + \arctan\left(\frac{1}{5}\right)}^{\frac{3\pi}{2}} \frac{1}{\pi} e^{-r^2} r \, dr \, d\theta$$

$u = -r^2$
$du = -2r\,dr$
$2r\,dr = \frac{du}{-2}$

$$= \frac{1}{\pi} \int e^{-r^2} r\,dr \left(\frac{3\pi}{2} - \pi - \arctan\left(\frac{1}{5}\right) \right)$$

$$= \frac{1}{\pi} \int e^{-r^2} r\,dr \left(\frac{1}{2}\pi - \arctan\left(\frac{1}{5}\right) \right)$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - \operatorname{arctg}\left(\frac{1}{5}\right) \right) \int (e^u) \cdot \frac{du}{-2}$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - \operatorname{arctg}\left(\frac{1}{5}\right) \right) \left(-\frac{1}{2} \right) \left(\frac{1}{e^{\infty}} - \frac{1}{e^0} \right)$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{2} - \operatorname{arctg}\left(\frac{1}{5}\right) \right)$$

$$P(z_1 \geqslant 0) = P(z_1 \geqslant 0 \mid B_{1/2} \geqslant 0) \cdot P(B_{1/2} \geqslant 0) + P(z_1 \geqslant 0, B_{1/2} < 0)$$

$$= (1)(0.5) + \frac{1}{2\pi} \left(\frac{\pi}{2} - \arctan\left(\frac{1}{5}\right) \right)$$

$$P(z_1 \geqslant 0) = \frac{1}{2} + \frac{1}{2\pi} \left(\frac{\pi}{2} - \arctan\left(\frac{1}{5}\right) \right)$$

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Exercise 1.3 (Case #3)

For $B_{1/2} \geq 0$, $Y = 1$, $Z_1 = B_{1/2} + (B_1 - B_{1/2}) = B_1$

$$P(Z_1 \geq 0, B_{1/2} \geq 0) = P(B_1 \geq 0, B_{1/2} \geq 0)$$

$$= \int_0^{\infty} P(B_1 \geq 0 \mid B_{1/2} = x)\, dP(B_{1/2} = x)$$

(Same as in Case #1)

$$P(B_1 \geq 0 \mid B_{1/2} = x) = \int_{-x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}\, dy$$

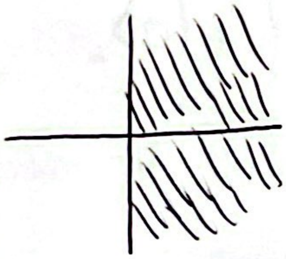
$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_0^{\infty}\int_{-x}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)}\, dy\, dx$$

x goes from $0 \to \infty$

$x > 0$

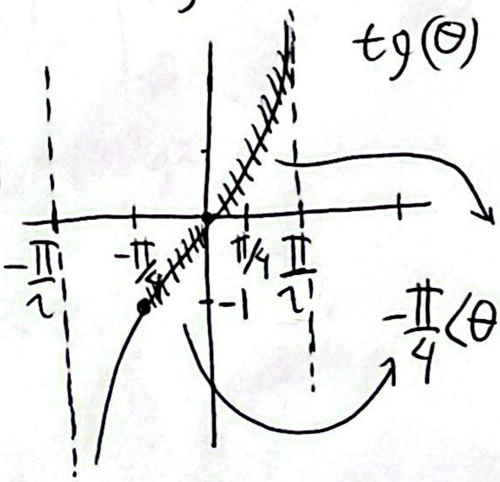
$\cos\theta > 0$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



$y \geq -x$

$tg\theta \geq -1$



$-\frac{\pi}{4} < \theta < \frac{\pi}{2}$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_0^{\infty}\int_{-\pi/4}^{\pi/2} \frac{1}{\pi} e^{-r^2} r\, dr\, d\theta = \int \frac{1}{\pi} e^{-r^2} r\, dr \left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\pi}\right)\left(\frac{3\pi}{4}\right) \int e^{u} \cdot \frac{du}{-2} = -\frac{3}{8}\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right) = \frac{3}{8}$$

$$P(Z_1 \geq 0 \mid B_{1/2} < 0) = \frac{P(Z_1 \geq 0, B_{1/2} < 0)}{P(B_{1/2} < 0)} = \frac{3/\cancel{8}\,4}{1/\cancel{2}} = \frac{3}{4}$$

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For $B_{1/2} < 0,\ Y = -1,\ Z_1 = B_{1/2} - (B_1 - B_{1/2}) = 2B_{1/2} - B_1$

$Z_1 \geq 0,\quad B_{1/2} = x$

$$2B_{1/2} - B_1 \geq 0$$
$$B_1 - 2B_{1/2} \leq 0$$
$$B_1 - B_{1/2} < B_{1/2} = x$$
$$B_1 - B_{1/2} < x$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_{-\infty}^{0} P(Z_1 \geq 0 \mid B_{1/2} = x)\, dP(B_{1/2} = x)$$

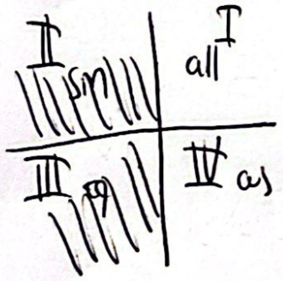
$$P(Z_1 \geq 0 \mid B_{1/2} = x) = P(B_1 - B_{1/2} < x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-y^2}\, dy$$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_{-\infty}^{0} \int_{-\infty}^{x} \frac{1}{\pi} e^{-(x^2+y^2)}\, dy\, dx$$

x goes from $-\infty$ to 0

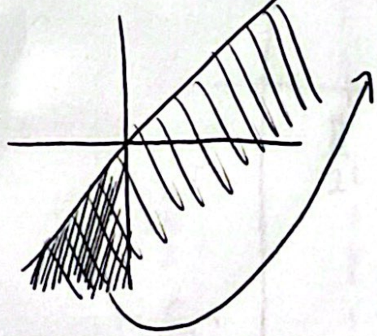
$x < 0$

$\cos\theta < 0$



$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

② $y \leq x$



overlap with ①:

$\frac{5\pi}{4} < \theta < \frac{3\pi}{2}$

$$P(Z_1 \geq 0, B_{1/2} < 0) = \int_0^{\infty} \int_{5\pi/4}^{\frac{3\pi}{2}} \frac{1}{\pi} e^{-r^2}\, r\, dr\, d\theta = \frac{1}{\pi} \int e^{-r^2} r\, dr \left(\frac{3\pi}{2} - \frac{5\pi}{4}\right)$$

$$= \frac{1}{\pi} \int e^{-r^2} dr \left(\frac{1}{4}\pi\right) = \frac{1}{4} \int e^{-r^2} dr$$

$u = -r^2$
$du = -2r\,dr$
$r\,dr = \frac{du}{-2}$

$P(t_1 \geq 0, B_{1/2} < 0) = \frac{1}{4} \int e^u \cdot \frac{du}{-2} = -\frac{1}{8}\left(\frac{1}{e^{\infty}} - \frac{1}{e^0}\right)$
$= \frac{1}{8}$

$P(t_1 \geq 0) = P(t_1 \geq 0, B_{1/2} \geq 0) + P(t_1 \geq 0, B_{1/2} < 0)$
$= \frac{3}{8} + \frac{1}{8}$

$P(t_1 \geq 0) = \frac{1}{2}$
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Stochastic Calculus – Homework 3
Matheus Raka Pradnyatama

Exercise 2

$$\langle Z \rangle_t = \int_0^t A_s^2 ds = \int_0^{\frac{1}{2}} A_s^2 ds + \int_{\frac{1}{2}}^t A_s^2 ds$$

$$\langle Z \rangle_t = \int_0^{\frac{1}{2}} 1^2 ds + \int_{\frac{1}{2}}^t Y^2 ds$$

$$\langle Z \rangle_t = s \Big|_{s=0}^{s=0.5} + Y^2 s \Big|_{s=0.5}^{s=t}$$

$$\langle Z \rangle_t = 0.5 + Y^2(t - 0.5)$$

Exercise 2.1) Case #1

If $B_{0.5} \geq 0$, then $Y = 0$:

$$\langle Z \rangle_t = 0.5 + 0(t - 0.5) = 0.5$$

If $B_{0.5} < 0$, then $Y = 1$:

$$\langle Z \rangle_t = 0.5 + 1(t - 0.5) = t$$

Conclusion:

For $B_{0.5} \geq 0$: $\langle Z \rangle_t = 0.5$
For $B_{0.5} < 0$: $\langle Z \rangle_t = t$

Exercise 2.2) Case #2
Same as in Exercise 2.1)

$$\langle Z \rangle_t = 0.5 + Y^2(t - 0.5)$$

If $B_{0.5} \geq 0$, then $Y = 0$:

$$\langle Z \rangle_t = 0.5 + 0(t - 0.5) = 0.5$$

If $B_{0.5} < 0$, then $Y = -5$:

$$\langle Z \rangle_t = 0.5 + (-5)^2(t - 0.5) = 0.5 + 25(t - 0.5) = 0.5 + 25t - 12.5$$
$$\langle Z \rangle_t = 25t - 12$$

Conclusion:
For $B_{0.5} \geq 0$: $\langle Z \rangle_t = 0.5$
For $B_{0.5} < 0$: $\langle Z \rangle_t = 25t - 12$

Exercise 2.3) Case #3
Same as in Exercise 2.1)

$$\langle Z \rangle_t = 0.5 + Y^2(t - 0.5)$$

If $B_{0.5} \geq 0$, then $Y = 1$:

$$\langle Z \rangle_t = 0.5 + 1(t - 0.5) = 0.5 + t - 0.5 = t$$

If $B_{0.5} < 0$, then $Y = -1$:

$$\langle Z \rangle_t = 0.5 + (-1)^2(t - 0.5) = 0.5 + 1(t - 0.5) = 0.5 + t - 0.5 = t$$

Conclusion:
For $B_{0.5} \geq 0$: $\langle Z \rangle_t = t$
For $B_{0.5} < 0$: $\langle Z \rangle_t = t$

Exercise 3

For case #2, we have two values of Z_1:

For $B_{0.5} \geq 0 \rightarrow Y = 0$:

$$Z_1 = B_{0.5}$$

where $B_{0.5} \sim N(0, 0.5)$

$$Var(Z_1) = Var(B_{0.5}) = 0.5 \qquad (1)$$

For $B_{0.5} < 0 \rightarrow Y = -5$:

$$Z_1 = 6B_{0.5} - 5B_1$$

$$Var(6B_{0.5} - 5B_1) = Var(6B_{0.5}) + Var(5B_1) + 2(6)(-5) * Cov(B_{0.5}, B_1)$$

Based on
https://www.stat.uchicago.edu/~yibi/teaching/stat317/2021/Lectures/Lecture22.pdf,
outlining the covariance function of a Brownian motion, because of the independent increment principle, for $0 < s < t$,

$$Cov(B_s, B_t) = Var(B_s)$$

$$Var(6B_{0.5} - 5B_1) = 36Var(B_{0.5}) + 25Var(B_1) - 60 * Var(B_{0.5})$$

$$B_{0.5} \sim N(0, 0.5)$$

$$B_1 \sim N(0, 1)$$

$$Var(6B_{0.5} - 5B_1) = 36 * 0.5 + 25 * 1 - 60 * 0.5$$

$$Var(6B_{0.5} - 5B_1) = 13$$

$$Var(Z_1) = Var(6B_{0.5} - 5B_1) = 13 \qquad (2)$$

We need to prove those variances for each $B_{0.5}$.

For $0 \leq t < 0.5$,

$$A_t = 1$$

$$A_t^2 = 1$$

$$E[A_t^2] = 1$$

For $0.5 < t \leq 1$,

$$A_t = Y$$

$$A_t^2 = Y^2$$

$$E[A_t^2] = E[Y^2]$$

$$Var(Z_1) = \int_0^1 E[A_t^2]dt = \int_{t=0}^{t=\frac{1}{2}} E[A_t^2]dt + \int_{t=\frac{1}{2}}^{t=1} E[A_t^2]dt$$

$$Var(Z_1) = \int_0^{\frac{1}{2}} 1dt + \int_{\frac{1}{2}}^1 E[Y^2]dt$$

$$Var(Z_1) = \left(\frac{1}{2} - 0\right) + E[Y^2] * \left(1 - \frac{1}{2}\right)$$

$$Var(Z_1) = 0.5 + E[Y^2] * 0.5$$

For $B_{0.5} \geq 0 \rightarrow Y = 0$:

$$E[Y^2] = 0$$

$$Var(Z_1) = \int_0^1 E[A_t^2]dt = 0.5 + 0 * 0.5 = 0.5$$

Same as in (1), for $B_{0.5} \geq 0$

$$Var(Z_1) = Var(B_{0.5}) = 0.5 \qquad (1)$$

$$Var(Z_1) = \int_0^1 E[A_t^2]dt$$

For $B_{0.5} < 0 \rightarrow Y = -5$:

$$E[Y^2] = E[(-5)^2] = 25$$

$$Var(Z_1) = \int_0^1 E[A_t^2]dt = 0.5 + 25 * 0.5 = 13$$

Same as in (2), for $B_{0.5} < 0$,

$$Var(Z_1) = Var(6B_{0.5} - 5B_1) = 13 \qquad (2)$$

$$Var(Z_1) = \int_0^1 E[A_t^2]dt$$

We have verified the statement for both cases of $B_{0.5}$.