Homework 6 Stochastic - Matheus Raka

Exercise 1.1) (page 12 d)

$$T = \inf\{t : B_t = 0 \text{ or } B_t = 3\}$$

B_t is a standard Brownian Motion

$$E[B_0] = E[B_T] \rightarrow \text{optional Sampling theorem}$$

starts at x $\leftarrow$ $x = 0 \cdot P(B_T = 0) + (3) P(B_T = 3)$

$$P(B_T = 3) = \frac{x}{3} \quad //$$

$\rightarrow$ probability that Brownian motion reaches 3 before reaching 0

Exercise 1.2

$$X_t = x + \int_0^t B_s\, dB_s = x + \frac{1}{2}[B_t^2 - t] \quad \text{(page 104)}$$

$$d(X_t) = d\left(\frac{1}{2}[B_t^2 - t]\right) \quad , x \text{ is a constant}$$

$$f(t, B_t) = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

$$\dot{f} = -\frac{1}{2} \quad , f' = \frac{1}{2}\cdot 2\cdot B_t = B_t \quad , f'' = 1$$

$$d(X_t) = f'\, dB_t + \left[\dot{f} + \frac{1}{2}f''\right] dt \quad \rightarrow \text{Ito's formula II}$$

$$= B_t\, dB_t + \left(-\frac{1}{2} + \frac{1}{2}\cdot 1\right) dt$$

$$d(X_t) = B_t\, dB_t + 0\, dt$$

drift term is 0 $\rightarrow$ X_t is a local martingale.

Since B_t is a standard Brownian motion, X_t is a continuous martingale.

at $t=0$,

$$X_0 = x + \int_0^0 B_s\, dB_s = x + 0 = x$$

Since X_t is a continuous martingale, starting at x, we can apply the same method as 1.1

$$E[X_0] = E[X_t]$$

$$x = 0\cdot P(X_t = 0) + 3\cdot P(X_t = 3)$$

$$P(X_t = 3) = \frac{x}{3} \quad \rightarrow \text{Prob. that } X_t \text{ reaches 3 before reaching 0}$$

Exercise 2.1) page 131-132

for $T = T(r,R) = \min\{t : X_t = r \text{ or } X_t = R\}$

for $r \leq x \leq R$, $\phi(x) = P(X_T = R \mid X_0 = x)$

$$\phi(x) = \frac{x^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}}, \quad a \neq \frac{1}{2}$$

in our case, $X_0 = 1$, $a < 1/2$,

$$P(r,R) = P(X_T = R \mid X_0 = 1) = \frac{1^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}} = \frac{1 - r^{1-2a}}{R^{1-2a} - r^{1-2a}}$$

$$P(0,R) = \lim_{r \downarrow 0} P(r,R) = \lim_{r \to 0^+} P(r,R) = \frac{1-0}{R^{1-2a} - 0} = \frac{1}{R^{1-2a}}$$

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Exercise 2.2

$$\lim_{R \to \infty} P(0,R) = \lim_{R \to \infty} \frac{1}{R^{1-2a}} = \frac{1}{\infty^{1-2a}} = \frac{1}{\infty} = 0$$

the denominator diverges, the limit goes to 0 as $R \to \infty$

The probability that the Bessel process reaches ∞ is 0.
the Bessel process will eventually reach 0 with probability one..

Exercise 2.3) page 132

For $a = \frac{1}{2}$, $\phi(x) = P(X_T = R \mid X_0 = x) = \frac{\ln(x) - \ln(r)}{\ln(R) - \ln(r)}$

For $a = \frac{1}{2}$, $X_0 = 1$,

$$P(r,R) = \frac{\ln(1) - \ln(r)}{\ln(R) - \ln(r)} = \frac{-\ln(r)}{\ln(R) - \ln(r)}$$

$$P(0,R) = \lim_{r \downarrow 0} \frac{-\ln(r)}{\ln(R) - \ln(r)} = \frac{-(-\infty)}{\ln(R) - (-\infty)} = \frac{\infty}{\infty} \Big\} \text{indeterminate}$$

L'hopital's rule: if $\lim_{x \to a} \frac{f(x)}{g(x)}$ = indeterminate, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$$P(0,R) = \lim_{r \downarrow 0} \frac{-\ln(r)}{\ln(R) - \ln(r)} = \lim_{r \to 0^+} \frac{-1/r}{-1/r} = \lim_{r \downarrow 0} \frac{1/r}{1/r} = 1$$

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here, $r \to 0^+$, $T = \min\{t : X_t = r \text{ or } R\}$

The process starts at 1 ($X_0 = 1$). The lowest possible value of X_t is 0. The process can hit 0 or R, and the probability that $X_T = R$ is 1 ($P(0,R) = 1$). This means that X_t is always positive.

With probability one, $X_t > 0$ for all t.

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Homework 6 – Stochastic Calculus

Exercise 2.4

For $a = \frac{1}{2}, X_0 = 1,$

$$\rho(r,\infty) = \lim_{R\to\infty} \rho(r,R) = \lim_{R\to\infty} \frac{\ln(1) - \ln(r)}{\ln(R) - \ln(r)} = \lim_{R\to\infty} \frac{-\ln(r)}{\ln(R) - \ln(r)} = \frac{-\ln(r)}{\infty} \to 0$$

$$\rho(r,\infty) = \lim_{R\to\infty} \rho(r,R) \to 0$$

As $R \to \infty$, the denominator diverges, and the limit goes to 0.

This means that for

$$T = T_{r,R} = \min\{t: X_t = r \; or \; R\}$$

The probability that $X_t = \infty$ is 0.

$$\rho(r,\infty) = 0$$

which means that X_t will never reach ∞.

This also means that process X_t will be below r for some t and it can also be above r for some t.

With probability one, for all $r > 0, X_t < r$ for some t.

Exercise 3.11

$f(t, B_t) = X_t = 2e^{B_t} \cdot e^{-t}$

$\dot{f}(t, B_t) = -1 \cdot 2e^{B_t - t}$

$f'(t, B_t) = f''(t, B_t) = 2e^{B_t - t}$. Using Ito's formula II

$$dX_t = f'(t, B_t)\,dB_t + \left[\dot{f}(t, B_t) + \frac{1}{2} f''(t, B_t)\right] dt$$

$$= \left[-2e^{B_t - t} + e^{B_t - t}\right] dt + 2e^{B_t - t}\, dB_t$$

$$dX_t = -e^{B_t - t}\, dt + 2e^{B_t - t}\, dB_t = -\frac{X_t}{2}\, dt + X_t\, dB_t$$

$m(X_t) = -\frac{X_t}{2}$ //

$\sigma(X_t) = X_t$ //

these functions do not depend on t. therefore, X_t is a time-homogeneous diffusion

Exercise 3.2

$$m(x) = -\frac{1}{2}x, \quad \sigma(x) = x, \quad r = 2$$

Feynman-Kac formula: page 135

$$\dot{\phi}(t,x) = -m(t,x)\,\phi'(t,x) - \frac{1}{2}(\sigma(t,x))^2\phi''(t,x) + r(t,x)\,\phi(t,x)$$

$$\dot{\phi}(t,x) = \frac{x}{2}\cdot\phi'(t,x) - \frac{1}{2}x^2\phi''(t,x) + 2\,\phi(t,x)$$ → PDE

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for $0 \leqslant t \leqslant T$, with terminal condition $\phi(T,x) = (x-3)_+$

↳ $F(x)$

Exercise 3.3

PDE:

$$\dot{\phi}(t,x) = \frac{x}{2}\phi'(t,x) - \frac{1}{2}x^2\phi''(t,x) + 2\,\phi(t,x)$$

for $0 \leqslant t \leqslant T$, with terminal condition $\phi(T,x) = x^2 e^{-x}$

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Exercise 4.1 Page 24: martingale M_n is square integrable if for each n, $E[M_n^2] < \infty$

Page 89: Variance Rule: $Var[Z_t] = E[Z_t^2] = \int_0^t E[A_s^2]\,ds$

$\alpha = 1/4$

$$E[Z_t^2] = \int_0^t \frac{1}{(1-s)^{1/4 \cdot 2}} ds = \int_0^t \frac{1}{(1-s)^{1/2}} ds = \int_0^t \frac{1}{u^{1/2}} \cdot (-du) = -\int u^{-1/2} du$$

$u = 1-s$

$du = -ds$

$ds = -du$

$$E(Z_t^2) = -\frac{1}{1/2} u^{1/2} \Big|_{s=0}^{s=t} = -2(1-s)^{1/2} \Big|_0^t = -2\left[(1-t)^{1/2} - 1\right]$$

$$E(Z_t^2) = 2\left[1-(1-t)^{1/2}\right] < \infty \text{ for } 0 \leq t < 1$$

Since $E(Z_t^2) < \infty$ for $0 \leq t < 1$, Z_t is square integrable martingale

$$\left.\begin{array}{l} \text{For } t=1,\ E(Z_t^2) = 2[1-0] = 2 \\ \text{For } t=0,\ E[Z_t^2] = 2[1-1^{1/2}] = 0 \end{array}\right\} 0 \leq E(Z_t^2) \leq 2 < \infty$$

$$Var(Z_t) = E[Z_t^2] \leq 2 < \infty$$

$$Var(Z_t) \leq 2 \rightarrow C = 2$$

Exercise 4.2

$\alpha = 1$

$Z_t = \int_0^t \frac{1}{1-s} dB_s$ $\quad \begin{cases} u = 1-s \\ du = -ds \\ ds = -du \end{cases}$

$E[Z_t^2] = \int \frac{1}{(1-s)^2} ds = \int u^{-2}(-du) = -\int u^{-2} du = u^{-1} \Big|_{s=0}^{s=t}$

$E[Z_t^2] = \frac{1}{1-s} \Big|_{s=0}^{s=t} = \frac{1}{1-t} - \frac{1}{1} = \frac{1}{1-t} - 1$

For $t=0$: $E[Z_t^2] = 1-1 = 0$

For $t = 0.999$: $E[Z_t^2] = \frac{1}{1-0.999} - 1 = 999$

as $t \to 1^-$, $E[Z_t^2] \to \infty$, meaning Z_t is not square integrable here

Since $E[Z_t^2]$ is between 0 and ∞ for $0 \leq t < 1$,
then with probability one, there will exist $t < 1$ with $Z_t = 1$

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