Problem Set 8 – Stochastic Calculus
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Discrete values = Integer values (whole numbers: 1, 2, 3, 4, 5)

$\boldsymbol{\mu_1}$

$\mu_1$is Poisson distribution with lambda 1 → discrete distribution
Support: $[0, \infty)$

$\mu_1(V) = 0$ for $V = \begin{Bmatrix} \emptyset \\ negative\ values \\ non-integer \end{Bmatrix}$

$\boldsymbol{\mu_2}$

μ_2 is uniform distribution on the interval $[0,1]$ → continuous distribution
Support: $[0, 1]$

$\mu_2(V) = 0$ for $V = \begin{Bmatrix} \emptyset \\ values\ not\ in\ [0,1] \\ discrete\ values \end{Bmatrix}$

$\boldsymbol{\mu_3}$

$\mu_3 = \mu_1 + \mu_2$

$\mu_3(V) = 0$ for $V = \begin{Bmatrix} \emptyset \\ negative\ values \end{Bmatrix}$

$\boldsymbol{\mu_4}$

μ_4 is not a probability measure, but a counting measure
$\mu_4(V) = 2$ if there are 2 integers in V
$\mu_4(V) = 0$ if there are 0 integers in V
Support: $\{-\infty, \infty\}$ but must be integers

$\mu_4(V) = 0$ for $V = \begin{Bmatrix} \emptyset \\ no\ integers \end{Bmatrix}$

$\boldsymbol{\mu_5}$

If $V = [1, 5]$, $\mu_5(V) = 5 - 1 = 4$
If $V = [5]$, $\mu_5(V) = 0$ since it's an individual point → 0 length

$\mu_5(V) = 0$ for $V = \begin{Bmatrix} \emptyset \\ discrete\ values(individual\ points) \end{Bmatrix}$

$\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_2}$
If $\mu_1(V) = 0$ for $V = \{non - integer\ in\ [0,1]\}$, $\mu_2(V) \neq 0$, $\mu_2 \not\ll \mu_1$
If $\mu_2(V) = 0$ for $V = \{values\ not\ in\ [0,1]\}$, $\mu_1(V) \neq 0$, $\mu_1 \not\ll \mu_2$

Say $V = [0, 1]$.
This contains non-integer such as 0.1, 0.2, 0.3, ..., 0.8, 0.9, 0.99, ...
$\mu_1(V) = 0$ for $V = \{non - integer\ in\ [0,1]\}$
$\mu_2(\Omega \backslash V) = 0$ for $V = \{non - integer\ in\ [0,1]\}$
$\Omega \backslash V$ means everything else outside V
Therefore, $\mu_1 \perp \mu_2$

$\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_3}$
If $\mu_1(V) = 0$ for $V = \{non - integer\ in\ [0,1]\}$, $\mu_3(V) \neq 0$, $\mu_3 \not\ll \mu_1$

If $\mu_3(V) = 0$ for $V = \emptyset$, $\mu_1(V) = 0$,
If $\mu_3(V) = 0$ for $V = \{negative\ values\}$, $\mu_1(V) = 0$,

$$\mu_1 \ll \mu_3$$

There will always be an overlap between μ_1 and μ_3 because μ_3 derives its value from μ_1.
Therefore, $\cancel{\mu_1 \perp \mu_3}$

$\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_4}$
If $\mu_1(V) = 0$ for $V = \{negative\ integer\ values\}$, $\mu_4(V) \neq 0$, $\mu_4 \not\ll \mu_1$

If $\mu_4(V) = 0$ for $V = \{0.3, 1.4\} = \{non - integer\ values\}$, $\mu_1(V) = 0$
If $\mu_4(V) = 0$ for $V = \{no\ integers\}$, $\mu_1(V) = 0$
If $\mu_4(V) = 0$ for $V = \emptyset$, $\mu_1(V) = 0$

$$\mu_1 \ll \mu_4$$

$\boldsymbol{\mu_1}$ and $\boldsymbol{\mu_5}$
If $\mu_1(V) = 0$ for $V = \{negative\ values\}$, $\mu_5(V) \neq 0$, $\mu_5 \not\ll \mu_1$
If $\mu_5(V) = 0$ for $V = \{discrete\ values\}$, $\mu_1(V) \neq 0$, $\mu_1 \not\ll \mu_5$

Suppose $E = \{0, 1, 2, 3, ...\} = \{set\ of\ individual\ discrete\ points\}$

$$\mu_1(E) \neq 0$$

However, $\mu_5(E) = 0$

$$\Omega \backslash E = \{sets\ of\ continuous\ intervals\} = \{[0.1, 0.9], [1.1, 1.9], ...\}$$

$$\mu_5(\Omega \backslash E) \neq 0$$

$$\mu_1(\Omega \backslash E) = 0$$

Since there is an event $E \in \mathcal{F}$ with,

$$\mu_5(E) = 0$$

$$\mu_1(\Omega \backslash E) = 0$$

We can conclude that:

$$\mu_1 \perp \mu_5$$

μ_2 and μ_3
If $\mu_2(V) = 0$ for $V = \{values\ not\ in\ [0,1]\}$, $\mu_3(V) \neq 0$, $\mu_3 \not\ll \mu_2$

If $\mu_3(V) = 0$ for $V = \{negative\ values\}$, $\mu_2(V) = 0$
If $\mu_3(V) = 0$ for $V = \emptyset$, $\mu_2(V) = 0$

$$\mu_2 \ll \mu_3$$

μ_2 and μ_4
If $\mu_2(V) = 0$ for $V = \{integer\ values\ not\ in\ [0,1]\}$, $\mu_4(V) \neq 0$, $\mu_4 \not\ll \mu_2$
If $\mu_4(V) = 0$ for $V = \{0.3, 0.4\} = \{non-integer\ values\ in\ [0,1]\}$, $\mu_2(V) \neq 0$, $\mu_2 \not\ll \mu_4$

Suppose $E = \{[0.1, 0.2],\ [0.3, 0.5], \dots\} = \{sets\ of\ continuous\ intervals\ in\ [0,1]\ \}$

$$\mu_2(E) \neq 0$$

However, $\mu_4(E) = 0$ (because there are no integers)

$$\Omega \backslash E = \{set\ of\ integers\ in\ [0,1]\} = \{[0, 1]\}$$
$$\mu_4(\Omega \backslash E) \neq 0$$
$$\mu_2(\Omega \backslash E) = 0$$

Since there is an event $E \in \mathcal{F}$ with,

$$\mu_4(E) = 0$$
$$\mu_2(\Omega \backslash E) = 0$$

We can conclude that:

$$\mu_2 \perp \mu_4$$

μ_2 and μ_5

Example: $V = [5, 8]$, $\mu_5(V) = 3$
If $\mu_2(V) = 0$ for $V = \{interval\ of\ discrete\ values\}$, $\mu_5(V) \neq 0$, $\mu_5 \not\ll \mu_2$

If $\mu_5(V) = 0$ for $V = \{3,4,5\} = \{individual\ points\}$, $\mu_2(V) = 0$
If $\mu_5(V) = 0$ for $V = \emptyset$, $\mu_2(V) = 0$

$$\mu_2 \ll \mu_5$$

μ_3 and μ_4
If $\mu_3(V) = 0$ for $V = [-5, -7] = \{negative\ values\}$, $\mu_4(V) \neq 0$, $\mu_4 \not\ll \mu_3$
If $\mu_4(V) = 0$ for $V = [0.3, 0.4] = \{no\ integers\}$, $\mu_3(V) \neq 0$, $\mu_3 \not\ll \mu_4$
Not singular

μ_3 and μ_5
If $\mu_3(V) = 0$ for $V = [-5, -7] = \{negative\ values\}$, $\mu_5(V) \neq 0$, $\mu_5 \not\ll \mu_3$
If $\mu_5(V) = 0$ for $V = [5] = \{individual\ point\}$, $\mu_3(V) \neq 0$, $\mu_3 \not\ll \mu_5$
Not singular

μ_4 and μ_5

If $\mu_4(V) = 0$ for $V = [0.3, 10.4] = \{no\ integers\}$, $\mu_5(V) \neq 0$, $\mu_5 \not\ll \mu_4$
If $\mu_5(V) = 0$ for $V = [5] = \{individual\ point\}$, $\mu_4(V) \neq 0$, $\mu_4 \not\ll \mu_5$

Suppose $E = \{\dots, -1, 0, 1, 2, \dots\} = \{set\ of\ individual\ integer\ points\}$

$$\mu_4(E) \neq 0$$

However, $\mu_5(E) = 0$

$$\Omega \backslash E = \{sets\ of\ continuous\ intervals\} = \{[0.1, 0.9], [1.1, 1.9], \dots\}$$

$$\mu_5(\Omega \backslash E) \neq 0$$

$$\mu_4(\Omega \backslash E) = 0 \text{ (because there is no integer)}$$

Since there is an event $E \in \mathcal{F}$ with,

$$\mu_5(E) = 0$$

$$\mu_4(\Omega \backslash E) = 0$$

We can conclude that:

$$\mu_4 \perp \mu_5$$

Homework 8 Stochastic Matheus Raka

2.1) $dS_t = S_t[3dt + dB_t] \rightarrow m = 3, \sigma = 1$

page 172: $d\tilde{S}_t = \sigma_t \tilde{S}_t \, dW_t$

page 166: $\tilde{S}_t = \dfrac{S_t}{R_t} = e^{-r \cdot t} S_t$

$d\tilde{S}_t = e^{-r \cdot t} S_t \, dW_t = e^{-0.05t} S_t \, dW_t$ → SDE //

2.2) page 172: V is a contingent claim if $V \geqslant 0$ ① and $E_Q[\tilde{V}^2] < \infty$ ②

$S_t \sim$ geometric Brownian motion

S_2 will be non-negative → S_2^2 will be non-negative

$V = S_2^2 \geqslant 0$ → satisfies $V \geqslant 0$ // ①

$\tilde{V} = e^{-r \cdot 2} \cdot V = e^{-2r} \cdot S_2^2$

$\tilde{V}^2 = e^{-4r} S_2^4 = e^{-4(0.05)} S_2^2 = e^{-0.2} S_2^2$

$E_Q[\tilde{V}^2] = e^{-0.2} E_Q[S_2^2] < \infty$

because $E_Q[S_2^2]$ will be a finite number $< \infty$

$E_Q[\tilde{V}^2] < \infty$ // ②

V is a contingent claim //

2.3) $\tilde{V}_t = E_Q[\tilde{V} \mid F_t]$

$V_T = S_T^2$

$\tilde{V}_T = e^{-r.T} \cdot S_T^2$

$\tilde{S}_T = e^{-r.T} \cdot S_T$

$S_T = e^{r.T} \tilde{S}_T$

$S_T^2 = e^{r.T.2} \cdot \tilde{S}_T^2$

$\tilde{V}_T = e^{-r.T} \cdot e^{2r.T} \cdot \tilde{S}_T^2 = e^{r.T} \tilde{S}_T^2$

$\tilde{V}_t = E_Q[e^{r.T} \cdot \tilde{S}_T^2 \mid F_t] = e^{r.T} E_Q[\tilde{S}_T^2 \mid F_t]$

$d\tilde{S}_t = \sigma_t \cdot \tilde{S}_t \, dW_t = \tilde{S}_t \, dW_t$

If $f(\tilde{S}_t) = \tilde{S}_t^2$, $f' = 2\tilde{S}_t$, $f'' = 2$

$d\langle \tilde{S} \rangle_t = \tilde{S}_t^2 \, dt$

$$d(\tilde{S}_t^2) = f' \cdot d\tilde{S}_t + \frac{1}{2} \cdot f'' \cdot d\langle \tilde{S} \rangle_t$$

$$= 2\tilde{S}_t \cdot [\tilde{S}_t \, dW_t] + \frac{1}{2} \cdot 2 \cdot \tilde{S}_t^2 \, dt$$

$$= 2\tilde{S}_t^2 \, dW_t + \tilde{S}_t^2 \, dt$$

Ignoring dW_t term: $d\tilde{S}_t^2 = \tilde{S}_t^2 \, dt$

$$\frac{d\tilde{S}_t^2}{\tilde{S}_t^2} = dt$$

$$\boxed{\int_{\tilde{S}_t^2}^{\tilde{S}_T^2} \frac{d\tilde{S}}{\tilde{S}} = \int_t^T dt}$$

$$\ln(\tilde{S}_T^2) - \ln(\tilde{S}_t^2) = T - t$$

$$\ln\left(\frac{\tilde{S}_T^2}{\tilde{S}_t^2}\right) = T - t$$

$$\frac{\tilde{S}_T^2}{\tilde{S}_t^2} = e^{T-t}$$

$$\tilde{S}_T^2 = \tilde{S}_t^2 e^{T-t}$$

2.3) (continued)

$$\tilde{V}_t = e^{\cdot r \cdot T} E_Q[\tilde{S}_T^2 | F_t] = e^{r \cdot T} E_Q[\tilde{S}_t^2 e^{T-t} | F_t]$$

$$\tilde{V}_t = e^{r \cdot T + T - t} \cdot \tilde{S}_t^2 = e^{rT + T - t} e^{-2rt} \cdot S_t^2$$

$$\tilde{V}_t = S_t^2 \exp\{rT + T - t - 2rt\}$$

$$\tilde{V}_t = S_t^2 \exp\{0.05 \times 2 + 2 - t - 2(0.05)t\}$$

$$\tilde{V}_t = S_t^2 \exp\{2.1 - 1.1t\} \quad //$$

$$\cancel{2.4) \; d\tilde{V}_t = e^{2.1-1.1t} \quad + e^{2.1-1.1t}\underbrace{[2\tilde{S}_t^3 dW_t + \tilde{S}_t^2 dt]}_{d(S_t^3)}}$$

2.4) $\tilde{V}_t = e^{rT+T} \cdot e^{-t} \cdot \tilde{S}_t^2$

$$d\tilde{V}_t = e^{rT+T}\left[e^{-t} \cdot d(\tilde{S}_t^2) + (-1) e^{-t} \cdot \tilde{S}_t^2 \, dt\right]$$

$$d\tilde{V}_t = e^{rT+T} [e^{-t}][2\tilde{S}_t^2 dW_t + \cancel{\tilde{S}_t^2 dt} - \cancel{\tilde{S}_t^2 dt}]$$

$$d\tilde{V}_t = e^{rT+T-t} \cdot 2\tilde{S}_t^2 \, dW_t \quad //$$

$$rT + T = 0.05(2) + 2 = 2.1$$

$$d\tilde{V}_t = 2e^{2.1-t} \tilde{S}_t^2 \, dW_t \rightarrow \text{SDE}$$

//

2.5) $d\tilde{V}_t = A_t\, dW_t = 2\tilde{S}_t^2 e^{rT+T-t}$

$$\tilde{S}_t = S_t e^{-r\cdot t}$$
$$\tilde{S}_t = S_t^2 e^{-2rt}$$

$$a_t = \frac{A_t}{\sigma_t \cdot \tilde{S}_t} = \frac{2\tilde{S}_t^{\cancel{2}} e^{rT+T-t}}{\tilde{S}_t} = 2\tilde{S}_t e^{rT+T-t}$$

$$a_t = 2S_t \cdot e^{-rt} \cdot e^{rT+T-t} = 2S_t\, e^{rT+T-t-rt}$$

//

$$b_t = \tilde{V}_t - \frac{A_t}{\sigma_t} = \tilde{S}_t^2 e^{rT+T-t} - 2\tilde{S}_t^2 e^{rT+T-t} = -\tilde{S}_t^2 e^{rT+T-t}$$

$$b_t = -S_t^2 \cdot e^{-2rt} e^{rT+T-t} = -S_t^2 e^{rT+T-t-2rt}$$

//

2.6) $V_t = a_t S_t + b_t \cdot R_t$

$$V_t = 2S_t e^{rT+T-t-rt} \cdot S_t + (-S_t^2)(e^{rT+T-t-2rt}) \cdot e^{rt}$$

$$V_t = 2S_t^2 e^{rT+T-t-rt} - S_t^2 e^{rT+T-t-rt}$$

$$V_t = S_t^2 \exp\{rT+T-t-rt\}$$

//

$$rT+T-t-rt = (0.05)(2)+2-(1+0.05)t$$
$$= 2.1 - 1.05t$$

$$V_t = S_t^2 \exp\{2.1 - 1.05t\}$$

//

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Exercise 3: S.8)

3.1) $d\tilde{S}_t = e^{-0.05t} S_t \, dW_t \rightarrow$ SDE

3.2) $S_t \sim$ geometric Brownian motion

S_2 will be non-negative $\rightarrow$ S_2^3 will be nonnegative

$V = S_2^3 \geq 0 \rightarrow V \geq 0$

$\tilde{V} = e^{-r.2} \cdot S_2^3 = e^{-2r} S_2^3$

$\tilde{V}^2 = e^{-4r} S_2^6 = e^{-0.2} S_2^6$

$E_Q[\tilde{V}^2] = e^{-0.2} E_Q[S_2^6] < \infty$

because $E_Q[S_2^6]$ will be a finite number $< \infty$

Because $V \geq 0$, and $E_Q[\tilde{V}^2] < \infty$,

V is a contingent claim. //

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3.3) $\tilde{V}_t = E_Q[\tilde{V} \mid F_t]$

$V_T = S_T^3$

$\tilde{V}_T = e^{-r.T} S_T^3$

$\tilde{S}_T = e^{-r.T} . S_T$

$S_T^3 = e^{3rT} . \tilde{S}_T^3$

$\tilde{V}_T = e^{-r.T} . e^{3rT} . \tilde{S}_T^3 = \tilde{S}_T^3 . e^{2r.T}$

$\tilde{V}_t = E_Q[e^{2rT} . \tilde{S}_T^3 \mid F_t] = e^{2rT} E_Q[\tilde{S}_T^3 \mid F_t]$

$f = \tilde{S}_t^3 \quad , \quad f' = 3\tilde{S}_t^2 \quad , \quad f'' = 6\tilde{S}_t \quad , \quad d\langle \tilde{S} \rangle_t = \tilde{S}_t^2 dt$

$d(\tilde{S}_t^3) = f' d\tilde{S}_t + \frac{1}{2} . f'' . d\langle \tilde{S} \rangle_t$

$= 3\tilde{S}_t^2 [\tilde{S}_t dW_t] + \frac{1}{2} . 6\tilde{S}_t . \tilde{S}_t^2 dt$

$= 3\tilde{S}_t^3 dW_t + 3\tilde{S}_t^3 dt$

$\frac{d\tilde{S}_t^3}{\tilde{S}_t^3} = 3 dt$ (ignoring dW_t term)

$\ln(\tilde{S}_T^3) - \ln(\tilde{S}_t^3) = 3(T-t)$

$\ln\left(\frac{\tilde{S}_T^3}{\tilde{S}_t^3}\right) = 3(T-t)$

$\frac{\tilde{S}_T^3}{\tilde{S}_t^3} = e^{3(T-t)}$

$\tilde{S}_T^3 = \tilde{S}_t^3 . e^{3(T-t)}$

$2rT + 3T - 3t = 2(0.05)2 + 3(2) - 3t$

$= 6.2 - 3t$

$\tilde{V}_t = e^{2rT} . E_Q[\tilde{S}_t^3 . e^{3(T-t)} \mid F_t] = \tilde{S}_t^3 e^{2rT + 3T - 3t}$ //

$\tilde{V}_t = \tilde{S}_t^3 \exp\{6.2 - 3t\}$ //

3.4) $\tilde{V}_t = \tilde{S}_t^3 \cdot e^{2rT+3T} e^{-3t}$

$d\tilde{V}_t = e^{2rT+3T}\left[e^{-3t}\cdot(-3)\,\tilde{S}_t^3 dt + e^{-3t}\cdot d(\tilde{S}_t^3)\right]$

$d\tilde{V}_t = e^{2rT+3T-3t}\left[-3\tilde{S}_t^3\,dt + 3\tilde{S}_t^3\,dW_t + 3\tilde{S}_t^3\,dt\right]$

$d\tilde{V}_t = 3\tilde{S}_t^3 \cdot e^{2rT+3T-3t}\,dW_t$ //

$d\tilde{V}_t = 3\tilde{S}_t^3 \cdot e^{6.2-3t}\,dW_t$ //

3.5) $A_t = 3\tilde{S}_t^3 \cdot e^{2rT+3T-3t}$

$a_t = \dfrac{A_t}{\sigma_t \cdot \tilde{S}_t} = 3\tilde{S}_t^2 e^{2rT+3T-3t} = 3S_t^2 e^{2rT+3T-3t-2rt}$ //

$\tilde{S}_t = e^{-r\cdot t} S_t \rightarrow \tilde{S}_t^2 = e^{-2rt} S_t^2 \quad , \quad \tilde{S}_t^3 = e^{-3rt} S_t^3$

$b_t = \tilde{V}_t - \dfrac{A_t}{\sigma_t} = \tilde{S}_t^3 e^{2rT+3T-3t} - 3\tilde{S}_t^3 e^{2rT+3T-3t}$

$b_t = -2\tilde{S}_t^3 e^{2rT+3T-3t} = -2S_t^3 e^{2rT+3T-3t-3rt}$ //

3.6) $V_t = a_t \cdot S_t + b_t \cdot R_t \rightarrow e^{rt}$

$V_t = 3S_t^3 e^{2rT+3T-3t-2rt} + (-2S_t^3)(e^{2rT+3T-3t-3rt}) \cdot e^{rt}$

$V_t = S_t^3 \exp\{2rT+3T-3t-2rt\}$ //

$2rT+3T-3t-2rt = 6.2 - t(3+2(0.05)) = 6.2-3.1t$

$V_t = S_t^3 \exp\{6.2-3.1t\}$

//

Exercise 4: S.g

4.1) $d\tilde{S}_t = e^{-0.05t} S_t \, dW_t$ // → SDE

4.2) $s \geq 0$ since $0 \leq s \leq 2$

$S_t \sim$ geometric Brownian motion → $S_t \geq 0$

$s \cdot S_t \geq 0$ for $0 \leq s \leq 2$

$V = \int_0^2 s \cdot S_s \, ds \geq 0$ → $V \geq 0$

$\tilde{V} = e^{-r \cdot 2} \int_0^2 s S_s \, ds$

$\tilde{V}^2 = e^{-4r} \left(\int_0^2 s S_s \, ds \right)^2$ ↗ finite number

$E_Q[\tilde{V}^2] = e^{-4r} \left(E_Q\left[\left(\int_0^2 s \cdot S_s \, ds \right)^2 \right] \right) < \infty$

Since $V \geq 0$ and $E_Q[\tilde{V}^2] < \infty$,

V is a contingent claim

4.3) $\tilde{V}_t = E_Q[\tilde{V}_T \mid F_t]$

$\tilde{V}_T = e^{-r.T} \cdot \int_0^T s \cdot S_s \, ds$

$\tilde{V}_t = E_Q\left[e^{rT} \cdot \int_0^T s.S_s \, ds\right] = e^{-r.T} E_Q\left[\int_0^T s.S_s \, ds \mid F_t\right]$

$e^{r.T}\tilde{V}_t = \int_0^t s.S_s \, ds + \int_t^T E_Q[s.S_s \mid F_t] \, ds$

martingale

$E_Q[s.S_s \mid F_t] = s.E_Q[\tilde{S}_s \cdot e^{r.s} \mid F_t] = s.e^{rs} E[\tilde{S}_s \mid F_t]$

$= s.e^{rs}.\tilde{S}_t$

$\int_t^T E_Q[s.S_s \mid F_t] = \int_t^T s.e^{rs}.\tilde{S}_t \, ds = \left(\frac{s}{r}e^{rs}\Big|_{s=t}^{s=T} - \int_0^T \frac{1}{r}e^{rs}.ds\right)\tilde{S}_t$

$u = s \quad du = ds \quad \Big| \quad dV = e^{rs} ds \quad V = \frac{1}{r}e^{rs}$

$= \left(\frac{T}{r}e^{Tr} - \frac{t}{r}e^{tr} - \frac{1}{r^2}e^{rs}\Big|_{s=t}^{s=T}\right)\tilde{S}_t$

$= \left(\frac{T}{r}e^{Tr} - \frac{t}{r}e^{tr} - \frac{1}{r^2}\left[e^{rT} - e^{rt}\right]\right)\tilde{S}_t$

$\tilde{V}_t = e^{-r.T}\int_0^t s.S_s \, ds + e^{-r.T}\tilde{S}_t\left[\frac{T}{r}e^{Tr} - \frac{t}{r}e^{tr} - \frac{1}{r^2}\left[e^{rT} - e^{rt}\right]\right]$

where $r = 0.05$, $T = 2$

4.4) $d\tilde{S}_t = \sigma.\tilde{S}_t \, dW_t = \tilde{S}_t \, dW_t$

ignoring first term,

$d\tilde{V}_t = e^{-r.T}\left[\frac{T}{r}e^{Tr} - \frac{t}{r}e^{tr} - \frac{1}{r^2}\left[e^{rT} - e^{rt}\right]\right]\tilde{S}_t \, dW_t$

//

$r = 0.05$, $T = 2$

d

4.5) $a_t = \frac{A_t}{\sigma_t \cdot \tilde{S}_t} = e^{-r \cdot T}\left\{\frac{T}{r}e^{Tr} - \frac{te^{tr}}{r} - \frac{1}{r^2}\left[e^{rT} - e^{rt}\right]\right\}$ //

$b_t = \tilde{V}_t - \frac{A_t}{\sigma_t} = e^{-rT} \cdot \int_0^t s \cdot S_s \, ds$ //

$r = 0.05, \quad T = 2$

4.6) $V_t = a_t S_t + \overbrace{R_t}^{e^{r \cdot t}} \cdot b_t \quad , R_0 = 1$

$V_t = S_t \cdot e^{-rT}\left\{\frac{T}{r}e^{tr} - \frac{te^{tr}}{r} - \frac{1}{r^2}\left[e^{rT} - e^{rt}\right]\right\} +$

$e^{rt} \cdot e^{-rT}\int_0^t s \cdot S_s \, ds$ //

$rT = (0.05)(2) = 0.1 \quad , \quad \frac{T}{r} = \frac{2}{0.05} = 40 \quad , \quad r^2 = 0.05^2$

$V_t = S_t \, e^{-0.1}\left\{40e^{0.1} - \frac{te^{0.05t}}{0.05} - \frac{1}{0.05^2}\left[e^{0.1} - e^{0.05t}\right]\right\} +$

$e^{0.05t - 0.1}\int_0^t s \, S_s \, ds$ //

Exercise 5

5.1) $z = e^{aN+y}$

$$\ln(z) = aN + y$$

$$aN = \ln(z) - y$$

$$N = \frac{-y+\ln(z)}{a} = -\frac{y}{a} + \frac{\ln(z)}{a}$$

$$\frac{d}{dz}N = \frac{1}{a}\cdot\frac{1}{z}$$

Source: https://ocw.mit.edu/courses/18-s096-topics-in-mathematics-with-applications-in-finance-fall-2013/94dc1d08ef12065198102ae23b2e4a8f_MIT18_S096F13_lecnote3.pdf

$$g(z) = \phi(N)\cdot\left(\frac{d}{dz}N\right) \therefore$$

$$g(z) = \frac{1}{az}\cdot\phi\left(\frac{-y+\ln(z)}{a}\right) \text{//}$$

5.2) $\int_x^\infty (z-x)\, g(z)\, dz = \int_x^\infty (z-x) \frac{1}{az} \cdot \phi\left(\frac{-y+\ln(z)}{a}\right) dz$

$$= \int_{z=x}^\infty \frac{1}{a} \phi(w)\, dz - \int_{z=x}^\infty \frac{x}{az} \cdot \phi(w)\, dz$$

$$w = \frac{-y+\ln(z)}{a} = -\frac{y}{a} + \frac{\ln(z)}{a}$$

$aw = -y + \ln(z)$

$z = e^{a \cdot w + y}$

$dz = a e^{aw+y} dw$

for $z = x$, $w = \frac{-y+\ln(x)}{a}$

$$\int_x^\infty (z-x) g(z)\, dz = \int_{\frac{\ln(x)-y}{a}}^\infty \frac{1}{a} \cdot e^{aw+y} \cdot a\, \phi(w)\, dw - \int \frac{x}{az} \cdot a e^{aw+y} \phi(w)\, dw$$

$$= \int e^{aw+y} \phi(w)\, dw - \int x \cdot \phi(w)\, dw$$

$$\int_{\frac{\ln(x)-y}{a}}^\infty x \cdot \phi(w)\, dw = x \cdot \left[1 - \Phi\left(\frac{\ln(x)-y}{a}\right)\right] = x \cdot \Phi\left(\frac{y-\ln(x)}{a}\right)$$

$$\int_b^\infty e^{aw} \cdot \phi(w)\, dw = e^{a^2/2} \cdot \Phi(-b+a)$$

$$\int_{b=\frac{\ln(x)-y}{a}}^\infty e^{aw+y} \cdot \phi(w)\, dw = e^{y} \cdot e^{a^2/2} \left(\Phi\left(\frac{y-\ln(x)}{a} + a\right)\right)$$

$$\int_x^\infty (z-x)\, g(z)\, dz = e^{y+\frac{a^2}{2}} \Phi\left(\frac{y-\ln(x)+a^2}{a}\right) - x \cdot \Phi\left(\frac{y-\ln(x)}{a}\right)$$

//

||

5.3) In example 5.5.1, we have:

$$z = e^{aN+y} \quad \text{where} \quad a = \sigma\sqrt{T-t}\ ,\ y = \ln(\tilde{S}_t) - \frac{a^2}{2}$$

the density of z: $g(z) = \frac{1}{az}\phi\left(\frac{-y+\ln(z)}{a}\right)$ + is correct

$$\tilde{V}_t = \int_{\tilde{K}}^{\infty} (z-\tilde{K})\, g(z)\, dz$$

From earlier, we have:

$$\int_x^{\infty} (z-x)\, g(z)\, dz = e^{y+\frac{a^2}{2}}\phi\left(\frac{y-\ln(x)+a^2}{a}\right) - x\,\phi\left(\frac{y-\ln(x)}{a}\right)$$

$$\tilde{V}_t = e^{\ln\tilde{S}_t - \frac{a^2}{2} + \frac{a^2}{2}} \cdot \phi\left(\frac{\ln(\tilde{S}_t) - \ln(\tilde{K}) - \frac{a^2}{2} + a^2}{a}\right) -$$

$$\tilde{K}\,\phi\left(\frac{\ln(\tilde{S}_t) - \ln(\tilde{K}) - a^2/2}{a}\right)$$

$$\tilde{V}_t = \tilde{S}_t\,\phi\left(\frac{\ln(\tilde{S}_t/\tilde{K}) + (a^2/2)}{a}\right) - \tilde{K}\,\phi\left(\frac{\ln(\tilde{S}_t/\tilde{K}) - a^2/2}{a}\right)$$

Which is also correctly stated in example 5.5.1